Posets

Yuriy Drozd

Institute of Mathematics Tereschenkivska 3, r. 229

https://www.imath.kiev.ua/∼drozd/Pos-beam.pdf
https://www.imath.kiev.ua/∼drozd/Pos-art.pdf

Kyiv 2020

Contents

1 Bisected posets

Definition. Let $\mathbf{S}$ be a *poset* (partially ordered set).

1. A *representation of the poset* $\mathbf{S}$ *over a field* $\Bbbk$ is, by definition, a monotone map $V : \mathbf{S} \to \mathsf{Sub}(V_0)$, where V_0 is a finite dimensional vector space over $\Bbbk$ and $\mathsf{Sub}(V_0)$ is the set of its subspaces.

   Monotone means that if $i \leqslant j$ in $\mathbf{S}$, then $V(i) \subseteq V(j)$.

2. A *morphism* $\varphi : V \to W$, where $W : \mathbf{S} \to \mathsf{Sub}(W_0)$ is a homomorphism of vector spaces $\varphi : V_0 \to W_0$ such that $\varphi(V(i)) \subseteq N(i)$ for every $i \in \mathbf{S}$.

We denote by $\mathrm{rep}(\mathbf{S}, \Bbbk)$ (or $\mathrm{rep}(\mathbf{S})$ if $\Bbbk$ is known) the category of representations of the poset $\mathbf{S}$ over the field $\Bbbk$.

We set $V(i)^- = \sum_{j<i} V(j)$ and choose a basis in $V(i) \bmod V^-(i)$. that is a set of vectors $\{\, u^i_k \mid 1 \leqslant k \leqslant d(i) \,\}$ such that their cosets $u^i_k + V^-(i)$ form a basis in $V(i)/V^-(i)$. Here $d(i) = \dim V(i)/V^-(i)$.

We also choose a basis $v_1, v_2, \dots, v_d$ in V_0. Let $u_k^i = \sum j = 1^d c_{jk}^i v_j$. Then the representation V is described by the set of matrices $\left\{ (c_{jk}^i) \right\} i \in \mathbf{S}$, which we also denote by $V(i)$.

If we change the basis in V_0, the matrices $V(i)$ are replaced by $S^{-1}V(i)$ for an invertible matrix S. For the change of basis in $V(i) \bmod V^-(i)$, one has to remember that if we add a vector from some $V(j)$ $(j < i)$ to some u_k^i, the coset $u_k^i + V^-(i)$ does not change. Therefore, if we change a basis in $V(i)$, the corresponding matrix changes in the way $V'(i) = V(i)S_i + \sum_{j<i} V(j)S_{ji}$ for an invertible S_i of size $d(i) \times d(i)$ and arbitrary S_{ji} of size $d(j) \times d(i)$.

As all matrices $V(i)$ have the same number of rows, they usually write them as one matrix divided into several vertical blocks corresponding to the elements of $\mathbf{S}$. If $j < i$, they draw an arrow from the j-th block to the i-th block, symbolizing that we can add the columns of the j-th to those of the i-th block.

Sometimes the same is presented as a quiver with the vertices from $\mathbf{S}$ and one new vertex "0" and arrows from i to 0 for every i. One also shows the "extra transformations" by dashed arrows from i to j if $j < i$ (they correspond to the matrices S_{ji}). Obviously, if $\mathbf{S}$ is *discrete* ($i \leqslant j$ if and only if $i = j$), we have representations of a quiver.

Here is a simple example

It gives the «matrix version» of the definition of $\operatorname{rep}(\mathbf{S}, \Bbbk)$, which was used in the original paper of Nazarova–Roiter [5]. We use more «invariant» language of linear maps, which is known to be equivalent to that of matrices. For convenience, we fix a symbol 0, supposing that $0 \notin \mathbf{S}$, and denote $\hat{\mathbf{S}} = \mathbf{S} \cup \{0\}$.

Definition. 1. A *representation of the poset* $\mathbf{S}$ *over a field* $\Bbbk$ is, by definition, a set $V \cup \mathfrak{v}$ of vector spaces $V = \left\{ V(i) \mid i \in \hat{\mathbf{S}} \right\}$ and linear maps $\mathfrak{v} = \{ \mathfrak{v}(i) : V(i) \to V(0) \mid i \in \mathbf{S} \}$.

2. The *dimension* $\mathbf{dim}\, V$ of a representation V (always in the second sense) is the map $\hat{\mathbf{S}} \to \mathbb{N}$: $i \mapsto \dim V(i)$.

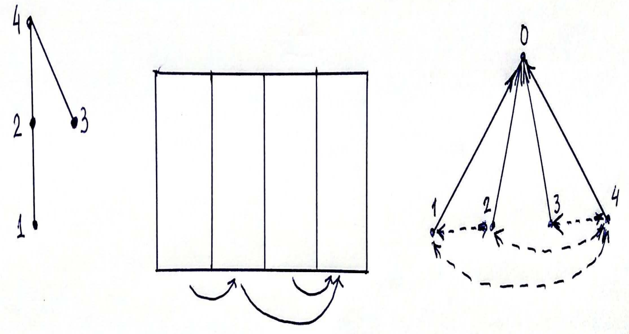
3. A *morphism* $\Phi : (V, \mathfrak{v}) \to (W, \mathfrak{w})$ is a set of linear maps $\Phi = \left\{ \Phi_i : V(i) \to W(i) \mid i \in \hat{\mathbf{S}} \right\} \cup \{ \Phi_{ji} : V(i) \to W(j) \mid i, j \in \mathbf{S},\ j < i \}$ such that $\Phi_0 \mathfrak{v}(i) = \mathfrak{w}(i)\Phi_i + \sum_{j<i} \mathfrak{w}(j)\Phi_{ji}$ for all $i \in \mathbf{S}$.

Definition. 4. The product of morphisms $\Phi\Psi$, where $\Psi : U \to V$, is defined by the rules

$$(\Phi\Psi)_i = \Phi_i \Psi_i,$$
$$(\Phi\Psi)_{ji} = \Phi_{ji}\Psi_i + \Phi_j \Psi_{ji} + \sum_{j<q<i} \Phi_{jq}\Psi_{qi}.$$

(We recommend the reader to explain these formulae.)

Compare this definition to that of representations of quivers.



4
2
3
1
0
1
2
3
4

Exercise 1. Prove that the morphism Φ is invertible (i.e. an isomorphism) if and only if all maps Φ_i are invertible. Calculate the morphism Φ^{-1}.

Remark. Actually, one can note that two proposed definitions are not equivalent. Namely, when we construct the matrices $V(i)$ in the first definition, their columns are linear independent (so $V(i)$ is of rank $d(i)$). It means that in the corresponding «matrix representations» from the second definition all maps $\mathfrak{v}(i)$ are injective.

Evidently, this change is negligible. Indeed, if a map $\mathfrak{v}(i)$ has a nonzero kernel, the representation $(V, \mathfrak{v})$ has a trivial direct summand E_i such that $E_i(j) = 0$ if $j \neq i$ and $E_i(i) = \Bbbk$.

In what follows, we use the second approach, which is more convenient for our considerations.

1.1 Tits form and reflections

Just as for quivers, an important role plays the *Tits form.*

Definition. The *Tits form* of the poset $\mathbf{S}$ is the quadratic form $\mathsf{Q}_{\mathbf{S}} : \mathbb{R}^{\hat{\mathbf{S}}} \to \mathbb{R}$ defined by the rule $\mathsf{Q}_{\mathbf{S}}(\mathbf{x}) = \sum_{i \in \hat{\mathbf{S}}} x_i^2 + \sum_{\substack{i,j \in \mathbf{S} \\ j<i}} x_i x_j - x_0 \sum_{i \in \mathbf{S}} x_i$.

The geometrical meaning of this form is just the same as for quivers. Namely, if $\mathbf{d} = \mathbf{dim}\, V$, its negative part, $\mathsf{Q}_{\mathbf{S}}^- = d_0 \sum_{i \in \mathbf{S}} d_i$ is the dimension of the space $\operatorname{rep}(\mathbf{d}, \mathbf{S})$ of matrices describing V, while the positive part, $\mathsf{Q}_{\mathbf{S}}^+ = \sum_{i \in \hat{\mathbf{S}}} d_i^2 + \sum_{\substack{i,j \in \mathbf{S} \\ j<i}} d_i d_j$ is the dimension of the group $\mathsf{G}(\mathbf{d}, \mathbf{S})$ acting on the set $\operatorname{rep}(\mathbf{d}, \mathbf{S})$ so that isomorphic reprsentations are just those belonging to the same orbit.

Just as for quivers, it immediatley implies

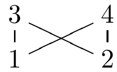
Proposition. *If* $\operatorname{rep}(\mathbf{d}, \mathbf{S})$ *consists of a finite number of isomorphism classes, then* $\mathsf{Q}_{\mathbf{S}}(\mathbf{d}) > 0$.

Corollary. *If* $\mathbf{S}$ *is* representation finite*, that is has only finitely many non-isomorphic indecomposable representations, then its Tits form is* weakly positive*, that is* $\mathsf{Q}_{\mathbf{S}}(\mathbf{x}) > 0$ *for every* $\mathbf{x} > 0$.

We denote it by $\mathsf{Q}_{\mathbf{S}} \rhd 0$.

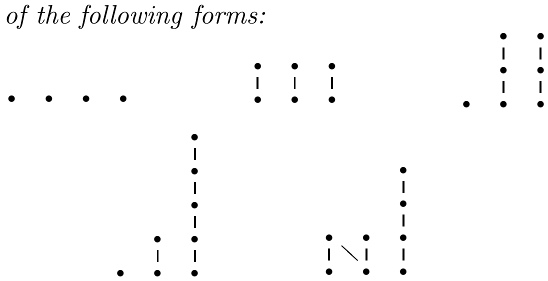
Remark.

1. Actually, Proposition 11 only implies that $\mathsf{Q}_{\mathbf{S}}(\mathbf{x}) > 0$ if $\mathbf{x} \in \mathbb{Q}^{\hat{\mathbf{S}}}$ and $\mathbf{x} > 0$. To prove it for all vectors from $\mathbb{R}^{\hat{\mathbf{S}}}$, one needs some additional considerations. As we do not use this result, we propose the reader to find it in [1, Supplement, Prop. 4].

2. Note that for quivers $\mathsf{Q}_\Gamma \rhd 0$ implies that $\mathsf{Q}_\Gamma > 0$. It is not the case for posets. For instance, let $\mathbf{S}$ be1 the poset . (It means that all comparable pairs are $1 < 3$, $1 < 4$, $2 < 3$, $3 < 4$.) Then $\mathsf{Q}_{\mathbf{S}}(\mathbf{x}) = \sum_{i=0}^4 x_i^2 + (x_1 + x_2)(x_3 + x_4) - x_0 \sum_{i=1}^4 x_i$. One can verify that $\mathsf{Q}_{\mathbf{S}} \rhd 0$, but $\mathsf{Q}_{\mathbf{S}}(0, 1, 1, -1, -1) = 0$.

Due to M. Kleiner [4], we know when the Tits form is weakly positive.

Theorem (Kleiner's criterion)**.** $\mathsf{Q}_{\mathbf{S}} \rhd 0$ *if and only if* $\mathbf{S}$ *does not contain a subposet of one of the following forms:*



These posets are usually called, respectively, $\{1,1,1,1\}$, $\{2,2,2\}$, $\{1,3,3\}$, $\{1,2,5\}$ and $\{\mathsf{N},4\}$.

Let $\mathsf{B}_{\mathbf{S}}(\mathbf{x},\mathbf{y})$ be the symmetric bilinear form associated with $\mathsf{Q}_{\mathbf{S}}$, that is
$\mathsf{B}_{\mathbf{S}}(\mathbf{x},\mathbf{y}) = \sum_{i \in \hat{\mathbf{S}}} x_i y_i + \frac{1}{2} \sum_{\substack{i,j \in \mathbf{S} \\ j < i}} (x_i y_j + x_j y_i) - \frac{1}{2} \sum_{i \in \mathbf{S}} (x_0 y_i + x_i y_0)$.

Jiust as for quivers, we can define *reflections* s_i by the same rule:

$$s_i \mathbf{x} = \mathbf{x} - 2\mathsf{B}_{\mathbf{S}}(\mathbf{x}, \mathbf{e}_i)\mathbf{e}_i.$$

New i-th coordinate of $s_i\mathbf{x}$ is

$$x_i' = x_0 - \sum_{j \lessgtr i} x_j \quad \text{if } i \neq 0,$$

$$x_0' = \sum_{i \in \mathbf{S}} x_i - x_0,$$

where $i \lessgtr j$ means that $i, j \in \mathbf{S}$ and $i \leqslant j$ or $j \leqslant i$.

Again, we have the following properties, which we propose the readers to prove themselves.

Proposition. *1.* $\mathsf{B}_{\mathbf{S}}(s_i\mathbf{x}, s_i\mathbf{y}) = \mathsf{B}_{\mathbf{S}}(\mathbf{x},\mathbf{y})$*, in particular,* $\mathsf{Q}_{\mathbf{S}}(s_i\mathbf{x}) = \mathsf{Q}_{\mathbf{S}}(\mathbf{x})$ *for all* $\mathbf{x}, \mathbf{y}$*).*

2. If $\mathbf{x} > 0$ *and* $\mathsf{Q}_{\mathbf{S}}(\mathbf{x}) > 0$*, there is* $i \in \hat{\mathbf{S}}$ *such that* $s_i\mathbf{x} < \mathbf{x}$*.*

3. If $\mathsf{Q}_{\mathbf{S}} \rhd 0$*,* $\mathbf{x} \in \mathbb{N}^{\mathbf{S}}$*,* $\mathsf{Q}_{\mathbf{S}}(\mathbf{x}) = 1$ *and* $\mathbf{x} \neq \mathbf{e}_i$*, then* $|x_i - x_i'| \leqslant 1$ *and* $s_i\mathbf{x} > 0$*.*

4. If $\mathsf{Q}_{\mathbf{S}} \rhd 0$*,* $\mathbf{x} \in \mathbb{N}^{\mathbf{S}}$*,* $\mathsf{Q}_{\mathbf{S}}(\mathbf{x}) = 1$*, there is a sequence of elements* $i_1, i_2, \dots, i_k \in \hat{\mathbf{S}}$ *such that* $s_{i_k} \dots s_{i_2} s_{i_1} \mathbf{x} = \mathbf{e}_j$ *for some* $j \in \mathbf{S}$*.*

1.2 Bisected posets and Tits form

It seems that there is no possibility to realize these reflections on representations of posets (except s_0, see [1]). Such realization is possible if we extend the framework and introduce “representations of bisected” posets [2].

Definition. A *bisected poset* (or *bisposet*) is a poset $\mathbf{S}$ together with its *bisection* into two disjoint subsets $\mathbf{S} = \mathbf{S}^- \sqcup \mathbf{S}^+$ such that if $i < j$ and $i \in \mathbf{S}^+$, then $j \in \mathbf{S}^+$ too.

We write $i \ll j$ (or $j \gg i$) if $i \in \mathbf{S}^-$, $j \in \mathbf{S}^+$ and $i < j$. We write $i \lessdot j$ (or $j \gtrdot i$) if $i < j$ and either both $i, j \in \mathbf{S}^-$ or both $i, j \in \mathbf{S}^+$.

We define representations of bisposets, positive and negative elements and the corresponding reflections.

Definition. 1. A *representation of the bisposet* $\mathbf{S} = \mathbf{S}^- \sqcup \mathbf{S}^+$ *over a field* $\Bbbk$ is, by definition, a set $V \cup \mathfrak{v}$ of vector spaces $V = \left\{ V(i) \mid i \in \hat{\mathbf{S}} \right\}$ and linear maps $\mathfrak{v} = \{ \mathfrak{v}(i) \mid i \in \mathbf{S} \}$, where $\mathfrak{v}(i) : V(i) \to V(0)$ if $i \in \mathbf{S}^-$, $\mathfrak{v}(i) : V(0) \to V(i)$ if $i \in \mathbf{S}^+$, such that $\mathfrak{v}(j)\mathfrak{v}(i) = 0$ if $i \ll j$.

2. The *dimension* $\mathbf{dim}\, V$ of a representation V is the map $\hat{\mathbf{S}} \to \mathbb{N}$: $i \mapsto \dim V(i)$.

3. A *morphism* $\Phi : (V, \mathfrak{v}) \to (W, \mathfrak{w})$ is a set of linear maps $\Phi = \left\{ \Phi_i : V(i) \to W(i) \mid i \in \hat{\mathbf{S}} \right\} \cup \{ \Phi_{ji} : V(i) \to W(j) \mid i, j \in \mathbf{S},\ j \lessdot i \}$ such that

$$\Phi_0 \mathfrak{v}(i) = \mathfrak{w}(i)\Phi_i + \sum_{j \lessdot i} \mathfrak{w}(j)\Phi_{ji} \text{ for all } i \in \mathbf{S}^-,$$

$$\mathfrak{w}(i)\Phi_0 = \Phi_i \mathfrak{v}(i) + \sum_{j \gtrdot i} \Phi_{ij}\mathfrak{v}(j) \text{ for all } i \in \mathbf{S}^+.$$

Schematically, it can be represented by the picture

$$\begin{tikzcd}
i_1 & i_2 \arrow[l, dotted] & \cdots & i_k \\
& 0 \arrow[ul] \arrow[u] \arrow[urr] & & \\
j_1 \arrow[ur] \arrow[uur, dashed] \arrow[r, dotted] & j_2 \arrow[u] & \cdots & j_l \arrow[ull]
\end{tikzcd}$$

Here the vertices i_r are from $\mathbf{S}^+$, the vertices j_r are from $\mathbf{S}^-$. The dotted arrows are from bigger elements to smaller ones. They show the extra transformations described by the maps Φ_{ji}. The dashed lines are from smaller elements to bigger ones. They show the relations $\mathfrak{v}(i)\mathfrak{v}(j) = 0$. This picture is useful to see which vertices are positive (like i_1) and which are negative (like j_1). Certainly, if $\mathbf{S}$ is discrete, we obtain again representations of a quiver.

The *Tits form* of a bisected poset is the same as for a poset without bisection:

$$\mathsf{Q}_{\mathbf{S}}(\mathbf{x}) = \sum_{i\in\hat{\mathbf{S}}} x_i^2 + \sum_{i<j} x_i x_j - x_0 \sum_{i\in\mathbf{S}} x_i.$$

This time we set $\mathsf{Q}^-_{\mathbf{X}} = x_0 \sum_{i\in\mathbf{S}} x_i - \sum_{i\ll j} x_i x_j$ and $\mathsf{Q}^+_{\mathbf{S}} = \sum_{i\in\hat{\mathbf{S}}} x_i^2 + \sum_{i\lessdot j} x_i x_j$. Thus $\mathsf{Q}_{\mathbf{S}}(\mathbf{x}) = \mathsf{Q}^+_{\mathbf{S}}(\mathbf{x}) - \mathsf{Q}^-_{\mathbf{S}}(\mathbf{x})$. Again, if $\mathbf{d} = \mathbf{dim}\, V$, the form $\mathsf{Q}^+_{\mathbf{S}}(\mathbf{d})$ is the dimension of the group acting on the variety $\mathrm{rep}(\mathbf{d},\mathbf{S})$ of representations of dimension $\mathbf{d}$. Note that this time $\mathrm{rep}(\mathbf{d},\mathbf{S})$ is not a vector space, but an algebraic variety, since we have non-linear equations $\mathfrak{v}(j)\mathfrak{v}(i) = 0$ for $i \ll j$. This product is a matrix of size $d_j \times d_i$. Hence this equation impose $d_i d_j$ relations on the elements of the matrices $\mathfrak{v}(i)$ and $\mathfrak{v}(j)$. It is known from algebraic geometry that every equation diminishes the dimension of a variety at most on 1. Therefore, the dimension of the variety $\mathrm{rep}(\mathbf{d},\mathbf{S})$ is at least $\mathsf{Q}^-_{\mathbf{S}}(\mathbf{d})$.

Therefore, the same geometrical observations give the next results.

Proposition. *If* $\mathrm{rep}(\mathbf{d},\mathbf{S})$ *consists of a finite number of isomorphism classes, then* $\mathsf{Q}_{\mathbf{S}}(\mathbf{d}) > 0$.

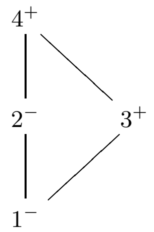
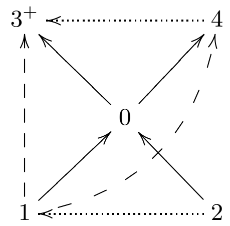
Corollary. *If* $\mathbf{S}$ *is* representation finite*, that is has only finitely many non-isomorphic indecomposable representations, then its Tits form is weakly positive.*

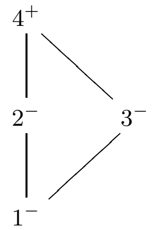
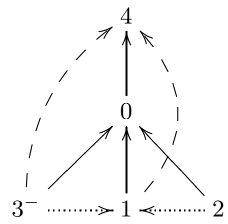
We will prove the inverse result, just as for quivers, using the "categorification" of reflections.

First, we define positive and negative elements $i \in \hat{\mathbf{S}}$ following the picture on page 18. "Positive" means that neither arrow starts at i, "negative" means that neither arrow ends at i.

Definition.
1. An element $i \in \hat{\mathbf{S}}$ is called *positive* if either it is a minimal element of $\mathbf{S}^+$ or $i = 0$ and $\mathbf{S}^+ = \emptyset$.
2. An element $i \in \hat{\mathbf{S}}$ is called *negative* if either it is a maximal element of $\mathbf{S}^-$ or $i = 0$ and $\mathbf{S}^- = \emptyset$.
3. If i is a minimal element in $\mathbf{S}^+$, we define $s_i\mathbf{S}$ as the same poset but with bisection $s_i\mathbf{S}^+ = \mathbf{S}^+ \setminus \{i\}$, $s_i\mathbf{S}^- = \mathbf{S}^- \cup \{i\}$.
4. If i is a maximal element in $\mathbf{S}^-$, we define $s_i\mathbf{S}$ as the same poset but with bisection $s_i\mathbf{S}^- = \mathbf{S}^- \setminus \{i\}$, $s_i\mathbf{S}^+ = \mathbf{S}^+ \cup \{i\}$.
5. If $\mathbf{S}^+ = \emptyset$, we define $s_0\mathbf{S}$ as the same poset, but with $s_0\mathbf{S}^+ = \mathbf{S}$, $s_0\mathbf{S}^- = \emptyset$.
6. If $\mathbf{S}^- = \emptyset$, we define $s_0\mathbf{S}$ as the same poset, but with $s_0\mathbf{S}^- = \mathbf{S}$, $s_0\mathbf{S}^+ = \emptyset$.

Example:

$\mathbf{S} =$  or 

$s_3\mathbf{S} =$  or 

1.3 Reflection functors

Now we define reflection functors $s_i^{\pm}$ on representations of a bisposet. First, we introduce some notations.

Definition.

$$
\begin{aligned}
& i^{\pm} = \left\{ j \in \mathbf{S}^{\pm} \mid j \lessgtr i \right\} \quad \text{for } i \in \mathbf{S}; \\
& 0^{\pm} = \mathbf{S}^{\pm}; \\
& i^{\sharp} = i^{+} \cup i^{-}; \\
& x_i^{\pm} = \sum\nolimits_{j \in i^{\pm}} x_j; \\
& x_i^{\sharp} = \sum\nolimits_{j \in i^{\sharp}} x_j; \\
& x_i' = x_0 - x_i^{\sharp} \quad \text{for } i \in \mathbf{S}; \\
& x_0' = x_0^{\sharp} - x_0; \\
& V^{\pm}(i) = \bigoplus\nolimits_{j \in i^{\pm}} V(j); \\
& \mathfrak{v}^{+}(i) : V(0) \to V^{+}(i) \quad \text{with components} \quad \mathfrak{v}(j) \quad (j \in i^{+}); \\
& \mathfrak{v}^{-}(i) : V^{-}(i) \to V(0) \quad \text{with components} \quad \mathfrak{v}(j) \quad (j \in i^{-}).
\end{aligned}
$$

Definition. Let i be a minimal element of $\mathbf{S}^{+}$ (hence positive). Then $j \in i^{+}$ if and only if $i < j$ and $j \in i^{-}$ if and only if $j < i$. Therefore, $\operatorname{Im} \mathfrak{v}^{-}(i) \subseteq \operatorname{Ker} \mathfrak{v}^{+}(i)$. We define $V' = s_i V$ as follows. We fix a *section* $\eta : V(0)/\operatorname{Im} \mathfrak{v}^{-}(i) \to V(0)$, that is a linear map such that its composition with the natural surjection $V(0) \to V(0)/\operatorname{Im} \mathfrak{v}^{-}(i)$ is identity.

- $V'(j) = V(j)$ and $\mathfrak{v}'(j) = \mathfrak{v}(j) =$ for $j \neq i$.

- $V'(i) = \operatorname{Ker} \mathfrak{v}^+(i)/\operatorname{Im} \mathfrak{v}^-(i)$.

- $\mathfrak{v}'(i) : V'(i) \to V(0)$ is the composition $V'(i) \hookrightarrow V(0)/\operatorname{Im} \mathfrak{v}^-(i) \xrightarrow{\eta} V(0)$.

Definition. Let i be a maximal element of $\mathbf{S}^-$ (hence negative). Then $j \in i^+$ if and only if $i < j$ and $j \in i^-$ if and only if $j < i$. Therefore, $\operatorname{Im} \mathfrak{v}^-(i) \subseteq \operatorname{Ker} \mathfrak{v}^+(i)$. We define $V' = s_i V$ as follows. We fix a *retraction* $\mu : V(0) \to \operatorname{Ker} \mathfrak{v}_i^-$, that is a linear map such that the composition of the embedding $\operatorname{Ker} \mathfrak{v}^+(i) \hookrightarrow V(0)$ with μ is identity.

- $V'(j) = V(j)$ and $\mathfrak{v}'(j) = \mathfrak{v}(j) =$ for $j \neq i$.

- $V'(i) = \operatorname{Ker} \mathfrak{v}^+(i)/\operatorname{Im} \mathfrak{v}^-(i)$.

- $\mathfrak{v}'(i) : V(0) \to V'(i)$ is the composition $V(0) \xrightarrow{\mu} \operatorname{Ker}^+(i) \twoheadrightarrow V'(i)$.

Definition. Let 0 be positive, i.e. $\mathbf{S}^+ = \emptyset$. We define $V' = s_0 V$ as follows.

- $V'(i) = V(i)$ for $i \in \mathbf{S}$.
- $V'(0) = \operatorname{Ker} \mathfrak{v}^-(0)$.
- $\mathfrak{v}'(i) : V'(0) \to V(i)$ is the composition $V'(0) \hookrightarrow V^-(0) \twoheadrightarrow \mathfrak{v}(i)$.

Let 0 be negative, i.e. $\mathbf{S}^- = \emptyset$. We define $V' = s_0 V$ as follows.

- $V'(i) = V(i)$ for $i \in \mathbf{S}$.
- $V'(0) = \operatorname{Coker} \mathfrak{v}^+(0)$.
- $\mathfrak{v}'(i) : V(i) \to V'(0)$ is the composition $V(i) \hookrightarrow V^+(0) \twoheadrightarrow V'(0)$.

One can easily see that if $\mathbf{S}$ is discrete, these definitions coincide with reflections for quivers.

One can prove that the resulting representation $s_i V$ does not depend on the choice of the section η or a retraction μ. Moreover, consider the category $\operatorname{rep}^i(\mathbf{S})$ which has the same objects but the set of morphisms from V to W is defined as $\operatorname{Hom}^i_{\mathbf{S}}(V, W) = \operatorname{Hom}_{\mathbf{S}}(V, W)/\mathbf{I}_i(V, W)$, where $\mathbf{I}_i(V, W) = \operatorname{Hom}_{\mathbf{S}}(\mathbf{E}_i, W) \operatorname{Hom}_{\mathbf{S}}(V, \mathbf{E}_i)$ (actually, I_i consists of morphisms that factors through direct sums of copies of $\mathbf{E}_i$). In particular, $\mathbf{E}_i \simeq 0$ in this category, so representations that differs on direct summands $\mathbf{E}_i$ become isomorphic in $\operatorname{rep}^i(\mathbf{S})$. Then s_i is extended to a functor $\operatorname{rep}^i(\mathbf{S}) \to \operatorname{rep}^i(s_i\mathbf{S})$ and $s_i s_i V \simeq V$ for all V which have no direct summands $\mathbf{E}_i$. We will not present here the proof. It can be found in [2].

Now we formulate and prove the main theorem.

Both in the formulation and in the proof we use some geometrical facts (mainly about dimensions of algebraic varieties). They can be found in most books on algebraic geometry, for instance, in [3] (Chapter 3 is devoted to the theory of dimensions and in Section 3.6 there are its applications to the study of actions of algebraic groups on algebraic varieties).

Theorem. *Suppose that* $\mathsf{Q}_{\mathbf{S}} \rhd 0$.

1. $\mathbf{S}$ *is representation finite.*
2. *There is an indecomposable representation of dimension* $\mathbf{d}$ *if and only if* $\mathsf{Q}_{\mathbf{S}}(\mathbf{d}) = 1$. *In this case all indecomposable representations of dimension* $\mathbf{d}$ *are isomorphic.*
3. $\mathrm{Hom}_{\mathbf{S}}(V,V) = \Bbbk$ *for every indecomposable representation* V.
4. *If* V *is indecomposable and* $\mathbf{dim}\, V = \mathbf{d}$, *the orbit of* V *is open and dense in the variety* $\mathrm{rep}(\mathbf{d},\mathbf{S})$.
5. *If* $\mathsf{Q}_{\mathbf{S}}(\mathbf{d}) = 1$, *the variety* $\mathrm{rep}(\mathbf{d},\mathbf{S})$ *is irreducible and* $\dim \mathrm{rep}(\mathbf{d},\mathbf{S}) = \mathsf{Q}^-_{\mathbf{S}}(\mathbf{d})$.

First, we show that (2) implies all other assertions. Indeed, if $\mathsf{Q}_{\mathbf{S}} \rhd 0$, there is only finitely many vectors $\mathbf{d} \in \mathbb{N}^{\hat{\mathbf{S}}}$ such that $\mathsf{Q}_{\mathbf{S}}(\mathbf{d}) = 1$ (explain it!), whence (1). Let X be an irreducible component of $\mathrm{rep}(\mathbf{d},\mathbf{S})$ and $\mathbf{d} = \dim V$, where V is indecomposable. Then $\dim X \geqslant \mathsf{Q}^-(\mathbf{d})$. As X is the union of finitely many orbits, one of them, say Y, is dense in X and $\dim Y = \dim X$. Since $Y = \mathbf{G}(\mathbf{d},\mathbf{S})/H$, where $H = \mathrm{Aut}\, W$ is the stabilizer of a representation $W \in Y$, and $\mathrm{Aut}\, W$ is open in $\mathrm{End}_{\mathbf{S}}\, W$, $\dim Y = \dim \mathbf{G}(\mathbf{d},\mathbf{S}) - \dim H = \mathsf{Q}^+(\mathbf{d}) - \dim \mathrm{End}_{\mathbf{S}}\, W$. Therefore, $\mathsf{Q}^+(\mathbf{d}) - \dim \mathrm{End}_{\mathbf{S}}\, W = \dim X$. But $\mathsf{Q}^+(\mathbf{d}) - \mathsf{Q}^-(\mathbf{d}) = 1$ by (2), hence $\dim X \geqslant \mathsf{Q}^+(\mathbf{d}) - 1$. Thus $\dim \mathrm{End}_{\mathbf{S}}\, W = 1$ and W is indecomposable. By (2), $W \simeq V$. So there is a unique component, i.e. $\mathrm{rep}(\mathbf{d},\mathbf{S})$ is irreducible, the orbit of V is dense in it and $\mathrm{End}_{\mathbf{S}}\, V = \Bbbk$, that is, we have (3-5).

2 Bibliography

Bibliography

References

[1] Дрозд Ю.А. Преобразования Кокстера и представления частично упорядоченных множеств. Функц. анализ приложения 8:3 (1974), 34–42.

[2] Drozd Yu.A. Representations of bisected posets and reflection functors. Algebras and Modules. II, CMS Conference Proceedings, vol. 24, Amer. Math. Soc., 1998, 153–165.

[3] Ю.А. Дрозд. Вступ до алгебричної геометрії. ВНТЛ–Класика, 2004 https://www.imath.kiev.ua/∼drozd/agukr.pdf

[4] Клейнер М.М. Частично упорядоченные множества конечного типа. Записки научн. сем. ЛОМИ 28 (1972), 32–41.

[5] Назарова Л.А., Ройтер А.В. Представления частично упорядоченных множеств. Записки научн. сем. ЛОМИ 28 (1972), 5–31.

Papers [1, 2] are available on my page `https://www.imath.kiev.ua/∼drozd/publ.html`.

Papers [4, 5] can be freely downloaded from MathNet.ru.