Minors and Categorical Resolutions

Igor Burban, Yuriy Drozd, and Volodymyr Gavran

Abstract We define minors of non-commutative schemes and study their properties. It is then applied to the study of a special class of non-commutative schemes, called quasi-hereditary, and to a construction of categorical resolutions for singular curves (maybe, non-commutative). In the rational case, this categorical resolution is realized by a finite dimensional quasi-hereditary algebra.

Keywords Bilocalization • Categorical resolution • Derived categories • Minors • Non-commutative schemes • Quasi-hereditary schemes

1 Introduction

When one compares the category of representations of the *Kronecker quiver* $\bullet \rightrightarrows \bullet$ ("matrix pencils") and the category of coherent sheaves over the projective line $\mathbb{P}^1$, one sees an astonishing resemblance. Indeed, the Auslander–Reiten quiver (describing the subcategory of indecomposable objects) of the first category looks like

$$\underbrace{\bullet \overset{x}{\underset{y}{\rightrightarrows}} \bullet \overset{x}{\underset{y}{\rightrightarrows}} \bullet \rightrightarrows \cdots}_{\text{Preprojective component}} \quad \underbrace{\boxed{\begin{array}{l}\text{Tubular family}\\ \{\,\mathcal{T}_\lambda \mid \lambda \in \mathbb{P}^1(\Bbbk)\,\}\end{array}}}_{\text{Regular part}} \quad \underbrace{\cdots \rightrightarrows \bullet \overset{x}{\underset{y}{\rightrightarrows}} \bullet \overset{x}{\underset{y}{\rightrightarrows}} \bullet}_{\text{Preinjective component}}$$

I. Burban
Mathematical Institute of the University of Cologne, Weyertal 86-90, 50931 Köln, Germany
e-mail: burban@math.uni-koeln.de

Y. Drozd (✉) • V. Gavran
Institute of Mathematics of the National Academy of Sciences of Ukraine, Tereschenkivska 3, 01601 Kiev, Ukraine
e-mail: y.a.drozd@gmail.com; vlgvrn@gmail.com

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while that of the second is

$$\underbrace{\cdots \rightrightarrows \bullet \overset{x}{\underset{y}{\rightrightarrows}} \bullet \overset{x}{\underset{y}{\rightrightarrows}} \bullet \rightrightarrows \cdots}_{\text{Vector bundles}} \quad \underbrace{\boxed{\begin{array}{l}\text{Tubular family}\\ \{\,\mathcal{T}_\lambda \mid \lambda \in \mathbb{P}^1(\Bbbk)\,\}\end{array}}}_{\text{Skyscraper sheaves}}$$

both with relations $xy = yx$. The *tubular family* here means a set of disjoint subcategories $\mathcal{T}_\lambda$ (*tubes*) parametrized by the points of the projective line and such that every $\mathcal{T}_\lambda$ is equivalent to the category of indecomposable finite dimensional modules over the algebra of formal power series $\Bbbk[[t]]$. Except the products of arrows, there are only morphisms "from the left to the right," also similar in both cases. Note that if we move the preinjective component of the first quiver to the very left and join it with the preprojective component by the arrows $\rightrightarrows$, we obtain the second quiver.

This resemblance has now a rather simple explanation. Namely, the vector bundle $\mathcal{G} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ is a so-called *tilting object* of the category $\operatorname{Coh}\mathbb{P}^1$. It means that $\operatorname{Ext}^i_{\mathbb{P}^1}(\mathcal{G},\mathcal{G}) = 0$ for all $i > 0$ and, for every nonzero morphism $f : \mathcal{F} \to \mathcal{F}'$ of coherent sheaves, $\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{G}, f) \neq 0$ (equivalently, $\mathcal{G}$ generates the derived category $\mathscr{D}(\operatorname{Coh}\mathbb{P}^1)$). Then it is known that the derived functor $\operatorname{RHom}_{\mathbb{P}^1}(\mathcal{G}, \_)$ establishes an equivalence of the derived categories $\mathscr{D}(\operatorname{Coh}\mathbb{P}^1)$ and $\mathscr{D}(\boldsymbol{A}\text{-mod})$, where $\boldsymbol{A} = \operatorname{End}(\mathcal{G})^{\mathrm{op}}$. In our case, $\boldsymbol{A}$ is just the path algebra of the Kronecker quiver. Moreover, since both categories $\boldsymbol{A}$-mod and $\operatorname{Coh}\mathbb{P}^1$ are *hereditary* (i.e., of global dimension 1), every indecomposable object of the derived category is just a shift of a module.

Actually, Beilinson [3] proved that the category $\operatorname{Coh}\mathbb{P}^n$ has a tilting sheaf $\mathcal{G} = \bigoplus_{i=-n}^{0} \mathcal{O}_{\mathbb{P}^n}(i)$, hence is equivalent to the category of representations of the finite dimensional algebra $\boldsymbol{A} = \operatorname{End}(\mathcal{G})^{\mathrm{op}}$, which can be explicitly described. Afterwards analogues of these results were proved for a wide class of projective varieties. In particular, Hille and Perling [21] constructed a tilting vector bundle for any smooth rational surface.

Later on Greuel and the second author [18] noticed that there is a resemblance between the categories of vector bundles over a class of singular curves and the categories of representations of some finite dimensional algebras. In particular, it is so for a nodal cubic C and the algebra $\boldsymbol{A}$ with the quiver

$$\bullet \overset{a}{\underset{b}{\rightrightarrows}} \bullet \overset{c}{\underset{d}{\rightrightarrows}} \bullet$$

and relations $da = cb = 0$. Certainly, this correspondence could not be a corollary of an equivalence of derived categories, since the algebra $\boldsymbol{A}$ is of global dimension 2, while $\mathcal{O}_C$ is of infinite global dimension. An explanation of this phenomenon was given by the first two authors [7]. For this purpose they considered a sheaf of non-commutative algebras $\mathcal{A}$ (called *Auslander sheaf*), which was already of global dimension 2, and constructed a tilting sheaf over $\mathcal{A}$ such that its endomorphism algebra was just the algebra mentioned in [18]. The category of coherent sheaves over the initial curve turned to be a *Serre quotient* of the category of coherent sheaves over $\mathcal{A}$ by a semi-simple subcategory; hence, their indecomposable objects were almost the same.

This paper is devoted to a generalization of the results of Burban and Drozd [7] to all singular curves. Namely, we construct for every curve X a sheaf of $\mathcal{O}_X$-algebras $\mathcal{R}$, such that $\mathcal{R}$ is of finite global dimension and there is a functor $\mathsf{F} : \operatorname{Coh}\mathcal{R} \to \operatorname{Coh}X$, which defines $\operatorname{Coh}X$ as a *bilocalization* (i.e., both localization and colocalization) of $\operatorname{Coh}\mathcal{R}$. The same is certainly true for their derived categories. Moreover, $\mathcal{R}$ has rather special properties analogous to those of *quasi-hereditary* algebras from [12, 14]. We call $\mathcal{R}$ the *König's resolution* of the curve X, since the idea of its construction goes back to the König's paper [23]. If X is rational, $\mathcal{R}$ has a tilting complex $\mathcal{T}$ which establishes an equivalence between the derived category of $\operatorname{Coh}\mathcal{R}$ and that of a finite dimensional quasi-hereditary algebra. Altogether, this construction can be considered as a *categorical resolution* of the category $\mathscr{D}(\operatorname{Coh}X)$ in the sense of [24]. If the curve X is Gorenstein, this categorical resolution is *weakly crepant* in the sense of [24]. We also show that this construction can also be applied to *non-commutative curves*.

The main tool in our considerations is the notion of *minors* of non-commutative schemes studied in Sect. 3. For the affine case (i.e., for rings), it was introduced in [15]. A minor $\mathcal{B}$ of a sheaf of algebras $\mathcal{A}$ is the endomorphism sheaf of a locally projective sheaf of $\mathcal{A}$-modules. Then the category $\operatorname{Qcoh}\mathcal{B}$ is a bilocalization of $\operatorname{Qcoh}\mathcal{A}$ and the same is true for their derived categories. We establish the main features of these bilocalizations and specialize them to the most important case arising as *endomorphism construction* (Example 3.14). The general properties of localizations and colocalizations used here are gathered in Sect. 2. In Sect. 4 we apply this technique to a special class of non-commutative schemes called *quasi-hereditary*. This notion generalizes that of quasi-hereditary algebras and has a lot of similar features. In particular, a quasi-hereditary non-commutative scheme is always of finite global dimension, and its derived category has good semi-orthogonal decompositions (see Corollary 4.23). In Sect. 5 we study some general properties of non-commutative curves and their minors, especially related with Cohen–Macaulay (or, the same, torsion-free) modules. In Sect. 6 we construct the König's resolution and prove that it is quasi-hereditary. We also show that in the commutative case, the functors of direct and inverse image arising from the normalization of the curve are actually compositions of the functors arising from the König's resolution. Finally, in Sect. 7 we construct a tilting complex for the König's resolution of a rational singular curve (maybe, non-commutative). It gives a categorical resolution of $\mathscr{D}(\operatorname{Qcoh}X)$ by a quasi-hereditary finite dimensional algebra. We also consider, as an example, the case when all singularities of a curve are of ADE types in the sense of Arnold [2].

Most results of Sects. 1–5 are contained in [8]. Sections 6 and 7 generalize the results of [9] to the non-commutative situation.

2 Bilocalizations

We recall here some general facts concerning localizations and bilocalizations of abelian and triangular categories. Their proofs are gathered in [8, Sect. 2].

Theorem 2.1 *Let* $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ *be an exact functor between abelian categories,* $\mathsf{F}^* : \mathcal{B} \to \mathcal{A}$ *be its right (left) adjoint such that the natural morphism* $\mathsf{FF}^* \to \mathbb{1}_{\mathscr{B}}$ *(respectively,* $\mathbb{1}_{\mathscr{B}} \to \mathsf{FF}^*$*) is an isomorphism. Let* $\mathscr{C} = \ker \mathsf{F}$.

1. $\mathscr{C}$ *is a thick subcategory in* $\mathscr{A}$ *and* $\mathsf{F} = \bar{\mathsf{F}}\Pi_{\mathscr{C}}$, *where* $\Pi_{\mathscr{C}} : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ *is the natural functor to the Serre quotient and* $\bar{\mathsf{F}} : \mathscr{A}/\mathscr{C} \to \mathscr{B}$ *is an equivalence. The quasi-inverse functor to* $\bar{\mathsf{F}}$ *is* $\Pi_{\mathscr{C}}\mathsf{F}^*$.
2. F^* *is a full embedding and its essential image* $\operatorname{Im}\mathsf{F}^*$ *coincides with the right (respectively, left) orthogonal subcategory to* $\mathscr{C}$, *i.e., the full subcategory*

$$\mathscr{C}^{\perp} = \left\{ A \in \operatorname{Ob}\mathscr{A} \mid \operatorname{Hom}_{\mathscr{A}}(C, A) = \operatorname{Ext}^1_{\mathscr{A}}(C, A) = 0 \text{ for all } C \in \mathscr{C} \right\}$$

(respectively,

$${}^{\perp}\mathscr{C} = \left\{ A \in \operatorname{Ob}\mathscr{A} \mid \operatorname{Hom}_{\mathscr{A}}(A, C) = \operatorname{Ext}^1_{\mathscr{A}}(A, C) = 0 \text{ for all } C \in \mathscr{C} \right\}).$$

3. $\mathscr{C} = {}^{\perp}(\mathscr{C}^{\perp})$ *(respectively,* $\mathscr{C} = ({}^{\perp}\mathscr{C})^{\perp}$*).*
4. *The embedding functor* $\mathsf{I} : \mathscr{C} \to \mathscr{A}$ *has a right (respectively, left) adjoint.*

In this case they say that F is a *localizing functor*, $\mathscr{C}$ is a *localizing subcategory*, and $\mathscr{B} \simeq \mathscr{A}/\mathscr{C}$ is a *localization* of the category $\mathscr{A}$ (respectively, F is a *colocalizing functor*, $\mathscr{C}$ is a *colocalizing subcategory*, and $\mathscr{B} \simeq \mathscr{A}/\mathscr{C}$ is a *colocalization* of the category $\mathscr{A}$).

Theorem 2.2 *Let* $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ *be an exact functor between triangulated categories,* $\mathsf{F}^* : \mathcal{B} \to \mathcal{A}$ *be its right (left) adjoint such that the natural morphism* $\mathsf{FF}^* \to \mathbb{1}_{\mathscr{B}}$ *(respectively,* $\mathbb{1}_{\mathscr{B}} \to \mathsf{FF}^*$*) is an isomorphism. Let* $\mathscr{C} = \ker \mathsf{F}$.

1. $\mathscr{C}$ *is a thick subcategory in* $\mathscr{A}$ *and* $\mathsf{F} = \bar{\mathsf{F}}\Pi_{\mathscr{C}}$, *where* $\Pi_{\mathscr{C}} : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ *is the natural functor to the Verdier quotient and* $\bar{\mathsf{F}} : \mathscr{A}/\mathscr{C} \to \mathscr{B}$ *is an equivalence. The quasi-inverse functor to* $\bar{\mathsf{F}}$ *is* $\Pi_{\mathscr{C}}\mathsf{F}^*$.
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3. $\mathscr{C} = {}^{\perp}(\mathscr{C}^{\perp})$ *(respectively,* $\mathscr{C} = ({}^{\perp}\mathscr{C})^{\perp}$*).*
4. *The embedding functor* $\mathsf{I} : \mathscr{C} \to \mathscr{A}$ *has a right (respectively, left) adjoint, which induces an equivalence* $\mathscr{A} / \operatorname{Im} \mathsf{F}^* \xrightarrow{\sim} \mathscr{C}$*.*[1]

In this case they say that F is a *localizing functor*, $\mathscr{C}$ is a *localizing subcategory*, and $\mathscr{B} \simeq \mathscr{A}/\mathscr{C}$ is a *localization* of the category $\mathscr{A}$ (respectively, F is a *colocalizing functor*, $\mathscr{C}$ is a *colocalizing subcategory*, and $\mathscr{B} \simeq \mathscr{A}/\mathscr{C}$ is a *colocalization* of the category $\mathscr{A}$).

Theorem 2.3 *Suppose that an exact functor* $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ *between abelian (or triangulated) categories has both left adjoint* F^* *and right adjoint* $\mathsf{F}^!$*. Then* F *is a localizing functor if and only if it is a colocalizing functor.*

In this case we say that F is a *bilocalizing functor*, its kernel $\mathscr{C} = \ker \mathsf{F}$ is a *bilocalizing subcategory*, and $\mathscr{B}$ is a *bilocalization* of the category $\mathscr{A}$.

If $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ is an exact functor between abelian categories, we denote by DF the functor between the derived categories $\mathscr{D}\mathscr{A} \to \mathscr{D}\mathscr{B}$ which acts on complexes componentwise. It is both right and left derived functor of F.

Theorem 2.4 *Let* $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ *be a localizing (colocalizing) functor between abelian categories,* $\mathscr{C} = \ker \mathsf{F}$*. Suppose that right (left) adjoint* F^* *of* F *has right (respectively, left) derived functor. Then* DF *is also a localizing (respectively, colocalizing) functor,* RF^* *is its right adjoint (respectively,* LF^* *is its left adjoint),*

$$\ker \mathsf{DF} = \mathscr{D}_{\mathscr{C}}\mathscr{A} = \{ \mathcal{F}^\bullet \in \mathscr{D}\mathscr{A} \mid H^n(\mathcal{F}^\bullet) \in \mathscr{C} \text{ for all } n \}$$

and $\mathscr{D}\mathscr{B} \simeq \mathscr{D}\mathscr{A} / \mathscr{D}_{\mathscr{C}}\mathscr{A}$*.*

Remark 2.5 If $\mathscr{A}$ is a Grothendieck category, a right derived functor always exists, so Theorem 2.4 can always be applied. We do not know any natural "categorical" conditions for the existence of a left adjoint, though it is the case in the situation that we consider nearby.

We recall that a *semi-orthogonal* decomposition $\langle \mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_m \rangle$ of a triangulated category $\mathscr{A}$ is a sequence of subcategories $(\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_m)$ such that

1. $\operatorname{Hom}_{\mathscr{A}}(A, B[m]) = 0$ if $A \in \mathscr{T}_i$, $B \in \mathscr{T}_j$ and $i > j$.
2. For every $A \in \mathscr{A}$, there is a sequence of morphisms

$$0 = T_m \xrightarrow{f_m} T_{m-1} \xrightarrow{f_{m-1}} \dots T_2 \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 = A$$

such that $\operatorname{cone}(f_i) \in \mathscr{T}_i$ $(1 \le i \le m)$ [25].

[1] In the case of abelian categories the functor $\mathscr{A} / \operatorname{Im} \mathsf{F}^* \to \mathscr{C}$ induced by the right (respectively, left) adjoint of I need not be an equivalence.

In particular, if $m = 2$, it means that there is an exact triangle $T_2 \to A \to T_1$, where $T_1 \in \mathscr{T}_1$, $T_2 \in \mathscr{T}_2$.

Corollary 2.6 *Let* $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ *be a localizing (colocalizing) functor between triangulated categories,* F^* *be its right (respectively, left) adjoint. There is a semi-orthogonal decomposition* $(\operatorname{Im}\mathsf{F}^*, \ker\mathsf{F})$ *(respectively,* $(\ker\mathsf{F}, \operatorname{Im}\mathsf{F}^*)$*) of the category* $\mathscr{A}$.

3 Minors

In this paper a *non-commutative scheme* is a pair $(X, \mathcal{A})$, where X is a scheme (called the *commutative background* of the non-commutative scheme) and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras, which is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules. Sometimes we say "non-commutative scheme $\mathcal{A}$" not mentioning its commutative background X. We denote by X_{cl} the set of closed points of X. If $\mathcal{A} = \mathcal{O}_X$, we sometimes say that it is a *usual scheme*. We denote by $\mathcal{A}$-Mod (respectively, by $\mathcal{A}$-mod) the category of quasi-coherent (respectively, coherent) sheaves of $\mathcal{A}$-modules. We call objects of this category just $\mathcal{A}$-modules (respectively, coherent $\mathcal{A}$-modules).

A non-commutative scheme $(X, \mathcal{A})$ is said to be *affine* (*separated, quasi-compact*) if so is its commutative background X. It is said to be *reduced* if $\mathcal{A}$ has no nilpotent ideals. If X is noetherian and $\mathcal{A}$ is a coherent $\mathcal{O}_X$-module, we say that this non-commutative scheme is *noetherian*. We say that $(X, \mathcal{A})$ is *quasi-projective* if there is an ample $\mathcal{O}_X$-module $\mathcal{L}$. Note that then X is indeed a quasi-projective scheme over the ring $\boldsymbol{R} = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes n})$. In this paper we always suppose that the considered non-commutative schemes are **separated** and **quasi-compact**. In this case $\mathcal{A}$-Mod is a Grothendieck category. In particular, every quasi-coherent $\mathcal{A}$-module has an injective envelope. We denote by $\mathcal{A}$-Inj the full subcategory of $\mathcal{A}$-Mod formed by injective modules.

A *morphism* of non-commutative schemes $f : (Y, \mathcal{B}) \to (X, \mathcal{A})$ is a pair $(f_X, f^\#)$, where $f_X : Y \to X$ is a morphism of schemes and $f^\#$ is a morphism of $f_X^{-1}\mathcal{O}_X$-algebras $f_X^{-1}\mathcal{A} \to \mathcal{B}$. In what follows we usually write f instead of f_X. Such morphism defines the functor of inverse image $f^* : \mathcal{A}\text{-Mod} \to \mathcal{B}\text{-Mod}$ which maps an $\mathcal{A}$-module $\mathcal{M}$ to the $\mathcal{B}$-module $\mathcal{B} \otimes_{f^{-1}\mathcal{A}} f^{-1}\mathcal{M}$. As the map f_X is separated and quasi-compact, the functor of direct image $f_* : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$ is also well-defined (cf. [20, § 0.1 and § 1.9.2]). Moreover, f^* maps coherent modules to coherent ones. Note that f^* and f_* do not coincide with $(f_X)^*$ and $(f_X)_*$. It is guaranteed only if $\mathcal{B} = f_X^*\mathcal{A}$, for instance, if Y is an open subset of X and $\mathcal{B} = \mathcal{A}|_Y$.

We call a non-commutative scheme $(X, \mathcal{A})$ *central* if $\operatorname{center}(\mathcal{A}) = \mathcal{O}_X$. Actually, we can only consider central non-commutative schemes as the following evident results show.

Proposition 3.7 *Let* $\mathcal{C} = \operatorname{center}(\mathcal{A})$, $\tilde{X} = \operatorname{spec}\mathcal{C}$, $\nu_X : \tilde{X} \to X$ *be the corresponding affine morphism, and* $\tilde{\mathcal{A}} = \nu_X^{-1}(\mathcal{A})$. *Then* ν_X *extends to a morphism* $\nu : (\tilde{X}, \tilde{\mathcal{A}}) \to (X, \mathcal{A})$ *and* ν_* *induces equivalences* $\tilde{\mathcal{A}}\text{-Mod} \to \mathcal{A}\text{-Mod}$ *and* $\tilde{\mathcal{A}}\text{-mod} \to \mathcal{A}\text{-mod}$.

We denote by $\mathsf{lp}\,\mathcal{A}$ the full subcategory of $\mathcal{A}$-mod consisting of *locally projective* modules, i.e., such coherent modules $\mathcal{P}$ that all localizations $\mathcal{P}_x$ are projective $\mathcal{A}_x$-modules. We say that $\mathcal{A}$ *has enough locally projective modules* if for every coherent $\mathcal{A}$-module $\mathcal{M}$, there is an epimorphism $\mathcal{P} \to \mathcal{M}$, where $\mathcal{P}$ is locally projective. It is the case, for instance, if the non-commutative scheme is quasi-projective.

We denote by $\mathscr{D}\mathcal{A}$ the derived category $\mathscr{D}(\mathcal{A}\text{-Mod})$, with subscripts $^{+,-,b}$ denoting its full subcategories consisting, respectively, of left-, right-, and two-sided bounded complexes. We also denote by $\mathsf{Perf}\mathcal{A}$ the full subcategory of *small* objects from $\mathscr{D}\mathcal{A}$, i.e., such complexes $\mathcal{F}^\bullet$ that $\mathrm{Hom}_{\mathscr{D}\mathcal{A}}(\mathcal{F}^\bullet, \coprod_i \mathcal{G}_i^\bullet) \simeq \coprod_i \mathrm{Hom}_{\mathscr{D}\mathcal{A}}(\mathcal{F}^\bullet, \mathcal{G}_i^\bullet)$ for any coproduct $\coprod_i \mathcal{G}_i^\bullet$. As X is separated and quasi-compact, small objects in $\mathscr{D}\mathcal{A}$ are just *perfect complexes*, i.e., complexes $\mathcal{F}^\bullet$ such that for every $x \in X$ the complex $\mathcal{F}_x$ is isomorphic to a finite complex of locally projective coherent modules. Moreover, $\mathsf{Perf}\mathcal{A}$ generates $\mathscr{D}\mathcal{A}$, i.e., for every complex $\mathcal{G}^\bullet$, there is a nonzero morphism from a perfect complex to $\mathcal{G}^\bullet$. It is well-known in affine and commutative cases and the proof in general case is quite analogous [8, Theorem 3.14].

Definition 3.8 Let $\mathcal{P}$ be a locally projective $\mathcal{A}$-module, $\mathcal{B} = (\mathcal{E}nd_{\mathcal{A}}\mathcal{P})^{\mathrm{op}}$. We call the non-commutative scheme $(X, \mathcal{B})$ a *minor* of the non-commutative scheme $(X, \mathcal{A})$.

This notion is just a globalization of the corresponding notion from [15].

We consider $\mathcal{P}$ as *right* $\mathcal{B}$-module and denote $\mathcal{P}^\vee = \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \mathcal{A})$; it is a right $\mathcal{A}$-module. It is known that for every $\mathcal{P} \in \mathsf{lp}\,\mathcal{A}$, the natural map $\mathcal{P} \to \mathcal{P}^{\vee\vee}$ is an isomorphism and $\mathcal{E}nd_{\mathcal{A}}\mathcal{P}^\vee \simeq \mathcal{E}nd_{\mathcal{A}}\mathcal{P} \simeq \mathcal{P}^\vee \otimes_{\mathcal{A}} \mathcal{P}$. The following functors play the crucial role in this paper:

$$
\begin{aligned}
\mathsf{F} &= \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \_) \simeq \mathcal{P}^\vee \otimes_{\mathcal{A}} \_ : \mathcal{A}\text{-Mod} \to \mathcal{B}\text{-Mod},\\
\mathsf{F}^* &= \mathcal{P} \otimes_{\mathcal{B}} \_ : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}, \qquad (1)\\
\mathsf{F}^! &= \mathcal{H}om_{\mathcal{B}}(\mathcal{P}^\vee, \_) : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}.
\end{aligned}
$$

The functor F is exact and both $(\mathsf{F}^*, \mathsf{F})$ and $(\mathsf{F}, \mathsf{F}^!)$ are adjoint pairs of functors. If the non-commutative scheme $(X, \mathcal{A})$ is noetherian, so is also $(X, \mathcal{B})$ and the functors $\mathsf{F}, \mathsf{F}^*, \mathsf{F}^!$ map coherent sheaves to coherent. Note that if $\mathcal{I}$ is an injective $\mathcal{B}$-module, then $\mathsf{F}^!(\mathcal{I})$ is an injective $\mathcal{A}$-module. We denote by $\mathcal{P}$-Inj the image $\mathsf{F}^!(\mathcal{B}\text{-Inj})$. We also denote by $\mu_{\mathcal{P}}$ the natural map $\mathcal{P} \otimes_{\mathcal{B}} \mathcal{P}^\vee \to \mathcal{A}$ such that $\mu(p \otimes f) = f(p)$ and $\mathcal{I}_P = \mathrm{Im}\,\mu_{\mathcal{P}}$. If $\mathcal{P} = \mathcal{A}e$, where e is an idempotent, then $\mathcal{P}^\vee \simeq e\mathcal{A}$, $(\mathcal{E}nd_{\mathcal{A}}\mathcal{P})^{\mathrm{op}} \simeq e\mathcal{A}e$ and $\mathcal{I}_P = \mathcal{A}e\mathcal{A}$.

The following result plays a crucial role in this paper:

Theorem 3.9

1. F *is a bilocalizing functor and its kernel* $\mathscr{C} = \ker \mathsf{F}$ *consists of the modules* $\mathcal{M}$ *such* $\mathcal{I}_P\mathcal{M} = 0$*, so can be identified with* $\mathcal{A}/\mathcal{I}_{\mathcal{P}}$-Mod.

2. $\operatorname{Im}\mathsf{F}^* = {}^{\perp}\mathscr{C}$ *consists of all* $\mathcal{A}$*-modules* $\mathcal{M}$ *such that for every* $x \in X$ *there is an exact sequence* $\mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{M} \to 0$*, where* $\mathcal{P}_0$ *and* $\mathcal{P}_1$ *are multiples (maybe infinite) of* $\mathcal{P}_x$*. We denote this subcategory by* $\mathcal{P}$-Mod*.*
3. $\operatorname{Im}\mathsf{F}^! = \mathscr{C}^{\perp}$ *consists of all* $\mathcal{A}$*-modules* $\mathcal{M}$ *such that there is an exact sequence* $0 \to \mathcal{M} \to \mathcal{I}_0 \to \mathcal{I}_1$*, where* $\mathcal{I}_0$ *and* $\mathcal{I}_1$ *belong to* $\mathcal{P}$-Inj*. We denote this subcategory by* $\mathcal{P}^{\mathrm{Inj}}$-Mod*.*

Proof The results of the preceding section show that it is enough to prove the following statements:

Proposition 3.10

1. *The natural morphism* $\phi : \mathbb{1}_{\mathcal{B}\text{-Mod}} \to \mathsf{FF}^*$ *is an isomorphism.*
2. $\operatorname{Im}\mathsf{F}^* = \mathcal{P}$-Mod.
3. $\operatorname{Im}\mathsf{F}^! = \mathcal{P}^{\mathrm{Inj}}$-Mod.
4. $\ker\mathsf{F} = \{\,\mathcal{M} \mid \mathcal{I}_P\mathcal{M} = 0\,\}$.

Proof Evidently, all claims are local, so we can suppose that $X = \operatorname{spec}\boldsymbol{R}$ for a commutative ring $\boldsymbol{R}$; $\mathcal{A} = \boldsymbol{A}^{\sim}$ is the sheafification of an $\boldsymbol{R}$-algebra $\boldsymbol{A}$, $\mathcal{P} = P^{\sim}$, where P is a finitely generated projective $\boldsymbol{A}$-module; and $\mathcal{B} = \boldsymbol{B}^{\sim}$, where $\boldsymbol{B} = \operatorname{End}_{\boldsymbol{A}} P$. Then we can replace $\mathcal{A}$-Mod, $\mathcal{B}$-Mod, and $\mathcal{P}$-Mod, respectively, by $\boldsymbol{A}$-Mod, $\boldsymbol{B}$-Mod, and P-Mod, where P-Mod is the full subcategory of $\boldsymbol{A}$-Mod consisting of all modules M such that there is an exact sequence $P_1 \to P_0 \to M \to 0$, where P_0 and P_1 are multiples of P.

Certainly, $\phi(\boldsymbol{B})$ is an isomorphism. Hence $\phi(F)$ is an isomorphism for every free $\boldsymbol{B}$-module F. For every $\boldsymbol{B}$-module M, there is an exact sequence $F_1 \to F_1 \to M \to 0$ with free modules F_0, F_1. It induces a commutative diagram with exact rows

$$\begin{array}{ccccccc}
F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
{\scriptstyle \phi(F_1)}\big\downarrow & & {\scriptstyle \phi(F_0)}\big\downarrow & & {\scriptstyle \phi(M)}\big\downarrow & & \\
\mathsf{FF}^*(F_1) & \longrightarrow & \mathsf{FF}^*(F_0) & \longrightarrow & \mathsf{FF}^*(M) & \longrightarrow & 0
\end{array}$$

As $\phi(F_1)$ and $\phi(F_2)$ are isomorphisms, so is $\phi(M)$. It proves (*1*).

Moreover, we have an exact sequence $\mathsf{F}^*(F_1) \to \mathsf{F}^*(F_0) \to \mathsf{F}^*(M) \to 0$, where $\mathsf{F}^*(F_i)$ are multiples of $\mathsf{F}^*(\boldsymbol{B}) = P$, so $\mathsf{F}^*(M) \in P$-Mod. On the contrary, let we have an exact sequence $P_1 \to P_0 \to N \to 0$, where P_i are multiples of P. Consider the natural morphism $\psi : \mathsf{F}^*\mathsf{F} \to \mathbb{1}_{\mathcal{A}\text{-Mod}}$. Obviously, $\psi(P)$ is an isomorphism, so $\psi(P_i)$ are also isomorphisms. Again we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathsf{F}^*\mathsf{F}(P_1) & \longrightarrow & \mathsf{F}^*\mathsf{F}(P_0) & \longrightarrow & \mathsf{F}^*\mathsf{F}(N) & \longrightarrow & 0 \\
{\scriptstyle \psi(P_1)}\big\downarrow & & {\scriptstyle \psi(P_0)}\big\downarrow & & {\scriptstyle \psi(N)}\big\downarrow & & \\
P_1 & \longrightarrow & P_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

It implies that $\psi(N)$ is an isomorphism, hence $N \in \operatorname{Im} \mathsf{F}^*$. It proves (*2*). The proof of (*3*) is quite analogous.

To prove (*4*), note that $I_P P = P$ [10, Proposition VII.3.1], where $I_P = \operatorname{Im} \mu_P$. Let $M \notin \ker \mathsf{F}$ and $f : P \to M$ be a nonzero homomorphism. Then $I_P M \supseteq \operatorname{Im} f \neq 0$. On the contrary, if $I_P M \neq 0$, there is an element $z \in M$, elements $p_i \in P$, and homomorphisms $f_i : P \to \boldsymbol{A}$ such that $\sum_i f_i(p_i)z \neq 0$. Denote by g the homomorphism $\boldsymbol{A} \to M$ mapping 1 to z and set $g_i = gf_i$. Then at least one of g_i is nonzero, so $M \notin \ker \mathsf{F}$. □

As the functor F is exact, it induces a functor $\mathsf{DF} : \mathscr{D}\mathcal{A} \to \mathscr{D}\mathcal{B}$ acting on complexes componentwise. It is both left and right derived of F. There are also left derived functor LF^* and right derived functor $\mathsf{RF}^!$, both $\mathscr{D}\mathcal{B} \to \mathscr{D}\mathcal{A}$ [29, Sect. 6]. Moreover, it follows from [29] that both $(\mathsf{LF}^*, \mathsf{DF})$ and $(\mathsf{DF}, \mathsf{RF}^!)$ are adjoint pairs (see [8] for details). Obviously, DF maps $\mathscr{D}^\sigma \mathcal{A}$ to $\mathscr{D}^\sigma \mathcal{B}$, where $\sigma \in \{+, -, b\}$; LF^* maps $\mathscr{D}^- \mathcal{B}$ to $\mathscr{D}^- \mathcal{A}$ and $\mathsf{RF}^!$ maps $\mathscr{D}^+ \mathcal{B}$ to $\mathscr{D}^+ \mathcal{A}$.

Theorem 3.11

1. DF *is a bilocalizing functor and* $\ker \mathsf{DF} \simeq \mathscr{D}_{\mathcal{A}/\mathcal{I}_\mathcal{P}} \mathcal{A}$, *where* $\mathscr{D}_{\mathcal{A}/\mathcal{I}_\mathcal{P}} \mathcal{A}$ *is the full subcategory of* $\mathscr{D}\mathcal{A}$ *consisting of complexes with cohomologies annihilated by* $\mathcal{I}_\mathcal{P}$ *(i.e., belonging to* $(\mathcal{A}/\mathcal{I}_\mathcal{P})$-Mod*).*
2. LF^* *maps* $\mathsf{Perf}\mathcal{B}$ *to* $\mathsf{Perf}\mathcal{A}$*.*
3. $\operatorname{Im} \mathsf{LF}^* = {}^\perp \mathscr{D}_{\mathcal{A}/\mathcal{I}_\mathcal{P}} \mathcal{A}$ *coincides with the full subcategory* $\mathscr{D}\mathcal{P}_\to$ *of* $\mathscr{D}\mathcal{A}$ *consisting of complexes quasi-isomorphic to K-flat complexes* $\mathcal{F}^\bullet$ *such that for every component* $\mathcal{F}^i$ *and every point* $x \in X$, *the localization* $\mathcal{F}^i_x$ *is a direct limit of modules from* $\operatorname{add} \mathcal{P}_x$. *The same is true if we replace* $\mathscr{D}$ *by* $\mathscr{D}^-$.
4. $\operatorname{Im} \mathsf{RF}^! = \mathscr{D}_{\mathcal{A}/\mathcal{I}_\mathcal{P}} \mathcal{A}^\perp$ *coincides with the full subcategory* $\mathscr{D}\mathcal{P}^{\mathrm{Inj}}$ *of* $\mathscr{D}\mathcal{A}$ *consisting of complexes quasi-isomorphic to K-injective complexes* $\mathcal{I}^\bullet$ *such that every component* $\mathcal{I}^i$ *belongs to* $\mathsf{F}^!(\mathcal{B}\text{-Inj})$. *The same is true if we replace* $\mathscr{D}$ *by* $\mathscr{D}^+$.

Note that the condition (*4*) can be verified locally at every point $x \in X$.

We recall that a complex $\mathcal{F}^\bullet$ is said to be *K-flat* (*K*-injective) if for every acyclic complex of right (respectively, left) $\mathcal{A}$-modules $\mathcal{C}^\bullet$, the complex $\mathcal{F}^\bullet \otimes_\mathcal{A} \mathcal{C}^\bullet$ (respectively, $\mathcal{H}om_\mathcal{A}(\mathcal{C}^\bullet, \mathcal{F}^\bullet)$) is also acyclic [29].

Proof (*1*) follows from the results of the previous section.

As $\mathcal{P}$ is coherent and locally projective, the functor DF preserves arbitrary coproducts. Therefore its left adjoint LF^* maps small objects to small ones, which gives (*2*).

(*3*) It follows from [1] that for every complex $\mathcal{M}^\bullet$ of $\mathcal{B}$-modules, there is a quasi-isomorphic K-flat complex $\mathcal{F}^\bullet$ with flat components. Then $\mathsf{LF}^*(\mathcal{M}^\bullet) = \mathsf{F}^*(\mathcal{F}^\bullet)$. By Bourbaki [5, Chap. X, § 1, Theorem 1], every localization $\mathcal{F}^i_x$ is a direct limit $\varinjlim_n \mathcal{L}^i_n$, where all $\mathcal{L}^i_n$ are finitely generated and projective, hence belong to $\operatorname{add} \mathcal{B}_x$. As F^* preserves direct limits and $\mathsf{F}^*(\mathcal{B}) \simeq \mathcal{P}$, $\mathsf{F}^*(\mathcal{F}_i) \simeq \varinjlim_n \mathsf{F}^*(\mathcal{L}^i_n)$ and $\mathsf{F}^*(\mathcal{L}^i_n)$ belongs to $\operatorname{add} \mathcal{P}_x$. Therefore, $\mathsf{F}^*(\mathcal{M}^\bullet) \in \mathscr{D}\mathcal{P}_\to$.

On the contrary, let $\mathcal{N}^\bullet \in \mathscr{D}\mathcal{P}_\to$. We can suppose that this complex is K-flat and every localization $\mathcal{N}^i_x$ is a direct limit $\varinjlim_n \mathcal{P}^i_n$, where $\mathcal{P}^i_n \in \operatorname{add}\mathcal{P}_x$. Then the complex $\mathsf{F}(\mathcal{N}^\bullet)$ is also K-flat, so $(\mathsf{LF}^*)(\mathsf{F}(\mathcal{N}^\bullet)) \simeq \mathsf{F}^*\mathsf{F}(\mathcal{N}^\bullet)$. As the natural map $\mathsf{F}^*\mathsf{F}(\mathcal{P}) \to \mathcal{P}$ is an isomorphism, the same is true for $\mathsf{F}^*\mathsf{F}(\mathcal{P}^i_n) \to \mathcal{P}^i_n$, hence also for $\mathsf{F}^*\mathsf{F}(\mathcal{N}^i_x) \to \mathcal{N}^i_x$. Therefore, the map $(\mathsf{LF}^*)(\mathsf{DF})(\mathcal{N}^\bullet) \to \mathcal{N}^\bullet$ is an isomorphism and $\mathcal{N} \in \operatorname{Im}\mathsf{LF}^*$.

The proof of (*4*) is quite analogous. □

Corollary 3.12 *There are semi-orthogonal decompositions* $(\ker\mathsf{DF}, \operatorname{Im}\mathsf{LF}^*)$ *and* $(\operatorname{Im}\mathsf{RF}^!, \ker\mathsf{DF})$ *of the category* $\mathscr{D}\mathcal{A}$.

Note that the subcategories $\operatorname{Im}\mathsf{LF}^*$ and $\operatorname{Im}\mathsf{RF}^!$ are equivalent (both are equivalent to $\mathscr{D}\mathcal{B}$) but usually do not coincide.

The following special case is rather important:

Theorem 3.13 *Suppose that the ideal* $\mathcal{I} = \mathcal{I}_\mathcal{P}$ *is flat as right* $\mathcal{A}$*-module. Set* $\mathcal{Q} = \mathcal{A}/\mathcal{I}$. *Then* $\ker\mathsf{DF} = \mathscr{D}_\mathcal{Q}\mathcal{A} \simeq \mathscr{D}\mathcal{Q}$.

Proof Let $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a quasi-isomorphism. As $\mathcal{I}$ is flat, then $\mathcal{I}\otimes_\mathcal{A}\mathcal{F}^\bullet \to \mathcal{I}\otimes_\mathcal{A}\mathcal{G}^\bullet$ is also a quasi-isomorphism. Therefore, $\mathcal{F}^\bullet\otimes_\mathcal{A}\mathcal{Q} \to \mathcal{G}^\bullet\otimes_\mathcal{A}\mathcal{Q}$ is also a quasi-isomorphism. In particular, if $\mathcal{G}^\bullet$ consists of $\mathcal{Q}$-modules, we get a quasi-isomorphism $\mathcal{F}^\bullet\otimes_\mathcal{A}\mathcal{Q} \to \mathcal{G}^\bullet$. It implies that we can identify $\mathscr{D}\mathcal{Q}$ with the full triangulated subcategory of $\mathscr{D}\mathcal{A}$. Obviously $\mathscr{D}\mathcal{Q} \subseteq \mathscr{D}_\mathcal{Q}\mathcal{A}$. Moreover, $\mathcal{I}^2 = \mathcal{I}$. Let $\mathcal{F}^\bullet \in \mathscr{D}_\mathcal{Q}\mathcal{A}$. We can suppose that $\mathcal{F}^\bullet$ is K-flat. Its tensor product with the exact sequence $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{Q} \to 0$ gives an exact sequence of complexes $0 \to \mathcal{I}\otimes_\mathcal{A}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \mathcal{Q}\otimes_\mathcal{A}\mathcal{F}^\bullet \to 0$. As $\mathcal{I}$ is flat, $H^*(\mathcal{I}\otimes_\mathcal{A}\mathcal{F}^\bullet) \simeq \mathcal{I}\otimes_\mathcal{A}H^*(\mathcal{F}^\bullet)$. Since $\mathcal{I}\otimes_\mathcal{A}\mathcal{Q} \simeq \mathcal{I}/\mathcal{I}^2 = 0$, also $\mathcal{I}\otimes_\mathcal{A}\mathcal{M} = 0$ for every $\mathcal{Q}$-module $\mathcal{M}$. Therefore $H^*(\mathcal{I}\otimes_\mathcal{A}\mathcal{F}^\bullet) = 0$, i.e., $\mathcal{I}\otimes_\mathcal{A}\mathcal{F}^\bullet$ is acyclic, whence $\mathcal{F}^\bullet$ is quasi-isomorphic to $\mathcal{Q}\otimes_\mathcal{A}\mathcal{F}$, which belongs to $\mathscr{D}\mathcal{Q}$. □

Example 3.14 (Endomorphism Construction) Let $\mathcal{F}$ be a coherent $\mathcal{A}$-module and $\mathcal{A}_\mathcal{F} = (\mathcal{E}nd_\mathcal{A}(\mathcal{A}\oplus\mathcal{F}))^{\mathrm{op}}$. We identify $\mathcal{A}_\mathcal{F}$ with the algebra of matrices

$$\mathcal{A}_\mathcal{F} = \begin{pmatrix} \mathcal{A} & \mathcal{F} \\ \mathcal{F}' & \mathcal{E} \end{pmatrix},$$

where $\mathcal{F}' = \operatorname{Hom}_\mathcal{A}(\mathcal{F}, \mathcal{A})$ and $\mathcal{E} = (\operatorname{End}_\mathcal{A}\mathcal{F})^{\mathrm{op}}$. Then $\mathcal{P} = \mathcal{P}_\mathcal{F} = \left(\begin{smallmatrix}\mathcal{A}\\ \mathcal{F}'\end{smallmatrix}\right)$ is a locally projective $\mathcal{A}_\mathcal{F}$-module and $\mathcal{A} \simeq (\operatorname{End}_{\mathcal{A}_\mathcal{F}}\mathcal{P})^{\mathrm{op}}$. Hence $\mathcal{A}$ is a minor of $\mathcal{A}_\mathcal{F}$, thus $\mathcal{A}$-Mod and $\mathscr{D}\mathcal{A}$ are bilocalizations, respectively, of $\mathcal{A}_\mathcal{F}$-Mod and $\mathscr{D}\mathcal{A}_\mathcal{F}$. The corresponding functors are

$$\begin{aligned}
\mathsf{F}_\mathcal{F} &= \operatorname{Hom}_{\mathcal{A}_\mathcal{F}}(\mathcal{P}_\mathcal{F}, _),\\
\mathsf{F}^*_\mathcal{F} &= \mathcal{P}_\mathcal{F}\otimes_\mathcal{A}_\,,\\
\mathsf{F}^!_\mathcal{F} &= \operatorname{Hom}_\mathcal{A}(\mathcal{P}^\vee_\mathcal{F}, _)
\end{aligned}$$

and their derived functors. Note that $\mathcal{P}^\vee \simeq (\mathcal{A}\ \mathcal{F}) \simeq \mathcal{A}\oplus\mathcal{F}$ as $\mathcal{A}$-$\mathcal{A}_\mathcal{F}$-bimodule and $\mathcal{I}_\mathcal{P}$ is the ideal of matrices

$$\mathcal{I}_\mathcal{P} = \begin{pmatrix} \mathcal{A} & \mathcal{F} \\ \mathcal{F}' & \mathcal{I}'_\mathcal{F} \end{pmatrix}$$

where $\mathcal{I}'_\mathcal{F}$ is the image of the map $\mu' : \mathcal{F}'\otimes_\mathcal{A}\mathcal{F} \to \mathcal{E}$ such that $\mu'(f'\otimes f)(v) = f'(v)f$ for all $f, v \in \mathcal{F}, f' \in \mathcal{F}'$. Therefore, $\ker \mathsf{F}_\mathcal{F} \simeq (\mathcal{E}/\mathcal{I}'_\mathcal{F})$-Mod and $\ker \mathsf{DF}_\mathcal{F} = \mathscr{D}_{\mathcal{E}/\mathcal{I}'_\mathcal{F}}\mathcal{A}_\mathcal{F}$.

This construction is especially convenient when $\mathcal{A}$ is strongly Gorenstein in the sense of the following definition:

Definition 3.15 A noetherian non-commutative scheme $(X, \mathcal{A})$ is said to be *strongly Gorenstein* if X is equidimensional, $\mathcal{A}$ is a Cohen–Macaulay $\mathcal{O}_X$-module, and $\operatorname{inj.dim}_\mathcal{A} \mathcal{A} = \dim X$.

Such non-commutative schemes possess almost all usual properties of Cohen–Macaulay rings and ("usual") schemes, and their proofs are quite analogous to those from [6] (see [8, Sect. 5] for details). We need here the *Cohen–Macaulay duality*. For a noetherian non-commutative scheme $(X, \mathcal{A})$ denote by $\mathrm{CM}\,\mathcal{A}$ the full subcategory of $\mathcal{A}$-mod consisting of *maximal Cohen–Macaulay $\mathcal{A}$-modules*, i.e., such coherent $\mathcal{A}$-modules $\mathcal{M}$ that each localization $\mathcal{M}_x$ is a maximal Cohen–Macaulay $\mathcal{O}_{X,x}$-module. Let $^* : \mathcal{A}\text{-mod} \to \mathcal{A}^{\mathrm{op}}\text{-mod}$ be the functor mapping $\mathcal{M}$ to $\mathcal{H}om_\mathcal{A}(\mathcal{M}, \mathcal{A})$. If $\mathcal{A}$ is strongly Gorenstein, so is also $\mathcal{A}^{\mathrm{op}}$, and * defines an exact duality between $\mathrm{CM}\,\mathcal{A}$ and $\mathrm{CM}\,\mathcal{A}^{\mathrm{op}}$. It means that, for every $\mathcal{M} \in \mathrm{CM}\,\mathcal{A}$, $\mathcal{E}xt^i_\mathcal{A}(\mathcal{M}, \mathcal{A}) = 0$ for $i > 0$ and the natural map $\mathcal{M} \to \mathcal{M}^{**}$ is an isomorphism. It also implies that the natural map $\mathcal{M}\otimes_\mathcal{A}\mathcal{L} \to \mathcal{H}om_\mathcal{A}(\mathcal{M}, \mathcal{L})$ is an isomorphism for every locally projective $\mathcal{A}$-module $\mathcal{L}$.

Theorem 3.16 *In the situation of Example 3.14, let $\mathcal{A}$ be strongly Gorenstein and has enough locally projective modules and $\mathcal{F} \in \mathrm{CM}\,\mathcal{A}$. Then the restrictions of the functors $\mathsf{LF}^*_\mathcal{F}$ and $\mathsf{RF}^!_\mathcal{F}$ onto $\mathsf{Perf}\mathcal{A}$ coincide. Thus these restrictions are both left and right adjoint to $\mathsf{DF}_\mathcal{F}$.*

Proof Note first that under the given conditions $\mathsf{F}^*_\mathcal{F}(\mathcal{L}) \simeq \mathsf{F}^!_\mathcal{F}(\mathcal{L})$ for every locally projective $\mathcal{A}$-module $\mathcal{L}$. As $\mathcal{A}$ has enough locally projective modules, any complex from $\mathsf{Perf}\mathcal{A}$ is quasi-isomorphic to a finite complex $\mathcal{L}^\bullet$ such that all $\mathcal{L}^i$ are from $\mathsf{lp}\,\mathcal{A}$. Then $\mathsf{LF}^*_\mathcal{F}(\mathcal{L}^\bullet) = \mathsf{F}^*_\mathcal{F}(\mathcal{L}^\bullet)$. On the other hand, $\mathsf{R}^k\mathsf{F}^!_\mathcal{F}(\mathcal{L}^i) = \mathcal{E}xt^k_\mathcal{A}(\mathcal{P}_\mathcal{F}, \mathcal{L}^i) = 0$ for $k \neq 0$. Therefore, $\mathsf{RF}^!_\mathcal{F}(\mathcal{L}^\bullet) = \mathsf{F}^!_\mathcal{F}(\mathcal{L}^\bullet) \simeq \mathsf{F}^*_\mathcal{F}(\mathcal{L}^\bullet)$. □

4 Quasi-Hereditary Schemes

In this section we generalize the notions of quasi-hereditary algebras and orders [12, 22] to non-commutative schemes. It is closely related with minors and bilocalizations. We start from the following facts. Let $(X, \mathcal{A})$ be a non-commutative scheme, $\mathcal{M}$ be an $\mathcal{A}$-module. We call $\sup\{\, i \mid \mathcal{E}xt^i_{\mathcal{A}}(\mathcal{M}, \_) \neq 0 \,\}$ the *local projective dimension* of the $\mathcal{A}$-module $\mathcal{M}$ and denote it by $\mathrm{lp.dim}_{\mathcal{A}}\,\mathcal{M}$. If the non-commutative scheme $(X, \mathcal{A})$ is noetherian and the module $\mathcal{M}$ is coherent, then $\mathrm{lp.dim}_{\mathcal{A}}\,\mathcal{M} = \sup\{\, \mathrm{pr.dim}_{\mathcal{A}_x}\,\mathcal{M}_x \mid x \in X \,\}$.

Lemma 4.17 (Cf. [8, Lemma 4.9]) *Let $(X, \mathcal{A})$ be a non-commutative scheme, $\mathcal{P}$ be a coherent locally projective $\mathcal{A}$-module, $\mathcal{B} = (\mathcal{E}nd_{\mathcal{A}}\,\mathcal{P})^{\mathrm{op}}$, and $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{I}_{\mathcal{P}}$. Suppose that $\mathcal{P}$ is flat as right $\mathcal{B}$-module,*

$$\mathrm{lp.dim}_{\mathcal{A}}\,\mathcal{I}_{\mathcal{P}} = d,$$

$$\mathrm{gl.dim}\,\mathcal{B} = n,$$

$$\mathrm{gl.dim}\,\bar{\mathcal{A}} = m.$$

Then $\mathrm{gl.dim}\,\mathcal{A} \le \max\{\, m + d + 2,\ n \,\}$.

Proof Let $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{I}_{\mathcal{P}}$. Then $\mathrm{lp.dim}_{\mathcal{A}}\,\bar{\mathcal{A}} = d + 1$. The spectral sequence $\mathrm{Ext}^p_{\bar{\mathcal{A}}}(\mathcal{M}, \mathcal{E}xt^q_{\mathcal{A}}(\bar{\mathcal{A}}, \_)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{A}}(\mathcal{M}, \_)$ implies that $\mathrm{pr.dim}_{\mathcal{A}}\,\mathcal{M} \le m + d + 1$ for every $\bar{\mathcal{A}}$-module $\mathcal{M}$. Consider the functors $\mathsf{F} = \mathcal{H}om_{\mathcal{A}}(\mathcal{P}, \_)$ and $\mathsf{F}^* = \mathcal{P} \otimes_{\mathcal{B}} \_$. As the morphism $\mathsf{F}\mathsf{F}^*\mathsf{F} \to \mathsf{F}$ arising from the adjunction is an isomorphism, the kernel and the cokernel of the natural map $\alpha : \mathsf{F}^*\mathsf{F}\mathcal{M} \to \mathcal{M}$ are annihilated by F, so are actually $\bar{\mathcal{A}}$-modules. It implies that $\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \simeq \mathrm{Ext}^i_{\mathcal{A}}(\mathsf{F}^*\mathsf{F}\mathcal{M}, \mathcal{N})$ if $i > m + d + 2$, so $\mathrm{pr.dim}_{\mathcal{A}}\,\mathcal{M} \le \max\{\, m + d + 2,\ \mathrm{pr.dim}_{\mathcal{A}}\,\mathsf{F}^*\mathsf{F}\mathcal{M} \,\}$. As both functors F and F^* are exact, $\mathrm{Ext}^i_{\mathcal{A}}(\mathsf{F}^*\_, \_) \simeq \mathrm{Ext}^i_{\mathcal{B}}(\_, \mathsf{F}\_)$, so $\mathrm{pr.dim}_{\mathcal{A}}\,\mathsf{F}^*\mathsf{F}\mathcal{M} \le n$. □

This result motivates the following definitions:

Definition 4.18 (Cf. [8, Definition 4.9])

1. Let $(X, \mathcal{A})$ and $(X, \mathcal{B})$ be two non-commutative schemes. A *relating chain* between $\mathcal{A}$ and $\mathcal{B}$ is a sequence $(\mathcal{A}_1, \mathcal{P}_1, \mathcal{A}_2, \mathcal{P}_2, \ldots,\ \mathcal{P}_r, \mathcal{A}_{r+1})$, where $\mathcal{A}_1 = \mathcal{A}$; $\mathcal{A}_{r+1} = \mathcal{B}$; every $\mathcal{P}_i$ $(1 \le i \le r)$ is a coherent locally projective $\mathcal{A}_i$-module which is also flat as right $\mathcal{B}_i$-module, where $\mathcal{B}_i = (\mathcal{E}nd_{\mathcal{A}_i}\,\mathcal{P}_i)^{\mathrm{op}}$; and $\mathcal{A}_{i+1} = \mathcal{A}_i/\mathcal{I}_{\mathcal{P}_i}$ for $1 \le i \le r$.
2. The relating chain is said to be *flat* if, for every $1 \le i \le r$, $\mathcal{I}_{\mathcal{P}_i}$ is flat as right $\mathcal{A}_i$-module. Note that it is the case if the natural map $\mu_i : \mathcal{P}_i \otimes_{\mathcal{B}_i} \mathcal{P}_i^\vee \to \mathcal{A}_i$ is a monomorphism.
3. The relating chain is said to be *heredity* if, for every $1 \le i \le r$, $\mathcal{I}_{\mathcal{P}_i}$ is locally projective as left $\mathcal{A}_i$-module. In this case μ_i is a monomorphism (it can be proved as in [14, Statement 7]), so this chain is flat.
4. If the relating chain is heredity and all non-commutative schemes $\mathcal{B}_i$ are hereditary, i.e., $\mathrm{gl.dim}\,\mathcal{B}_i \le 1$, we say that the non-commutative scheme $\mathcal{A}$ is *quasi-hereditary* of level r. (Thus quasi-hereditary of level 0 means hereditary.)

The following result is obvious:

Proposition 4.19 *If $(\mathcal{A}_1, \mathcal{P}_1, \mathcal{A}_2, \mathcal{P}_2, \ldots, \mathcal{P}_r, \mathcal{A}_{r+1})$ is a relating chain between $\mathcal{A}$ and $\mathcal{B}$, then $(\mathcal{A}_1^{\mathrm{op}}, \mathcal{P}_1^\vee, \mathcal{A}_2^{\mathrm{op}}, \mathcal{P}_2^\vee, \ldots, \mathcal{P}_r^\vee, \mathcal{A}_{r+1}^{\mathrm{op}})$ is a relating chain between $\mathcal{A}^{\mathrm{op}}$ and $\mathcal{B}^{\mathrm{op}}$ with the same endomorphism algebras $\mathcal{B}_i$.*

Note that if $\mathcal{A}$ is noetherian, so are all $\mathcal{A}_i$ and $\mathcal{B}_i$. As for noetherian non-commutative schemes all flat coherent modules are locally projective, we obtain the following corollary:

Corollary 4.20 *If a noetherian non-commutative scheme* $(X, \mathcal{A})$ *is quasi-hereditary, so is also* $(X, \mathcal{A}^{\mathrm{op}})$.

We fix a relating chain $(\mathcal{A}_1, \mathcal{P}_1, \mathcal{A}_2, \mathcal{P}_2, \dots, \mathcal{P}_r, \mathcal{A}_{r+1})$ between $\mathcal{A}$ and $\mathcal{B}$ and keep the notations of Definition 4.18(1). Lemma 4.17 immediately implies an estimate for global dimensions.

Corollary 4.21 *Let* $\operatorname{gl.dim} \mathcal{B}_i \le n$ *and* $\operatorname{lp.dim}_{\mathcal{A}_i} \mathcal{I}_{\mathcal{P}_i} \le d$ *for all* $1 \le i \le r$. *Then* $\operatorname{gl.dim} \mathcal{A} \le r(d+2) + \max\{\operatorname{gl.dim} \mathcal{B},\, n-d-2\}$. *If this relating chain is heredity, then* $\operatorname{gl.dim} \mathcal{A} \le \operatorname{gl.dim} \mathcal{B} + 2r$.

Using Corollary 3.12, Theorem 3.13, and induction, we obtain the following result:

Corollary 4.22 *If this relating chain is flat, there are semi-orthogonal decompositions* $(\mathscr{T}, \mathscr{T}_r, \dots, \mathscr{T}_1)$ *and* $(\mathscr{T}_1', \mathscr{T}_2', \dots, \mathscr{T}_r', \mathscr{T})$ *of* $\mathscr{D}\mathcal{A}$ *such that* $\mathscr{T}_i \simeq \mathscr{T}_i' \simeq \mathscr{D}\mathcal{B}_i$ $(1 \le i \le r)$ *and* $\mathscr{T} \simeq \mathscr{D}\mathcal{B}$.

Note that, as a rule, $\mathscr{T}_i \ne \mathscr{T}_i'$.

Corollary 4.23 *If a non-commutative scheme* $\mathcal{A}$ *is quasi-hereditary of level r, then* $\operatorname{gl.dim} \mathcal{A} \le 2r+1$, *and there are semi-orthogonal decompositions* $(\mathscr{T}, \mathscr{T}_r, \dots, \mathscr{T}_1)$ *and* $(\mathscr{T}_1', \mathscr{T}_2', \dots, \mathscr{T}_r', \mathscr{T})$ *of* $\mathscr{D}\mathcal{A}$ *such that* $\mathscr{T}_i \simeq \mathscr{T}_i'$ $(1 \le i \le r)$ *and all categories* $\mathscr{T}_i$, *as well as* $\mathscr{T}$, *are equivalent to derived categories of some hereditary non-commutative schemes.*

The following result is evident:

Proposition 4.24 *If there is a heredity relating chain between* $\mathcal{A}$ *and* $\mathcal{B}$ *such that all* $\mathcal{B}_i$ *are hereditary and* $\mathcal{B}$ *is quasi-hereditary, then* $\mathcal{A}$ *is quasi-hereditary too.*

Corollary 4.25 *Consider the endomorphism construction of Example 3.14 (with the same notations). Suppose that* $\mathcal{F}$ *is flat as right* $\mathcal{E}$*-module,* $\mathcal{F}'$ *is locally projective as left* $\mathcal{E}$*-module and the natural map* $\mu_{\mathcal{F}} : \mathcal{F} \otimes_{\mathcal{E}} \mathcal{F}' \to \mathcal{A}$ *is a monomorphism. If both* $\mathcal{E}$ *and* $\bar{\mathcal{A}} = \mathcal{A} / \operatorname{Im} \mu_{\mathcal{F}}$ *are quasi-hereditary, so is* $\mathcal{A}_{\mathcal{F}}$.

Proof Let $\tilde{\mathcal{P}} = \begin{pmatrix} \mathcal{F} \\ \mathcal{E} \end{pmatrix}$. Then $\mathcal{I}_{\tilde{\mathcal{P}}}$ is the ideal of matrices

$$\begin{pmatrix} \mathcal{F} \otimes_{\mathcal{E}} \mathcal{F}' & \mathcal{F} \\ \mathcal{F}' & \mathcal{E} \end{pmatrix}.$$

Its first row is $\mathcal{F} \otimes_{\mathcal{E}} \begin{pmatrix} \mathcal{F}' & \mathcal{E} \end{pmatrix}$ and its first column is $\begin{pmatrix} \mathcal{F} \\ \mathcal{E} \end{pmatrix} \otimes_{\mathcal{E}} \mathcal{F}'$. Under the prescribed conditions, the first one is flat as right $\mathcal{A}_{\mathcal{F}}$-module and the second one is locally projective as left $\mathcal{A}_{\mathcal{F}}$-module. Therefore $(\mathcal{A}_{\mathcal{F}}, \tilde{\mathcal{P}}, \bar{\mathcal{A}})$ is a heredity relating chain relating between $\mathcal{A}$ and $\bar{\mathcal{A}}$, so we can apply Proposition 4.24. ◻

One can show that the definition of quasi-hereditary non-commutative schemes is indeed a generalization of the well-known definition for semiprimary rings [12, 14].

Theorem 4.26 *Let $(X, \mathcal{A})$ be affine: $X = \operatorname{spec} \boldsymbol{R}$, $\mathcal{A} = \boldsymbol{A}^{\sim}$, and the ring $\boldsymbol{A}$ be semiprimary. This non-commutative scheme is quasi-hereditary in the sense of Definition 4.18(4) if and only if the ring $\boldsymbol{A}$ is quasi-hereditary in the sense of [12].*

Proof Recall that a semiprimary ring $\boldsymbol{A}$ is called quasi-hereditary if there is a chain of ideals $0 = I_0 \subseteq I_1 \subset I_2 \subset \cdots \subset I_r \subset I_{r+1} = \boldsymbol{A}$ such that the following conditions hold for $\bar{I}_i = I_i/I_{i-1}$ as for an ideal in $\boldsymbol{A}_i = \boldsymbol{A}/\boldsymbol{I}_{i-1}$:

1. $\bar{I}_i^2 = \bar{I}_i$. (As $\boldsymbol{A}_i$ is semiprimary, it means that $\bar{I}_i = \boldsymbol{A}_i e_i \boldsymbol{A}_i$ for some idempotent e_i.)
2. $\bar{I}_i(\operatorname{rad} \boldsymbol{A}_i)\bar{I}_i = 0$. (It means that $\operatorname{rad}(e_i \boldsymbol{A}_i e_i) = 0$.)
3. $\bar{I}_i$ is projective as $\boldsymbol{A}_i$-module. (Under condition (2) it is equivalent to the claim that the map $\boldsymbol{A}_i e_i \otimes_{e_i A_i e_i} e_i \boldsymbol{A}_i \to \bar{I}_i$ is bijective, see [14, Statement 7].)

In other words, it means that $(\boldsymbol{A} = \boldsymbol{A}_1, P_1, \boldsymbol{A}_2, P_2, \ldots, \boldsymbol{A}_r, P_r, \boldsymbol{A}_{r+1})$, where $P_i = \boldsymbol{A}_i e_i$ and $\boldsymbol{A}_{r+1} = \boldsymbol{A}/I_r$, is a heredity relating chain between $\boldsymbol{A}$ with semisimple endomorphism rings $\boldsymbol{B}_i$ and semisimple ring $\boldsymbol{A}_{r+1}$. Thus $\boldsymbol{A}$ is a quasi-hereditary affine non-commutative scheme. On the contrary, let $\boldsymbol{A}$ be quasi-hereditary as an affine non-commutative scheme. To show that $\boldsymbol{B}$ is a quasi-hereditary ring, we can use induction and the following result.

Recall that a ring $\boldsymbol{A}$ is said to be *triangular* if it has a set of idempotents $\{e_1, e_2, \ldots, e_m\}$ such that $\sum_{i=1}^m e_i = 1$, $e_i \boldsymbol{A} e_j = 0$ if $i > j$ and $\boldsymbol{A}_i = e_i \boldsymbol{A} e_i$ are *prime rings*, i.e., $IJ \neq 0$ for any two nonzero ideals of $\boldsymbol{A}_i$. If $\boldsymbol{A}$ is semiprimary, then $\boldsymbol{A}_i$ are simple artinian rings. For instance, every semiprimary hereditary ring is triangular [16].

Lemma 4.27 *Let $\boldsymbol{A}$ be a semiprimary ring, $I = \boldsymbol{A}e\boldsymbol{A}$ be an idempotent ideal such that I is projective as $\boldsymbol{A}$-module, $\boldsymbol{A}/I$ is quasi-hereditary, and $\boldsymbol{E} = e\boldsymbol{A}e$ is triangular. Then $\boldsymbol{A}$ is quasi-hereditary. In particular, any triangular semiprimary ring is quasi-hereditary.*

Proof According to [13], it is enough to find a heredity chain of ideals $0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_m = \boldsymbol{E}$ in $\boldsymbol{E}$ such that each factor $I_i e/I_{i-1}e$ is projective as $\boldsymbol{E}_i = \boldsymbol{E}/I_{i-1}$-module. Since $\boldsymbol{E}$ is triangular, it can be considered as an algebra of triangular matrices:

$$\boldsymbol{E} = \begin{pmatrix} \boldsymbol{E}_{11} & \boldsymbol{E}_{12} & \boldsymbol{E}_{13} & \ldots & \boldsymbol{E}_{1m} \\ 0 & \boldsymbol{E}_{22} & \boldsymbol{E}_{23} & \ldots & \boldsymbol{E}_{2m} \\ 0 & 0 & \boldsymbol{E}_{33} & \ldots & \boldsymbol{E}_{3m} \\ & & \ldots\ldots\ldots & & \\ 0 & 0 & 0 & \ldots & \boldsymbol{E}_{mm} \end{pmatrix},$$

where all rings $\boldsymbol{E}_{ii}$ are simple artinian. Let e_j $(1 \le j \le m)$ be the standard diagonal idempotents in this matrix ring, $\varepsilon_i = \sum_{j=1}^i e_j$ and $I_i = \boldsymbol{E}\varepsilon_i\boldsymbol{E}$. Then I_i is the ideal of matrices such that their first $m - i$ rows are zero. Therefore, $\boldsymbol{E}/I_{i-1}$ is the matrix ring obtained from $\boldsymbol{E}$ by crossing out the first $i-1$ rows and columns. Evidently, $0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_m = \boldsymbol{E}$ is a heredity chain of ideals in $\boldsymbol{E}$. One easily sees that $e_j(I_iM) = 0$ for any $\boldsymbol{E}_i$-module M and any $j > i$. Then there is an epimorphism $k\boldsymbol{E}_ie_i \to I_iM$ for some k. As the module $\boldsymbol{E}_ie_i$ is semisimple, this epimorphism splits, so I_iM is projective. In particular, $I_ie/I_{i-1}e$ is projective, so $\boldsymbol{A}$ is quasi-hereditary.

□

Just in the same way, one can show that if $X = \operatorname{spec} \boldsymbol{R}$, where $\boldsymbol{R}$ is a discrete valuation ring and $\mathcal{A} = \boldsymbol{A}^{\sim}$, where $\boldsymbol{A}$ is a semiprime $\boldsymbol{R}$-order, then the non-commutative scheme $(X, \mathcal{A})$ is quasi-hereditary if and only if $\boldsymbol{A}$ is a quasi-hereditary $\boldsymbol{R}$-order in the sense of [22].

5 Non-commutative Curves

We call a *curve* a noetherian excellent reduced scheme such that all its irreducible components are of dimension 1. We call a *non-commutative curve* a reduced non-commutative scheme $(X, \mathcal{A})$ such that X is a curve and $\mathcal{A}$ is a torsion-free finitely generated $\mathcal{O}_X$-module. We can suppose, without loss of generality, that the $\mathcal{O}_X$-module $\mathcal{A}$ is sincere. In this section $(X, \mathcal{A})$ always denotes a non-commutative curve and we suppose that $\mathcal{A}$ is a sincere $\mathcal{O}_X$-module. We denote by X_{reg} and X_{sng}, respectively, the subsets of regular and singular points of X. As X is excellent and reduced, the set X_{sng} is finite.

If $(X, \mathcal{A})$ is a non-commutative curve, the category $\mathrm{CM}\,\mathcal{A}$ consists of coherent $\mathcal{A}$-modules which are *torsion-free* as $\mathcal{O}_X$-modules. These modules can be defined locally. Namely, let $\mathcal{K} = \mathcal{K}_X$ be the sheaf of rational functions on X. Set $\mathcal{K}\mathcal{M} = \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{M}$. Then $\mathcal{K}\mathcal{A}$ is a sheaf of semisimple $\mathcal{K}$-algebras and $\mathcal{A}$ is an $\mathcal{O}_X$*-order* in $\mathcal{K}\mathcal{A}$, i.e., an $\mathcal{O}_X$-subalgebra in $\mathcal{K}\mathcal{A}$ which is coherent as $\mathcal{O}_X$-module and generates $\mathcal{K}\mathcal{A}$ as $\mathcal{K}$-module. If $\mathcal{V}$ is a coherent $\mathcal{K}\mathcal{A}$-module and $\mathcal{M} \subset \mathcal{V}$ is its coherent $\mathcal{A}$-submodule which generates $\mathcal{V}$ as $\mathcal{K}$-module, we say that $\mathcal{M}$ is an $\mathcal{A}$*-lattice* in $\mathcal{V}$. Then $\mathcal{M} \in \mathrm{CM}\,\mathcal{A}$ and conversely, every $\mathcal{M} \in \mathrm{CM}\,\mathcal{A}$ is a lattice in $\mathcal{K}\mathcal{M}$. If $\mathcal{M}$ is a lattice in $\mathcal{V}$, then $\mathcal{M}_x$ is a lattice in $\mathcal{V}_x$ and $\mathcal{M}$ is completely defined by the set of lattices $\{\, \mathcal{M}_x \mid x \in X_{\mathrm{cl}} \,\}$. The following result is an immediate consequence of its affine variant, which can be proved like in [4, Chap. 7, § 4, Théorème 3]:

Proposition 5.28

1. *If $\mathcal{M}$ and $\mathcal{N}$ are lattices in $\mathcal{V}$, then $\mathcal{M}_x = \mathcal{N}_x$ for almost all $x \in X_{\mathrm{cl}}$.*
2. *Let $\mathcal{M}$ be a lattice in $\mathcal{V}$, $S \subseteq X_{\mathrm{cl}}$ be a finite set, and for every $x \in S$, let $N(x)$ be an $\mathcal{A}_x$-lattice in $\mathcal{V}_x$. Then there is a lattice $\mathcal{N}$ in $\mathcal{V}$ such that $\mathcal{N}_x = N(x)$ for every $x \in S$ and $\mathcal{N}_x = \mathcal{M}_x$ for every $x \notin S$.*

Using this proposition, one can prove the following properties of non-commutative curves (see [8] for details):

Proposition 5.29 *Let $(X, \mathcal{A})$ be a non-commutative curve.*

1. *$\mathcal{A}$ has enough locally projective modules.*
2. *There is a* canonical $\mathcal{A}$-module*, i.e., such module $\omega_{\mathcal{A}}$ from $\mathrm{CM}\,\mathcal{A}$ that $\mathrm{inj.dim}_{\mathcal{A}}\,\omega_{\mathcal{A}} = 1$ and $\mathcal{E}nd_{\mathcal{A}}\,\omega_{\mathcal{A}} \simeq \mathcal{A}^{\mathrm{op}}$ (hence $\omega_{\mathcal{A}}$ is indeed an $\mathcal{A}$-bimodule). Moreover, also $\mathrm{inj.dim}_{\mathcal{A}^{\mathrm{op}}}\,\omega_{\mathcal{A}} = 1$ and $\omega_{\mathcal{A}}$ is isomorphic (as $\mathcal{A}$-bimodule) to an ideal of $\mathcal{A}$.*
3. *The functor $^* : \mathcal{M} \mapsto \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \omega_{\mathcal{A}})$ defines an exact duality between $\mathrm{CM}\,\mathcal{A}$ and $\mathrm{CM}\,\mathcal{A}^{\mathrm{op}}$. It means that, for every $\mathcal{M} \in \mathrm{CM}\,\mathcal{A}$, the natural map $\mathcal{M} \to \mathcal{M}^{**}$ is an isomorphism and $\mathcal{E}xt^i_{\mathcal{A}}(\mathcal{M}, \omega_{\mathcal{A}}) = 0$ if $i > 0$.*

Actually, one can choose for $\omega_{\mathcal{A}}$ the module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \omega_X)$, and we always do so. Then $\mathcal{M}^*$ is identified with $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \omega_X)$. Note also that $\mathcal{A}$ is strongly Gorenstein if and only if $\mathcal{A}$ is itself a canonical $\mathcal{A}$-module.

Let $\mathcal{B}$ be a minor of the non-commutative curve $\mathcal{A}$, i.e., $\mathcal{B} = \mathcal{E}nd_{\mathcal{A}}\,\mathcal{P}$ for some coherent locally projective $\mathcal{A}$-module $\mathcal{P}$, and let $\mathsf{F}, \mathsf{F}^*, \mathsf{F}^!$ denote the corresponding functors (see formulae (1) on page 77).

$$\begin{tikzcd} \mathcal{A}\text{-Mod} \arrow[rr, "\mathsf{F}"] & & \mathcal{B}\text{-Mod} \arrow[ll, bend right, "\mathsf{F}^*"'] \arrow[ll, bend left, "\mathsf{F}^!"] \end{tikzcd}$$

Obviously F and $\mathsf{F}^!$ map torsion-free modules to torsion-free. It is not true for F^*, so we modify it, setting $\mathsf{F}^\dagger(\mathcal{M}) = (\mathsf{F}^*(\mathcal{M}))^{**}$. We also set $\mathcal{P}' = (\mathcal{P}^\vee)^*$. Then $\mathrm{inj.dim}_{\mathcal{A}}\,\mathcal{P}' = 1$ and $\mathcal{E}xt^1_{\mathcal{A}}(\mathcal{M}, \mathcal{P}') = 0$ for every $\mathcal{M} \in \mathrm{CM}\,\mathcal{A}$; such $\mathcal{A}$-lattices are called *locally injective $\mathcal{A}$-lattices*. In affine case they are indeed injective in the exact category $\mathrm{CM}\,\mathcal{A}$. After this modification, we have results about the categories of torsion-free modules quite analogous to Theorem 3.9.

Theorem 5.30

1. *The functors $\mathsf{F}^!$ and F define an equivalence of $\mathrm{CM}\,\mathcal{B}$ and $\mathrm{CM}'\,\mathcal{P}$, where $\mathrm{CM}'\,\mathcal{P}$ is the full subcategory of $\mathrm{CM}\,\mathcal{A}$ consisting of all modules $\mathcal{M}$ such that for every point $x \in X$ there is an exact sequence $0 \to \mathcal{M}_x \to Q \to N \to 0$, where Q is a multiple of the $\mathcal{A}_x$-module $\mathcal{P}'_x$ and $N \in \mathrm{CM}\,\mathcal{A}_x$.*
2. *The restriction of the functor $\mathsf{F}^\dagger$ onto $\mathrm{CM}\,\mathcal{B}$ is left adjoint to the restriction of F onto $\mathrm{CM}\,\mathcal{A}$. Moreover, if $\mathcal{M} \in \mathrm{CM}\,\mathcal{B}$, the natural map $\mathsf{F}\mathsf{F}^\dagger(\mathcal{M}) \to \mathcal{M}$ is an isomorphism, and the functors $\mathsf{F}^\dagger$ and F define an equivalence of the categories $\mathrm{CM}\,\mathcal{B}$ and $\mathrm{CM}\,\mathcal{P}$, where $\mathrm{CM}\,\mathcal{P}$ is the full subcategory of $\mathrm{CM}\,\mathcal{A}$ consisting of all modules $\mathcal{M}$ such that for every point $x \in X$ there is an epimorphism $n\mathcal{P}_x \to \mathcal{M}$.*

Proof

1. This statement is local, so we can suppose that $X = \mathrm{spec}\,\boldsymbol{R}$, where $\boldsymbol{R}$ is an excellent local reduced ring of Krull dimension 1, $\mathcal{A} = \boldsymbol{A}^\sim$ for some $\boldsymbol{R}$-order $\boldsymbol{A}$, i.e., an $\boldsymbol{R}$-algebra $\boldsymbol{A}$ without nilpotent ideals which is finitely generated and Cohen–Macaulay as an $\boldsymbol{R}$-module, $\mathcal{P} = P^\sim$ for some finitely generated

projective $\boldsymbol{A}$-module P. Moreover, we can suppose that P is sincere as $\boldsymbol{A}$-module. Then $\mathcal{B} = \boldsymbol{B}^{\sim}$, where $\boldsymbol{B} = \operatorname{End}_{\boldsymbol{A}} P$. If $M \in \mathrm{CM}\,\boldsymbol{B}$, there is an exact sequence $m\boldsymbol{B} \to n\boldsymbol{B} \to M^* \to 0$, which gives an exact sequence:

$$0 \to M \to n\boldsymbol{B}^* \to m\boldsymbol{B}^*. \tag{2}$$

We denote by ψ the natural morphism $\mathbb{1}_{\boldsymbol{A}\text{-Mod}} \to \mathsf{F}^!\mathsf{F}$. One easily sees that $\mathsf{F}(P') \simeq \boldsymbol{B}^*$ and $\mathsf{F}^!(\boldsymbol{B}^*) \simeq P'$, so $\psi(P')$ is an isomorphism. The exact sequence (2) gives an exact sequence $0 \to \mathsf{F}^!(M) \to nP' \to mP'$, which shows that $\mathsf{F}^!(M) \in \mathrm{CM}'\,P$.

Let now $M \in \mathrm{CM}'\,P$. An exact sequence $0 \to M \to nP' \to N \to 0$, where $N \in \mathrm{CM}\,\boldsymbol{A}$, gives an exact sequence $0 \to \mathsf{F}(M) \to \mathsf{F}(nP') \to \mathsf{F}(N) \to 0$. For any $\boldsymbol{A}$-module N, $\psi(N)$ is the homomorphism:

$$h : N \to \mathcal{H}om_{\boldsymbol{B}}(P^\vee, \mathcal{H}om_{\boldsymbol{A}}(P, N)) \simeq \mathcal{H}om_{\boldsymbol{A}}(P \otimes_{\boldsymbol{B}} P^\vee, N)$$

such that $h(u)(\alpha \otimes \gamma) = \gamma(\alpha)u$. Tensoring with $\boldsymbol{KA}$, we obtain the map $\boldsymbol{K}N \to \mathcal{H}om_{\boldsymbol{KA}}(\boldsymbol{K}P \otimes_{\boldsymbol{KB}} \boldsymbol{K}P^\vee, \boldsymbol{K}N)$. As $\boldsymbol{KA}$ is semi-simple and $\boldsymbol{K}P$ is sincere, the natural map $\boldsymbol{K}P \otimes_{\boldsymbol{KB}} \boldsymbol{K}P^\vee \to \boldsymbol{A}$ is surjective; therefore, $\boldsymbol{K}\psi(N)$ is injective. If N is torsion-free, hence embeds into $\boldsymbol{K}N$, it implies that $\psi(N)$ is injective. So we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\psi(M)"] & nP' \arrow[r] \arrow[d, "\psi(nP')"] & N \arrow[d, "\psi(N)"] \\
0 \arrow[r] & \mathsf{F}^!\mathsf{F}(M) \arrow[r] & \mathsf{F}^!\mathsf{F}(nP') \arrow[r] & \mathsf{F}^!\mathsf{F}(N).
\end{tikzcd}$$

Since $\psi(nP')$ is an isomorphism and $\psi(N)$ is a monomorphism, $\psi(M)$ is an isomorphism. As the natural map $\mathsf{F}\mathsf{F}^! \to \mathbb{1}_{\boldsymbol{B}\text{-Mod}}$ is an isomorphism, it proves the statement (*1*).

2. If $\mathcal{M} \in \mathrm{CM}\,\mathcal{B}$ and $\mathcal{N} \in \mathrm{CM}\,\mathcal{A}$, then also $\mathsf{F}\mathcal{N} \in \mathrm{CM}\,\mathcal{B}$, so

$$\mathcal{H}om_{\mathcal{A}}(\mathsf{F}^\dagger\mathcal{M}, \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{A}}(\mathsf{F}^*\mathcal{M}, \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathsf{F}\mathcal{N}),$$

which proves the first claim. Consider now the functors

$$\mathsf{F}^{\mathrm{op}} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}^\vee, \_) : \mathrm{Qcoh}\,\mathcal{A}^{\mathrm{op}} \to \mathrm{Qcoh}\,\mathcal{B}^{\mathrm{op}},$$
$$(\mathsf{F}^{\mathrm{op}})^! = \mathrm{Hom}_{\mathcal{B}}(\mathcal{P}, \_) : \mathrm{Qcoh}\,\mathcal{B}^{\mathrm{op}} \to \mathrm{Qcoh}\,\mathcal{A}^{\mathrm{op}}.$$

As we have just proved, they establish an equivalence between the categories $\mathrm{CM}\,\mathcal{B}^{\mathrm{op}}$ and $\mathrm{CM}'\,\mathcal{P}^\vee$, where $\mathrm{CM}'\,\mathcal{P}^\vee$ consists of all right $\mathcal{A}$-modules $\mathcal{N}$ such that for every point $x \in X$ there is an exact sequence $0 \to \mathcal{N}_x \to Q \to N' \to 0$, where Q is a multiple of $\mathcal{P}_x^*$ and $N' \in \mathrm{CM}\,\mathcal{A}_x$. Equivalently, there is an epimorphism

$Q^* \to \mathcal{N}_x^*$, i.e., $\mathcal{N}^* \in \mathrm{CM}\,\mathcal{P}$. On the other hand,

$$(\mathsf{F}^{\mathrm{op}})^! \mathcal{M}^* = \mathrm{Hom}_{\mathcal{B}}(\mathcal{P}, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \omega_X)) \simeq$$
$$\simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P} \otimes_{\mathcal{B}} \mathcal{M}, \omega_X) = (\mathcal{P} \otimes_{\mathcal{B}} \mathcal{M})^* = (\mathsf{F}^{\dagger} \mathcal{M})^*.$$

Therefore, the statement (2) follows by duality. □

6 König's Resolution

A non-commutative curve $(X, \mathcal{A}')$, where $\mathcal{A} \subseteq \mathcal{A}' \subset \mathcal{K}\mathcal{A}$, is called an *over-ring* of the non-commutative curve $(X, \mathcal{A})$. If $\mathcal{A}$ has no proper over-rings, it is called *normal*. Since X is excellent and $\mathcal{A}$ is reduced, the set of over-rings of $\mathcal{A}$ satisfies the maximality condition, i.e., there are no infinite ascending chains of over-rings. (It follows, for instance, from [27, Chap. 5] or from [17].) In particular, there is always a normal over-ring of $\mathcal{A}$. In non-commutative case, such a normal over-ring is usually not unique, though all of them are locally conjugate inside $\mathcal{K}\mathcal{A}$ [27, Theorem 18.7], and every normal non-commutative curve is hereditary [27, Theorem 18.1]. Thus every non-commutative curve has a hereditary over-ring, and usually a lot of them. Actually, there is one "special" hereditary over-ring which plays an important role in this section.

Let $(X, \mathcal{A})$ be a non-commutative curve. Consider the ideal $\mathcal{J} = \mathcal{J}_A$ defined by its localizations as follows:

$$\mathcal{J}_x = \begin{cases} \mathcal{A}_x & \text{if } \mathcal{A}_x \text{ is hereditary,} \\ \operatorname{rad} \mathcal{A}_x & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}^\sharp = \mathcal{E}nd_{\mathcal{A}^{\mathrm{op}}} \mathcal{J}$ (the endomorphism algebra of $\mathcal{J}$ as of right $\mathcal{A}$-module). Note that $\mathcal{A}^\sharp$ can (and will) be identified with the over-ring of $\mathcal{A}$ such that its x-localization coincides with $\{ \lambda \in \mathcal{K}\mathcal{A}_x \mid \lambda \mathcal{J}_x \subseteq \mathcal{J}_x \}$ for each $x \in X_{\mathrm{cl}}$. It is known [27, Theorem 39.14] that $\mathcal{A}^\sharp = \mathcal{A}$ if and only if $\mathcal{A}$ is hereditary. So there is a chain of over-rings of $\mathcal{A}$:

$$\mathcal{A} = \mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots \subset \mathcal{A}_n \subset \mathcal{A}_{n+1} = \tilde{\mathcal{A}},$$

where $\mathcal{A}_{i+1} = \mathcal{A}_i^\sharp$ for $1 \le i \le n$ and $\tilde{\mathcal{A}}$ is hereditary. We call n the *level* of $\mathcal{A}$. For instance, a usual (commutative) curve over an algebraically closed field is of level 1 if and only if all its singular points are simple nodes or cusps. (The derived categories of such curves were investigated in [7].)

Consider the endomorphism algebra $\mathcal{R} = \mathcal{R}_{\mathcal{A}} = (\mathcal{E}nd_{\mathcal{A}} \bigoplus_{i=1}^{n+1} \mathcal{A}_i)^{\mathrm{op}}$. We call it the *König's resolution* of the non-commutative curve $\mathcal{A}$, since it is analogous to that considered in [23] (though does not coincide with it even in case if orders over discrete valuation rings) and has analogous properties.

We identify $\mathcal{R}_{\mathcal{A}}$ with the ring of matrices:

$$\mathcal{R} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \dots & \mathcal{A}_{1n} & \mathcal{A}_{1,n+1} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \dots & \mathcal{A}_{2n} & \mathcal{A}_{2,n+1} \\ & \dots & \dots & \dots & \\ \mathcal{A}_{n1} & \mathcal{A}_{n2} & \dots & \mathcal{A}_{nn} & \mathcal{A}_{n,n+1} \\ \mathcal{A}_{n+1,1} & \mathcal{A}_{n+1,2} & \dots & \mathcal{A}_{n+1,n} & \mathcal{A}_{n+1,n+1} \end{pmatrix},$$

where $\mathcal{A}_{ij} = \mathcal{H}om_{\mathcal{A}}(\mathcal{A}_i, \mathcal{A}_j)$. Note that $\mathcal{A}_{ij} = \mathcal{A}_j$ if $i \le j$ and $\mathcal{A}_{i+1,i} = \mathcal{J}_{\mathcal{A}_i}$. We denote by e_j the standard diagonal idempotents in $\mathcal{R}$ and set $\mathcal{P} = \mathcal{R}e_1, \tilde{\mathcal{P}} = \mathcal{R}e_{n+1}$. Then $(\mathcal{E}nd_{\mathcal{R}}\,\mathcal{P})^{\mathrm{op}} \simeq \mathcal{A}$, so $\mathcal{A}$ is a minor of $\mathcal{R}$ and the categories $\mathcal{A}$-Mod and $\mathscr{D}\mathcal{A}$ are bilocalization, respectively, of $\mathcal{R}$-Mod and $\mathscr{D}\mathcal{R}$. The corresponding functors are $\mathsf{F} = \mathcal{H}om_{\mathcal{R}}(\mathcal{P}, \_)$ and its left derived functor LF. In the same way, $\tilde{\mathcal{A}} \simeq (\mathcal{E}nd_{\mathcal{R}}\,\tilde{\mathcal{P}})^{\mathrm{op}}$ is a minor of $\mathcal{R}$, so the categories $\tilde{\mathcal{A}}$-Mod and $\mathscr{D}\tilde{\mathcal{A}}$ are bilocalization, respectively, of $\mathcal{R}$-Mod and $\mathscr{D}\mathcal{R}$. The corresponding functors are $\tilde{\mathsf{F}} = \mathcal{H}om_{\mathcal{R}}(\tilde{\mathcal{P}}, \_)$ and its left derived functor $\mathsf{L}\tilde{\mathsf{F}}$. Thus we have a diagram of bilocalizations

$$\begin{tikzcd} \tilde{\mathcal{A}}\text{-Mod} \arrow[r, bend left, "\tilde{\mathsf{F}}^*"] \arrow[r, bend right, "\tilde{\mathsf{F}}^!"'] & \mathcal{R}\text{-Mod} \arrow[l, "\tilde{\mathsf{F}}" description] \arrow[r, "\mathsf{F}" description] & \mathcal{A}\text{-Mod} \arrow[l, bend right, "\mathsf{F}^*"'] \arrow[l, bend left, "\mathsf{F}^!"] \end{tikzcd} \tag{3}$$

Since $\tilde{\mathcal{A}}$ is an over-ring of $\mathcal{A}$, there is a morphism $\nu : (X, \tilde{\mathcal{A}}) \to (X, \mathcal{A})$. According to Proposition 3.7, we can replace here $(X, \tilde{\mathcal{A}})$ by $(\tilde{X}, \tilde{\mathcal{A}})$ where $\tilde{X} = \operatorname{spec}(\operatorname{center}(\tilde{\mathcal{A}}))$. In case of "usual" schemes, when $\mathcal{A} = \mathcal{O}_X$, $\tilde{X}$ is the normalization of X and ν is the normalization map. The morphism ν induces the functor of direct image $\nu_* : \tilde{\mathcal{A}}$-Mod $\to \mathcal{A}$-Mod and its left and right adjoints ν^* and $\nu^!$, which are functors $\mathcal{A}$-Mod $\to \tilde{\mathcal{A}}$-Mod. It so happens that these functors, maybe up to twist, are compositions of the functors from the diagram (3).

Theorem 6.31

1. $\mathsf{F}\tilde{\mathsf{F}}^* \simeq \nu_*$ *and* $\tilde{\mathsf{F}}\mathsf{F}^! \simeq \nu^!$.
2. $\tilde{\mathsf{F}}\mathsf{F}^* \simeq \mathcal{C}\otimes_{\tilde{\mathcal{A}}}\nu^*\_$ *and* $\mathsf{F}\tilde{\mathsf{F}}^! \simeq \nu_*(\mathcal{C}'\otimes_{\tilde{\mathcal{A}}}\_)$*, where* $\mathcal{C} = \mathcal{H}om_{\mathcal{A}}(\tilde{\mathcal{A}}, \mathcal{A}) = \mathcal{A}_{n+1,1}$ *is the conductor of* $\tilde{\mathcal{A}}$ *in* $\mathcal{A}$ *and* $\mathcal{C}' = \mathcal{H}om_{\tilde{\mathcal{A}}}(\mathcal{C}, \tilde{\mathcal{A}})$ *is its dual* $\tilde{\mathcal{A}}$*-module.*

Proof We verify the equalities (*1*). Indeed, since $e_1\tilde{\mathcal{P}} = \tilde{\mathcal{A}}$ as $\mathcal{A}$-$\tilde{\mathcal{A}}$-bimodule,

$$\mathsf{F}\tilde{\mathsf{F}}^*(\mathcal{M}) = \mathcal{H}om_{\mathcal{R}}(\mathcal{P}, \tilde{\mathcal{P}}\otimes_{\tilde{\mathcal{A}}}\mathcal{M}) \simeq e_1\tilde{\mathcal{P}}\otimes_{\tilde{\mathcal{A}}}\mathcal{M} \simeq \mathcal{M}$$

considered as $\mathcal{A}$-module, which is just $\nu_*(\mathcal{M})$. Also

$$\begin{aligned} \tilde{\mathsf{F}}\mathsf{F}^!(\mathcal{N}) &= \mathcal{H}om_{\mathcal{R}}(\tilde{\mathcal{P}}, \mathcal{H}om_{\mathcal{A}}(\mathcal{P}^\vee, \mathcal{N})) \simeq e_{n+1}\,\mathcal{H}om_{\mathcal{A}}(\mathcal{P}^\vee, \mathcal{N}) \\ &\simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{P}^\vee e_{n+1}, \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{A}}(\tilde{\mathcal{A}}, \mathcal{N}) = \nu^!(\mathcal{N}). \end{aligned}$$

The equalities (*2*) are proved analogously (see also [9]). □

Theorem 6.32 *Let $\varepsilon_k = \sum_{j=k}^{n+1} e_j$, $\mathcal{I}_k = \mathcal{R}\varepsilon_k\mathcal{R}$, $\mathcal{Q}_k = \mathcal{R}/\mathcal{I}_{k+1}$, and $\mathcal{P}_k = \mathcal{Q}_k e_k$. Then $(\mathcal{R}, \tilde{\mathcal{P}}, \mathcal{Q}_n, \mathcal{P}_n, \mathcal{Q}_{n-1}, \mathcal{P}_{n-1}, \dots, \mathcal{P}_2, \mathcal{Q}_1)$ is a heredity relating chain between $\mathcal{R}$ and $\mathcal{Q}_1 \simeq \mathcal{A}/\mathcal{J}_{\mathcal{A}}$. Moreover, $(\mathcal{E}nd_{\mathcal{Q}_k} \mathcal{P}_i)^{\mathrm{op}} \simeq \mathcal{A}_k/\mathcal{J}_{\mathcal{A}_k}$ is a semi-simple algebra, so $\mathcal{R}$ is a quasi-hereditary non-commutative scheme of level n and $\operatorname{gl.dim} \mathcal{R} \le 2n$.*

Proof A straightforward calculation shows that $\mathcal{I}_k$ is the ideal of matrices:

$$\mathcal{I}_k = \begin{pmatrix} \mathcal{A}_{k1} & \mathcal{A}_{k2} & \dots & \mathcal{A}_{k,k-1} & \mathcal{A}_k & \mathcal{A}_{k+1} & \dots & \mathcal{A}_{n+1} \\ \mathcal{A}_{k1} & \mathcal{A}_{k2} & \dots & \mathcal{A}_{k,k-1} & \mathcal{A}_k & \mathcal{A}_{k+1} & \dots & \mathcal{A}_{n+1} \\ \multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ \mathcal{A}_{k1} & \mathcal{A}_{k2} & \dots & \mathcal{A}_{k,k-1} & \mathcal{A}_k & \mathcal{A}_{k+1} & \dots & \mathcal{A}_{n+1} \\ \mathcal{A}_{k+1,1} & \mathcal{A}_{k+1,2} & \dots & \mathcal{A}_{k+1,k-1} & \mathcal{A}_{k+1,k} & \mathcal{A}_{k+1} & \dots & \mathcal{A}_{n+1} \\ \multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ \mathcal{A}_{n+1,1} & \mathcal{A}_{n+1,2} & \dots & \mathcal{A}_{n+1,k-1} & \mathcal{A}_{n+1,k} & \mathcal{A}_{n+1,k+1} & \dots & \mathcal{A}_{n+1} \end{pmatrix}$$

Hence, $\mathcal{Q}_k$ is the algebra of $k \times k$ matrices (a_{ij}), where $a_{ij} \in \mathcal{A}_{ij}/\mathcal{A}_{k+1,j}$. In particular, $a_{ik} \in \mathcal{A}_k/\mathcal{A}_{i+1,k} = \mathcal{A}_k/\mathcal{J}_{\mathcal{A}_k}$ and this algebra is semi-simple. Therefore, $(\mathcal{E}nd_{\mathcal{Q}_k} \mathcal{P}_k)^{\mathrm{op}} \simeq e_k\mathcal{Q}_k e_k = \mathcal{A}_{kk}/\mathcal{A}_{k+1,k}$ is semi-simple. Obviously, $\mathcal{I}_{\mathcal{P}_k} = \mathcal{Q}_k e_k \mathcal{Q}_k = \mathcal{I}_{k+1}/\mathcal{I}_k$, hence $\mathcal{Q}_{k-1} \simeq \mathcal{Q}_k/\mathcal{I}_{\mathcal{P}_k}$, so we have indeed a relating chain. Moreover, $\mathcal{I}_k$ is obviously projective as right $\mathcal{R}$-module, hence $\mathcal{I}_k/\mathcal{I}_{k+1}$ is projective as right $\mathcal{Q}_k$-module and this relating chain is heredity. As $\tilde{\mathcal{A}} = (\mathcal{E}nd_{\mathcal{R}} \tilde{\mathcal{P}})^{\mathrm{op}}$ is hereditary and all $(\mathcal{E}nd_{\mathcal{Q}_k} \mathcal{P}_k)^{\mathrm{op}}$ are semi-simple, $\mathcal{R}$ is quasi-hereditary and $\operatorname{gl.dim} \mathcal{R} \le 2n$. □

Thus the functor $\mathsf{DF} : \mathscr{D}\mathcal{R} \to \mathscr{D}\mathcal{A}$ defines a *categorical resolution* of the derived category $\mathscr{D}\mathcal{A}$ in the sense of [24]. If $\mathcal{A}$ is strongly Gorenstein, Theorem 3.13 shows that this resolution is even *weakly crepant*, i.e., the restrictions of its left and right adjoint functors coincide on perfect complexes (small objects in $\mathscr{D}\mathcal{A}$).

We denote by $\bar{\mathcal{A}}_k$ the semi-simple algebra $\mathcal{A}_k/\mathcal{J}_{\mathcal{A}_k}$.

Corollary 6.33 *The derived category $\mathscr{D}\mathcal{R}$ has two semi-orthogonal decompositions: $\mathscr{D}\mathcal{R} = \langle \mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_n, \mathscr{T} \rangle$ and $\mathscr{D}\mathcal{R} = \langle \mathscr{T}', \mathscr{T}'_n, \dots, \mathscr{T}'_2, \mathscr{T}'_1 \rangle$, where $\mathscr{T} \simeq \mathscr{T}' \simeq \mathscr{D}\tilde{\mathcal{A}}$ and $\mathscr{T}_k \simeq \mathscr{T}'_k \simeq \mathscr{D}\bar{\mathcal{A}}_k$.*

Remark 6.34 Note that usually $\mathscr{T} \ne \mathscr{T}'$ as well as $\mathscr{T}_k \ne \mathscr{T}'_k$ for $k > 1$, though $\mathscr{T}_1 = \mathscr{T}'_1 = \mathscr{D}(\mathcal{R}/\mathcal{I}_2)$ naturally embedded into $\mathscr{D}\mathcal{R}$.

7 Tilting on Rational Curves

We say that a non-commutative curve $(X, \mathcal{A})$ is *rational* if X is a rational projective curve over an algebraically closed field $\Bbbk$ and $\mathcal{A}$ is central. Since the Brauer group of the field of rational functions $\Bbbk(t)$ is trivial [26], then $\mathcal{K}\mathcal{A} \simeq \mathrm{Mat}(m, \mathcal{K})$ for some

m. In this case the structure of hereditary non-commutative curves is well-known (see, for instance, [11] or [8]). Namely, if such a curve is connected, then $X = \mathbb{P}^1$, and up to Morita equivalence, this curve is given by a function $\mathbf{r} : X_{\rm cl} \to \mathbb{N}$ such that $\mathbf{r}(x) = 1$ for almost all points. A representative $\mathcal{H}(\mathbf{r})$ of the Morita class defined by this function can be defined as follows. Choose $m \in \mathbb{N}$ such that $m \geq \mathbf{r}(x)$ for all $x \in X_{\rm cl}$ and choose partitions $m = \sum_{k=1}^{\mathbf{r}(x)} m_{xk}$ for every x. Set $\hat{m}_{xk} = \sum_{l=1}^{k} m_{xl}$. Let $\mathcal{H}_x$ be the subalgebra in $\mathrm{Mat}(m, \mathcal{O}_{X,x})$ consisting of all matrices (a_{ij}) such that $a_{ij}(x) = 0$ if $i \leq \hat{m}_{xk}$ and $j > \hat{m}_{xk}$ for some k. Then $\mathcal{H}(\mathbf{r})$ is the subsheaf of $\mathrm{Mat}(m, \mathcal{O}_X)$ such that its x-stalk equals $\mathcal{H}_x$.

It is also known that $\mathcal{H}(\mathbf{r})$ has a *tilting module*, i.e., a coherent $\mathcal{H}(\mathbf{r})$-module $\mathcal{T}$ such that $\mathrm{pr.dim}\,\mathcal{T} < \infty$, $\mathrm{Ext}^q_{\mathcal{H}(\mathbf{r})}(\mathcal{T}, \mathcal{T}) = 0$ for all $q > 0$ and $\mathcal{T}$ generates the derived category $\mathscr{D}\mathcal{H}(\mathbf{r})$. Namely, let $\mathcal{H} = \mathcal{H}(\mathbf{r})$, $\mathcal{L} = \mathcal{O}_X^m$ considered as $\mathcal{H}$-module and $\mathbf{S} = \{\, x \in X_{\rm cl} \mid \mathbf{r}(x) > 1 \,\}$. If $\mathbf{S} = \{\, x_1, x_2, \dots, x_s \,\}$ with $s > 1$, we suppose that $x_1 = (1 : 0)$, $x_2 = (0 : 1)$ and $x_i = (1 : \lambda_i)$ for $1 < i \leq s$, where $\lambda \in \Bbbk \setminus \{\, 0, 1 \,\}$, and set $r_i = \mathbf{r}(x_i)$. If $\#(\mathbf{S}) = 1$, we set $s = 2$, $r_1 = \mathbf{r}(x_1)$, $r_2 = 1$. If $\mathbf{S} = \emptyset$, then $\mathcal{H} = \mathrm{Mat}(m, \mathcal{O}_X)$ is Morita equivalent to $\mathcal{O}_X$, so $\mathcal{L} \oplus \mathcal{L}(1)$ is a tilting sheaf for $\mathcal{H}$. In this case we also set $s = 2$, $r_1 = r_2 = 1$. Consider the submodule $\mathcal{L}(x, k) \subseteq \mathcal{L}$ such that $\mathcal{L}(x, k)_y = \mathcal{L}_y$ for $y \neq x$ and $\mathcal{L}(x, k)_x$ consists of all vectors $(a_i)_{1 \leq i \leq k}$ such that $a_i(x) = 0$ for $i \leq \hat{m}_k$ and set $\mathcal{T} = \mathcal{L} \oplus \mathcal{L}(1) \oplus \big(\bigoplus_{\mathbf{r}(x) > 1} \bigoplus_{k=1}^{\mathbf{r}(x)-1} \mathcal{L}(x, k) \big)$.

Theorem 7.35 (See [8][2])

1. *$\mathcal{T}$ is a tilting module for $\mathcal{H}$.*
2. *$(\mathcal{E}nd_{\mathcal{H}}\, \mathcal{T})^{\rm op} \simeq \boldsymbol{R}(\mathbf{r}, \boldsymbol{\lambda})$, where $\boldsymbol{R}(\mathbf{r}, \boldsymbol{\lambda})$ is the* canonical algebra *defined by the sequences $\mathbf{r} = (r_1, r_2, \dots, r_s)$ and $\boldsymbol{\lambda} = (\lambda_3, \dots, \lambda_s)$, i.e., the algebra given by the quiver*

$$\begin{tikzcd}
 & \bullet \arrow[r, "\alpha_{21}"] & \bullet & \cdots & \bullet \arrow[r, "\alpha_{r_1-1,1}"] & \bullet \arrow[dr, "\alpha_{r_1 1}"] & \\
\bullet \arrow[ur, "\alpha_{11}"] \arrow[r, "\alpha_{12}"'] \arrow[ddr, "\alpha_{1s}"'] & \bullet \arrow[r, "\alpha_{22}"'] & \bullet & \cdots & \bullet \arrow[r, "\alpha_{r_2-1,2}"'] & \bullet \arrow[r, "\alpha_{r_2 2}"'] & \bullet \\
 & & & \vdots & & & \\
 & \bullet \arrow[r, "\alpha_{2s}"'] & \bullet & \cdots & \bullet \arrow[r, "\alpha_{r_s-1,s}"'] & d\bullet \arrow[uur, "\alpha_{r_s s}"'] &
\end{tikzcd}$$

with relations $\alpha_j = \alpha_1 + \lambda_j \alpha_2$ for $3 \leq j \leq s$, where $\alpha_j = \alpha_{r_j j} \dots \alpha_{2j} \alpha_{1j}$ [28, Sect. 3.7].

Note that if $s = 2$, it is just the quiver algebra of the quiver $\tilde{A}_{r_1, r_2}$; if, moreover, $r_1 = r_2 = 1$, it is the Kronecker algebra. Note also that any canonical algebra is triangular, hence quasi-hereditary.

[2] It also follows from [19], since $\mathcal{H}(\mathbf{r})$ is Morita equivalent to the weighted projective line $C(\mathbf{r}, \mathbf{S})$.

Obviously, if a rational hereditary non-commutative scheme $(X, \mathcal{H})$ is not connected, it splits into a direct product of connected hereditary non-commutative schemes. Therefore it has a tilting module $\mathcal{T}$ such that $(\mathcal{E}nd_{\mathcal{H}})^{\mathrm{op}}$ is a direct product of canonical algebras.

Let now $(X, \mathcal{A})$ be a rational non-commutative curve, $\mathcal{R}$ be its König's resolution. We use the notations of the preceding section. The hereditary non-commutative curve $\tilde{\mathcal{A}}$ has a tilting module $\mathcal{T}$ such that $(\mathcal{E}nd_{\tilde{\mathcal{A}}}\,\mathcal{T})^{\mathrm{op}} = \boldsymbol{R}$ is a direct product of canonical algebras. Then $\tilde{\mathcal{T}} = \tilde{\mathsf{F}}(\mathcal{T})$ generates $\operatorname{Im}\tilde{\mathsf{F}}$ and $\operatorname{Ext}^q_{\mathcal{H}(\mathbf{r})}(\mathcal{T}, \mathcal{T}) = 0$ for all $q > 0$. As $\langle \ker\tilde{\mathsf{F}}, \operatorname{Im}\tilde{\mathsf{F}} \rangle$ is a semi-orthogonal decomposition of $\mathscr{D}\mathcal{R}$, also $\operatorname{pr.dim}\tilde{\mathcal{T}} < \infty$. As $\mathcal{Q}$ generates $\mathscr{D}\mathcal{Q}$, which can be identified with $\ker\tilde{\mathsf{F}}$, $\mathcal{Q}\oplus\tilde{\mathcal{T}}$ generates $\mathscr{D}\mathcal{R}$. Note that $\dim\operatorname{supp}\mathcal{Q} = 0$; therefore, $\operatorname{Ext}^q_{\mathcal{R}}(\mathcal{Q}, \mathcal{M}) = H^0(X, \mathcal{E}xt^q_{\mathcal{R}}(\mathcal{Q}, \mathcal{M}))$ for every quasi-coherent module $\mathcal{M}$. A locally projective resolution of $\mathcal{Q}$ is $0 \to \tilde{\mathcal{I}} \xrightarrow{\iota} \mathcal{R} \to \mathcal{Q} \to 0$. Thus $\operatorname{pr.dim}_{\mathcal{R}}\mathcal{Q} = 1$. Moreover, $\operatorname{Ext}^1_{\mathcal{R}}(\mathcal{Q}, \mathcal{N}) = 0$ for any $\mathcal{Q}$-module $\mathcal{N}$, because $\tilde{\mathcal{I}}^2 = \tilde{\mathcal{I}}$ and $\tilde{\mathcal{I}}\mathcal{N} = 0$, thus $\operatorname{Hom}_{\mathcal{R}}(\tilde{\mathcal{I}}, \mathcal{N}) = 0$. Obviously, $\operatorname{Hom}_{\mathcal{R}}(\mathcal{Q}, \mathcal{T}) = 0$. It implies the following result:

Theorem 7.36 $\mathcal{T}^+ = \mathcal{Q}[-1]\oplus\tilde{\mathcal{T}}$ *is a* tilting complex *for* $\mathcal{R}$, *i.e., it belongs to* $\mathsf{Perf}\mathcal{R}$, *generates* $\mathscr{D}\mathcal{R}$ *and* $\operatorname{Hom}_{\mathscr{D}\mathcal{R}}(\mathcal{T}^+, \mathcal{T}^+[k]) = 0$ *if* $k \neq 0$. *Therefore* $\mathscr{D}\mathcal{R} \simeq \mathscr{D}\boldsymbol{E}$, *where* $\boldsymbol{E} = (\mathcal{E}nd_{\mathscr{D}\mathcal{R}}\,\mathcal{T}^+)^{\mathrm{op}}$.

Note that $\boldsymbol{E}$ can be considered as the algebra of triangular matrices:

$$\boldsymbol{E} = \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{T} \\ 0 & \boldsymbol{R} \end{pmatrix}, \tag{4}$$

where $\boldsymbol{R} = (\mathcal{E}nd_{\tilde{\mathcal{A}}}\,\mathcal{T})^{\mathrm{op}}$ is a direct product of canonical algebras and $\boldsymbol{T} = \operatorname{Ext}^1_{\mathcal{R}}(\mathcal{Q}, \tilde{\mathcal{T}}) \simeq \operatorname{Hom}_{\mathcal{R}}(\tilde{\mathcal{I}}, \tilde{\mathcal{T}})/\iota^*\operatorname{Hom}_{\mathcal{R}}(\mathcal{R}, \tilde{\mathcal{T}})$. Note that $\tilde{\mathcal{I}} \simeq \bigoplus_{i=1}^{n+1}\tilde{\mathsf{F}}\mathcal{A}_{n+1,i}$, whence $\boldsymbol{T} \simeq \bigoplus_{i=1}^{n+1}\operatorname{Hom}_{\tilde{\mathcal{A}}}(\mathcal{A}_{n+1,i}, \mathcal{T})$.

Corollary 7.37 *For every rational non-commutative curve* $(X, \mathcal{A})$*, there is a finite dimensional quasi-hereditary algebra* $\boldsymbol{E}$ *and a bilocalizing functor* $\mathscr{D}\boldsymbol{E} \to \mathscr{D}\mathcal{A}$.

Proof In the triangular presentation (4) of the algebra $\boldsymbol{E}$, let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $I = \boldsymbol{E}e\boldsymbol{E} = \begin{pmatrix} 0 & \boldsymbol{T} \\ 0 & \boldsymbol{R} \end{pmatrix}$ is projective as $\boldsymbol{E}$-module and $e\boldsymbol{E}e \simeq \boldsymbol{R}$ is triangular. Hence $\boldsymbol{E}$ is quasi-hereditary by Lemma 4.27.

Thus every rational non-commutative curve has a categorical resolution by a finite dimensional quasi-hereditary algebra. If the curve is strongly Gorenstein, this resolution is weakly crepant. In particular, it is the case for "usual" (commutative) rational curves. Note that $\boldsymbol{Q} = \prod_x \boldsymbol{Q}_x$, where x runs through all points such that $\mathcal{A}_x$ is not hereditary (in the commutative case through singular points of X).

Example 7.38 (See [9, Sect. 8]) We consider the input $\boldsymbol{Q}_x$ for *simple singularities* of (usual) plain curves in the sense of [2]. We present it as a quiver with relations.

1. If x is of type A_m, $m > 2$, then

$$\boldsymbol{Q}_x = \begin{tikzcd} 1 \arrow[r, bend right, "\beta_1"'] & 2 \arrow[l, bend right, "\alpha_1"'] \arrow[r, bend right, "\beta_2"'] & 3 \arrow[l, bend right, "\alpha_2"'] \arrow[r, dotted, no head] & (n-1) \arrow[r, bend right, "\beta_{n-1}"'] & n \arrow[l, bend right, "\alpha_{n-1}"'] \end{tikzcd}$$

$$\begin{aligned} &\beta_k\alpha_k = \alpha_{k+1}\beta_{k+1} \quad \text{if} \quad 1 \le k < n-1, \\ &\beta_{n-1}\alpha_{n-1} = 0, \end{aligned}$$

where $n = \left[\frac{m+1}{2}\right]$. Note that for $m \le 2$, the algebra $\boldsymbol{Q}_x$ is semisimple.

2. If x is of type D_m, $m \ge 4$, then

$$\boldsymbol{Q}_x = \begin{tikzcd} 1 \arrow[r, "\beta_1"] \arrow[r, bend right, "\beta'"'] & 2 \arrow[l, bend right, "\alpha_1"'] \arrow[r, bend right, "\beta_2"'] & 3 \arrow[l, bend right, "\alpha_2"'] \arrow[r, bend right, "\beta_3"'] & 4 \arrow[l, bend right, "\alpha_3"'] \arrow[r, dotted, no head] & (n-1) \arrow[r, bend right, "\beta_{n-1}"'] & n \arrow[l, bend right, "\alpha_{n-1}"'] \end{tikzcd}$$

$$\begin{aligned} &\beta_k\alpha_k = \alpha_{k+1}\beta_{k+1} \quad \text{if} \quad 1 \le k < n-1, \\ &\beta_{n-1}\alpha_{n-1} = 0, \\ &\beta'\alpha_1 = 0, \\ &\beta_2\beta' = 0, \end{aligned}$$

where $n = \left[\frac{m}{2}\right]$.

3. If x is of type E_6, $\boldsymbol{Q}_x$ is the same as for D_4, and if x is of type E_7 or E_8, $\boldsymbol{Q}_x$ is the same as for D_6.

4. Finally, we consider a "global" example, where X has two irreducible rational components X_1, X_2 and three singular points $x_1 \in X_1$ of type E_6, $x_2 \in X_1 \cap X_2$ of type D_7, and $x_3 \in X_2$ of type A_5. Then the algebra $\boldsymbol{E}$ has the quiver

$$\begin{tikzcd} & & 1_1 \arrow[r, bend left, "\alpha_1"] & 2_1 \arrow[l, "\beta'_1"] \arrow[l, bend left, "\beta_1"] & \\ -1_1 \arrow[r, bend left, "\xi_1"] \arrow[r, bend right, "\eta_1"'] & 0_1 \arrow[ur, bend left, "\gamma_{11}"] \arrow[ur, "\gamma_{12}"] \arrow[ur, bend right, "\gamma_{13}"] \arrow[dr, "\gamma_{23}"] & & & \\ & & 1_2 \arrow[r, bend left, "\alpha_{21}"] & 2_2 \arrow[l, "\beta'_2"] \arrow[l, bend left, "\beta_{21}"] \arrow[r, bend left, "\alpha_{22}"] & 3_2 \arrow[l, bend left, "\beta_{22}"] \\ -1_2 \arrow[r, bend left, "\xi_2"] \arrow[r, bend right, "\eta_2"'] & 0_2 \arrow[ur, bend left, "\gamma_{21}"] \arrow[ur, "\gamma_{22}"] \arrow[dr, "\gamma_{32}"] \arrow[dr, bend right, "\gamma_{31}"'] & & & \\ & & 1_3 \arrow[r, bend left, "\alpha_{31}"] & 2_3 \arrow[l, bend left, "\beta_{31}"] \arrow[r, bend left, "\alpha_{32}"] & 3_3 \arrow[l, bend left, "\beta_{32}"] \end{tikzcd}$$

It consists of "local parts" (formed by the vertices $1_i, 2_i, 3_i$) and "Kronecker parts" (formed by the vertices $0_j, -1_j$) arising from the components of $\tilde{X}$.

One can also explicitly describe the relations for the arrows γ_{ij} between these parts (they depend on the positions of the preimages of singular points on the components of $\tilde{X}$).

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