Matrix Problems

Yuriy Drozd

Institute of Mathematics Tereschenkivska 3, r. 229

`https://www.imath.kiev.ua/~drozd/Qui-beam.pdf`

Kyiv 2020

There can be (and definitely are) misprints and even mistakes in this text. I will be grateful for all who tell me about them, especially those in mathematical formulae. Please send your comments to: `y.a.drozd@gmail.com`

Contents

1 Quivers

Quivers and representations

A *quiver* (or *oriented graph*, or *orgraph*) is just a set of points related by arrows, like

$$
\begin{tikzcd}
& \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[d] \\
\bullet \arrow[ur] \arrow[rr, bend left] \arrow[rr, bend right] & & \bullet \arrow[loop right]
\end{tikzcd}
$$

Usually, these points and arrows have their names (mainly, letters or numbers). Then the point are often replaces by their names, like

$$
\begin{tikzcd}
& 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[d, "d"] \\
3 \arrow[ur, "c"] \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] & & 4 \arrow[loop right, "g"]
\end{tikzcd}
\tag{1}
$$

As you see, there can be *multiple arrow*, like f_1 and f_2, and *loops*, like g.

Here is a formal definition.

. A *quiver* Γ is a triple $(\operatorname{Ver}\Gamma, \operatorname{Arr}\Gamma, \iota)$, where $\operatorname{Ver}\Gamma$ and $\operatorname{Arr}\Gamma$ are sets and ι is a map $\operatorname{Arr}\Gamma \to \operatorname{Ver}\Gamma \times \operatorname{Ver}\Gamma$.

Usually, the elements of $\operatorname{Ver}\Gamma$ are called the *vertices* of the quiver Γ, while the elements of $\operatorname{Arr}\Gamma$ are called the *arrows* of Γ.

We will write $\iota(a) = (\iota_0(a), \iota_1(a))$ and call $\iota_0(a)$ the *source* of the arrow a and $\iota_1(a)$ its *target*. If $\iota_0(a) = x$, $\iota_1(a) = y$, they usually write $a : x \to y$ or $x \xrightarrow{a} y$.

If both sets $\operatorname{Ver}\Gamma$ and $\operatorname{Arr}\Gamma$ are finite, they say that Γ is a *finite quiver*.

. A *path* p in the quiver Γ is a sequence of arrows $p = a_l \dots a_2 a_1$ such that $\iota_0(a_{i+1}) = \iota_1(a)$ for each $i = 1, 2, \dots, l-1$:

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \dots \bullet \xrightarrow{a_l} \bullet$$

The number l is called the *length* of the path p and denoted by $\ell(p)$.

The source $i_0(a_1)$ us called the *source of the path* p and denoted by $\iota_0(p)$.

The target $i_1(a_l)$ us called the *target of the path* p and denoted by $\iota_1(p)$.

Just as for arrows, we write $p : x \to y$ or $x \xrightarrow{p} y$ if $x = \iota_0(p)$ and $y = \iota_1(p)$.

Certainly, every arrow is a path (of length 1).

If $\iota_1(p) = \iota_0(p)$, they say that p is a *cycle*. For instance, any loop is a cycle. For instance in the graph above

$$\begin{tikzcd}
 & 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[d, "d"] \\
3 \arrow[ur, "c"] \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] & & 4 \arrow[loop right, "g"]
\end{tikzcd}$$

there are paths $gda : 1 \to 4$ (of length 3) and $g^3 f_2 : 3 \to 4$ (of length 4). (Here and further we write g^3 instead of ggg and so on). The path $ab : 2 \to 2$ is a cycle (of length 2).

It is convenient to consider, for every vertex x, *the empty path* $\emptyset_x : x \to x$ *at the vertex* x, which contains no arrows and is of length zero. By definition, it is a cycle.

There is an imporatant operation of *composition* of paths.

. A pair of paths (p, q) is said to be *composable* if $\iota_0(p) = \iota_1(q)$:

$$\bullet \underbrace{\xrightarrow{b_1} \bullet \xrightarrow{b_2} \bullet \dots \bullet \xrightarrow{b_m}}_{q} \bullet \underbrace{\xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \dots \bullet \xrightarrow{a_l}}_{p} \bullet$$

If this pair is composable, the *composition* pq is obtained by their concatenation: $pq = a_l \dots a_2 a_1 b_m \dots b_2 b_1$.

Obviously, $\iota_0(pq) = \iota_0(q)$, $\iota_1(pq) = \iota_1(p)$ and $\ell(pq) = \ell(p) + \ell(q)$.

Note that $p\emptyset_x = \emptyset_y p = p$ for any path $p : x \to y$.

Exercise 1. Let Γ is a finite graph. Prove that the following conditions are equivalent:

1. The set of paths in Γ is finite.
2. There is a number L such that $\ell(p) \leqslant L$ for every path p in Γ.
3. There are no cycles in Γ.

2 Representations

The main notion of our lectures is that of *representations* of a quiver over a field.

. 1. A *representation of a quiver* Γ *over a field* $\Bbbk$ is a maps M, which maps each vertex x to a vector space $M(x)$ over the field $\Bbbk$ and every arrow $a : x \to y$ to a linear map $M(a) : M(x) \to M(y)$.

2. The representation M is said to be *pointwise finite dimensional* if all spaces $M(x)$ $(x \in \operatorname{Ver}\Gamma)$ are finite dimensional. If the quiver Γ is finite, they say in this case that M is *finite dimensional*.

Consider some examples.

Example. 1. If Γ consists of one arrow (not a loop) $1 \xrightarrow{a} 2$, its representation consists of two vector spaces $M(1), M(2)$ and a linear map $M(a) : M(1) \to M(2)$.

2. If Γ consists of a loop $1 \circlearrowright a$, its representations consists of a vector space $M(1)$ and a linear map $M(a) : M(1) \to M(1)$.

3. The *Kronecker quiver* is that of the form $1 \overset{a}{\underset{b}{\rightrightarrows}} 2$. Its representation consists of two vector spaces $M(1), M(2)$ and two linear maps $M(a)$ and $M(b)$, both $M(1) \to M(2)$.

If M is a pointwise finite dimensional representation, one can choose a basis in every space $M(x)$ $(x \in \operatorname{Ver}\Gamma)$ and present linear maps $M(a)$ $(a \in \operatorname{Arr}\Gamma)$ by their matrices. If $a : x \to y$, the corresponding matrix is of size $\dim`M(y) \times \dim M(x)$. In this way we obtain a *matrix representation* of the quiver Γ.

For instance, in Example 1 $(1 \xrightarrow{a} 2)$ above a matrix representation consists of one matrix A of size $\dim M(2) \times \dim M(1)$.

In Example 2 ($1 \circlearrowright a$) a matrix representation consists of one square matrix A of size $\dim M(1) \times \dim M(1)$.

In Example 3 ($1 \overset{a}{\underset{b}{\rightrightarrows}} 2$) a matrix representation consists of two matrices (A, B) of the same size $\dim M(2) \times \dim M(1)$.

Exercise 2. Control question: How do these matrices change if we choose other bases?

Now we define *morphisms* between representations.

. Let M and N be representations of a quiver Γ over a field $\Bbbk$. A *morphism* $\varphi : M \to N$ is a set of linear maps $\varphi(x) : M(x) \to N(x)$, where x runs through the vertices of Γ such that $\varphi(y)M(a) = N(a)\varphi(x)$ for every arrow $a : x \to y$.

In other words, for every arrow $a : x \to y$ the diagram

$$\begin{tikzcd} M(x) \arrow[r, "M(a)"] \arrow[d, "\varphi(x)"'] & M(y) \arrow[d, "\varphi(y)"] \\ N(x) \arrow[r, "N(a)"] & N(y) \end{tikzcd}$$

must be commutative, that is both composite maps $M(x) \to N(y)$ coincide.

. 1. If $\psi : L \to M$ is another morphism of representations, the *product* (or *composition*) $\varphi\psi : L \to N$ is defined as the set of maps $\varphi(x)\psi(x) : L(x) \to N(x)$.

   Check that it is indeed a morphism $L \to N$.

2. The *identity* (or *unit*) morphism $1_M : M \to M$ is the set of identity maps $1_M(x) = 1_{M(x)} : M(x) \to M(x)$.

   Obviously $\varphi 1_M = 1_N \varphi$ for every morphism $\varphi : M \to N$.

3. The morphism $\varphi : M \to N$ is called an *isomorphism* if there is an *inverse morphism* $\varphi^{-1} : N \to M$ such that $\varphi\varphi^{-1} = 1_N$ and $\varphi^{-1}\varphi = 1_M$. Sometimes it is expressed by writing $\varphi : M \xrightarrow{\sim} N$.

   As usually, an inverse mophism, if exists, is unique (check it).

4. If there is an isomorphism $\varphi : M \to N$, they write $M \simeq N$ and call these representations *isomorphic*.

   e

Obviously, φ is an isomorphism if and only if all maps $\varphi(x)$ are bijections. Then $\varphi^{-1} = \{ \varphi(x)^{-1} \}$ (check that it is a morphism $N \to M$).

If we choose bases and use matrix representations, a morphism is given by a set of matrices $\Phi(x)$ of size $\dim N(x) \times \dim M(x)$ such that, if $a : x \to y$ is presented by a matrix A, $\Phi(y)A = A\Phi(x)$.

In particular, ϕ is an isomorphism if and only if all matrices $\Phi(x)$ are invertible square matrices. Then the inverse morphism is given by the set $\{ \Phi(x)^{-1} \}$.

. The set of morphisms $M \to N$ is denoted by $\mathrm{Hom}_\Gamma(M, N)$. It has the structure of a vector space over the same field $\Bbbk$. Namely the *sum* of morphisms $\varphi, \psi : M \to N$ is defined pointwise, by the maps $(\varphi + \psi)(x) = \varphi(x) + \psi(x)$ for all vertices x. The product $\lambda\varphi$, where $\lambda \in \Bbbk$ is also defined pointwise: $(\lambda\varphi)(x) = \lambda\varphi(x)$.

Obviously, if the representations M and N are finite dimensional, the space $\mathrm{Hom}_\Gamma(M, N)$ is finite dimensional as well.

Example. In Example 1 abobe, when Γ is $1 \xrightarrow{a} 2$, a morphism $\varphi : M \to N$ is a pair of linear maps $\varphi(1) : M(1) \to N(1)$ and $\varphi(2) : M(2) \to N(2)$ such that $\varphi(2)M(a) = N(a)\varphi(1)$.

It is an isomorphism if and only if both these maps are bijective. The inverse morphism is given by the maps $\varphi(1)^{-1}$ and $\varphi(2)^{-1}$.

If we consider matrix representations, $A = M(a)$ and $B = N(a)$, a mophism is a pair of matrices $\Phi(1), \Phi(2)$ such that $\Phi(2)A = B\Phi(1)$.

In particular, these representations are isomorphic if and only if $B = S_2AS_1^{-1}$ for some invertible matrices $S_1 = \Phi(1)$ and $S_2 = \Phi(2)$.

Exercise 3. What are morphisms of representations in the Examples 2 $\left(\begin{tikzcd} 1 \arrow[loop right, "a"] \end{tikzcd}\right)$ and Example 3 $\left(\begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 2 \end{tikzcd}\right)$ above?

In particular, when two matrix representations are isomorphic?

Exercise 4. Let Γ is a *chain*: $0 \xrightarrow{a_1} 1 \xrightarrow{a_2} 2 \dots (n-1) \xrightarrow{a_n} n$.

- What is a representation (a matrix representation) of Γ?
- What is a morphism of representations (of matrix representations)?
- When two representations (matrix representations) are isomorphic?
- The same questions for a *cycle*:

$$\begin{tikzcd} 0 \arrow[r, "a_1"] & 1 \arrow[r, "a_2"] & 2 & \dots & (n-1) \arrow[r, "a_n"] & n \arrow[lllll, bend left=10, "a_0"] \end{tikzcd}$$

. 1. Let M be a pointwise finite dimensional representation of a quiver Γ over a field $\Bbbk$. The *dimension* (sometimes the *vector dimension*) of M is the function $\mathbf{dim}(M) : \operatorname{Ver}\Gamma \to \mathbb{N}$ such that $\mathbf{dim}(M)(x) = \dim_{\Bbbk} M(x)$.

2. For any function $\mathbf{d} : \operatorname{Ver}\Gamma \to \mathbb{N}$ we define by $\operatorname{rep}(\mathbf{d}, \Gamma, \Bbbk)$ the set of representations of the quiver Γ of dimension $\mathbf{d}$.

Exercise 5. Let Σ_n be the quiver $\begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[loop right, "b"] \end{tikzcd}$. For any n-tuple of square matrices

$\mathbf{A} = (A_1, A_2, \dots, A_n)$ of size $d \times d$ define the matrix representation $M_{\mathbf{A}}$ of Σ_n setting

$$M_{\mathbf{A}}(a) = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_n \end{pmatrix}, M_{\mathbf{A}}(b) = \begin{pmatrix} I & I & 0 & \dots & 0 & 0 \\ 0 & I & I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I & I \\ 0 & 0 & 0 & \dots & 0 & I \end{pmatrix},$$

where I is the unit $d \times d$ matrix.

Prove that $M_{\mathbf{A}} \simeq M_{\mathbf{B}}$, where $\mathbf{B} = (B_1, B_2, \dots, B_n)$ is another n-tuple of matrices, if and only if there is an invertible matrix S such that $B_i = SA_iS^{-1}$ for all $i = 1, 2, \dots, n$.

Exercise 6. Let $\mathbf{K}_{n+1}$ be the quiver
$$\begin{tikzcd} 1 \arrow[r, bend left=40, "a_0"] \arrow[r, bend left=10, "a_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=20, "a_n"'] & 2 \end{tikzcd}$$
($n+1$ arrows), Σ_n be the quiver with one vertex $*$ and n loops $b_1, b_2, \dots, b_n$. For every representation M of Σ_n define the representation $\widetilde{M}$ of $\mathbf{K}_{n+1}$ setting $\widetilde{M}(1) = \widetilde{M}(2) = M(*)$, $\widetilde{M}(a_i) = M(b_i)$ if $1 \leqslant i \leqslant n$ and $\widetilde{M}(a_0) = 1_{M(*)}$ (identity map).

Prove that $M \simeq N$ if and only if $\widetilde{M} \simeq \widetilde{N}$.

An important notion is that of the *direct sum* of representations.

. The *direct sum* of representations M and N of the quiver Γ is the representation $M \oplus N$ such that

- $(M \oplus N)(x) = M(x) \oplus N(x)$ for every vertex x.
- $(M \oplus N)(a) = M(a) \oplus N(a)$ for every arrow $a : x \to y$.

Recall that, by definition, the direct sum of maps $M(a) \oplus N(a)$ is the map $M(x) \oplus N(x) \to M(y) \oplus N(y)$ such that $(M(a) \oplus N(a))(u, v) = (M(a)(u), N(a)(v))$ for all $u \in M(x)$, $v \in N(x)$.

In the matrix form, if we choose a basis in every space $(M \oplus N)(x)$ as the union of bases of $M(x)$ and $N(x)$, the matrix $(M \oplus N)(a)$ is the direct sum of the matrices $M(a)$ and $N(a)$, that is

$$(M \oplus N)(a) = \begin{pmatrix} M(a) & 0 \\ 0 & N(a) \end{pmatrix}$$

Just in the same way one defines direct sums of several representations $M_1 \oplus M_2 \oplus \ldots \oplus M_k$. In the matrix form

$$(M_1 \oplus M_2 \oplus \ldots \oplus M_k)(a) = \begin{pmatrix} M_1(a) & 0 & \ldots & 0 \\ 0 & M_2(a) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & M_k(a) \end{pmatrix}$$

If $M \simeq N \oplus N'$, we write $N \mathrel{\in\!\!\!\!+} M$ or $M \mathrel{\ni\!\!\!\!+} N$ and say that N is a (direct) summand of M.

For any quiver Γ there is a *trivial* (or *zero*) representation that maps every vertex to the zero vector space 0 and every arrows to the unique map $0 \to 0$. We also denote it by 0.

Obviously, $M \oplus 0 \simeq M$ for any representation M.

If there are non-trivial representations M_1 and M_2 such that $M \simeq M_1 \oplus M_2$, they say that the representation M is *decomposable*. Otherwise it is said to be *indecomposable*.

One easily sees that every finite dimensional representation can be decomposed into a direct sum of indecomposable representations (explain why). Further we shall prove that such a decomposition is unique up to isomorphism and permutation of the summands. Thus, to know all finite dimensional representations, one only has to know indecomposable ones.

.

- We denote by $\operatorname{ind}(\Gamma, \Bbbk)$ the set of isomorphism classes of indecomposable representations of the quiver Γ over the field $\Bbbk$ and by $\operatorname{ind}(\mathbf{d}, \Gamma, \Bbbk)$ its subset consisting of representations of dimension $\mathbf{d}$.

- If the set $\operatorname{ind}(\Gamma, \Bbbk)$ is finite, they say that the quiver Γ is *representation finite* over the field $\Bbbk$, otherwise they say that it is *representation infinite*.

Further we shall see that these properties do not depend on the field $\Bbbk$: if Γ is representation finite over some field, it is representation finite over any field.

Our nearest goal is to find a criterion for a quiver Γ to be representation finite.

3 Tits form

Tits form and Gabriel theorem

Starting from this section, we suppose that all quivers are *finite* (if the opposite is not explicitly declared). We introduce a notion that plays a very important role in the study of representations of quivers.

.

- For a quiver Γ and a set S we denote by S^Γ the set of functions $\operatorname{Ver}\Gamma \to S$.

  In particular, a dimension of a representation of Γ is an element of $\mathbb{N}^\Gamma$.

- The *Tits form* of the quiver Γ is the quadratic form $\mathsf{Q}_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}$ such that
$$\mathsf{Q}_\Gamma(\mathbf{x}) = \sum_{i \in \operatorname{Ver}\Gamma} \mathbf{x}(i)^2 - \sum_{a \in \operatorname{Arr}\Gamma} \mathbf{x}(\iota_0(a))\mathbf{x}(\iota_1(a)).$$

Example. 1. For the quiver Γ: $1 \xrightarrow{a} 2$, $\mathbb{R}^\Gamma = \mathbb{R}^2$ and $\mathsf{Q}_\Gamma(x_1, x_2) = x_1^2 + x_2^2 - x_1x_2$.

2. For the quiver Γ: $1 \circlearrowright a$, $\mathbb{R}^\Gamma = \mathbb{R}$ and $\mathsf{Q}_\Gamma(x) = x^2 - x^2 = 0$.

3. For the Kronecker quiver Ga: $\begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 2 \end{tikzcd}$, $\mathbb{R}^\Gamma = \mathbb{R}^2$ and $\mathsf{Q}_\Gamma(x_1, x_2) = x_1^2 + x_2^2 - 2x_1x_2$.

For the graph Γ from the very first example

$$\begin{tikzcd}
 & 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[d, "d"] \\
3 \arrow[ur, "c"] \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] & & 4 \arrow[loop right, "g"]
\end{tikzcd}$$

the Tits form is $x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - x_1x_3 - x_2x_4 - 2x_3x_4$.

Exercise 7. Calculate Tits forms for the chain

$$0 \xrightarrow{a_1} 1 \xrightarrow{a_2} 2 \ldots (n-1) \xrightarrow{a_n} n$$

and for the cycle

$$\begin{tikzcd}
0 \arrow[r, "a_1"] & 1 \arrow[r, "a_2"] & 2 & \ldots & (n-1) \arrow[r, "a_n"] & n \arrow[lllll, bend left, "a_0"]
\end{tikzcd}$$

Are these forms positive definite? non-negative definite?

Tits form has a natural geometric meaning. In what follows we use some facts from the dimension theory in algebraic geometry. For references, we recommend the books [AG] or [Shafarevich].

Considering matrix representations, we can identify the representations from $\operatorname{rep}(\mathbf{d}, \Gamma, \Bbbk)$ with the collection of matrices $\{\, M(a) \mid a \in \operatorname{Arr}\Gamma \,\}$, where $M(a)$ is of size $\mathbf{d}(y) \times \mathbf{d}(x)$ if $a : x \to y$. Altogether, such collections form the affine space over the field $\Bbbk$ of dimension $\mathsf{Q}_\Gamma^-(\mathbf{d}) = \sum_{a \in \operatorname{Arr}\Gamma} \mathbf{d}(\iota_0(a))\mathbf{d}(\iota_1(a))$. It is just the negative part of the Tits form calculated at $\mathbf{d}$.

An isomorphism of representations $M \to N$ in the matrix form is given by a collection of matrices $\{\, \Phi(x) \mid x \in \operatorname{Ver}\Gamma \,\}$, where $\Phi(x)$ is of size $\mathbf{d}(x) \times \mathbf{d}(x)$. Taking into account the conditions $\det \Phi(x) \neq 0$, we see that the set of such collections is an open subset in the affine space of dimension $\mathsf{Q}_\Gamma^+(\mathbf{d}) = \sum_{x \in \operatorname{Ver}\Gamma} \mathbf{d}(x)^2$. It is just the positive part of the Tits form calculated at $\mathbf{d}$. Moreover, it is a *principle open set* (i.e. given by one inequality condition $\prod_{x \in \operatorname{Ver}\Gamma} \det \Phi(x) \neq 0$). Hence it is an affine variety $\mathsf{GL}(\mathbf{d}, \Gamma, \Bbbk)$ of dimension $\mathsf{Q}_\Gamma^+(\mathbf{d})$.

The variety $\mathsf{GL}(\mathbf{d}, \Gamma, \Bbbk)$ is an algebraic group under the multiplication of morphisms, and it acts regularly on the affine space $\operatorname{rep}(\mathbf{d}, \Gamma, \Bbbk)$:

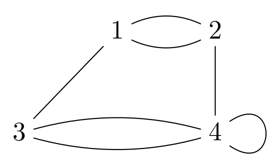
$$\big(\Phi(x)\big) \cdot \big(M(a)\big) = \big(\Phi(\iota_1(a)) M(a) \Phi(\iota_0(a))^{-1}\big).$$

The isomorphism classes of representations are just the *orbits* of this action.

Every such orbit is the image of $\mathsf{GL}(\mathbf{d},\Gamma,\Bbbk)$ under the regular map $\big(\Phi(x)\big)\mapsto\big(\Phi(\iota_1(a))M(a)\Phi(\iota_0(a))^{-1}\big)$, where M is some (arbitrary) representation from this orbit. As it is known, the dimension of this image is not bigger than the dimension of $\mathsf{GL}(\mathbf{d},\Gamma,\Bbbk)$, i.e. of $\mathsf{Q}^+_\Gamma(\mathbf{d})$ (see, for instance, [AG, Prop. 3.6.6]).

Moreover, the 1-dimensional subgroup D consisting of such collections Δ_λ ($\lambda\in\Bbbk$) that $\Delta_\lambda(x)=\lambda I_{\mathbf{d}(x)}$ acts trivially. Therefore, actually the acting group is $\mathsf{GL}(\mathbf{d},\Gamma,\Bbbk)/\mathsf{D}$ and the dimensions of the orbits are strictly less than $\mathsf{Q}^+(\mathbf{d},\Gamma,\Bbbk)$ (ibid.).

Note that the Tits form does not depend on the *orientation.* Namely for every quiver Γ we can consider the non-oriented graph $|\Gamma|$. It has the same vertices and its edges are in one-to-one correspondence with the arrows of Γ: the edge corresponding to an arrow $a:x\to y$ has the ends x and y. For instance, for the very first example Γ the graph $|\Gamma|$ is



Obviously, if $|\Gamma|=|\Gamma'|$, then $\mathsf{Q}_\Gamma=\mathsf{Q}_{\Gamma'}$. So we can speak about the Tits form of a (non-oriented) graph.

A *subquiver* Γ' of a quiver Γ is a pair of subsets $\operatorname{Ver}\Gamma'\subseteq\operatorname{Ver}\Gamma$, $\operatorname{Arr}\Gamma'\subseteq\operatorname{Arr}\Gamma$ such that if $a\in\operatorname{Arr}\Gamma'$, $a:x\to y$, then $x,y\in\operatorname{Ver}\Gamma'$.

For instance, in our favorite example 1, there are subgraphs

$$\Gamma_1 = \begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \qquad \Gamma_2 = \begin{tikzcd} 3 \arrow[r, "f_1"] & 4 \arrow[loop right, "g"] \end{tikzcd}$$

$$\Gamma_3 = \begin{tikzcd} & 1 \\ 3 \arrow[ur, "c"] \arrow[r, "f_2"'] & 4 \end{tikzcd} \qquad \Gamma_4 = \begin{tikzcd} 2 \arrow[r, "d"] & 4 \arrow[loop right, "g"] \end{tikzcd}$$

The subquiver $\Gamma'\subseteq\Gamma$ is said to be *full* if every arrow $a:x\to y$ such that $x,y\in\operatorname{Ver}\Gamma'$ belongs to $\operatorname{Arr}\Gamma'$.

In the examples above, the subquivers Γ_1 and Γ_4 are full, while Γ_2 and Γ_3 are not.

1. A *partition* of a quiver Γ is a pair of its non-empty full subquivers $\{\Gamma_1,\Gamma_2\}$ such that $\operatorname{Ver}\Gamma_1\cup\operatorname{Ver}\Gamma_2=\operatorname{Ver}\Gamma$, $\operatorname{Ver}\Gamma_1\cap\operatorname{Ver}\Gamma_2=\emptyset$ and $\operatorname{Arr}\Gamma=\operatorname{Arr}\Gamma_1\cup\operatorname{Arr}\Gamma_2$ (not that $\operatorname{Arr}\Gamma_1\cap\operatorname{Arr}\Gamma_2=\emptyset$ follows from the preceding conditions).

2. If Γ has no partitions, it is said to be *connected*, otherwise *disconnected.*

If $\{\Gamma_1, \Gamma_2\}$ is a partition of Γ, we write $\Gamma = \Gamma_1 \sqcup \Gamma_2$.

Quite in the same way one can define partitions into several components $\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \cdots \sqcup \Gamma_k$.

Exercise 8. Let $\Gamma = \Gamma_1 \sqcup \Gamma_2$.

Show that a representation M of Γ is the same as a pair of representations M_1 of Γ_1 and M_2 of Γ_2. Moreover, $M \simeq N$ if and only if $M_1 \simeq N_1$ and $M_2 \simeq N_2$.

In particular, Γ is representations finite if and only if both Γ_1 and Γ_2 are so.

Thus in what follows we can only consider connected quivers.

4 Gabriel theorem

.

- A connected quiver Γ is called *Dynkin* if its Tits form is *positive definite*, i.e. $\mathsf{Q}_\Gamma(\mathbf{x}) > 0$ for any $\mathbf{x} \neq 0$. In this case we write $\mathsf{Q}_\Gamma > 0$.
- A connected quiver Γ is called *Euclidean* if it is not Dynkin, but its Tits form is *non-negative definite*, i.e. $\mathsf{Q}_\Gamma(\mathbf{x}) \geqslant 0$ for any $\mathbf{x}$. In this case we write $\mathsf{Q}_\Gamma \geqslant 0$.

Note that if $|\mathbf{x}| = (|x_1|, \dots, |x_n|)$, one easily verifies that $\mathsf{Q}_\Gamma(|\mathbf{x}|) \leqslant \mathsf{Q}_\Gamma(\mathbf{x})$, so one can always only check the vectors with non-negative coordinates.

Exercise 9. Prove that $\mathsf{Q}_\Gamma > 0$ if and only if $\mathsf{Q}_\Gamma(\mathbf{d}) > 0$ for every non-zero $\mathbf{d} \in \mathbb{N}^\Gamma$ (that is, for every dimension of non-trivial representations of Γ).

Prove the same for non-negative definiteness.

Another important definition is that of *roots* of Tits forms.

. Let $\mathsf{Q} = \mathsf{Q}_\Gamma$ be the Tits form of a quiver without loops, $\mathbf{d} \in \mathbb{N}^\Gamma$.

1. If $\mathsf{Q}(\mathbf{d}) = 1$, the vector $\mathbf{d}$ is called a (positive) *real root* of the form Q.
2. Suppose that $\mathsf{Q} \geqslant 0$. If $\mathbf{d} \neq 0$ and $\mathsf{Q}(\mathbf{d}) = 0$, the vector $\mathbf{d}$ is called a (positive) *imaginary root* of the form Q.

Set $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 at the i-th place). If there are no loops at the vertex i, then $\mathsf{Q}(\mathbf{e}_i) = 1$, so it is a real root of Q. These roots are called the *simple roots*.

Imaginary roots are also defined in general case, but this definition is rather complicated. Since we do not use it, we do not present it here. One can find it in [Stekolshchik, p. 40].

Example. Let Γ be $\begin{array}{ccc} & 1 & \\ & \downarrow & \\ 2 \rightarrow & 4 & \leftarrow 3 \end{array}$ (the quiver of type D_4).

Then $\mathsf{Q}_\Gamma(\mathbf{x}) = \sum_{i=1}^4 x_i^2 - x_4(x_1+x_2+x_3) = (x_4 - \frac{1}{2}\sum_{i=1}^3 x_i)^2 + \frac{1}{4}\Big((x_1-x_2)^2 + (x_1-x_3)^2 + (x_2-x_3)^2\Big) + \frac{1}{4}\sum_{i=1}^3 x_i^2$.

If $\mathbf{x} \in \mathbb{N}^\Gamma$ and $\mathsf{Q}_\Gamma(\mathbf{x}) = 1$, one can easily prove (do it) that
$x_i \leqslant 1$ for $1 \leqslant i \leqslant 3$,
$x_4 \leqslant 2$,
if $x_4 = 2$, then $x_1 = x_2 = x_3 = 1$,
if $x_4 = 0$, then at most one other coordinate equals 1.

It gives the list of roots of Q_Γ, except the simple root $\mathbf{e}_i$ $(1 \leqslant i \leqslant 4)$:
$(1,0,0,1)$, $(0,1,0,1)$, $(0,0,1,1)$,
$(1,1,0,1)$, $(1,0,1,1)$, $(0,1,1,1)$,
$(1,1,1,1)$,
$(1,1,1,2)$.

Now we can formulate the main result about representation finite quivers.

Theorem 1 (Gabriel theorem)**.** *Let Γ be a finite quiver, $\Bbbk$ be a field, $\mathbf{d} \in \mathbb{N}^\Gamma$.*

1. *Γ is representation finite over $\Bbbk$ if and only if $\mathsf{Q}_\Gamma > 0$.*

2. *There is an indecomposable representation M of dimension $\mathbf{d}$ if and only if $\mathsf{Q}_\Gamma(\mathbf{d}) = 1$, that is $\mathbf{d}$ is a real root of Q_Γ.*

3. *If M, N are indecomposable representations, $M \simeq N$ if and only if $\mathbf{dim}\, M = \mathbf{dim}\, N$.*

Thus, in the finite case $M \mapsto \mathbf{dim}\, M$ is a one-to-one correspondence between $\operatorname{ind}(\Gamma, \Bbbk)$ and the set of (positive) real roots of Q_Γ.

Using the geometrical meaning of the Tits form, one can immediately prove the *necessity* in the item 1 of this theorem in the case when the field $\Bbbk$ is algebraically closed.

Indeed, if $\mathsf{Q}_\Gamma(\mathbf{d}) = \mathsf{Q}_\Gamma^+(\mathbf{d}) - \mathsf{Q}_\Gamma^-(\mathbf{d}) \leqslant 0$ for some nonzero $\mathbf{d} \in \mathbb{N}^\Gamma$, then

$$\dim \operatorname{rep}(\mathbf{d}, \Gamma, \Bbbk) = \mathsf{Q}_\Gamma^-(\mathbf{d}) \geqslant \mathsf{Q}_\Gamma^+(\mathbf{d}) = \dim \mathsf{GL}(\mathbf{d}, \Gamma, \Bbbk) > \dim O(M),$$

where $O(M)$ is the orbit of any matrix representation $M \in \operatorname{rep}(\mathbf{d}, \Gamma, \Bbbk)$ under the action of the group $\mathsf{GL}(\mathbf{d}, \Gamma, \Bbbk)$.

As $\operatorname{rep}(\mathbf{d}, \Gamma, \Bbbk)$ is the union of the orbits, there must be infinitely many of them, that is infinitely many isomorphism classes of representations of dimension $\mathbf{d}$ (see, for instance, [AG, Cor. 3.6.9]). Certainly, it immediately implies that there are infinitely many isomorphism classes of indecomposable representations (explain why).

Therefore, if Γ is representation finite, $\mathsf{Q}_\Gamma(\mathbf{d}) > 0$ for every nonzero $\mathbf{d} \in \mathbb{N}^\Gamma$.
As we have seen in Exercise 9 above, it implies that $\mathsf{Q}_\Gamma > 0$.

Actually, almost the same proof holds valid if the field $\Bbbk$ is infinite. Indeed, let $\mathbb{K}$ be the algebraically closure of $\Bbbk$. Set

$$\begin{aligned} \mathbb{A} &= \mathrm{rep}(\mathbf{d},\Gamma,\Bbbk), & \tilde{\mathbb{A}} &= \mathrm{rep}(\mathbf{d},\Gamma,\mathbb{K}),\\ G &= \mathsf{GL}(\mathbf{d},\Gamma,\Bbbk), & \tilde{G} &= \mathsf{GL}(\mathbf{d},\Gamma,\mathbb{K}). \end{aligned}$$

$\tilde{\mathbb{A}}$ is an affine space over $\mathbb{K}$ and $\mathbb{A}$ is the set of points of $\tilde{\mathbb{A}}$ with coordinates in $\Bbbk$.

If Γ is representation finite over $\Bbbk$, $\mathbb{A} = O_1 \cup O_2 \cup \cdots \cup O_r$ for some orbits of the group G.

It is known (see, for instance, [AG, Ex. 1.1.3 (4)]) that $\mathbb{A}$ is dense in $\tilde{\mathbb{A}}$, whence, taking closures

$$\tilde{\mathbb{A}} = \overline{\mathbb{A}} = \overline{O_1 \cup O_2 \cup \cdots \cup O_r} = \overline{O_1} \cup \overline{O_2} \cup \cdots \cup \overline{O_r} \subseteq \overline{\tilde{O}_1} \cup \overline{\tilde{O}_2} \cup \cdots \cup \overline{\tilde{O}_r},$$

where $\tilde{O}_i$ are orbits of the group $\tilde{G}$.

Since $\dim X = \dim \overline{X}$ for every subvariety X, it implies again that $\mathsf{Q}_\Gamma(\mathbf{d}) > 0$.

We propose a proof of Gabriel theorem for finite fields. It is based on a calculation of the number of orbits in $\mathrm{rep}(\mathbf{d},\Gamma,\Bbbk)$.

Exercise 10. Let $\Bbbk$ be a finite field with q elements, Γ be a finite quiver. For $\mathbf{d} \in \mathbb{N}^\Gamma$, set $|\mathbf{d}| = \sum_{i\in \mathrm{Ver}\,\Gamma} \mathbf{d}(i)$.

1. Find the number of elements in $\mathrm{rep}(\mathbf{d},\Gamma,\Bbbk)$ and in $\mathsf{GL}(\mathbf{d},\Gamma,\Bbbk)$.
2. Deduce that if $\mathsf{Q}_\Gamma(\mathbf{d}) \leqslant 0$, there is at least $q^{|\mathbf{d}|}$ non-isomorphic representations of dimension $\mathbf{d}$.
3. Prove that if there are only r non-isomorphic indecomposable representation of Γ over $\Bbbk$, the number of orbits in $\mathrm{rep}(\mathbf{d},\Gamma,\Bbbk)$ is smaller that a polynomial of $|\mathbf{d}|$.
4. Deduce that $\mathsf{Q}_\Gamma > 0$ if Γ is representation finite over $\Bbbk$.

Exercise 11. Let M, N be representations of the same dimension of a finite quiver Γ over an infinite field $\Bbbk$. If $\mathbb{f}$ is an extension of the field $\Bbbk$, we can consider M and N as representations of Γ over $\mathbb{f}$ (for instance, considering matrix representations). Prove that if M and N are isomorphic as representations of Γ over $\mathbb{f}$, they are also isomorphic as representations of Γ over $\Bbbk$.

Hints:

- Present homomorphisms $M \to N$ as solutions of a system of linear equations with coefficients in $\Bbbk$.
- If $\Phi_1, \Phi_2, \ldots, \Phi_r$ is a basis of the space of such solutions over $\Bbbk$, it is also that over $\mathbb{f}$.
- Consider the product $D(x_1, x_2, \ldots, x_r)$ of the determinants of the matrices presenting the "general solution" $\sum_{i\in \mathrm{Ver}\,\Gamma} x_i\Phi_i$.

- If $M \simeq N$ over $\mathbb{f}$, then $D(x_1, x_2, \ldots, x_r) \neq 0$, hence there are $\lambda_1, \lambda_2, \ldots, \lambda_r \in \Bbbk$ such that $D(\lambda_1, \lambda_2, \ldots, \lambda_r) \neq 0$, whence $M \simeq N$ over $\Bbbk$.

Note that this result is valid for finite fields too (it can be deduced from the Krull–Schmidt theorem, which holds for finite dimensional representations).

The Dynkin quivers can be completely described.

First, one easily sees that if there is a cycle

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[rrrrr, dash, bend right] & 2 \arrow[r, dash] & 3 & \dots & (k-1) \arrow[r, dash] & k \end{tikzcd}$$

in the non-oriented graph $|\Gamma|$ and $\mathbf{x} = (\underbrace{1, 1, \ldots, 1}_{k \text{ times}}, 0, \ldots, 0)$, then $\mathsf{Q}_\Gamma(\mathbf{x}) = 0$. Therefore, if Γ is a Dynkin quiver, there are no cycles in $|\Gamma|$. In particular, there are neither loops nor multiple edges in $|\Gamma|$. Hence, $|\Gamma|$ is a tree.

We propose to the reader to prove the next description as a (not very easy) exercise.

Exercise 12. Γ is a Dynkin quiver if and only if the graph $|\Gamma|$ is one of the following:

$$A_n: \quad 1 - 2 - 3 \cdots (n-1) - n$$

$$D_n: \quad \begin{tikzcd}[row sep=small] 1 \arrow[dr, dash] & & & \\ & 3 - 4 \cdots (n-1) - n & & (n \geqslant 4) \\ 2 \arrow[ur, dash] & & & \end{tikzcd}$$

$$E_6: \quad \begin{tikzcd}[row sep=small] & & 6 \arrow[d, dash] & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \end{tikzcd}$$

$$E_7: \quad \begin{tikzcd}[row sep=small] & & 7 \arrow[d, dash] & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \end{tikzcd}$$

$$E_8: \quad \begin{tikzcd}[row sep=small] & & 8 \arrow[d, dash] & & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] & 7 \end{tikzcd}$$

There is also a description of Euclidean quivers. We also propose the reader to prove it himself. First of all, the following important remark.

Exercise 13. Let a quiver Γ be Euclidean, i be any its vertex and Γ' be the full subgraph of Γ such that $\operatorname{Ver}\Gamma' = \operatorname{Ver}\Gamma \setminus \{i\}$. Then $\mathsf{Q}_{\Gamma'} > 0$, so Γ' is a disjoint union of Dynkin quivers (maybe, just a Dynkin quiver).

Corollary: The rank of the matrix of the Tits form of a Euclidean quiver with n vertices equals $n-1$. (Use the Silvester criterion).

Hint: Let $\mathsf{Q}_{\Gamma'}(\mathbf{x}') = 0$ for some $\mathbf{x}' \in \mathbb{N}^{\Gamma'}$. Then $\mathsf{Q}(k\mathbf{x}') = 0$ for any k, so we can suppose that all coordinates of $\mathbf{x}'$ are at least 2. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where $x_i = 1$ and other coordinates are the same as in $\mathbf{x}'$. Then $\mathsf{Q}(\mathbf{x}) < \mathsf{Q}(\mathbf{x}')$.

Now, the description.

Exercise 14. Γ is a Euclidean quiver if and only if the graph $|\Gamma|$ is one of the following:

$$\widetilde{A}_n: \quad \begin{tikzcd}[column sep=small] 1 \arrow[r, dash] \arrow[rrrr, dash, bend right] & 2 \arrow[r, dash] & 3 \cdots & n \arrow[r, dash] & (n+1) \end{tikzcd}$$

$$\widetilde{D}_n: \quad \begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr, dash] & & & & n \\ & 3 \arrow[r, dash] & 4 \cdots & (n-1) \arrow[ur, dash] \arrow[dr, dash] & \\ 2 \arrow[ur, dash] & & & & (n+1) \end{tikzcd} \qquad (n \geqslant 4)$$

$$\widetilde{E}_6: \quad \begin{tikzcd}[column sep=small, row sep=small] & & 7 \arrow[d, dash] & & \\ & & 6 \arrow[d, dash] & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \end{tikzcd}$$

$$\widetilde{E}_7: \quad \begin{tikzcd}[column sep=small, row sep=small] & & & 8 \arrow[d, dash] & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] & 7 \end{tikzcd}$$

$$\widetilde{E}_8: \quad \begin{tikzcd}[column sep=small, row sep=small] & & 9 \arrow[d, dash] & & & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] & 7 \arrow[r, dash] & 8 \end{tikzcd}$$

5 Reflections

Reflections

So, we have proved that if Γ is representation finite, then $\mathsf{Q}_\Gamma > 0$. To prove the converse as well as to prove the claims about indecomposable representations, we have to study the notion of *reflections*.

In this section we denote by Q the Tits form of some quiver Γ without loops. If $\operatorname{Ver}\Gamma = \{1, 2, \dots, n\}$, it is of the form

$$\mathsf{Q}(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} x_i^2 - \sum_{i<j} c_{ij} x_i x_j, \quad \text{where } c_{ij} \in \mathbb{N}.$$

We denote by $\mathsf{B}(\mathbf{x}, \mathbf{y})$ the associated symmetric bilinear form, i.e. such that $\mathsf{Q}(\mathbf{x}) = \mathsf{B}(\mathbf{x}, \mathbf{x})$. It is

$$\mathsf{B}(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) = \sum_{i=1}^{n} x_i y_i - \frac{1}{2} \sum_{i \neq j} c_{ij} x_i y_j,$$

where we set $c_{ij} = c_{ji}$ if $i > j$.

Note that if $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the natural basis of $\mathbb{R}^n$, that is $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 at the i-th place), then

$$\begin{aligned} \mathsf{Q}(\mathbf{e}_i) &= 1, \\ \mathsf{B}(\mathbf{e}_i, \mathbf{e}_j) &= -\frac{1}{2} c_{ij} \quad \text{if } i \neq j, \\ \mathsf{B}(\mathbf{x}, \mathbf{e}_i) &= x_i - \frac{1}{2} \sum_{j \neq i} c_{ij} x_j. \end{aligned}$$

Note also that if $\mathsf{Q} > 0$, then $|c_{ij}| \leqslant 1$ for all $i \neq j$: otherwise $\mathsf{Q}(\mathbf{e}_i + \mathbf{e}_j) \leqslant 0$.

Now we define the reflections in $\mathbb{R}^n$ with respect to the form Q.

. The *reflection at the vertex i with respect to the form* Q is the linear map $s_i : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$s_i(\mathbf{x}) = \mathbf{x} - 2\mathsf{B}(\mathbf{x}, \mathbf{e}_i)\mathbf{e}_i.$$

Obviously, if $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $s_i\mathbf{x} = (x'_1, x'_2, \dots, x'_n)$, then $x'_j = x_j$ if $i \neq j$. The preceding calculation implies that $x'_i = \sum_{j \neq i} c_{ij}x_j - x_i$.

We denote $\delta_i(\mathbf{x}) = x_i - x'_i = 2\mathsf{B}(\mathbf{x}, \mathbf{e}_i) = 2x_i - \sum_{j \neq i} c_{ij}x_j$.

Consider some simple properties of reflections. We propose to prove them yourself.

Exercise 15. Check that

1. $\mathsf{Q}(s_i\mathbf{x}) = \mathsf{Q}(\mathbf{x})$, moreover, $\mathsf{B}(s_i\mathbf{x}, s_i\mathbf{y}) = \mathsf{B}(\mathbf{x}, \mathbf{y})$.
2. $\mathsf{Q}(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n} \delta_i(\mathbf{x})\mathbf{x}_i$.
3. $s_i^2 = \mathrm{Id}$ (identity map).
4. If $i \neq j$ and $c_{ij} = 0$, then $s_is_j = s_js_i$.
5. If $i \neq j$ and $c_{ij} = 1$, then $s_is_js_i = s_js_is_j$ (equivalently, $(s_is_j)^3 = \mathrm{Id}$).

. 1. The subgroup $W(\Gamma) \subseteq \mathsf{GL}(n, \mathbb{R})$ generated by all reflections $s_1, s_2, \dots, s_n$ is called the *Weyl group* of the quiver Γ.

2. The map $C = s_n \dots s_2s_1$ is called the *Coxeter transformation.*

   Note that the Coxeter transformation depends on the chosen order of coordinates (i.e. of the vertices of Γ).

Example. If Γ is a chain $1 \to 2 \to 3 \dots (n-1) \to n$, then $W(\Gamma) \simeq \mathbf{S}_{n-1}$ (the permutation group).

Indeed, it is known that $\mathbf{S}_{n-1}$ is generated by the transpositions $t_i = (i, i+1)$ $(1 \leqslant i < n)$ with the defining relations $t_i^2 = 1$, $t_it_j = t_jt_i$ if $|i-j| > 1$ and $(t_it_{i+1})^3 = 1$ $(1 \leqslant i < n)$.
These are just the relations for s_i from the preceding exercise.
Hence the map $t_i \mapsto s_i$ defines a homomorphism $\varphi : \mathbf{S}_{n-1} \to W(\Gamma)$.
One easily verifies (do it) that $\operatorname{Ker}\varphi = \{1\}$.
(Just recall which are the normal subgroups of $\mathbf{S}_{n-1}$).

Proposition. *Suppose that* $\mathsf{Q} > 0$.

1. *The set* $\mathbf{R} = \{\, \mathbf{x} \in \mathbb{Z}^n \mid \mathsf{Q}(\mathbf{x}) = 1 \,\}$ *is finite. In particular, the set* $\mathbf{R}^+$ *of positive real roots is finite.*

2. *The Weyl group $W(\Gamma)$ is finite.*

Proof. (1) The set $\{ \mathbf{x} \in \mathbb{R}^n \mid Q(\mathbf{x}) \leqslant 1 \}$ is bounded (why?), hence contains only finitely many vectors with integral coordinates.

(2) If $w \in W(\Gamma)$, then $w(\mathbf{e}_i) \in \mathbf{R}$, hence there are finitely many possibilities for it. As the linear map w is defined by the values $w(\mathbf{e}_i)$ $(1 \leqslant i \leqslant n)$, there are finitely many $w \in W(\Gamma)$. □

. Let $\mathbf{x} \in \mathbb{R}^\Gamma = \mathbb{R}^n$.

- The *support* $\operatorname{supp} \mathbf{x}$ of $\mathbf{x}$ is $\{ i \in \operatorname{Ver} \Gamma \mid x_i \neq 0 \}$.

- $\mathbf{x}$ is called *connected* if $\operatorname{supp} \mathbf{x}$ is a connected part of $|\Gamma|$.

Evidently, every real root of a positive definite Tits form is connected (explain it).

We write $\mathbf{x} \geqslant \mathbf{y}$ if $x_i \geqslant y_i$ for all i. If, moreover, $\mathbf{x} \neq \mathbf{y}$, we write $\mathbf{x} > \mathbf{y}$. In particular, $\mathbf{x} > 0$ means that all coordinates of $\mathbf{x}$ are non-negative and at least one of them is positive.

Using this notion, we can precise one of the preceding claims. We propose to prove it as an exercise.

Exercise 16. Prove that if $\mathbf{x}$ is connected and neither $\mathbf{x} \geqslant 0$ nor $\mathbf{x} \leqslant 0$, then $\mathsf{Q}(|\mathbf{x}|) < \mathsf{Q}(\mathbf{x})$.

Deduce that if $\mathsf{Q} > 0$ and $\mathbf{x}$ is a real root, then either $\mathbf{x} > 0$ or $\mathbf{x} < 0$.

Consider more properties of reflections.

Proposition. *Let Γ be a Dynkin quiver,* $\operatorname{Ver} \Gamma = \{ 1, 2, \dots, n \}$ *and* $\mathbf{x} \in \mathbb{R}^\Gamma = \mathbb{R}^n$.

1. *If $\mathbf{x} > 0$, there is i such that $s_i \mathbf{x} < \mathbf{x}$.*

2. *If $\mathbf{x} \neq 0$, then $C\mathbf{x} \neq \mathbf{x}$.*

3. *If $\mathbf{x} > 0$, there is $k \in \mathbb{N}$ such that $C^k \mathbf{x} \not> 0$.*

4. *If $\mathbf{x}$ is a real root and $\mathbf{x} \neq \pm \mathbf{e}_i$, then $|\delta_i(\mathbf{x})| \leqslant 1$.*

5. *If $\mathbf{x}$ is a positive real root and $\mathbf{x}_i \neq \mathbf{e}_i$, then $s_i(\mathbf{x}) > 0$.*

Proof:
(1) We know [15] that $\mathsf{Q}(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^n \delta_i(\mathbf{x}) x_i$, where $\delta_i(\mathbf{x}) = x_i - x'_i$.
As $\mathsf{Q}(\mathbf{x}) > 0$ and all $x_i \geqslant 0$, at least one of $\delta_i(\mathbf{x}) > 0$.
It means that $x_i > x'_i$, so $\mathbf{x} > \mathbf{x}'$.

(2) Note that every s_i only changes the i-th coordinate of $\mathbf{x}$. The claim (1) shows that at least one of s_i changes $\mathbf{x}$. Hence their product also changes $\mathbf{x}$.

(3) Suppose that $C^k\mathbf{x} > 0$ for all k. As $W(\Gamma)$ is finite, there is m such that $C^m = \mathrm{Id}$. Set $\mathbf{y} = \sum_{k=0}^{m-1} C^k\mathbf{x}$. Then $C\mathbf{y} = \mathbf{y}$ and $\mathbf{y} > 0$, which is impossible.

(4) Suppose that $|x_i - x_i'| \geqslant 2$. Then there is an integer y such that either $x_i' < y < x_i$ or $x_i < y < x_i'$. Consider all vectors $\widetilde{\mathbf{x}}$ such that $\widetilde{x}_j = x_j$ for all $j \neq i$. Then $Q(\widetilde{\mathbf{x}})$ is a quadratic polynomial q in $\widetilde{x}_i$: $\mathsf{q}(\widetilde{x}_i) = \widetilde{x}_i^2 + a\widetilde{x} + b$. Let $\mathbf{y}$ be such that $y_j = x_j$ for $j \neq i$ and $y_i = y$. As $\mathsf{q}(x_i) = \mathsf{q}(x_i')$ and y is in between, $\mathsf{Q}(\mathbf{y}) = \mathsf{q}(y) < \mathsf{q}(x_i) = \mathsf{Q}(\mathbf{x}) = 1$. Since it is an integer, $\mathsf{Q}(\mathbf{y}) \leqslant 0$, whence $\mathbf{y} = 0$. Therefore, $x_j = 0$ for $j \neq i$ and $1 = \mathsf{Q}(\mathbf{x}) = x_i^2$, so $x_i = \pm 1$ and $\mathbf{x} = \pm\mathbf{e}_i$.

(5) follows immediately from (4). Indeed, if $\mathbf{x} \neq \mathbf{e}_i$, there is $j \neq i$ such that $x_j > 0$. If $x_i = 0$, then $\mathsf{B}(\mathbf{x}, \mathbf{e}_i) \leqslant 0$, hence $x_i' \geqslant 0$. If $x_i > 0$, the claim (4) implies that $x_i' \geqslant x_i - 1 \geqslant 0$. □

Theorem 2. *Let Γ be a Dynkin quiver. Then the set of real roots $\mathbf{R}$ coincides with $\{\, w(\mathbf{e}_i) \mid 1 \leqslant i \leqslant n,\ w \in W(\Gamma) \,\}$.*

Proof. As $\mathsf{Q}(w(\mathbf{x})) = \mathsf{Q}(\mathbf{x})$ and $w(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$, all vectors $w(\mathbf{e}_i)$ are real roots.

On the contrary, let $\mathbf{x}$ be a real root and $\mathbf{x} > 0$. There is i such that $s_i\mathbf{x} < \mathbf{x}$.

Repeating such choice, we find a sequence of reflections $s_{i_1}, s_{i_2}, \dots, s_{i_{m+1}}$ such that $s_{i_k} \dots s_{i_2} s_{i_1}\mathbf{x} > 0$ for $1 \leqslant k \leqslant m$ but $s_{i_{m+1}} s_{i_m} \dots s_{i_2} s_{i_1}\mathbf{x} \not> 0$.

As we have already proved, it is only possible if $s_{i_m} \dots s_{i_2} s_{i_1}\mathbf{x} = \mathbf{e}_{i_{m+1}}$.

Therefore $\mathbf{x} = w(\mathbf{e}_{i_{m+1}})$, where $w = s_{i_1} s_{i_2} \dots s_{i_m}$.

If $\mathbf{x} < 0$, then $-\mathbf{x} > 0$, hence $-\mathbf{x} = w(\mathbf{e}_i)$ and $\mathbf{x} = w(-\mathbf{e}_i) = ws_i(\mathbf{e}_i)$. □

Exercise 17. Let Γ be a chain $1 \to 2 \to 3 \dots (n-1) \to n$, $\mathbf{e}_{ij}$, where $1 \leqslant i \leqslant j \leqslant n$ be the vectors from $\mathbb{R}^\Gamma = \mathbb{R}^n$ such that $\mathbf{v}_{ij}(k) = 1$ if $i \leqslant k \leqslant j$ and 0 otherwise (e.g. $\mathbf{e}_{ii} = \mathbf{e}_i$). Prove that $\mathbf{e}_{ij}$ are roots of Γ and there are no other positive roots.

Hint: Calculate the vectors $s_k\mathbf{e}_{ij}$ for all choices of i, j, k.

Exercise 18. Construct indecomposable representations E_{ij} such that $\mathbf{dim}\, E_{ij} = \mathbf{e}_{ij}$.

(By the Gabriel theorem, such representation is unique up to isomorphism.)

6 Reflection functors

The main idea of the proof of sufficiency and the claims about dimensions, proposed by Bernstein–Gelfand–Ponomarev [BGP], is the *categorification* of reflections. Namely they constructed operations on representations which reproduce the action of reflections on their dimensions.

So we are going to define the *reflection functors* $s_i^{\pm}$ on the category $\operatorname{rep}(\Gamma, \Bbbk)$. We still suppose that Γ has no loops.

First, some auxiliary definitions

1. A vertex i of Γ is said to be *positive* (respectively, *negative*) if $\iota_0(a) \neq i$ (respectively, $\iota_1(a) \neq i$) for any arrow a. It means that no arrow begins (respectively, ends) at the vertex i.

2. For a vertex i we define the quiver $s_i\Gamma$ such that

   - $\operatorname{Ver}(s_i\Gamma) = \operatorname{Ver}\Gamma$ and $\operatorname{Arr}(s_i\Gamma) = \operatorname{Arr}\Gamma$.
   - If $a : x \to y$ in Γ and neither $x \neq i$ nor $y \neq i$, then $a : x \to y$ in $s_i\Gamma$ too.
   - If $a : i \to x$ (respectively, $a : x \to i$) in Γ, then $a : x \to i$ (respectively, $a : i \to x$) in $s_i\Gamma$.

     In other words, we change orientation of all arrows with the source or target i.

   Obviously, if the vertex i was positive (respectively, negative) in Γ, it becomes negative (respectively, positive) in $s_i\Gamma$.

Example. In the quiver

$$\Gamma : \quad \begin{tikzcd} & 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[d, "d"] \\ 3 \arrow[ur, "c"] \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] & & 4 \end{tikzcd}$$

the vertex 4 is positive and the vertex 3 is negative.

Here are the quivers $s_1\Gamma$ and $s_3\Gamma$:

$$s_1\Gamma : \quad \begin{tikzcd} & 1 \arrow[r, bend right, "b"'] \arrow[dl, "c"'] & 2 \arrow[l, bend right, "a"'] \arrow[d, "d"] \\ 3 \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] & & 4 \end{tikzcd}$$

$$s_3\Gamma : \quad \begin{tikzcd} & 1 \arrow[r, bend left, "a"] \arrow[dl, "c"'] & 2 \arrow[l, bend left, "b"] \arrow[d, "d"] \\ 3 & & 4 \arrow[ll, bend right, "f_1"'] \arrow[ll, bend left, "f_2"] \end{tikzcd}$$

In the latter quiver 3 is a positive vertex.

. Let $i \in \operatorname{Ver}\Gamma$

- We denote by E_i the *simple representation* at this vertex, namely such that $E_i(i) = \Bbbk$, $E_i(j) = 0$ and $E_i(a) = 0$ for every arrow a.

- If the vertex i is positive, set $M^+(i) = \bigoplus_{a:x\to i} M(x)$ and consider the map $\pi_i(M) : M^+(i) \to M(i)$ such that its restriction onto $M(\iota_0(a))$ is $M(a)$.

- If the vertex i is negative, set $M^-(i) = \bigoplus_{a:i\to x} M(x)$ and consider the map $\varepsilon_i(M) : M(i) \to M^-(i)$ such that its projection onto $M(\iota_1(a))$ is $M(a)$.

In the preceding example 67,

$$\pi_4(M) : M(2) \oplus M(3) \oplus M(3) \xrightarrow{(\, M(d)\ M(f_1)\ M(f_2)\,)} M(4),$$

$$\varepsilon_3(M) : M(3) \xrightarrow{\begin{pmatrix} M(c) \\ M(f_1) \\ M(f_2) \end{pmatrix}} M(1) \oplus M(4) \oplus M(4).$$

Lemma 3. *Let $i \in \operatorname{Ver}\Gamma$, $M \in \operatorname{rep}(\Gamma, \Bbbk)$. Suppose that M has no direct summands isomorphic to E_i.*

1. *If i is positive, $\pi_i(M)$ is surjective.*

2. *If i is negative, $\varepsilon_i(M)$ is injective.*

Proof. (1) Suppose that $\pi_i(M)$ is not surjective and $\mathbf{v} \notin \operatorname{Im}\pi_i(M)$. Then $M(i) = \langle \mathbf{v} \rangle \oplus M'$, where $M' \supseteq \operatorname{Im}\pi_i(M)$.

Hence $M \simeq M_1 \oplus M_0$, where $M_1(j) = M(j)$ for $j \neq i$, $M_1(i) = M'$ and $M_1(a) = M(a)$ for all arrows a, while $M_0(i) = \langle \mathbf{v} \rangle$ and $M_0(j) = 0$ if $j \neq i$, so $M_0 \simeq E_i$.

(2) is proved analogously, considering $\mathbf{v} \in \operatorname{Ker}\varepsilon_i(M)$ (do it as an excercise). $\square$

Now we define *reflection functors* $s_i^{\pm}$.

. Let i be a positive vertex, $M \in \operatorname{rep}(\Gamma)$. We define the representation $M' = s_i^+ M \in \operatorname{rep}(s_i\Gamma, \Bbbk)$ as follows:

- $M'(j) = M(j)$ if $j \neq i$ and $M'(a) = M(a)$ if $\iota_1(a) \neq i$.

- $M'(i) = \operatorname{Ker} \pi_i(M)$.

  Recall that $\pi_i : M^+(i) = \bigoplus_{a:x\to i} M(x) \to M(i)$.

- If $a : x \to i$, then $M'(a) : M'(i) \to M(x)$ is the composition of the embedding $M'(i) \to M^+(i)$ and the projection $M^+(i) \to M(x)$.

Note that $\dim M'(i) = \dim M^+(i) - \dim \operatorname{Im} \pi_i(M)$. If M has no direct summands isomorphic to E_i, then, as we have seen, $\pi_i(M)$ is surjective, hence $\operatorname{Im} \pi_i(M) = M(i)$ and

$$\dim M'(i) = \sum_{a:x\to i} \dim M(x) - \dim M(i).$$

If we recall the definition of reflections 53, we see that if M has no direct summands isomorphic to E_i, then

$$\mathbf{dim}\, s_i^+ M = s_i\, \mathbf{dim}\, M.$$

. Let now i be a negative vertex, $M \in \operatorname{rep}(\Gamma)$. We define the representation $M' = s_i^- M \in \operatorname{rep}(s_i\Gamma, \Bbbk)$ as follows:

- $M'(j) = M(j)$ if $j \neq i$ and $M'(a) = M(a)$ if $\iota_0(a) \neq i$.

- $M'(i) = \operatorname{Coker} \varepsilon_i(M) = M^-(i)/\operatorname{Im} \varepsilon_i(M)$.

  Recall that $\varepsilon_i : M(i) \to M^-(i) = \bigoplus_{a:i\to x} M(x)$.

- If $a : i \to x$, then $M'(a) : M(x) \to M'(i)$ is the composition of the embedding $M(x) \to M^-(i)$ and the surjection $M^-(i) \to M'(i)$.

Just as above, one can prove that $\mathbf{dim}\, s_i^- M = s_i\, \mathbf{dim}\, M$ if M has no direct summands isomorphic to E_i (do it as an exercise).

Example. Let Γ be
$$\begin{tikzcd} & 1 \arrow[d, "a_1"] & \\ 2 \arrow[r, "a_2"] & 4 & 3 \arrow[l, "a_3"'] \end{tikzcd}$$
and the representation M be such that all $M(i) = \Bbbk$ and all $M(a_i)$ are identity. The vertex 4 is positive in Γ. We calculate $s_4^+ M$, which is a representation of the quiver $s_4\Gamma$:
$$\begin{tikzcd} & 1 & \\ 2 & 4 \arrow[l, "a_2"'] \arrow[u, "a_1"'] \arrow[r, "a_3"] & 3 \end{tikzcd}.$$

$M^+(4) = M(1) \oplus M(2) \oplus M(3) \simeq \Bbbk^3$ and $\pi_4(M) : \Bbbk^3 \to \Bbbk = M(4)$ is given by the matrix $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$.

Its kernel has a basis $\mathbf{u}_1, \mathbf{u}_2$, where $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. So $s_4^+ M(4) \simeq \Bbbk^2$ and $\mathbf{dim}\, s_4^+(M) = (1,1,1,2) = s_4(1,1,1,1)$.

The maps $s_4^+ M(a_i)$ just project the vectors $\mathbf{u}_1, \mathbf{u}_2$ to their i-th coordinates.

Therefore, in the matrix form $s_4^+ M(a_1) = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $s_4^+ M(a_2) = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $s_4^+ M(a_3) = \begin{pmatrix} -1 & -1 \end{pmatrix}$.

Theorem 4. *For every positive (or negative) vertex i and a nontrivial representation M of a quiver without loops*

1. $s_i^{\pm} M \not\ni E_i$.

   and, if $E_i \notin M$, then

2. $\mathbf{dim}\, s_i^{\pm} M = s_i\, \mathbf{dim}\, M$.

3. $s_i^{\mp} s_i^{\pm} M \simeq M$. *In particular,* $s_i^{\pm} M \neq 0$.

4. *If M is indecomposable, so is $s_i^{\pm} M$.*

We present the proof for a positive vertex. The proof for negative vertices is analogous and is proposed as an exercise.

(1) is evident, since, by the definitions 74, the map $(s_i^+ M)^-(i)$ is injective and the map $(s_i^- M)^+(i)$ is surjective. (2) was alreday checked.

(3) Let i be positive, $M' = s_i^+ M$. Then $M'(j) = M(j)$ if $j \neq i$, while $M'(i) = \operatorname{Ker} \pi_i(M)$, where $\pi_i(M) : M^+(i) = \bigoplus_{a:x\to i} M(x) \to M(i)$ has components $M(a) : M(x) \to M(a)$.

The map $M'(a) : M'(i) \to M'(x) = M(x)$ is the composition $M'(i) \hookrightarrow M^+(i) \twoheadrightarrow M(x)$, the first map being the embedding and the second one being the projection onto a direct summand.

Hence $M'^-(i) = M^+(i)$ and $\varepsilon_i(M') : M'(i) \to M'^-(i)$ is just the embedding $\operatorname{Ker} \pi_i(M) \hookrightarrow M^+(i)$.

As $E_i \notin M$, $\pi_i(M)$ is surjective, so $s_i^- M'(i) = M'^-(i)/\operatorname{Im} \varepsilon_i(M') = M^+(i)/\operatorname{Ker} \pi_i(M) = M(i)$.

The map $s_i^- M'(a) : M'(x) \to s_i^- M'(i)$ is the composition $M(x) \hookrightarrow M^+(i) \twoheadrightarrow M(i)$, which coincides with $M(a)$.

Therefore, $s_i^- M' \simeq M$.

(4) easily follows from (1–3). Indeed, if $s_i^{\pm} M = N_1 \oplus N_2$, neither N_1 nor N_2 contains E_i as a summand.

Therefore, neither $s_i^{\mp} N_1$ nor $s_i^{\mp} N_2$ is trivial.

Hence $M \simeq s_i^{\mp}(N_1 \oplus N_2) = s_i^{\mp}(N_1) \oplus s_i^{\mp}(N_2)$ is decomposable.

Exercise 19. Verify that if there are no arrows between the vertices i and j, then $s_i^{\pm} s_j^{\pm} = s_j^{\pm} s_i^{\pm}$.

7 Gabriel theorem - proof

Proof of Gabriel theorem

We are now able to prove the Gabriel theorem 1. A "*naive*" approach seems to be as follows:

1. We consider an indecomposable (and not simple) represenation M and its dimension $\mathbf{d}$. As $\mathsf{Q} > 0$, we know that there is a vertex i such that $\mathbf{d}_1 = s_i\mathbf{d} < \mathbf{d}$.

2. Set $M_1 = s_iM$ and find i_1 such that $\mathbf{d}_2 = s_{i_1}\mathbf{d}_1 < \mathbf{d}_1$. Then set $M_2 = s_{i_2}M_1$ etc.

3. Finally, we get $\mathbf{d}_{m+1} = \mathbf{e}_k$ for some k, whence $M_{m+1} = E_k$. Then $\mathbf{d}$ is obtained by reflections from $\mathbf{e}_k$ and M is obtained by reflections from E_k.

4. Now the proofs of all claims are amost evident.

Why doesn't this procedure work?

The answer is also evident: what does it mean s_iM? We know s_i^+M if i is positive and s_i^-M if i is negative. But nobody guarantees that our favourit vertex i is either positive or negative.

For instance, if Γ is $\begin{array}{ccccc} & & 1 & & \\ & & \uparrow & & \\ 2 & \to & 4 & \leftarrow & 3 \end{array}$ and $\mathbf{d} = (1, 1, 1, 2)$ the only vertex i such that $s_i\mathbf{d} < \mathbf{d}$ is 4, which is neither positive nor negative.

So we have do some extra work. Our instrument will be the *Coxeter transformation* $C = s_n \dots s_2s_1$. To use it, we actually need a special ordering of the vertices.

. 1. An ordering $\operatorname{Ver}\Gamma = \{ i_1, i_2, \dots, i_n \}$ of the vertices of the quiver Γ is called *positive* (respectively, *negative*) if the vertex i_1 is positive (respectively, negative) and for $1 \leqslant k < n$ the vertex i_{k+1} is positive (respectively, negative) in the quiver $s_k \dots s_2s_1\Gamma$.

2. If $\operatorname{Ver}\Gamma = \{ i_1, i_2, \dots, i_n \}$ is a positive (respectively, negative) ordering of the vertices of Γ, we set $C^+ = s_{i_n}^+ \dots s_{i_2}^+ s_{i_1}^+$ (respectively, $C^- = s_{i_n}^- \dots s_{i_2}^- s_{i_1}^-$) and call C^+ (respectively, C^-) the *Coxeter transformation* with respect to this ordering.

For instance, if Γ is $\begin{array}{ccccc} & & 1 & & \\ & & \uparrow & & \\ 2 & \to & 4 & \leftarrow & 3 \end{array}$, as above, the ordering $\{ 1, 4, 2, 3 \}$ is positive and the ordering $\{ 2, 3, 4, 1 \}$ is negative.

Obviously, if the ordering $\{ i_1, i_2, \dots, i_n \}$ is positive, the inverse ordering $\{ i_n, \dots, i_2, i_1 \}$ is negative and vice versa.

If Γ is a cycle

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \cdots (k-1) \arrow[r] & k \arrow[lll, bend left] \end{tikzcd}$$

it has neither positive nor negative vertices.

Actually, cycles are unique obstacles for positive (or negative) numerations.

Lemma 5. *If a quiver Γ contains no (oriented) cycles, there is a positive (as well as a negative) numeration of its vertices.*

Proof. We know that there are finitely many paths in Γ (see Exercise 1). Let p be a path of the maximal length.

Then its target i_1 must be positive: if $a : i_1 \to x$, then the path ap is longer.

Delete this vertex, i.e. consider the full subquiver Γ' with $\operatorname{Ver}\Gamma' = \operatorname{Ver}\Gamma \setminus \{i_1\}$. There are no cycles in Γ' as well and Γ' has less vertices.

So we can use induction and suppose that there is a positive numeration $\{ i_2, \dots, i_n \}$ of $\operatorname{Ver}\Gamma'$. Then $\{ i_1, i_2, \dots, i_n \}$ is a positive numeration of $\operatorname{Ver}\Gamma$. $\square$

Exercise 20. Let Γ and Γ' be finite quivers such that $|\Gamma| = |\Gamma'|$ and $|\Gamma|$ is a tree. There is a sequence $i_1, i_2, \dots, i_m$ of vertices of Γ such that $s_{i_m} \dots s_{i_2} s_{i_1} \Gamma = \Gamma'$.

(Here it is possible that $m > n$. Then we define $s_{i+qn} = s_i$.)

Sketch of the proof. One can suppose that there is only one arrow a in Γ such that $\iota_\Gamma(a) \neq \iota_{\Gamma'}(a)$. Let $a : x \to y$, $\bar\Gamma$ be the quiver obtained from Γ by deleting the arrow a. The same quiver is obtained by deleting a from Γ'. Then $\bar\Gamma = \bar\Gamma_1 \sqcup \bar\Gamma_2$, where $x \in \bar\Gamma_1$, $y \in \bar\Gamma_2$.

If neither the source nor the target of an arrow b from Γ_1 equals x, it has the same source and target in Γ, $s_x\Gamma$ and Γ'. Therefore, if $\operatorname{Ver}\Gamma_1 \setminus \{x\} = \{ i_1, \dots, i_k \}$, then $s_{i_k} \dots s_{i_1} s_x \Gamma = \Gamma'$. $\square$

If $|\Gamma|$ is not a tree, this assertion is not valid even if we replace equality by isomorphism.

Exercise 21. Let Γ be $\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 3 \arrow[r] & 4 \end{tikzcd}$ and Γ' be $\begin{tikzcd} 1 \arrow[d] & 2 \arrow[l] \arrow[d] \\ 3 \arrow[r] & 4 \end{tikzcd}$.

Prove that $\Gamma' \not\simeq s_{i_m} \dots s_{i_2} s_{i_1} \Gamma$ for any sequence of reflections.

So, let now Γ have no cycles and $\{1, 2, \dots, n\}$ be a positive numeration of its vertices. Let $C^+ = s_n^+ \dots s_2^+ s_1^+$. Theorem 4 implies that, if M is an indecomposable representation of Γ, either $C^+M = 0$ or C^+M is indecomposable and $\mathbf{dim}\, C^+M = C\, \mathbf{dim}\, M > 0$, where $C = s_n \dots s_2 s_1$ is the Coxeter transformation in $\mathbb{R}^\Gamma$. We set $s_{i+qn} = s_i$ for any i and q. Then the product $s_m \dots s_2 s_1$ is defined for every m. In particular, it coincides with C^k if $m = kn$.

Note that $s_n \dots s_2 s_1 \Gamma = \Gamma$, hence $C^\pm$ maps representations of the quiver Γ to representations of the same quiver.

If ${C^+}^k M = 0$ for some k, there is an integer m such that $s_{m-1}^+ \dots s_2^+ s_1^+ M \neq 0$, but $s_m^+ \dots s_2^+ s_1^+ M = 0$, whence $s_{m-1}^+ \dots s_2^+ s_1^+ M = E_m$ and $M \simeq s_1^- s_2^- \dots s_{m-1}^- E_m$. Note that in this case $\mathbf{dim}\, M = s_1 s_2 \dots s_{m-1} \mathbf{e}_m$ is a real root of the Tits form.

Moreover, if M' is another indecomposable representation of the same dimension, the same consideration shows that $M' \simeq s_1^- s_2^- \dots s_{m-1}^- E_m$ as well. Hence $M \simeq M'$.

Exercise 22. Let both $\{i_1, i_2, \dots, i_n\}$ and $\{j_1, j_2, \dots, j_n\}$ be positive (or negative) numerations of vertices of a quiver Γ. Prove that the corresponding Coxeter transformations coincide:

$$s_{i_n} \dots s_{i_2} s_{i_1} = s_{j_n} \dots s_{j_2} s_{j_1},$$

and as well

$$s_{i_n}^\pm \dots s_{i_2}^\pm s_{i_1}^\pm = s_{j_n}^\pm \dots s_{j_2}^\pm s_{j_1}^\pm$$

So we can speak about *positive* or *negative Coxeter transformation* for the quiver Γ.

(Use that $s_i s_j = s_j s_i$ if there are no arrows between i and j.)

Let now Γ be a Dynkin quiver and $\mathbf{d} = \mathbf{dim}\, M$, where M is an indecomposable representation of Γ. By Proposition 60, there is $k > 0$ such that $C^k \mathbf{d} \not> 0$. Then ${C^+}^k M = 0$, therefore, as we have seen, $M \simeq s_1^- s_2^- \dots s_{m-1}^- E_m$ for some m, $\mathbf{dim}\, M$ is a real root of Q_Γ and if $\mathbf{dim}\, M' \simeq \mathbf{dim}\, M$ for another indecomposable representation M', then $M' \simeq M$.

On the contrary, let $\mathbf{d} > 0$ be a real root of Q_Γ. By the same Proposition, $C^k \mathbf{x} \not> 0$ for some k, hence $s_m \dots s_2 s_1 \mathbf{x} \not> 0$ for some m.

If m is the smallest possible, then $s_{m-1} \dots s_2 s_1 \mathbf{d} = \mathbf{e}_m$ and $s_l \dots s_2 s_1 \mathbf{d} > 0$ for $l < m$. Therefore, $\mathbf{d} = s_1 s_2 \dots s_{m-1} \mathbf{e}_m$ and $\mathbf{d}_l = s_l s_{l+1} \dots s_{m-1} \mathbf{e}_m = s_1 s_2 \dots s_{l-1} \mathbf{d} > 0$ for $l < m$.

Then $M_l = s_l^+ s_{l+1}^+ \dots s_{m-1}^+ E_m$ is an indecomposable representation of dimension $\mathbf{d}_l$ for every $l < m$. In particular, M_1 is an indecomposable representation of dimension $\mathbf{d}$.

It accomplishes the proof of the Gabriel theorem.

Remark. In this proof we used reflections with respect to positive vertices. Certainly, the same proof works for negative vertices.

In particular, every representation of a Dynkin quiver can be obtained from some simple ones both by a sequence of positive reflections and by a sequence of negative reflections.

Exercise 23. Let Γ be
$$\begin{tikzcd} & 1 \arrow[d, "a_1"] & \\ 2 \arrow[r, "a_2"] & 4 & 3 \arrow[l, "a_3"'] \end{tikzcd}$$
and M is an indecomposable representation of dimension $\mathbf{d} = (1,1,1,2)$.

Verify that $C^+M = E_4$, whence $M \simeq C^-E_4$.

On the other hand, $C^-(M)$ is of dimension $(1,1,1,1)$ and only $s_3^- s_2^- s_1^- C^- M \simeq E_4$, whence $M \simeq C^+ s_1^+ s_2^+ s_3^+ E_4$.

In the latter case we have the product of 7 positive reflections. Prove that it is the smallest possible number.

8 Kronecker quiver

Kronecker quiver

Now we are going to describe representations of the Kronecker quiver $\mathbf{K} = 1 \underset{b}{\overset{a}{\rightrightarrows}} 2$, that is, diagrams of vector spaces and linear maps $K(1) \underset{B}{\overset{A}{\rightrightarrows}} K(2)$. Usually we identify A and B with the corresponding matrices.

Two pairs of matrices (A,B) and (A',B') give isomorphic representations if and only if there are invertible matrices S_1, S_2 such that $A' = S_2AS_1^{-1}$ and $B' = S_2BS_1^{-1}$. Equivalently, the matrices A' and B' are obtained from A and B by simultaneous elementary transformations.

The Tits form of this quiver is $\mathsf{Q}(x,y) = x^2+y^2-2xy = (x-y)^2$. Therefore, its real roots are $(n, n\pm 1)$ and imaginary roots are (n,n).

Let K be a representation of this quiver, $\mathbf{dim}\, K = (n,m)$, where $n = \dim K(1)$, $m = \dim K(2)$. Then the matrices A and B are of size $m \times n$. The reflections act on such dimension as follows:

$$s_1(n,m) = (2m-n, m), \quad s_2(n,m) = (n, 2n-m).$$

The ordering $\{1,2\}$ of the vertices is negative, while the ordering $\{2,1\}$ is positive. We set $C^+ = s_1s_2$ and $C^- = s_2s_1$.

For representations, we have Kronecker functors $C^+ = s_1^+s_2^+$ and $C^- = s_2^-s_1^-$.

Note that the reflected representations are representations of the reflected quiver Γ': $1 \underset{b}{\overset{a}{\leftleftarrows}} 2$ (it is both $s_1\Gamma$ and $s_2\Gamma$).

Recall that the reflection functor s_2^+ is constructed as follows. If K is the diagram $K(1) \underset{B}{\overset{A}{\rightrightarrows}} K(2)$, s_2^+K is the diagram $K(1) \underset{B'}{\overset{A'}{\leftleftarrows}} K'(2)$ such that

$$\begin{aligned}
&K'(2) = \operatorname{Ker}\Big(K(1) \oplus K(1) \xrightarrow{(A\ B)} K(2)\Big),\\
&A' \text{ is the composition } K'(2) \hookrightarrow K(1) \oplus K(1) \xrightarrow{\mathrm{pr}_1} K(1),\\
&B' \text{ is the composition } K'(2) \hookrightarrow K(1) \oplus K(1) \xrightarrow{\mathrm{pr}_2} K(1),
\end{aligned}$$

where pr_1 and pr_2 are the projections of the direct sum, respectively, onto the first and onto the second summand.

Analogously, if K is the diagram $K(1) \underset{B}{\overset{A}{\rightrightarrows}} K(2)$, s_1^-K is the diagram $K'(1) \underset{B'}{\overset{A'}{\leftleftarrows}} K(2)$ such that

$$\begin{aligned}
&K'(1) = K(2) \oplus K(2)/\operatorname{Im}\Big(K(1) \xrightarrow{\binom{A}{B}} K(2) \oplus K(2)\Big),\\
&A' \text{ is the composition } K(2) \xrightarrow{\mathrm{emb}_1} K(2) \oplus K(2) \twoheadrightarrow K'(1),\\
&B' \text{ is the composition } K(2) \xrightarrow{\mathrm{emb}_2} K(2) \oplus K(2) \twoheadrightarrow K'(1),
\end{aligned}$$

where emb_1 and emb_2 are the embeddings, respectively, of the first and of the second summand into the direct sum.

The technique that we use for the description of representations is quite different for the "square" case, of dimensions (n, n), and "rectangular" case, of dimensions (n, m) with $m \neq n$. First we consider the rectangular case.

We will use two special indecomposable representations:

$$P:\ \Bbbk \underset{\binom{0}{1}}{\overset{\binom{1}{0}}{\rightrightarrows}} \Bbbk^2\ ,\ \text{with } \mathbf{dim}\, P = \mathbf{p} = (1, 2),$$

$$Q:\ \Bbbk^2 \underset{(0\ 1)}{\overset{(1\ 0)}{\rightrightarrows}} \Bbbk\ ,\ \text{with } \mathbf{dim}\, Q = \mathbf{q} = (1, 2).$$

(Verify that they are indeed indecomposable.)

Note that

- $s_2\mathbf{p} = \mathbf{e}_1$, hence $s_2^+P \simeq E_1$ and $P \simeq s_2^-E_1$.

- $s_1\mathbf{q} = \mathbf{e}_2$, hence $s_1^- Q \simeq E_2$ and $Q \simeq s_1^+ E_2$.

Here in the final formulae E_i and $s_j^{\pm}$ are representations and reflection functors for the reflected quiver Γ', where the vertex 1 is positive and the vertex 2 is negative.

Theorem 6. *Suppose that K is an indecomposable representation of the Kronecker quiver of dimension $\mathbf{d} = (n, m)$ and $m \neq n$.*

1. *(n, m) is a real root of the Tits form, i.e. $m = n \pm 1$. On the contrary, if $\mathbf{d}$ is a real root, there is an indecomposable representation of dimension $\mathbf{d}$ which is unique up to isomorphism.*

2. *If $m > n$, there is $k > 0$ such that either $K \simeq {C^-}^k E_2$ or $K \simeq {C^-}^k P$.*

3. *If $m < n$, there is $k > 0$ such that either $K \simeq {C^+}^k E_1$ or $K \simeq {C^+}^k Q$.*

Proof. Suppose that $m > n$ (the case $m < n$ is quite analogous and we leave it to the reader). Set $K_r = s_r^+ \dots s_1^+ s_2^+ K$ and $\mathbf{d}_r = s_r \dots s_1 s_2 \mathbf{d}$, where $s_i = s_2$ if i is odd and $s_i = s_1$ if i is even.

One can easily calculate that

$$\mathbf{d}_r = \begin{cases} \big(rn - (r-1)m,\ (r+1)n - rm\big) & \text{if } r \text{ is odd,} \\ \big((r+1)n - rm,\ rn - (r-1)m\big) & \text{if } r \text{ is even.} \end{cases}$$

As $m > n$, $\mathbf{d}_r > \mathbf{d}_{r+1}$ for every r.

Therefore, there is $r > 0$ such that K_r is an idecomposable representation, while $K_{r+1} = 0$. We know that it is only possible if $K_r = E_i$. Obviously, $i = 2$ if r is even and $i = 1$ if r is odd.

Using inverse reflections, we get:

$$K \simeq s_2^- s_1^- \dots s_r^- E_i \simeq \begin{cases} {C^-}^k E_2 & \text{if } r = 2k, \\ {C^-}^k P & \text{if } r = 2k+1, \end{cases}$$

since $s_2^- E_1 \simeq P$.

In particular, K is defined up to isomorphism by its dimension, since $r = \min\{\, r \mid \mathbf{d}_{r+1} \not> 0 \,\}$.

On the other hand, one can check that $\mathbf{dim}\, {C^-}^k E_2 = (2k-1, 2k)$ and $\mathbf{dim}\, {C^-}^k P = (2k, 2k+1)$ (do it). Thus all real roots $(n, n+1)$ are dimensions of indecomposable representations, which accomplishes the proof. $\square$

Note that the reflected representations $s_1^- K$ and $s_2^+ K$ are actually representations of the reflected quiver $\Gamma' = s_1\Gamma = s_2\Gamma$ which is again the Kronecker quiver $2 \overset{a}{\underset{b}{\rightrightarrows}} 1$. Therefore, we can consider reflection functors as mappings $\operatorname{rep}(\Gamma, \Bbbk)$ to itself, just interchanges the vertices 1 and 2.

In particular, for the quiver Γ' there are reflection functors s_2^- and s_1^+ acting just as the functors s_1^- and s_2^+ act for the quiver Γ.

We propose as an exercise an explicit description of all representations corresponding to real roots.

Exercise 24. Let A_n, B_n are the following $(n+1)\times n$ matrices

$$A_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad B_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

$K_n^+ = (A_n, B_n)$ and $K_n^- = (A_n^\top, B_n^\top)$.

Prove that K_n^+ and K_n^- are indecomposable representations of the Kronecker quiver. It gives a description of all indecomposable rectangular representations.

Hint: Verify that if $\Phi = (\Phi(1), \Phi(2))$ is a morphism $K_n^+ \to K_n^+$, then $\Phi(1) = \lambda I_n$ and $\Phi(2) = \lambda I_{n+1}$ for some $\lambda \in \Bbbk$. Thus $\dim \operatorname{Hom}_\Gamma(K_n^+, K_n^+) = 1$, which is impossible for a decomposable representation (why?). The case of K_n^- is analogous.

Obviously $K_1^- = Q$ and $K_1^+ = P$.

Exercise 25. Prove that $s_2^+ K_n^+ \simeq K_{n-1}^+$ and $s_1^- K_n^+ \simeq K_{n+1}^+$, considered as representations of the reflected quiver Γ', whence $s_1^- K_n^- \simeq K_{n-1}^-$ and $s_2^+ K_n^- \simeq K_{n+1}^-$ as representations of Γ'.

Now we consider the case of imaginary roots (n, n). A representation K of this dimension is a pair of square matrices (A, B).

If A is invertible and $v_1, v_2, \dots, v_n$ is a basis of $K(1)$, then $Av_1, Av_2, \dots, Av_n$ is a basis of $K(2)$, so, with respect to to these bases, $A = I_n$ (unit matrix of size $n \times n$).

The pairs (I_n, B) and (I_n, B') give isomorphic representations if and only if $B' = SBS^{-1}$ for some invertible matrix S (why?).

Therefore, we can suppose that B is in the Frobenius normal form (or Jordan normal form if $\Bbbk$ is algebraically closed), that is $B = \bigoplus_i F(\varphi_i)$, where φ_i are unital polynomials over the field $\Bbbk$ and $F(\varphi)$ is the Frobenius cell corresponding to the polynomial φ.

Recall that if $\varphi(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1 t + a_0$, then

$$F(\varphi) = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}.$$

The representation (I_n, B) is indecomposable if and only if the matrix B is indecomposable under conjugation, thus if and only if $B = F(\varphi)$, where φ is a power of an irreducible polynomial.

If we use the Jordan normal form, then $B = J_n(\lambda)$, the Jordan $n \times n$ cell with the eigenvalue λ:

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

This representation is denoted by $K_n(\lambda)$.

The same considerations show that, if B is invertible and K is indecomposable, we can suppose that $B = I_n$ and $A = F(\varphi)$, where φ is a power of an irreducible polynomial.

Note that if $\varphi \neq t^n$, the matrix $F(\varphi)$ is invertible, so the only new indecomposable representation that we obtain is $(F(t^n), I_n) \simeq (J_n(0), I_n)$. We denote $K_n(\infty) = (J_n(0), I_n)$.

It so happens that these are the only possibilities.

Theorem 7. *If $K = (A, B)$ is an indecomposable representation of the Kronecker quiver of dimension (n, n), either A or B is invertible.*

Thus this representation is isomorphic to one of the following:

$$K(\varphi) = (I_n, F(\varphi)),$$
$$K_n(\infty) = (J_n(0), I_n).$$

Here φ runs through unital polynomials of degree n that are powers of irreducible polynomials over the field $\Bbbk$.

If $\Bbbk$ is algebraically closed, one can replace $K(\varphi)$, where $\varphi(t) = (t - \lambda)^n$, by $K_n(\lambda)$.

Together with Theorem 6, it accomplishes the description of indecomposable representations of the Kronecker quiver.

Another approach to this classification can be found in [Gantmacher, Ch. XII].

Proof. We use the induction by n. The case $n = 1$ is obvious. So we suppose that every representation of dimension $(n-1, n-1)$ is a direct sum of representations described in this theorem and of representations corresponding to real roots.

Let A be not invertible, i.e. there is a vector $\mathbf{v} \in K(1)$ such that $A\mathbf{v} = 0$. If $B\mathbf{v} = 0$, K has a direct summand isomorphic to E_1.

Hence $\mathbf{u} = B\mathbf{v} \neq 0$ and we can consider bases $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in $K(1)$ and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ in $K(2)$ such that $\mathbf{v}_1 = \mathbf{v}$ and $\mathbf{u}_1 = \mathbf{u}$.

Then the matrices A and B are of the form

$$A = \begin{pmatrix} 0 & \mathbf{a} \\ 0 & A' \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \mathbf{b} \\ 0 & B' \end{pmatrix}$$

for some vectors $\mathbf{a}, \mathbf{b}$ from $\Bbbk^{n-1}$ and a pair of $(n-1) \times (n-1)$ matrices (A', B').

One easily sees that if we replace the pair (A, B) by the pair $(S_2^{-1}AS_1, S_2^{-1}BS_1)$, where

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & T_1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & T_2 \end{pmatrix},$$

we obtain a pair of the same form, where the pair (A', B') changes to $(T_2^{-1}A'T_1, T_2^{-1}B'T_1)$. So we can treat the pair $K' = (A', B')$ as a representation of the same Kronecker quiver.

Using the induction, we can suppose that it is a direct sum of indecomposable representations $K(\varphi)$, $K_m(\infty)$ and $K_k^{\pm}$ of smaller dimensions.

As it is a square representation, the number of summands of the form K^- must be the same as the number of summands of the form K^+.

Suppose that K' has a direct summand $K(\varphi) = (I_m, B_1)$:

$$A = \begin{pmatrix} 0 & \mathbf{a}_1 & \mathbf{a}_2 \\ 0 & I_m & 0 \\ 0 & 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \mathbf{b}_1 & \mathbf{b}_2 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{pmatrix}.$$

Using elementary representations of rows, we can make $\mathbf{a}_1 = 0$.

Then, using elementary transformations of columns, we can make $\mathbf{b}_1 = \mathbf{b}_2 = 0$.

Then $K \simeq K(\varphi) + K''$ (what is K''?).

Just in the same way, if K' has a direct summand K_m^+, also $K \simeq K_m^+ \oplus K''$ (explain it). So K' can only contain direct summands $K_m(\infty) = (J_0(m), I_m)$, so $B' = I_{n-1}$. Then the whole matrix B is invertible as well, which accomplishes the proof. $\square$

How do the reflection functors act on these representations? We consider an example.

Example. Let $K = K_n(\lambda) = (I_n, J_n(\lambda))$ $(\lambda \neq 0)$. We calculate $K' = s_1^- K$. Recall that $K'(2) = K(2)$ and $K'(1) = K(2) \oplus K(2)/\operatorname{Im}\varepsilon$, where $\varepsilon = \begin{pmatrix} I_n \\ J_n(\lambda) \end{pmatrix} : K(1) \to K(2) \oplus K(2)$.

Let $u_1, u_2, \ldots, u_n$ be the chosen basis of $K(1)$ and $v_1, v_2, \ldots, v_n$ be the chosen basis of $K(2)$.

Then $\varepsilon(u_1) = (v_1, \lambda v_1)$ and $\varepsilon(u_i) = (v_i, v_{i-1} + \lambda v_i)$ if $1 < i \leqslant n$. As $\lambda \neq 0$, we can choose the basis $w_1, w_2, \ldots, w_n$ of $\operatorname{Im}\varepsilon$ taking for w_i the coset of $(v_i, 0)$.

The map $A' : K(2) \to K'(1)$ maps v_i just to the image in $K(2) \oplus K(2)/\operatorname{Im}\varepsilon$ of the pair $(v_i, 0)$ that is to w_i. Thus $A' = I_n$.

The map $B' : K(2) \to K'(1)$ maps v_i to the image in $K(2) \oplus K(2)/\operatorname{Im}\varepsilon$ of the pair $(0, v_i)$. One can check (do it) that

$$(0, v_i) = \sum_{j=0}^{i-1} (-1)^j \lambda^{-1-j} \varepsilon(u_{i-j}) - \sum_{j=0}^{i} (-1)^j \lambda^{-1-j} (v_{i-j}, 0).$$

The first sum belongs to $\operatorname{Im}\varepsilon$, whence

$$B' v_i = \sum_{j=0}^{i-1} (-\lambda)^{-1-j} w_{i-j},$$

that is

$$B' = \begin{pmatrix} \mu & \mu^2 & \mu^3 & \dots & \mu^{n-1} & \mu^n \\ 0 & \mu & \mu^2 & \dots & \mu^{n-2} & \mu^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \mu \end{pmatrix},$$

where $\mu = -\lambda^{-1}$.

One easily verifies that the Jordan normal form of the matrix B' is $J_n(\mu)$. Therefore,

$$s_1^- K_n(\lambda) \simeq K_n(-\lambda^{-1}).$$

Exercise 26. Prove that

$$\begin{aligned} &s_2^+ K_n(\lambda) \simeq K_n(-\lambda^{-1}) \text{ if } \lambda \neq 0, \\ &s_1^- K_n(0) \simeq s_2^+ K_n(0) \simeq K_n(\infty), \\ &s_1^- K_n(\infty) \simeq s_2^+ K_n(\infty) \simeq K_n(0), \end{aligned}$$

Therefore,

$$C^+ K_n(\lambda) \simeq C^- K_n(\lambda) \simeq K_n(\lambda) \text{ for any } \lambda \in \Bbbk \cup \{\infty\}.$$

9 Euclidean quivers

9.1 Roots and Weyl group

Euclidean quivers

Now we consider Euclidean quivers Γ, that is such that $|\Gamma|$ is of the forms presented on the next slide.

In these pictures we show the coordinates of the smallest imaginary root, i.e. the smallest vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_n)$ with natural coordinates such that $\mathsf{Q}_\Gamma(\boldsymbol{\omega}) = 0$.

We will see below that any imaginary root equals $k\boldsymbol{\omega}$ for some $k \in \mathbb{N}$.

We have also marked in all cases the vertex $\mathbf{o}$ that was added to the corresponding Dynkin quiver Γ_0 to obtain Γ. Note that always $\boldsymbol{\omega}(\mathbf{o}) = 1$.

$$\widetilde{A}_n: \quad 1 - 1 - 1 \cdots 1 - \check{1}$$

$$\widetilde{D}_n: \quad \begin{array}{ccccc} 1 & & & & 1 \\ & \diagdown & & \diagup & \\ & & 2 - 2 \cdots 2 & & \\ & \diagup & & \diagdown & \\ 1 & & & & \check{1} \end{array} \qquad (n \geqslant 4)$$

$$\widetilde{E}_6: \quad \begin{array}{ccccccccc} & & & & \check{1} & & & & \\ & & & & | & & & & \\ & & & & 2 & & & & \\ & & & & | & & & & \\ 1 & - & 2 & - & 3 & - & 2 & - & 1 \end{array}$$

$$\widetilde{E}_7: \quad \begin{array}{ccccccccccccc} & & & & & & 2 & & & & & & \\ & & & & & & | & & & & & & \\ \check{1} & - & 2 & - & 3 & - & 4 & - & 3 & - & 2 & - & 1 \end{array}$$

$$\widetilde{E}_8: \quad \begin{array}{ccccccccccccccc} & & & & 3 & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ 2 & - & 4 & - & 6 & - & 5 & - & 4 & - & 3 & - & 2 & - & \check{1} \end{array}$$

In this section we suppose that Γ is a Euclidean quiver, $\mathsf{Q} = \mathsf{Q}_\Gamma$ is its Tits form and B is the corresponding symmetric bilinear form.

Proposition. *Suppose that* $\mathsf{Q}(\mathbf{v}) = 0$.

1. *The vector* $\mathbf{v}$ *is in the kernel of* B*, that is* $\mathsf{B}(\mathbf{v}, \mathbf{x}) = 0$ *for every* $\mathbf{x}$.

2. $\mathbf{v} = \alpha\boldsymbol{\omega}$ *for some* $\alpha \in \mathbb{R}$.

Proof. (1) $\mathsf{Q}(k\mathbf{v} \pm \mathbf{e}_i) = k^2\mathsf{Q}(\mathbf{v}) + \mathsf{Q}(\mathbf{e}_i) \pm 2k\mathsf{B}(\mathbf{v}, \mathbf{e}_i) = 1 \pm 2k\mathsf{B}(\mathbf{v}, \mathbf{e}_i)$. If $\mathsf{B}(\mathbf{v}, \mathbf{e}_i) \neq 0$, $\mathsf{Q}(k\mathbf{v} \pm \mathbf{e}_i) < 0$ for some k, which is impossible. Therefore $\mathsf{B}(\mathbf{v}, \mathbf{e}_i) = 0$ for all basic vectors $\mathbf{e}_i$, whence $\mathsf{B}(\mathbf{v}, \mathbf{x}) = 0$ for every $\mathbf{x}$.

(2) If $\mathbf{v} \neq \alpha\boldsymbol{\omega}$, the vectors $\mathbf{v}, \boldsymbol{\omega}$ are linear independent, that is $\mathbf{u} = \alpha\mathbf{v} + \beta\boldsymbol{\omega} \neq 0$ for any real numbers α, β. One can choose α, β such that the vector $\mathbf{u}$ has a zero coordinate u_i. As both $\mathbf{v}$ and $\boldsymbol{\omega}$ are in the kernel of B, $\mathsf{Q}(\mathbf{u}) = 0$, which is impossible, since the Tits form of the quiver $\Gamma \setminus \{i\}$ is positive definite (Exercise 13). $\square$

Besides the symmetric bilinear form B defined above, we also use the *non-symmetric* form

$$\chi_\Gamma(\mathbf{x}, \mathbf{y}) = \sum_{i \in \operatorname{Ver}\Gamma} x_i y_i - \sum_{a: i \to j} x_i y_j.$$

Obviously, we also have that $\chi_\Gamma(\mathbf{x}, \mathbf{x}) = \mathsf{Q}(\mathbf{x})$. Note that this form depends on the orientation of Γ.

We set
$$\kappa_\Gamma(\mathbf{x}) = \chi_\Gamma(\boldsymbol{\omega}, \mathbf{x}) = \sum_{i \in \mathrm{Ver}\,\Gamma} w_i x_i - \sum_{a: i \to j} w_i x_j.$$
If Γ is fixed, we omit the index Γ and write χ and κ.

For instance, for the Kronecker quiver $1 \overset{a}{\underset{b}{\rightrightarrows}} 2$

$$\chi(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 - 2x_1 y_2,$$
$$\kappa(\mathbf{x}) = x_1 + x_2 - 2x_2 = x_1 - x_2.$$

. Let Γ be a Euclidean quiver, $M \in \mathrm{ind}(\Gamma, \Bbbk)$, $\mathbf{d} = \mathbf{dim}\, M$.
We call M

- *preprojective* if $\kappa_\Gamma(\mathbf{d}) < 0$,
- *preinjective* if $\kappa_\Gamma(\mathbf{d}) > 0$,
- *regular* if $\kappa_\Gamma(\mathbf{d}) = 0$.

Just in the same way, we define preprojective, preinjective and regular roots of the form Q.

Obviously, all imaginary roots are regular, but if Γ is not the Kronecker quiver, there are also real regular roots. It is clear, since $\dim \mathrm{Ker}\, \kappa = n - 1$ and $n > 2$, so $\mathrm{Ker}\, \kappa \neq \langle \boldsymbol{\omega} \rangle$.'

Example. Let Γ be
$$\begin{tikzcd} & 2 & \\ 3 & 1 \arrow[l] \arrow[u] \arrow[r] \arrow[d] & 5 \\ & 4 & \end{tikzcd}.$$

Then $\boldsymbol{\omega} = (2, 1, 1, 1, 1)$ $\kappa(\mathbf{x}) = 2x_1 - (x_2 + x_3 + x_4 + x_5)$. One can see that $\mathbf{d}_{23} = (1, 1, 1, 0, 0)$ is a real root, but $\kappa(\mathbf{d}) = 0$ so this root is regular. In the same way we define regular real roots $\mathbf{d}_{ij}$, where $i, j \in \{2, 3, 4, 5\}$ and $i \neq j$.

Exercise 27. Prove that all regular real roots are of the form $k\boldsymbol{\omega} + \mathbf{d}_{ij}$ for some $k, i.j$.

We will describe real roots of Euclidean quivers. First we establish the properties analogous to those of Dynkin quivers.

Proposition. *Let Γ be a Euclidean quiver, $\mathsf{Q} = \mathsf{Q}_\Gamma$ and $\mathbf{x} > 0$ be a vector from $\mathbb{R}^\Gamma = \mathbb{R}^n$.*

1. *If $\mathbf{x} \neq \alpha\boldsymbol{\omega}$, there is $i \in \mathrm{Ver}\,\Gamma$ such that $s_i \mathbf{x} < \mathbf{x}$ and $C\mathbf{x} \neq \mathbf{x}$.*

2. *If $\mathbf{x}$ is a real root, then $|\delta_i(\mathbf{x})| \leqslant 2$ and $|\delta_i(\mathbf{x})| = 2$ if and only if $\mathbf{x} = \pm\mathbf{e}_i + k\boldsymbol{\omega}$.*

3. *If* $\mathbf{x} > 0$ *is a real root and* $\mathbf{x} \neq \mathbf{e}_i$*, then* $s_i\mathbf{x} > 0$*.*

Proof. (1) If $\mathbf{x} \neq \alpha\boldsymbol{\omega}$, then $\mathsf{Q}(\mathbf{x}) > 0$ and we can proceed just as in the Dynkin case (see the proof of Proposition 60).

(2) Let x'_i be the i-th coordinate of $s_i\mathbf{x}$, y be an integer between x and x' and $\mathbf{y}$ be the vector obtained from $\mathbf{x}$ by changing x_i to y.

Then $\mathsf{Q}(\mathbf{y}) < \mathsf{Q}(\mathbf{x})$, whence $\mathsf{Q}(\mathbf{y}) = 0$ and $\mathbf{y} = k\boldsymbol{\omega}$.

Moreover, if $y' \neq y$ is another integer between x and x' and $\mathbf{y}'$ is the vector obtained from $\mathbf{x}$ by changing x_i to y', then also $\mathbf{y}' = k'\boldsymbol{\omega}$ and $\mathsf{Q}(\mathbf{y} - \mathbf{y}') = \mathsf{Q}((k - k')\mathbf{e}_i) = 0$, which is impossible.

Hence, if $|\delta_i| > 1$, we have that $\delta_i = \pm 2$, $y = x_i \pm 1$ and $\mathbf{x} = k\boldsymbol{\omega} \mp \mathbf{e}_i$.

(3) follows immediately from (2), just as in the Dynkin case (explain it). $\square$

Theorem 8. *Let* $\mathbf{x}$ *be a real root of the form* Q*.*

1. *Either* $\mathbf{x} > 0$ *or* $\mathbf{x} < 0$*.*

2. *There is an element* w *of the Weyl group* $W(\mathsf{Q})$ *and a vertex* i *such that* $\mathbf{x} = w(\mathbf{e}_i)$*.*

3. $\mathbf{x} = \mathbf{x}_0 + k\boldsymbol{\omega}$ *for some* $k \in \mathbb{Z}$ *and a real root* $\mathbf{x}$ *such that* $0 < \mathbf{x}_0 < \boldsymbol{\omega}$*. On the contrary, all such vectors are real roots.*

Proof. If the i-th coordinate of $\mathbf{x}$ is zero, $\mathbf{x}$ is a real root of the Tits form of the quiver $\Gamma \setminus \{i\}$, hence of some Dynkin subquiver of Γ. Then all assertions are known. Thus we suppose that $\operatorname{supp}\mathbf{x} = \operatorname{Ver}\Gamma$, i.e. all $x_i \neq 0$.

(1) Suppose that neither $\mathbf{x} > 0$ nor $\mathbf{x} < 0$ and present $\mathbf{x}$ as $\mathbf{y} + \mathbf{z}$, where $\mathbf{y} > 0$, $\mathbf{z} < 0$ and $\operatorname{supp}\mathbf{x} \cap \operatorname{supp}\mathbf{z} = \emptyset$.
Then $\mathsf{Q}(\mathbf{y}) \geqslant 1$, $\mathsf{Q}(\mathbf{z}) \geqslant 1$ and $\mathsf{B}(\mathbf{y}, \mathbf{z}) \geqslant 0$, whence $\mathsf{Q}(\mathbf{x}) = \mathsf{Q}(\mathbf{y}) + \mathsf{Q}(\mathbf{z}) + 2\mathsf{B}(\mathbf{y}, \mathbf{z}) \geqslant 2$, which is a contradiction.

(2) follows from (1) and the preceding proposition, just as in the Dynkin case.

(3) As $\mathsf{B}(\mathbf{x}, \boldsymbol{\omega}) = 0$, all vectors $\mathbf{x} - \boldsymbol{\omega}$ are also real roots, hence either positive or negative. If k be the greatest integer such that $\mathbf{x}_0 = \mathbf{x} - k\boldsymbol{\omega} > 0$, then $\mathbf{x}_0 - \boldsymbol{\omega} < 0$, so $0 < \mathbf{x}_0 < \boldsymbol{\omega}$.

$\square$

Corollary. *If* Γ *is a Euclidean quiver, the Weyl group* $W(\mathsf{Q})$ *is infinite.*

Proof. The set of real roots is infinite and all of them are obtained from a finite set $\{\, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \,\}$ by the action of the Weyl group. Therefore, this group is infinite. $\square$

Exercise 28. Let Γ be the Kronecker quiver. Prove that $W(\mathsf{Q})$ is the *infinite dihedral group* defined by the generators and relations as follows:

$$W(\mathsf{Q}) = \langle\, s_1, s_2 \mid s_1^2 = s_2^2 = 1 \,\rangle.$$

Hint:

We know that $s_1^2 = s_2^2 = 1$. It remains to check that no product of these elements, where any two neighbours are different, is identity.

How do such products act on the simple roots?

Let $\mathsf{N} = \langle \boldsymbol{\omega} \rangle$ and $\mathbf{V} = \mathbb{R}^\Gamma/\mathsf{N}$. Since $s_i\boldsymbol{\omega} = \boldsymbol{\omega}$, hence also $w\boldsymbol{\omega} = \boldsymbol{\omega}$ for all $w \in W$, we can consider the induced action of the Weyl group W on $\mathbf{V}$. We denote the image of W in $\operatorname{Aut}\mathbf{V}$ by $\overline{W}$.

We also denote by $\overline{\mathsf{R}}$ the image in $\mathbf{V}$ of the set R of real roots.

Corollary. *The group $\overline{W}$ is finite.*

Proof. Any element $\bar{w} \in \overline{W}$ maps $\overline{\mathsf{R}}$ into itself and is defined by its action on $\overline{\mathsf{R}}$ (even on the images of simple roots). As $\overline{\mathsf{R}}$ is finite, it gives the proof. □

In particular, the the image $\overline{C}$ in $\overline{W}$ of the Coxeter transformation C is of finite order.

It means that for every $\mathbf{x} \in \mathbb{R}^\Gamma$ there is $\partial(\mathbf{x}) \in \mathbb{R}$ such that $C^m\mathbf{x} = \mathbf{x} + \partial(\mathbf{x})\boldsymbol{\omega}$. Obviously, $\partial : \mathbb{R}^\Gamma \to \mathbb{R}$ is a linear form, called the *defect* of roots and representations.

9.2 Peprojective and preinjective representations

Recall that we have introduced the non-symmetric bilinear form $\chi_\Gamma(\mathbf{x},\mathbf{y}) = \sum_{i=1}^n x_iy_i - \sum_{a:i\to j} x_iy_j$ and the linear form $\kappa_\Gamma(\mathbf{x}) = \chi_\Gamma(\boldsymbol{\omega},\mathbf{x})$. In contrast to the quadratic form $\mathsf{Q}_\Gamma(\mathbf{x})$ and the symmetric bilinear form $\mathsf{B}_\Gamma(\mathbf{x},\mathbf{y})$, this form is usually not invariant under reflections, even if we take into account the reflection of the quiver.

For instance, if Γ is $1 \to 2 \to 3$, then one can check that

$$\begin{aligned}
&\chi_\Gamma(\mathbf{e}_1,\mathbf{e}_2) = -1, \text{ but } \chi_\Gamma(s_2\mathbf{e}_1, s_2\mathbf{e}_2) = 0,\\
&\chi_\Gamma(\mathbf{e}_1,\mathbf{e}_3) = 0, \text{ but } \chi_{s_2\Gamma}(s_2\mathbf{e}_1, s_2\mathbf{e}_3) = -1.
\end{aligned}$$

Nevertheless, there is one important case.

Lemma 9. *If the vertex i is positive or negative, then $\chi_{s_i\Gamma}(s_i\mathbf{x}, s_i\mathbf{y}) = \chi_\Gamma(\mathbf{x},\mathbf{y})$. In particular, $\kappa_{s_i\Gamma}(s_i\mathbf{x}) = \kappa_\Gamma(\mathbf{x})$.*

Proof. We prove it for a positive vertex and suppose that $i = 1$. The proof for a negative vertex is quite the same. We only have to verify this identity for pairs of basic vectors $\mathbf{e}_i, \mathbf{e}_j$. Note that $\chi_\Gamma(\mathbf{x}, \mathbf{x}) = \mathsf{Q}_\Gamma(\mathbf{x}, \mathbf{x})$, so we can suppose that $i \neq j$.

Note also that $\chi_\Gamma(\mathbf{x}, \mathbf{y}) + \chi_\Gamma(\mathbf{y}, \mathbf{x}) = 2\mathsf{B}_\Gamma(\mathbf{x}, \mathbf{y})$, so if the claim is valid for the pair of indices i, j, it is also valid for the pair j, i.

Recall that $s_1\mathbf{e}_1 = -\mathbf{e}_1$ and, if $i \neq 1$, $s_1\mathbf{e}_j = \mathbf{e}_j - 2\mathsf{B}(\mathbf{e}_j, \mathbf{e}_1)\mathbf{e}_1 = \mathbf{e}_j + c_{1j}\mathbf{e}_1$, where c_{ij} is the number of arrows $j \to i$ (there are no arrows $1 \to j$).

If $j = 1$, then $\chi_\Gamma(\mathbf{e}_1, \mathbf{e}_j) = 0$, since there are no arrows $1 \to j$. On the other hand, in the quiver $s_1\Gamma$ there are c_{1j} arrows $1 \to j$, whence also

$$\begin{aligned}
\chi_{s_1\Gamma}(s_1\mathbf{e}_1, s_1\mathbf{e}_j) &= \chi_{s_1\Gamma}(-\mathbf{e}_1, \mathbf{e}_j + c_{1j}\mathbf{e}_1) \\
&= -\chi_{s_1\Gamma}(\mathbf{e}_1, \mathbf{e}_j) - c_{1j}\chi_{s_1\Gamma}(\mathbf{e}_1, \mathbf{e}_1) \\
&= c_{1j} - c_{1j} = 0.
\end{aligned}$$

□

Let now $i \neq 1$ and $j \neq 1$. Then $\chi_\Gamma(\mathbf{e}_i, \mathbf{e}_j) = -c_{ji}$. On the other hand, $\chi_{s_1\Gamma}(\mathbf{e}_1, \mathbf{e}_j) = -c_{1j}$, while $\chi_{s_1\Gamma}(\mathbf{e}_i, \mathbf{e}_1) = 0$, whence

$$\begin{aligned}
&\chi_{s_1\Gamma}(s_1\mathbf{e}_i, s_1\mathbf{e}_j) = \chi_{s_1\Gamma}(\mathbf{e}_i + c_{1i}\mathbf{e}_1, \mathbf{e}_j + c_{1j}\mathbf{e}_1) \\
&\quad = \chi_{s_1\Gamma}(\mathbf{e}_i, \mathbf{e}_j) + c_{1j}\chi_{s_1\Gamma}(\mathbf{e}_i, \mathbf{e}_1) + c_{1i}\chi_{s_1\Gamma}(\mathbf{e}_1, \mathbf{e}_j) + c_{1i}c_{1j}\chi_{s_1\Gamma}(\mathbf{e}_1, \mathbf{e}_1) \\
&\quad = -c_{ji} - c_{1i}c_{1j} + c_{1i}c_{1j} = -c_{ji} = \chi_\Gamma(\mathbf{e}_i, \mathbf{e}_j).
\end{aligned}$$

which accomplishes the proof.

In what follows we suppose that $\{1, 2, \dots, n\}$ is a positive numeration of $\operatorname{Ver}\Gamma$ and denote by C the positive Coxeter transformation $C^+ = s_n \dots s_2 s_1$.

Then $\{\, n, n-1, \dots, 2, 1 \,\}$ is a negative numeration and $C^- = s_1 s_2 \dots s_n = C^{-1}$. We denote by $\overline{C}$ the image of C in $\operatorname{Aut}\mathbf{V}$.

The preceding Lemma immediately implies

Corollary. *The bilinear form χ_Γ and the linear form κ_Γ are invariant under the positive Coxeter transformation C^+, as well as under the negative Coxeter transformation C^-.*

Recall that we have also defined the linear form $\partial_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}$ by the rule $\partial_\Gamma(\mathbf{x})\boldsymbol{\omega} = C^m\mathbf{x} - \mathbf{x}$, where m is the order of the map $\overline{C}$ induced by C on the quotient $\mathbb{R}^\Gamma/\langle\boldsymbol{\omega}\rangle$.

As $C\boldsymbol{\omega} = \boldsymbol{\omega}$, the form ∂_Γ is also invariant under C: $\partial_\Gamma(C\mathbf{x}) = \partial_\Gamma(\mathbf{x})$. It implies the following result.

Proposition. *There is $\lambda \in \mathbb{R}$ such that $\partial_\Gamma = \lambda\kappa_\Gamma$.*

Proof. The operator C acts on the space of linear forms (the adjoint operator on the dual space of $\mathbb{R}^\Gamma$) and both κ and ∂ are nonzero and invariant.

We already know that $\boldsymbol{\omega}$ is a unique, up to a scalar, invariant vector of C in $\mathbb{R}^\Gamma$. Therefore, this operator also have a unique, up to scalar, invariant vector in the dual space (the matrices of these operators in dual bases are transposed, hence have the same Jordan form).

Hence these two forms only differ by a scalar. $\square$

We fix a Euclidean quiver Γ and omit the index Γ in the notations for $\mathsf{Q}, \mathsf{B}, \chi, \kappa, \partial$.

Evidently, both $\kappa(\mathbf{x})$ and $\partial(\mathbf{x})$ are rational. Hence coefficient λ in the previous proposition is also rational.

Actually, one can prove that $\lambda > 0$, though it depends on rather complicated calculations. We will use this fact without proof. Note that if $\lambda < 0$ all claims nearby would remain valid, one only had to interchange preprojective and preinjective roots.

Example. Let Γ be $\begin{tikzcd} & 2 \arrow[dr] & \\ 3 \arrow[ur] \arrow[rr] & & 1 \end{tikzcd}$. Then $\boldsymbol{\omega} = (1,1,1)$, $C = s_3s_2s_1$ and

$$\begin{aligned} s_3s_2s_1(1,0,0) &= s_3s_2(-1,0,0) = s_3(-1,-1,0) = (-1,-1,-2),\\ s_3s_2s_1(0,1,0) &= s_3s_2(1,1,0) = s_3(1,0,0) = (1,0,1),\\ s_3s_2s_1(0,0,1) &= s_3s_2(1,0,1) = s_3(1,2,1) = (1,2,2), \end{aligned}$$

thus $C = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 2 \\ -2 & 1 & 2 \end{pmatrix}$ and $C^2 = \begin{pmatrix} -2 & 0 & 3 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{pmatrix}$, whence

$$\begin{aligned} C^2\mathbf{e}_1 &= (-2,-3,-3) = \mathbf{e}_1 - 3\boldsymbol{\omega},\\ C^2\mathbf{e}_2 &= (0,1,0) = \mathbf{e}_2,\\ C^2\mathbf{e}_3 &= (3,3,4) = \mathbf{e}_3 + 3\boldsymbol{\omega}. \end{aligned}$$

Therefore, $m = 2$. Moreover, $\kappa(\mathbf{e}_1) = -1$, $\kappa(\mathbf{e}_2) = 0$, $\kappa(\mathbf{e}_3) = 1$, so $\partial = 3\kappa$.

Note that the simple root $\mathbf{e}_1$ is preprojective, $\mathbf{e}_2$ is regular and $\mathbf{e}_3$ is preinjective.

Exercise 29. Let Γ be $\begin{tikzcd} & 2 & \\ 3 & 1 \arrow[l] \arrow[u] \arrow[r] \arrow[d] & 5 \\ & 4 & \end{tikzcd}$.

Prove that $C\mathbf{e}_i - \mathbf{e}_i \notin \langle \boldsymbol{\omega} \rangle$, while $C^2\mathbf{e}_i = \mathbf{e}_i - \boldsymbol{\omega}$ if $i \neq 1$, and $C\mathbf{e}_1 = \mathbf{e}_1 + \boldsymbol{\omega}$, whence $C^2\mathbf{e}_1 = \mathbf{e}_1 + 2\boldsymbol{\omega}$. Therefore, $m = 2$, $\partial(\mathbf{e}_i) = -1$ if $i \neq 1$ and $\partial(\mathbf{e}_1) = 2$.

Note that $\kappa(\mathbf{e}_1) = 2$ too, whence $\partial = \kappa$.

Recall that a dimension $\mathbf{d}$ and an indecomposable representation M of this dimension are said to be *preprojective* if $\kappa(\mathbf{d}) < 0$ and *preinjective* if $\kappa(\mathbf{d}) > 0$. Equivalently, $\partial(\mathbf{d}) < 0$ or, respectively, $\partial(\mathbf{d}) > 0$.

We know that $C^m\mathbf{d} = \mathbf{d} + \partial(\mathbf{d})\boldsymbol{\omega}$, whence $C^{km}\mathbf{d} = \mathbf{d} + k\partial(\mathbf{d})$.

Therefore, if $\mathbf{d}$ is preprojective (preinjective), there is $k > 0$ (respectively, $k < 0$) such that $C^{km}\mathbf{d} \not> 0$.

Now we are in position to classify preprojective and preinjective indecomposable representations almost in the same way as we have done it for Dynkin quivers.

We use the following notations for every $k > 0$

$$\begin{aligned}
\overrightarrow{\sigma}_k &= s_{k-1} \dots s_2 s_1, \\
\overrightarrow{\sigma}_k^+ &= s_{k-1}^+ \dots s_2^+ s_1^+, \\
\overrightarrow{\sigma}_k^- &= s_1^- s_2^- \dots s_{k-1}^-, \\
\overrightarrow{\Gamma}_k &= \overrightarrow{\sigma}_k \Gamma,
\end{aligned}$$

and, dually,

$$\begin{aligned}
\overleftarrow{\sigma}_k &= s_{k+1} \dots s_{n-1} s_n, \\
\overleftarrow{\sigma}_k^- &= s_{k+1}^- \dots s_{n-1}^- s_n^-, \\
\overleftarrow{\sigma}_k^+ &= s_n^+ s_{n-1}^+ \dots s_{k+1}^-, \\
\overleftarrow{\Gamma}_k &= \overleftarrow{\sigma}_k \Gamma.
\end{aligned}$$

Note that $C = \overrightarrow{\sigma}_n$, hence $C^r = \overrightarrow{\sigma}_{nr}$.

Theorem 10. *Let M be an indecomposable representation of a Euclidean quiver Γ, $\mathbf{d} = \mathbf{dim}\, M$.*

1. *If M is preprojective, there is k such that $\mathbf{d} = \overrightarrow{\sigma}_k^{-1}\mathbf{e}_k$ and $M \simeq \overrightarrow{\sigma}_k^- E_k$, where E_k is the k-th simple representation of the quiver $\overrightarrow{\sigma}_k\Gamma$. In particular, $\mathbf{d}$ is a preprojective real root of the Tits form.*

2. *If M is preinjective, there is k such that $\mathbf{d} = \overleftarrow{\sigma}_k^{-1}\mathbf{e}_k$ and $M \simeq \overleftarrow{\sigma}_k^+ E_k$, where E_k is the k-th simple representation of the quiver $\overleftarrow{\sigma}_k\Gamma$. In particular, $\mathbf{d}$ is a preinjective real root of the Tits form.*

3. *Any preprojective or preinjective indecomposable representation is uniquely determined by its dimension.*

4. *Any preprojective or preinjective positive real root of the Tits form is a dimension of an indecomposable representation.*

Thus there is a one-to-one correspondence between preprojective (preinjective) positive real roots and preprojective (preinjective) indecomposable representations.

Proof. We give the proof for the preprojective case. The preinjective case is analogous.

If M is preprojective, $C^r\mathbf{d} = \overrightarrow{\sigma}_{nr}\mathbf{d} \not> 0$ for some r. Let $k \geqslant 0$ be the smallest integer such that $\overrightarrow{\sigma}_{k+1}\mathbf{d} \not> 0$.

Then $M' = \overrightarrow{\sigma}_k^{+}M \neq 0$ and $\overrightarrow{\sigma}_{k+1}^{+}M = s_kM' = 0$, hence $M' \simeq E_k$ and $M \simeq \overrightarrow{\sigma}_k^{-}E_k$. Therefore, $\mathbf{d} = \overrightarrow{\sigma}^{-1}\mathbf{e}_k$ is a real root.

Note that k is defined by $\mathbf{d}$, hence M is also uniquely defined by $\mathbf{d}$.

On the contrary, if $\mathbf{d}$ is a preprojective positive real root, let k be the smallest such that $\mathbf{d}' = \overrightarrow{\sigma}_k\mathbf{d} > 0$, while $\overrightarrow{\sigma}_{k+1}\mathbf{d} = s_k\mathbf{d}' < 0$.

Then $\mathbf{d}' = \mathbf{e}_k$ and $\mathbf{d} = \overrightarrow{\sigma}_k^{-1}\mathbf{e}_k = \mathbf{dim}\, M$, where $M = \overrightarrow{\sigma}_k^{-}E_k$ is an indecomposable representation. □

9.3 Principal and coprincipal representations

Now we introduce an important calss of representations.

For every vertex k we define a representation P_k as follows.

- $P_k(i)$ is the vector space whose basis is the set $\mathbf{P}_{ik}$ of all paths $k \to i$ (in particular, $P_k(k) = \langle \emptyset_k \rangle$ is 1-dimensional).

- If $p \in \mathbf{P}_{ik}$ and $a : i \to j$ is an arrow, then $P_k(a)p = ap \in P_{jk}$.

These representations are closely related with the transformations $\overrightarrow{\sigma}_k$. (138)

Proposition. $\overrightarrow{\sigma}_k^{+}P_k \simeq E_k$. *Therefore,* $P_k \simeq \overrightarrow{\sigma}_k^{-}E_k$, *where* E_k *is the simple representation of the quiver* $\overrightarrow{\sigma}_k\Gamma$.

During the proof we use representations P_k for different quivers. To mention the considered quiver, we write P_k^{Γ}.

Proof. $P_1 \simeq E_1$, since the vertex 1 is positive, so there are no nontrivial paths starting at it. So we suppose that $k \neq 1$.

We show that $s_1^{+}P_k^{\Gamma} = P_k^{s_1\Gamma}$.

Indeed, let $P' = s_1^{+}P^{\Gamma}$, $P'' = P_k^{s_1\Gamma}$. All spaces $P'(i)$ and $P''(i)$ $(i \neq 1)$ are the same $P_k^{\Gamma}(i)$ and the action of arrows $a : j \to i$ $(i, j \neq 1)$ is also the same. $P''(1) = 0$, since 1 is negative in the quiver $s_1\Gamma$, so there are no paths ending at 1.

On the other hand, if $a : i \to 1$ the map $P_k^{\Gamma}(a) : p \mapsto ap$ is an embedding $a^{+} : \mathbf{P}_{ik} \hookrightarrow \mathbf{P}_{1k}$, and if $a \neq b$ then $\operatorname{Im} a^{+} \cap \operatorname{Im} b^{+} = \emptyset$. Moreover, if $p : k \to 1$ and a is its last arrow, then $p \in \operatorname{Im} a^{+}$.

Therefore, $\varepsilon_1 : \bigoplus_{a:i\to 1} P_k^{\Gamma}(i) \to P_k^{\Gamma}(1)$ is an isomorphism, whence $P'(1) = \operatorname{Coker} \varepsilon_1 = 0$ and $P' = P''$.

Now, an obvious induction gives that $\overrightarrow{\sigma}_i^{+}P_k^{\Gamma} = P_k^{\overrightarrow{\sigma}_i\Gamma}$ for $i \leqslant k$.

In particular, $\overrightarrow{\sigma}_k^{+}P_k^{\Gamma} = P_k^{\overrightarrow{\sigma}_k\Gamma} \simeq E_k$, since the vertex k is positive in the quiver $\overrightarrow{\sigma}_k\Gamma$.

□

Dually, we define representations Q_k $(1 \leqslant k \leqslant n)$ as follows.

- $Q_k(i)$ is the vector space whose basis is the set $\mathbf{Q}_{ki}$ of all paths $i \to k$ (in particular, $Q_k(k) = \langle \emptyset_k \rangle$ is 1-dimensional).

- If $p \in Q_{ki}$ and $a : i \to j$ is an arrow, let $p = p'b$, where b is the first arrow in the path p. Then

$$Q_k(a)p = \begin{cases} p' & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition. *$\overleftarrow{\sigma}_k^- Q_k \simeq E_k$. Therefore, $Q_k \simeq \overleftarrow{\sigma}_k^+ E_k$, where E_k is the simple representation of the quiver $\overleftarrow{\sigma}_k \Gamma$.*

Exercise 30. Prove this proposition.

We call the representation P_k the *principal representation at the vertex k* and Q_k the *coprincipal representation at the vertex k.*

The next corollaries show important properties of these representations.

Corollary. *Let $M \in \operatorname{ind}(\Gamma, \Bbbk)$, $\mathbf{d} = \mathbf{dim}\, M$. The following conditions are equivalent:*

1. *$CM = 0$.*
2. *$\overrightarrow{\sigma}_{i+1}^+ M = 0$ for some $i \leqslant n$.*
3. *$M \simeq P_i$ for some i.*
4. *$C\mathbf{d} < 0$.*
5. *$\overrightarrow{\sigma}_{i+1} \mathbf{d} < 0$ for some $i \leqslant n$.*
6. *$\mathbf{d} = \mathbf{p}_i$ for some i.*

Proof. Note first that any of these conditions implies that M and $\mathbf{d}$ are preprojective. Hence $M \simeq \overrightarrow{\sigma}_i^- E_i$ for some i and $\overrightarrow{\sigma}_j^+ M \neq 0$ for $j < i$, as well as $\mathbf{d} = \overrightarrow{\sigma}_i^{-1} \mathbf{e}_i$ for some i and $\overrightarrow{\sigma}_j \mathbf{d} > 0$ for $j < i$.

Each of the conditions (1–6) means that $i \leqslant n$ (explain it). Therefore, all these conditions are equivalent. □

We propose as an exercise the dual proposition.

Corollary. *Let $M \in \operatorname{ind}(\Gamma, \Bbbk)$, $\mathbf{d} = \mathbf{dim}\, M$. The following conditions are equivalent::*

1. *$C^- M = 0$.*
2. *$\overleftarrow{\sigma}_{i+1}^- M = 0$ for some $i \leqslant n$.*

3. *$M \simeq Q_i$ for some i.*

4. *$C^{-1}\mathbf{d} < 0$.*

5. *$\overleftarrow{\sigma}_{i+1}\mathbf{d} < 0$ for some $i \leqslant n$.*

6. *$\mathbf{d} = \mathbf{q}_i$ for some i.*

Corollary.
- *Every preprojective representation is of the form $C^{-k}P_i$ for uniquely defined $i \leqslant n$ and $k \in \mathbb{N}$.*
- *Every preinjective representation is of the form C^kQ_i for uniquely defined $i \leqslant n$ and $k \in \mathbb{N}$.*

Proof. Again, we only consider the preprojective case.

Let M be preprojective. Then $M = \overrightarrow{\sigma}_r^- E_r$ for uniquely defined $r > 0$: namely, r is the smallest such that $\overrightarrow{\sigma}_{r+1}^+ M = 0$. It remains to present $r = nk+i$, where $i \leqslant n$, and note that then $\overrightarrow{\sigma}_r^- = C^{-k}\overrightarrow{\sigma}_i^-$. □

9.4 Subrepresentations and quotients

We need more results on the action of Coxeter transformations and on regular representations. First, we introduce *subrepresentations* and *quotient representations*.

. 1. A *subrepresentation* N of a representation M is a collection of subspaces $\{ N(i) \subseteq M(i) \mid i \in \operatorname{Ver}\Gamma \}$ such that $M(a) : N(i) \subseteq N(j)$ for every arrow $a : i \to j$. We denote by $N(a)$ the restriction of $M(a)$ onto $N(i)$ and consider it as a map $N(i) \to N(j)$. Thus we consider N as a representation of Γ. We write $N \subseteq M$.

2. If $N \subseteq M$ is a subrepresentation, we denote by $(M/N)(i)$ the quotient space $M(i)/N(i)$ and by $(M/N)(a)$ the induced map $M(i)/N(i) \to M(j)/N(j)$. Thus we obtain a representation M/N of Γ called the *quotient of M by N*.

Obviously, $\mathbf{dim}\, N \leqslant \mathbf{dim}\, M$ (strictly less if $N \subset M$) and $\mathbf{dim}\, M/N = \mathbf{dim}\, M - \mathbf{dim}\, N$.

Recall that if i is a positive vertex and $M' = s_i^+ M$, then $M'(i) = \operatorname{Ker}\Big(\bigoplus_{a:j\to i} M(j) \to M(i)\Big)$. It implies that if $N \subseteq M$, also $s_i^+ N \subseteq s_i^+ M$.

It gives several facts about the values of the form ∂ (or, equivalently, κ).

We denote by $\mathcal{R}(\Gamma, \Bbbk)$ the set of all direct sums of indecomposable regular representations of the quiver Γ and also call the representation from this set *regular*.

Corollary. *The following conditions are equivalent:*

1. *M is regular.*

2. $\partial(M) = 0$ *and* $\partial(N) \leqslant 0$ *for every* $N \subseteq M$.

3. $\partial(M) = 0$ *and* $\partial(M/N) \geqslant 0$ *for every* $N \subseteq M$.

Proof. Obviously, (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2). If $\partial(N) > 0$, then $\mathbf{dim}\, C^{rm}N = \mathbf{dim}\, N + r\partial(N)\boldsymbol{\omega} > \mathbf{dim}\, M$ for essentially big r. On the other hand, $C^{rm}N \subseteq C^{rm}M$, whence $\mathbf{dim}\, C^{rm}N \leqslant \mathbf{dim}\, C^{rm}M = \mathbf{dim}\, M$, a contradiction.

(2) $\Rightarrow$ (1). Let $M = \bigoplus_k M_k$, where all M_k are indecomposable. (2) implies that neither M_k is preinjective. As $\partial(M) = \sum_k \partial(M_k) = 0$ and $\partial(N) < 0$ for preprojective N, neither M_k can be preprojective too. $\square$

If $\varphi : M \to N$ is a morphism of representations, one easily sees that $\operatorname{Ker}\varphi = \{\, \operatorname{Ker}\varphi(i) \mid i \in \operatorname{Ver}\Gamma \,\}$ and $\operatorname{Im}\varphi = \{\, \operatorname{Im}\varphi(i) \mid i \in \operatorname{Ver}\Gamma \,\}$ are subrepresentaions, respectively, of M and N, calleed, respectively, the *kernel* and the *image* of φ. As usually, we define the *cokernel* $\operatorname{Coker}\varphi$ as $N/\operatorname{Im}\varphi$.

Corollary. *If M and N are regular, then* $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$, $\operatorname{Coker}\varphi$ *are also regular.*

Proof. As $\operatorname{Im}\varphi$ is a subrepresentation of N, $\partial(\operatorname{Im}\varphi) \leqslant 0$. As it is a quotient of M, $\partial(\operatorname{Im}\varphi) \geqslant 0$. Thus $\partial(\operatorname{Im}\varphi) = 0$. Every subrepresentation $N' \subseteq \operatorname{Im}\varphi$ is also a subrepresentation of N, hence $\partial(N') \leqslant 0$. Therefore, $\operatorname{Im}\varphi$ is regular.

As $\mathbf{dim}\operatorname{Im}\varphi + \mathbf{dim}\operatorname{Ker}\varphi = \mathbf{dim}\, M$, also $\partial(\operatorname{Ker}\varphi) = 0$. As $\partial(M') \leqslant 0$ for every $M' \subseteq \operatorname{Ker}\varphi$, $\operatorname{Ker}\varphi$ is regular.

Finally, $\operatorname{Coker}\varphi = N/\operatorname{Im}\varphi$, so $\partial(\operatorname{Coker}\varphi) = 0$. Every quotient L of $\operatorname{Coker}\varphi$ is also a quotient of N, hence $\partial(L) \geqslant 0$. Therefore, $\operatorname{Coker} L$ is regular. $\square$

Exercise 31. We call a representation M *preprojective* (*preinjective*) if all its indecomposable direct summands are preprojective (preinjective).

1. Prove that M is preprojective if and only if $\partial(N) < 0$ for every $N \subseteq M$.

2. Prove that M is preinjective if and only if $\partial(L) > 0$ for every quotient L of M.

In particular, every subrepresentaiton of a preprojective representation is preprojective and every quotient of a preinjective representation is preinjective.

Hint to (2): Verify that if L is a quotient of M and i is a negative vertex, then $s_i^- L$ is a quotient of $s_i^- M$.

If we use the language of *categories*, the last Corollary means that $\mathcal{R}(\Gamma, \Bbbk)$ is an *exact abelian subcategory* of the category $\operatorname{rep}(\Gamma, \Bbbk)$ of all representations. The next proposition shows that it is also *closed under extensions.*

Proposition. *Let $N \subseteq M$, $L = M/N$. If two of the representations L, M, N are regular, so is the third.*

Proof. If M and N are regular, so is $L = \operatorname{Coker}(N \hookrightarrow M)$.

If M and L are regular, so is $N = \operatorname{Ker}(M \twoheadrightarrow L)$.

Let N and L are regular. Then $\partial(M) = \partial(N) + \partial(L) = 0$.
If $M' \subseteq M$, then $M' \cap N = N' \subseteq N$ and $M'/N' \simeq (N + M')/N \subseteq L$.
Therefore, $\partial(N') \leqslant 0$ and $\partial(M'/N') \leqslant 0$, so $\partial(M') \leqslant 0$. $\square$

Exercise 32. Let the representation M be preprojective, N be regular and L be preinjective. Prove that

$$\operatorname{Hom}_\Gamma(L, N) = \operatorname{Hom}_\Gamma(L, M) = \operatorname{Hom}_\Gamma(N, M) = 0.$$

On the other hand, if the vertex i is positive (negative), the simple representation E_i is preprojective (preinjective), but $\operatorname{Hom}_\Gamma(E_i, M) \neq 0$ (respectively, $\operatorname{Hom}_\Gamma(M, E_i) \neq 0$) for every representation M such that $M(i) \neq 0$.

10 Homological algebra

10.1 Complexes and homology

Homological algebra

To consider the case of regular representations and regular roots (both real and imaginary) we need some results from *homological algebra*. We will give a survey of them in the next section. Perhaps, the best book to get acquainted to homological algebra is [Weibel]. A short introduction, enough for our purpose, is contained in [DK, Ch. 11]. I will present now the main results that we use.

We use the language of *categories* and *functors* (see [Weibel] or [DK, Ch. 8]). We denote by **A**-Mod the category of modules over a ring **A**.
Note that representations of a quiver Γ over a field $\Bbbk$ can be considered as modules over the *quiver algebra* $\Bbbk\Gamma$. The latter is a $\Bbbk$-vector space whose basis consists of all paths of this quiver and the product pq is defined as their composition pq if they are composable (i.e. $\iota_0(p) = \iota_1(q)$) and as 0 otherwise.

Namely, if M is a representation of Γ, we define the $\Bbbk\Gamma$-modiule $\widetilde{M}$ as $\bigoplus_{i \in \operatorname{Ver}\Gamma} M(i)$ with the action pv, where $p = a_k \dots a_2 a_1 : i \to j$ is a path and $v \in M(z)$, defined as $M(a_n) \dots M(a_2) M(a_1) v$ if $z = i$ and as 0 if $z \neq i$.

On the contrary, if $\widetilde{M}$ is a $\Bbbk\Gamma$-module, we obtain a representation M setting $M(i) = \emptyset_i \widetilde{M}$ and $M(a)v = av$ if $a : i \to j$ and $v \in \emptyset_i \widetilde{M}$ (since $\emptyset_j a = a$, then $av \in \emptyset_j \widetilde{M}$).

Obviously, it gives a bijection between $\Bbbk\Gamma$-modules and representations of Γ.

. 1. A *complex* of Λ-modules (or of representations of a quiver Γ) is a sequence of morphisms

$$M_\bullet : \quad \dots \to M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \to \dots$$

such that $d_n d_{n+1} = 0$ for all n. Equivalently, $\operatorname{Im} d_{n+1} \subseteq \operatorname{Ker} d_n$.

We often write d instead of d_n; in particular, the condition above is then written as $d^2 = 0$. The morphisms d_n are called the *differential* of the complex $M_\bullet$. If necessary, we write d_n^M to precise the complex.

2. The quotient $\operatorname{Ker} d_n / \operatorname{Im} d_{n-1}$ is called the *n-th homology* of this complex and denoted by $H_n(M_\bullet)$.

3. A complex $M_\bullet$ is called *exact at the place* M_n if $\operatorname{Im} d_{n+1} = d_n$ or, the same, $H_n(M_\bullet) = 0$. If it is exact at each place, this complex is called *exact* (or an *exact sequence*).

. A complex $M_\bullet$ is called

1. *Right bounded* if there is n_0 such that $M_n = 0$ for $n < n_0$.
2. *Left bounded* if there is n_0 such that $M_n = 0$ for $n < n_0$.
3. *Bounded* if it is both right and left bounded.

If M_n is right bounded, they usually write it as

$$\dots \to M n_0 + 2 \xrightarrow{d_{n_0+2}} M_{n_0+1} \xrightarrow{d_{n_0+1}} M^{n_0} \to 0$$

It is meant that all terms on the right are zero.

Analogous notations are used for left bounded and bounded complexes.

Example. 1. A sequence $0 \to M \xrightarrow{\alpha} N$ is exact if and only if α is injective.

2. A sequence $M \xrightarrow{\alpha} N \to 0$ is exact if and only if α is surjective.

3. A sequence $0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} L$ is exact if and only if α is injective and maps M bijectively onto $\operatorname{Ker} \beta$. Then we write $\alpha = \operatorname{Ker} \beta$.

4. A sequence $M \xrightarrow{\alpha} N \xrightarrow{\beta} L \to 0$ is exact if and only if $\beta\alpha = 0$, β is surjective and induces a bijection $M/\operatorname{Im}\alpha$ onto L. Then we write $\beta = \operatorname{Coker} \alpha$.

5. A sequence $0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} L \to 0$ is exact if and only if $\alpha = \operatorname{Ker} \beta$ and $\beta = \operatorname{Coker} \alpha$. Then we say that it is a *short exact sequence.*

We prove some important results about exact sequences.

Lemma 11 (4-lemma)**.** *Suppose that*

$$\begin{tikzcd} M_1 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_1"] & M_2 \arrow[r, "\alpha_2"] \arrow[d, "\gamma_2"] & M_3 \arrow[r, "\alpha_3"] \arrow[d, "\gamma_3"] & M_4 \arrow[d, "\gamma_4"] \\ N_1 \arrow[r, "\beta_1"] & N_2 \arrow[r, "\beta_2"] & N_3 \arrow[r, "\beta_3"] & N_4 \end{tikzcd}$$

is a commutative diagram with exact rows.

1. *If γ_2 and γ_4 are injective and γ_1 is surjective, then γ_3 is injecive.*
2. *If γ_1 and γ_3 are surjective and γ_4 is injective, then γ_2 is surjecive.*

We prove (1) and propose (2) as an exercise.
Proof.

$$\begin{tikzcd} M_1 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_1"] & M_2 \arrow[r, "\alpha_2"] \arrow[d, "\gamma_2"] & M_3 \arrow[r, "\alpha_3"] \arrow[d, "\gamma_3"] & M_4 \arrow[d, "\gamma_4"] \\ N_1 \arrow[r, "\beta_1"] & N_2 \arrow[r, "\beta_2"] & N_3 \arrow[r, "\beta_3"] & N_4 \end{tikzcd}$$

γ_2 and γ_4 are injective and γ_1 is surjective $\Rightarrow$ γ_3 is injective?

Let $x \in M_3$ and $\gamma_3 x = 0$. Then $\gamma_4\alpha_3 x = \beta_3\gamma_3 x = 0$, hence $\alpha_3 x = 0$ (γ_4 is injective) and $x = \alpha_2 y$ for some $y \in M_2$ (the upper row is exact).

$\beta_2\gamma_2 y = \gamma_3\alpha_2 y = \gamma_3 x = 0$, hence $\gamma_2 y = \beta_1 z$ for some $z \in N_1$ (the lower row is exact).

$z = \gamma_1 t$ for some $t \in M_1$ (γ_1 is surjective).

$\gamma_2\alpha_1 t = \beta_1\gamma_1 t = \beta_1 z = \gamma_2 y$, so $\alpha_1 t = y$ (γ_2 is injective).

Therefore, $x = \alpha_2 y = \alpha_2\alpha_1 t = 0$. $\square$

The most used case of this lemma is the following.

Corollary (5-lemma)**.** *Suppose that*

$$\begin{tikzcd} M_1 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_1"] & M_2 \arrow[r, "\alpha_2"] \arrow[d, "\gamma_2"] & M_3 \arrow[r, "\alpha_3"] \arrow[d, "\gamma_3"] & M_4 \arrow[r, "\alpha_4"] \arrow[d, "\gamma_4"] & M_5 \arrow[d, "\gamma_5"] \\ N_1 \arrow[r, "\beta_1"] & N_2 \arrow[r, "\beta_2"] & N_3 \arrow[r, "\beta_3"] & N_4 \arrow[r, "\beta_4"] & N_5 \end{tikzcd}$$

is a commutative diagram with exact rows.
If γ_i $(i = 1, 2, 4, 5)$ are isomorphisms, so is γ_3.

Lemma 12 (Snake lemma)**.** *Suppose that*

$$\begin{tikzcd} & M_1 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_1"] & M_2 \arrow[r, "\alpha_2"] \arrow[d, "\gamma_2"] & M_3 \arrow[r] \arrow[d, "\gamma_3"] & 0 \\ 0 \arrow[r] & N_1 \arrow[r, "\beta_1"] & N_2 \arrow[r, "\beta_2"] & N_3 & \end{tikzcd}$$

is a commutative diagram with exact rows.

There is a morphism $\delta : \operatorname{Ker}\gamma_3 \to \operatorname{Coker}\gamma_1$ *such that the sequence*

$$\operatorname{Ker}\gamma_1 \xrightarrow{\overline{\alpha_1}} \operatorname{Ker}\gamma_2 \xrightarrow{\overline{\alpha_2}} \operatorname{Ker}\gamma_3 \xrightarrow{\delta} \operatorname{Coker}\gamma_1 \xrightarrow{\overline{\beta_1}} \operatorname{Coker}\gamma_2 \xrightarrow{\overline{\beta_2}} \operatorname{Coker}\gamma_3.$$

is exact. Here $\overline{\alpha_i}$ *is the restriction of* α_i *onto* $\operatorname{Ker}\gamma_i$ *and* $\overline{\beta_i}$ *is induced by* β_i*:* $\overline{\beta_i}(x + \operatorname{Im}\gamma_i) = \beta_i x + \operatorname{Im}\gamma_{i+1}$ *(check that this definition is consistent).*

They call δ the *connecting morphism* for this diagram.

$$\begin{tikzcd}
 & M_1 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_1"] & M_2 \arrow[r, "\alpha_2"] \arrow[d, "\gamma_2"] & M_3 \arrow[r] \arrow[d, "\gamma_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "\beta_1"] & N_2 \arrow[r, "\beta_2"] & N_3 &
\end{tikzcd}$$

gives

$$\operatorname{Ker}\gamma_1 \xrightarrow{\overline{\alpha_1}} \operatorname{Ker}\gamma_2 \xrightarrow{\overline{\alpha_2}} \operatorname{Ker}\gamma_3 \xrightarrow{\delta ?} \operatorname{Coker}\gamma_1 \xrightarrow{\overline{\beta_1}} \operatorname{Coker}\gamma_2 \xrightarrow{\overline{\beta_2}} \operatorname{Coker}\gamma_3 \quad ?$$

1. *Constructing* δ. If $x \in \operatorname{Ker}\gamma_3$, choose $y \in M_2$ such that $\alpha_2 y = x$.
   Then $\beta_2\gamma_2 y = \gamma_3 x = 0$, hence $\gamma_2 y = \beta_1 z$ for a unique $z \in N_1$.
   Verify that another choice of y replaces z by $z + \gamma_1 u$ for some u.
   Thus we can set $\delta x = z + \operatorname{Im}\gamma_1 \in \operatorname{Coker}\gamma_1$.

2. *Exactness at* $\operatorname{Ker}\gamma_3$. If $x = \alpha_2 y$ for $y \in \operatorname{Ker}\gamma_2$, the previous construction gives $\delta x = 0$, so $\operatorname{Im}\overline{\alpha_2} \subseteq \operatorname{Ker}\delta$.

Let $\delta x = 0$, that is the constructed above element z is in $\operatorname{Im}\gamma_1$: $z = \gamma_1 u$. Then $\gamma_2 y = \beta_1 z = \gamma_2\alpha_2 u$, whence $y - \alpha_2 u \in \operatorname{Ker}\gamma_2$.

Therefore $x = \alpha_2 y = \alpha_2(y - \alpha_1 u) \in \operatorname{Im}\overline{\alpha_2}$ and $\operatorname{Ker}\delta = \operatorname{Im}\overline{\alpha_2}$.

3. *Exercise.* Prove the exactness at other terms. (The only nontrivial case is the term $\operatorname{Coker}\gamma_1$.)

Exercise 33. Let

$$\begin{tikzcd}
 & M_1 \arrow[r] \arrow[d, "\gamma_1"'] \arrow[ddr, "\xi_1"] & M_2 \arrow[r] \arrow[d, "\gamma_2"'] \arrow[ddr, "\xi_2"] & M_3 \arrow[r] \arrow[d, "\gamma_3"'] \arrow[ddr, "\xi_3"] & 0 & \\
0 \arrow[r] & N_1 \arrow[r] \arrow[ddr, "\eta_1"'] & N_2 \arrow[r] \arrow[ddr, "\eta_2"'] & N_3 \arrow[ddr, "\eta_3"'] & & \\
 & & M_1' \arrow[r] \arrow[d, "\gamma_1'"] & M_2' \arrow[r] \arrow[d, "\gamma_2'"] & M_3' \arrow[r] \arrow[d, "\gamma_3'"] & 0 \\
 & 0 \arrow[r] & N_1' \arrow[r] & N_2' \arrow[r] & N_3' &
\end{tikzcd}$$

be a commutative diagram with exact rows, $\delta : \operatorname{Ker}\gamma_3 \to \operatorname{Coker}\gamma_1$ and $\delta' : \operatorname{Ker}\gamma_3' \to \operatorname{Coker}\gamma_1'$ be the connecting morphisms. Prove that the induced diagram

$$\begin{tikzcd}
\operatorname{Ker}\gamma_3 \arrow[r, "\delta"] \arrow[d, "\overline{\xi_3}"'] & \operatorname{Coker}\gamma_1 \arrow[d, "\overline{\eta_1}"'] \\
\operatorname{Ker}\gamma_3' \arrow[r, "\delta'"] & \operatorname{Coker}\gamma_1'
\end{tikzcd}$$

is commutative.

. 1. A *subcomplex* of a complex $M_\bullet$ is a collection of submodules $N_\bullet = \{ N_n \subseteq M_n \mid n \in \mathbb{Z} \}$ such that $d_n(N_n) \subseteq N_{n-1}$ for all n. It is considered as complex, defining its differential d_n^N as the restriction of d_n^M onto N_n. We write $N_\bullet \subseteq M_\bullet$.

2. If $N_\bullet \subseteq M_\bullet$, the differential d_n induces a map $M_n/N_n \to M_{n-1}/N_{n-1}$. Thus, one defines the *quotient complex* $M_\bullet/N_\bullet$ such that $(M/N)_n = M_n/N_n$.

3. A *morphism* of complexes $\varphi : M_\bullet \to N_\bullet$ is a sequence of morphisms $\varphi_n : M_n \to N_n$ such that $d_n^N \varphi_n = \varphi_{n-1} d_n^M$ for all n.

   It means that the diagram

$$\begin{tikzcd}
\dots \arrow[r] & M_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\varphi_{n+1}"] & M_n \arrow[r, "d_n"] \arrow[d, "\varphi_n"] & M_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & \dots \\
\dots \arrow[r] & N_{n+1} \arrow[r, "d_{n+1}"] & N_n \arrow[r, "d_n"] & N_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

   is commutative.

The *kernel* $\operatorname{Ker}\varphi$ of a morphism $\varphi : M_\bullet \to N_\bullet$ is defined as the subcomplex $\operatorname{Ker}\varphi = \{ \operatorname{Ker}\varphi_n \} \subseteq M_\bullet$.

Analogously one defines $\operatorname{Im}\varphi \subseteq N_\bullet$ and $\operatorname{Coker}\varphi = N_\bullet/\operatorname{Im}\varphi$.

(Check that $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$ are indeed subcomplexes.)

One easily verifies that if $\varphi : M_\bullet \to N_\bullet$ is a morphism of complexes,

$$\varphi_n(\operatorname{Ker} d_n^M) \subseteq \operatorname{Ker} d_n^N$$

and

$$\varphi_n(\operatorname{Im} d_{n+1}^M) \subseteq \operatorname{Im} d_{n+1}^N$$

Hence φ induces the maps of quotients

$$H_n(\varphi) : H_n(M_\bullet) \to H_n(N_\bullet).$$

Since kernels and images are defined, we can speak about *exact sequences* of complexes. Now we formulate the main theorem about complexes and homologies.

Theorem 13 (LES-theorem). *Let $0 \to N_\bullet \xrightarrow{\alpha} M_\bullet \xrightarrow{\beta} L_\bullet \to 0$ be an exact sequence of complexes. There are morphisms $\delta_n : H_n(L_\bullet) \to H_{n-1}(N_\bullet)$ such that the sequence*

$$\begin{aligned} &\dots \to H_n(N_\bullet) \xrightarrow{H_n(\alpha)} H_n(M_\bullet) \xrightarrow{H_n(\beta)} H_n(L_\bullet) \xrightarrow{\delta_n} \\ &\xrightarrow{\delta_n} H_{n-1}(N_\bullet) \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(M_\bullet) \xrightarrow{H_{n-1}(\beta)} H_{n-1}(L_\bullet) \to \dots \end{aligned} \tag{LES}$$

is exact.

The sequence (LES) is called the *long exact sequence* of homologies and the morphisms δ_n are called the *connecting morphisms* for this exact sequence of complexes.

The proof of this theorem is based on the next consideration.

As $\operatorname{Im} d_{n+1} \subseteq \operatorname{Ker} d_n$, d_n induces a map

$$\overline{d_n} : M_n / \operatorname{Im} d_{n+1} \to \operatorname{Ker} d_{n-1}.$$

Moreover,

$$\operatorname{Ker} \overline{d_n} = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1} = H_n(M_\bullet),$$

and

$$\operatorname{Coker} \overline{d_n} = \operatorname{Ker} d_{n-1} / \operatorname{Im} d_n = H_{n-1}(M_\bullet).$$

Now, the exact sequence of complexes induces the commutative diagram with exact rows

$$\begin{tikzcd}
 & N_n / \operatorname{Im} d^N_{n+1} \arrow[r, "\overline{\alpha}"] \arrow[d, "\overline{d^N_n}"] & M_n / \operatorname{Im} d^M_{n+1} \arrow[r, "\overline{\beta}"] \arrow[d, "\overline{d^M_n}"] & L_n / \operatorname{Im} d^L_{n+1} \arrow[r] \arrow[d, "\overline{d^L_n}"] & 0 \\
0 \arrow[r] & \operatorname{Ker} d^N_{n-1} \arrow[r, "\overline{\alpha}"] & \operatorname{Ker} d^M_{n-1} \arrow[r, "\overline{\beta}"] & \operatorname{Ker} d^L_{n-1} &
\end{tikzcd}$$

(Check that the rows are indeed exact.)

If we apply to this diagram the Snake lemma 14, we just obtain the exact sequence 13, since the kernels of the first row are H_n and cokernels of the second row are H_{n-1} of the corresponding complexes.

Corollary. *If* $0 \to N_\bullet \xrightarrow{\alpha} M_\bullet \xrightarrow{\beta} L_\bullet \to 0$ *is an exact sequence of complexes and two of these complexes are exact, the third one is exact too.*

This corollary easily implies one more important property of exact sequences, the so called 3×3-*Lemma.*

We propose to prove it as an exercise.

Lemma 14 (3×3-Lemma)**.** *Let*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_2 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & L_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_3 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & L_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

be a commutative diagram with exact rows.

1. *If the first and the second columns are exact, the third column is exact too.*
2. *If the second and the third columns are exact, the first column is exact too.*
3. *If the first and the third columns are exact and the product of morphisms $M_1 \to M_2 \to M_3$ is zero, the second column is exact too.*

Exercise 34. Prove that if

$$\begin{tikzcd}
0 \arrow[r] & N_\bullet \arrow[r] \arrow[d, "\gamma_1"'] & M_\bullet \arrow[r] \arrow[d, "\gamma_2"'] & L_\bullet \arrow[r] \arrow[d, "\gamma_3"'] & 0 \\
0 \arrow[r] & N'_\bullet \arrow[r] & M'_\bullet \arrow[r] & L'_\bullet \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of complexes with exact rows, all diagrams

$$\begin{tikzcd}
H_n(L_\bullet) \arrow[r, "\delta_n"] \arrow[d, "H_n(\gamma_3)"'] & H_{n-1}(N) \arrow[d, "H_{n-1}(\gamma_1)"] \\
H_n(L'_\bullet) \arrow[r, "\delta'_n"'] & H_{n-1}(N'_\bullet)
\end{tikzcd}$$

are commutative

Hint: Use Exercise 33.

Rather often (especially when the complexes are left bounded), they use the "*upper notations*" setting $M^n = M_{-n}$ and $d^n = d_{-n} : M^n \to M^{n+1}$. Respectively, they write $H^n(M^\bullet) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$ for $H_{-n}(M_\bullet)$ and call it the *n-th cohomology* of the complex $M^\bullet$.

We propose the reader to rewrite the LES-theorem using the upper notations.

Now we consider an important notion of *homotopy* of complexes and their morphisms.

.

1. A morphism of complexes $\varphi : M_\bullet \to N_\bullet$ is called *homotopically trivial* if there are homomorphisms $\sigma_n : M_n \to N_{n+1}$ ($n \in \mathbb{Z}$) such that $\varphi_n = d'_{n+1}\sigma_n + \sigma_{n-1}d_n$ for all $n \in \mathbb{Z}$. (Here we denote by d' the differential of N.)

   If we omit indices, it is written as $\varphi = d\sigma + \sigma d$.

2. Two morphisms $\varphi, \psi : M_\bullet \to N_\bullet$ are called *homotopical* if $\varphi - \psi$ is homotopically trivial. Then we write $\varphi \sim \psi$. The collection $\{\, \sigma_n \mid n \in \mathbb{Z} \,\}$ is called a *homotopy* between φ and ψ.

(In particular, $\varphi \sim 0$ means that φ is homotopically trivial.)

Here is a picture explaining the notion of homotopical triviality:

$$\begin{tikzcd}
\dots \arrow[r] & M_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\varphi_{n+1}"'] \arrow[dl] & M_n \arrow[r, "d_n"] \arrow[d, "\varphi_n"'] \arrow[dl, "\sigma_n"'] & M_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"'] \arrow[dl, "\sigma_{n-1}"'] & \dots \arrow[dl] \\
\dots \arrow[r] & N_{n+1} \arrow[r, "d'_{n+1}"] & N_n \arrow[r, "d'_n"] & N_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

(Every vertical arrow equals the sum of two its bypasses arising from the neighbour triangles.)

Proposition. *If $\varphi \sim \psi$, then $H_n(\varphi) = H_n(\psi)$ for all $n \in \mathbb{Z}$.*

Proof. It is enough to show that if $\varphi \sim 0$, then $H_n(\varphi) = 0$.

Indeed, if $dx = 0$ for some $x \in M_n$, then $\varphi(x) = d\sigma x + \sigma dx = d\sigma x$, that is $\varphi(x) \in \operatorname{Im} d$, hence its class in $H_n(N)$ is zero. □

.

- Two complexes, $M_\bullet$ and $N_\bullet$ are called *homotopical* if there are morphisms $\varphi : M_\bullet \to N_\bullet$ and $\psi : N_\bullet \to M_\bullet$ such that $\psi\varphi \sim 1_{M_\bullet}$ and $\varphi\psi \sim 1_{N_\bullet}$. Then we write $M \sim N$.

- A complex $M_\bullet$ is said to be *contractible* if $1_{M_\bullet} \sim 0$, that is there are homomorphisms $\sigma_n : M_n \to M_{n+1}$ such that $d_{n+1}\sigma_n + \sigma_{n-1}d_n = 1$. Obviously, it means that $M \sim 0$.

The conditions on φ and ψ imply that $H_n(\varphi)$ and $H_n(\psi)$ are mutually inverse. Therefore, if $M \sim N$, then $H_n(M_\bullet) \simeq H_n(N_\bullet)$ for all n. In particular, if $M_\bullet$ is contractible, $H_n(M_\bullet) = 0$ for all n.

Proposition. *The following conditions are equivalent:*

1. *A complex $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ is contractible.*
2. *This complex is exact and there is a homomorphism $\alpha' : M \to N$ such that $\alpha'\alpha = 1_N$.*
3. *This complex is exact and there is a homomorphism $\beta' : L \to M$ such that $\beta\beta' = 1_L$.*
4. *$M = M' \oplus M''$, α maps N isomorphically onto M' and the restriction $\beta|_{M'}$ is an isomorphism $M' \xrightarrow{\sim} L$.*

If these conditions are satisfied, they say that $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ is a split exact sequence.

Remark. The existence of α' in (2) implies that α is injective, so the complex is exact at the place N.

The existence of β' in (3) implies that β is surective, so the complex is exact at the place L.

Proof. (1) ⇒ (2) and (1) ⇒ (3) by definition.

(2) ⇒ (4). Set $M' = \operatorname{Im}\alpha$ and $M'' = \operatorname{Ker}\alpha'$. Then α' maps $\operatorname{Im}\alpha$ isomorphically onto N.

If $x \in M' \cap M''$, then $x = \alpha(y) = \alpha\alpha'\alpha(y) = \alpha\alpha'(x) = 0$, so $M' \cap M'' = 0$.

On the other hand, for every $x \in M$, $x = \alpha\alpha'(x) + \big(x - \alpha\alpha'(x)\big)$ and $\alpha'\big(x - \alpha\alpha'(x)\big) = \alpha'(x) - \alpha'\alpha\alpha'(x) = \alpha(x) - \alpha(x) = 0$, so $M = M' \oplus M''$. As $\operatorname{Ker}\beta = \operatorname{Im}\alpha = M'$, β maps M'' isomorphically onto L.

The analogous proof of (3) ⇒ (4), as well as that of (4) ⇒ (1), is left to the reader as an exercise. □

10.2 Projective resolutions

.
- A *left resolution* $(L_\bullet, \pi)$ of a module M is a complex $L_\bullet$ such that $L_n = 0$ for $n < 0$, $H_n(L_\bullet) = 0$ for $n \neq 0$ and a epimorphism is given $\pi : L_0 \to M$ with $\operatorname{Ker} \pi = \operatorname{Im} d_1$ (hence $M \simeq H_0(L_\bullet)$).

- A *right resolution* $(R_\bullet, \varepsilon)$ of a module M is a complex $R_\bullet$ such that $R_n = 0$ for $n > 0$, $H_n(R_\bullet) = 0$ for $n \neq 0$ and a monomorphism is given $\varepsilon : M \to R_0$ with $\operatorname{Im} \varepsilon = \operatorname{Ker} d_0$ (hence $M \simeq H_0(R_\bullet)$).

Usually, they write right resolutions in the upper notation.

Thus, a left resolution is a sequence of homomorphisms

$$\cdots \to L_n \xrightarrow{d_n} L_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \to 0$$

which is exact at all places except 0, while $L_0 / \operatorname{Im} d_0 \simeq M$.

They often present this resolution as an exact sequence

$$\cdots \to L_n \xrightarrow{d_n} L_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{\pi} M \to 0$$

One only has to remember that the terms $\xrightarrow{\pi} M$ are not a part of the resolution.

Usually a right resolution is written using upper notations as a complex $R^\bullet$

$$0 \to R^0 \xrightarrow{d^0} R^1 \xrightarrow{d^1} \cdots \to R^n \xrightarrow{d_n} R^{n+1} \xrightarrow{d_{n+1}} \dots,$$

which is exact everywhere except the place R^0 and $\operatorname{Ker} d^0 \simeq M$.

Again, one often presents it as an exact sequence

$$0 \to M \xrightarrow{\varepsilon} R^0 \xrightarrow{d^0} R^1 \xrightarrow{d^1} \cdots \to R^n \xrightarrow{d_n} R^{n+1} \to \dots,$$

where one has to remember that the terms $M \xrightarrow{\varepsilon}$ are not a part of the resolution.

In what follows, we consider left resolutions and propose the reader to formulate himself analogous definitions and results for right resolutions.

. Let $(L_\bullet, \pi)$ be a left resolution of a module M, $(L'_\bullet, \pi')$ be a left resolution of a module M' and α be a homomorphism $M \to M'$. A *lifting of α onto the given resolutions* is a morphism of complexes $\varphi : L_\bullet \to L'_\bullet$ such that $\alpha\pi = \pi'\varphi_0$.

It means that the whole diagram

$$\begin{tikzcd}
\dots \arrow[r] & L_n \arrow[r, "d_n"] \arrow[d, "\varphi_n"] & L_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\varphi_{n-1}"] & \dots \arrow[r, "d_2"] & L_1 \arrow[r, "d_1"] \arrow[d, "\varphi_1"] & L_0 \arrow[r, "\pi"] \arrow[d, "\varphi_0"] & M \arrow[r] \arrow[d, "\alpha"] & 0 \\
\dots \arrow[r] & L'_n \arrow[r, "d'_n"] & L'_{n-1} \arrow[r, "d'_{n-1}"] & \dots \arrow[r, "d'_2"] & L'_1 \arrow[r, "d'_1"] & L'_0 \arrow[r, "\pi'"] & M' \arrow[r] & 0
\end{tikzcd}$$

is commutative.

The existence and uniqueness of such a lifting is usually a problem. Nevertheless, there is a special kind of resolutions that play a crucial role in homological algebra such that for them a lifting always exists and is unique up to homotopy.

. A module P is called *projective* if for every homomorphism $\alpha : P \to N$ and any epimorphism $\beta : M \to N$ there is a homomorphism $\alpha' : P \to M$ such that $\alpha = \beta\alpha'$

Schematically, they usually present this property by the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\alpha'"'] \arrow[d, "\alpha"] & \\ M \arrow[r, "\beta"] & N \arrow[r] & 0 \end{tikzcd}$$

with the exact row.

It is meant that the solid arrows are given and the existence of the dashed arrow is claimed.

Example. Any free module Λ^n is projective.

Indeed, if $e_1, e_2, \dots, e_n$ is a basis of Λ^n, a homomorphism $\alpha : \Lambda^n \to N$ is completely defined by the images $v_n = \alpha(e_n)$ and these elements can be chosen arbitrary. (In particular, $\operatorname{Hom}_\Lambda(\Lambda^n, N) \simeq N^n$.) Now, if $\beta : M \to N$ is surjective and $\alpha : \Lambda^n \to N$ maps e_i to v_i, find u_i such that $\beta(u_i) = v_i$ and define $\alpha' : \Lambda^n \to M$ such that $\alpha'(e_i) = u_i$. Then $\alpha = \beta\alpha'$.

As every module is isomorphic to a quotient of a free module, we obtain

Corollary. *For every module M there is an epimorphism $P \twoheadrightarrow M$, where P is projective.*

Exercise 35. Let $M = N \oplus L$. Prove that M is projective if and only if both N and L are projective.

Proposition. *The following conditions are equivalent:*

1. *P is projective.*
2. *If $M \xrightarrow{\pi} P$ is surjective, $M = \operatorname{Ker}\pi \oplus P'$, where $P' \simeq P$.*
3. *P is isomorphic to a direct summand of a free module.*

Proof. (1) $\Rightarrow$ (2). As P is projective and φ is surjective, there is $\pi' : P \to M$ such that $1_P = \pi\pi'$.

Hence the exact sequence $0 \to \operatorname{Ker}\varphi \to M \xrightarrow{\varphi} P \to 0$ is contractible and $M \simeq \operatorname{Ker}\varphi + P$.

(2) $\Rightarrow$ (3), since there is an epimorphism $\Lambda^n \twoheadrightarrow P$.

(3) $\Rightarrow$ (1) follows from the preceding exercise. $\square$

Exercise 36. Here is a sketch of the proof (2) $\Rightarrow$ (1) not using the fact that every module is a quotient of a free one.

If $\alpha : P \to N$ and $\beta : M \to N$ are given, consider the submodule $M' \subseteq M \oplus P$: $M' = \{\, (u, v) \mid \beta(u) = \alpha(v) \,\}$.

There are maps $\alpha' : M' \to M$: $(u,v) \mapsto u$ and $\beta' : M' \to P$: $(u,v) \mapsto v$. Moreover, $\alpha\beta' = \beta\alpha'$.

If β is surjective, so is β' (check it). Hence, $M' = \operatorname{Ker}\beta' \oplus P'$ and $P' \simeq P$.
Then there is a map $\gamma : P \to M'$ such that $\beta'\gamma = 1_P$, whence
$\beta\alpha'\gamma = \alpha\beta'\gamma = \alpha$, so (1) holds true.

. A *projective resolution* of a module M is a left resolution $P_\bullet$ of M such that all modules P_n are projective.

We will prove that every module has a projective resolution and such a resolution is unique up to homotopy.

Proposition. *Every module has a projective resolution.*

Proof. We construct modules P_n and homomorphisms d_n recursively. We know already that there is an epimorphism $\varphi : P_0 \twoheadrightarrow M$ with projective P_0. Let $K_1 = \operatorname{Ker}\varphi$. There is also an epimorphism $\varphi_1 : P_1 \to K_1$ with projective P_1. Denote by d_1 the composition of φ_1 with the embedding $K_1 \to P_0$. Then $M = P_0/\operatorname{Im} d_1$. Suppose now that P_k and d_k are constructed for $k < n$ such that $\operatorname{Im} d_{k+1} = \operatorname{Ker} d_k$ for $k < n-1$. Let $K_n = \operatorname{Ker} d_{n-1}$. There is an epimorphism $\varphi_n : P_n \twoheadrightarrow K_n$ with projective P_n. Then we only have to take for d_n the composition of φ_n with the embedding $K_n \to P_{n-1}$. $\square$

Theorem 15. *Let $(P_\bullet, \pi)$ be a projective resolution of M and $(L_\bullet, \pi')$ be a left resolution of N.*

1. *For every homomorphism $\alpha : M \to N$ there is a lifting to a morphism $\varphi : P_\bullet \to L_\bullet$.*

2. *If ψ is another lifting of α to these resolutions, $\varphi \sim \psi$.*

3. *If $P'_\bullet$ is another projective resolution of M, then $P' \sim P$.*

Proof. (1) We construct the components φ_n recursively. The differential of L is denored by d'.

Consider the composition $\alpha\pi : P_0 \to N$. As P_0 is projective and $\pi' : L_0 \to N$ is surjective, there is $\varphi_0 : P_0 \to L_0$ such that $\alpha\pi = \pi'\varphi_0$, i.e. the diagram

$$\begin{tikzcd} P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\pi"] \arrow[d, "\varphi_0"] & M \arrow[r] \arrow[d, "\alpha"] & 0 \\ & L_0 \arrow[r, "\pi'"] & N \arrow[r] & 0 \end{tikzcd}$$

is commutative.

Note that $\pi'\varphi_0 d_1 = \alpha\pi d_1 = 0$, hence actually $\varphi_0 d_1$ maps P_1 to $\operatorname{Ker}\pi' = \operatorname{Im} d'_1$. As P_1 is projective, there is $\varphi_1 : P_1 \to L_1$ such that $\varphi_0 d_1 = d'_1\varphi_1$, i.e. the diagram

$$\begin{tikzcd} P_1 \arrow[r, "d_1"] \arrow[d, "\varphi_1"] & P_0 \arrow[r, "\pi"] \arrow[d, "\varphi_0"] & M \arrow[r] \arrow[d, "\alpha"] & 0 \\ L_1 \arrow[r, "d'_1"] & L_0 \arrow[r, "\pi'"] & N \arrow[r] & 0 \end{tikzcd}$$

is commutative.

Suppose now that we have constructed $\varphi_k : P_k \to k$ for $k \leqslant n$ such that the diagram

$$\begin{tikzcd}
P_{n+1} \arrow[r, "d_{n+1}"] & P_n \arrow[r, "d_n"] \arrow[d, "\varphi_n"] & P_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\varphi_{n-1}"] & \dots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\varphi_1"] & P_0 \arrow[r, "\pi"] \arrow[d, "\varphi_0"] & M \arrow[r] \arrow[d, "\alpha"] & 0 \\
 & L_n \arrow[r, "d'_n"] & L_{n-1} \arrow[r, "d'_{n-1}"] & \dots \arrow[r, "d'_2"] & L_1 \arrow[r, "d'_1"] & 0 \arrow[r, "\pi'"] & M' \arrow[r] & 0
\end{tikzcd}$$

is commutative. Just as above, one checks that $\varphi_n d_{n+1}$ maps P_{n+1} to $\operatorname{Im} d'_{n+1}$. Therefore, there is $\varphi_{n+1} : P_{n+1} \to L_{n+1}$ such that $\varphi_n d_{n+1} = d'_{n+1}\varphi_{n+1}$, that is the extended diagram

$$\begin{tikzcd}
P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\varphi_{n+1}"] & P_n \arrow[r, "d_n"] \arrow[d, "\varphi_n"] & P_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\varphi_{n-1}"] & \dots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\varphi_1"] & P_0 \arrow[r, "\pi"] \arrow[d, "\varphi_0"] & M \arrow[r] \arrow[d, "\alpha"] & 0 \\
L_{n+1} \arrow[r, "d'_{n+1}"] & L_n \arrow[r, "d'_n"] & L_{n-1} \arrow[r, "d'_{n-1}"] & \dots \arrow[r, "d'_2"] & L_1 \arrow[r, "d'_1"] & L_0 \arrow[r, "\pi'"] & M' \arrow[r] & 0
\end{tikzcd}$$

is commutative. It accomplishes the construction. $\square$

(2) It is enough to show that if φ lifts the zero map, it is homotopically trivial. Again we construct the maps σ_n (176) recursively.

If φ lifts 0, then $\pi'\varphi_0 = 0 \cdot \pi = 0$, so $\operatorname{Im}\varphi_0 \subseteq \operatorname{Ker}\pi' = \operatorname{Im} d'_1$. Just as above, it implies that there is $\sigma_0 : P_0 \to L_1$ such that $\varphi_0 = d'_1\sigma_0$. Then $d'_1(\varphi_1 - \sigma_0 d_1) = \varphi_0 d_1 - d'_1\sigma_0 d_1 = 0$. Hence $\operatorname{Im}(\varphi_1 - \sigma_0 d_1) \subseteq \operatorname{Ker} d'_1 = \operatorname{Im} d'_2$ and there is $\sigma_1 : P_1 \to L_2$ such that $d'_2\sigma_1 = \varphi_1 - \sigma_0 d_1$, that is $\varphi_1 = \sigma_0 d_1 + d'_2\sigma_1$. Now, if σ_k are already constructed for $k < n$, then $\varphi_{n-1} = \sigma_{n-2}d_{n-1} + d'_n\sigma_{n-1}$, whence $d'_n\varphi_n = \varphi_{n-1}d_n = d'_n\sigma_{n-1}d_n$, so $\operatorname{Im}(\varphi_n - \sigma_{n-1}d_n) \subseteq \operatorname{Ker} d'_n = \operatorname{Im} d'_{n+1}$. Hence there is $\sigma_n : P_n \to L_{n+1}$ such that $\varphi_n - \sigma_{n-1}d_n = d'_{n+1}\sigma_n$, that is $\varphi_n = \sigma_{n-1}d_n + d'_{n+1}\sigma_n$. It accomplishes the construction. $\square$

(3) Let $(P'_\bullet, \pi')$ be another projective resolution of M. There is a lifting of 1_M to a morphism $\varphi : P_\bullet \to P'_\bullet$, as well as to a morphism $\varphi' : P'_\bullet \to P_\bullet$. Then $\varphi'\varphi$ is a lifting of 1_M to a morphism $P_\bullet \to P_\bullet$, just as the identity morphism of $P_\bullet$. Therefore, $\varphi'\varphi \sim 1_{P_\bullet}$. In the same way, $\varphi\varphi' \sim 1_{P'_\bullet}$, so $P_\bullet \sim P'_\bullet$. $\square$

We propose several exercises concerning *injective modules* and *injective resolutions.*

. A module Q is called *injective* if for every homomorphism $\alpha : N \to Q$ and for every monomorphism $\beta : N \to M$ there is $\alpha' : M \to Q$ such that $\alpha = \alpha'\beta$. Schematically:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\beta"] \arrow[d, "\alpha"] & M \arrow[dl, dashed, "\alpha'"] \\
 & Q &
\end{tikzcd}$$

Exercise 37. Prove that the following conditions are equivalent:

1. Q is injective.
2. If $\alpha : Q \hookrightarrow M$ is a monomorphism and Q is injective, then $M = \operatorname{Im}\alpha \oplus Q'$ and $\operatorname{Im}\alpha \simeq Q$.

It is known that every module can be embedded into an injective one. A proof, as well as examples of injective modules, see [Weibel, Sec. 2.3]. For finite dimensional algebras, in particular, for acyclic quivers, it follows from duality, see [DK, Sec. 9.1].

Exercise 38. Prove:

1. Every module M has an injective resolution, i.e. a right resolution $Q^\bullet$ such that all modules Q^n are injective.
2. If $Q^\bullet$ is a injective resolution of M, $R^\bullet$ is a right resolution of N and $\alpha : N \to M$, there is a lifting of α to a morphism of resolutions $\varphi : R^\bullet \to Q^\bullet$.
3. If $\psi : R^\bullet \to Q^\bullet$ is another lifting of α, then $\varphi \sim \psi$.
4. If $Q'^\bullet$ is another injective resolution of M, then $Q^\bullet \sim Q'^\bullet$.

10.3 Derived functors

Derived functors

Recall that a *functor* (or *covariant functor*) $F : \Lambda\text{-}\mathsf{Mod} \to \Lambda'\text{-}\mathsf{Mod}$ is a map sending every Λ-module M to a Λ'-module FM and every homomoprhism $\alpha : M \to N$ to a homomorphism $F\alpha : FM \to FN$, so that

1. $F1_M = 1_{FM}$ for every module M.
2. $F(\alpha\beta) = F\alpha \cdot F\beta$ as soon as the product $\alpha\beta$ is defined.

Analogously, a *contravariant functor* $F : \Lambda\text{-}\mathsf{Mod} \to \Lambda'\text{-}\mathsf{Mod}$ is a map sending every Λ-module M to a Λ'-module FM and every homomoprhism $\alpha : M \to N$ to a homomorphism $F\alpha : FN \to FM$, so that

1. $F1_M = 1_{FM}$ for every module M.
2. $F(\alpha\beta) = F\beta \cdot F\alpha$ as soon as the product $\alpha\beta$ is defined.

We always suppose that F is also *additive*, i.e. $F(\alpha + \beta) = F\alpha + F\beta$. Then $F0_M = 0_{FM}$.

The most important for us are the following examples.

Example. 1. For every Λ-module A the (covariant) functor h^A is defined as follows:

- $h^A(M) = \mathrm{Hom}_\Lambda(A, M)$.

- If $\alpha : M \to N$, then $h^\alpha = \alpha\cdot : \mathrm{Hom}_\Lambda(A, M) \to \mathrm{Hom}_\Lambda(A, N)$ maps β to $\alpha\beta$.

2. For every Λ-module A the contravariant functor h_A is defined as follows:

- $h_A(M) = \mathrm{Hom}_\Lambda(M, A)$.

- If $\alpha : M \to N$, then $h_\alpha = \cdot\alpha : \mathrm{Hom}_\Lambda(N, A) \to \mathrm{Hom}_\Lambda(M, A)$ maps β to $\beta\alpha$.

Such functors are called *representable* and they say that A *represents* the functor h^A or h_A.

Another important example of functors are reflections $s_i^\pm$. Indeed, given a morphism $\alpha : M \to N$, we have maps $\alpha(i) : M(i) \to N(i)$ and $\alpha^+(i) : M^+(i) \to N^+(i)$ with the components $\alpha(j) : M(j) \to N(j)$ for all arrows $a : j \to i$ so that the diagram

$$\begin{tikzcd}
M^+(i) \arrow[r, "\pi_i^M"] \arrow[d, "\alpha^+(i)"'] & M(i) \arrow[d, "\alpha(i)"'] \\
N^+(i) \arrow[r, "\pi_i^N"] & M(i)
\end{tikzcd}$$

is commutative Therefore, we obtain a unique map of kernels $\alpha'(i) : M'(i) \to N'(i)$, which are just $s_i^+M(i)$, so that the whole diagram

$$\begin{tikzcd}
0 \arrow[r] & M'(i) \arrow[r] \arrow[d, "\alpha'(i)"'] & M^+(i) \arrow[r, "\pi_i^M"] \arrow[d, "\alpha^+(i)"'] & M(i) \arrow[d, "\alpha(i)"'] \\
0 \arrow[r] & N'(i) \arrow[r] & N^+(i) \arrow[r, "\pi_i^N"] & M(i)
\end{tikzcd}$$

is commutative. Together with the "old" maps $\alpha(j)$ $(j \neq i)$ it gives a morphism $s_i^+\alpha : s_i^+M \to s_i^+N$. Analogously the action of s_i^- on morphisms can be defined (do it).

. 1. A functor F is called *left exact* if it preserves kernels, i.e. for every exact sequence $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L$ the sequence $0 \to FN \xrightarrow{F\alpha} FM \xrightarrow{F\beta} FL$ is also exact. In particular, it maps monomorphisms to monomorphisms.

2. A functor F is called *right exact* if it preserves cokernels, i.e. for every exact sequence $N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ the sequence $FN \xrightarrow{F\alpha} FM \xrightarrow{F\beta} FL \to$ is also exact. In particular, it maps epimorphisms to epimorphisms.

If a functor F is both right and left exact, it os called an *exact functor*.

Remark. Actually, one can prove that a functor F is left exact if and only if for every exact sequence $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ the sequence $0 \to FN \xrightarrow{F\alpha} FM \xrightarrow{F\beta} FL$ is also exact, and the same change can be done for right exactness.

As contravariant functors reverse arrows, the definitions in contravariant case are also reversed.

. 1. A contravariant functor F is called *left exact* if it maps cokernels to kernels, i.e. for every exact sequence $N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ the sequence $0 \to FL \xrightarrow{F\beta} FM \xrightarrow{F\alpha} FN$ is also exact. In particular, it maps epimorphisms to monomorphisms.

2. A contravariant functor F is called *right exact* if it maps kernels to cokernels, i.e. for every exact sequence $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L$ the sequence $FL \xrightarrow{F\beta} FM \xrightarrow{F\alpha} FN \to 0$ is also exact. In particular, it maps monomorphisms to epimorphisms.

Again, if F is both right and left exact, it os called an *exact functor.*

The preceding remark is also valid in contravariant case.

Representable functors give an important example of exactness.

Theorem 16. *Representable functors* $h^A = \operatorname{Hom}_\Lambda(A, \_\ )$ *and* $h_A = \operatorname{Hom}_\Lambda(\_\,, A)$ *are left exact. Moreover,*

1. *A sequence* $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L$ *is exact if and only if so is the sequence*

$$0 \to \operatorname{Hom}_\Lambda(A, N) \xrightarrow{\alpha\cdot} \operatorname{Hom}_\Lambda(A, M) \xrightarrow{\beta\cdot} \operatorname{Hom}_\Lambda(A, L).$$

   for any module A.

2. *A sequence* $N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ *is exact if and only if so is the sequence*

$$0 \to \operatorname{Hom}_\Lambda(L, A) \xrightarrow{\cdot\beta} \operatorname{Hom}_\Lambda(M, A) \xrightarrow{\cdot\alpha} \operatorname{Hom}_\Lambda(N, A).$$

   for any module A.

We prove the claim for the contravariant functor h_A and leave the covariant case as an exercise.

$N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0 \;\Rightarrow\; 0 \to \operatorname{Hom}_\Lambda(L, A) \xrightarrow{\cdot\beta} \operatorname{Hom}_\Lambda(M, A) \xrightarrow{\cdot\alpha} \operatorname{Hom}_\Lambda(N, A)$

Let the first sequence is exact. If $\gamma\beta = 0$ for some $\gamma : L \to A$, then $\gamma = 0$, since β is surjective. Thus $\operatorname{Ker}(\cdot\beta) = 0$.

For every $\gamma : L \to A$, $(\cdot\alpha)(\cdot\beta)(\gamma) = \gamma\beta\alpha = 0$, since $\operatorname{Im}\alpha = \operatorname{Ker}\beta$. Thus $\operatorname{Im}(\cdot\beta) \subseteq \operatorname{Ker}(\cdot\alpha)$.

If $\xi : M \to A$ and $(\cdot\alpha)(\xi) = \xi\alpha = 0$, then $\operatorname{Ker}\xi \supseteq \operatorname{Im}\alpha = \operatorname{Ker}\beta$. Hence ξ can be decomposed as a product $M \to M/\operatorname{Ker}\beta \to A$. As β induces an isomorphism $M/\operatorname{Ker}\beta \simeq L$, ξ can be presented as $\xi'\beta$, that is $\xi \in \operatorname{Im}(\cdot\beta)$ and $\operatorname{Ker}(\cdot\alpha) = \operatorname{Im}(\cdot\beta)$. □

$0 \to \operatorname{Hom}_\Lambda(L, A) \xrightarrow{\cdot\beta} \operatorname{Hom}_\Lambda(M, A) \xrightarrow{\cdot\alpha} \operatorname{Hom}_\Lambda(N, A) \;\Rightarrow\; N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$

Take $A = L/\operatorname{Im}\beta$, $\gamma : L \twoheadrightarrow A$ the natural surjection. Then $\gamma\beta = 0$, i.e. $\gamma \in \operatorname{Ker}(\cdot\beta) = 0$, so $\gamma = 0$, that is β is surjective.

Take $A = L$, $\gamma = 1_L$. Then $\beta\alpha = (\cdot\alpha)(\cdot\beta)(1_L) = 0$, since $\operatorname{Im}(\cdot\beta) = \operatorname{Ker}(\cdot\alpha)$. Hence $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\beta$.

Take $A = M/\operatorname{Im}\alpha$, $\xi : M \twoheadrightarrow A$ the natural surjection. Then $(\cdot\alpha)(\xi) = \xi\alpha = 0$, so $\xi \in \operatorname{Ker}(\cdot\alpha) = \operatorname{Im}(\cdot\beta)$, i.e. $\xi = \xi'\beta$ for some $\xi' : L \to A$.If $x \in \operatorname{Ker}\beta$, then $\xi(x) = \xi'\beta(x) = 0$, hence $x \in \operatorname{Ker}\xi = \operatorname{Im}\alpha$. Therefore, $\operatorname{Ker}\beta = \operatorname{Im}\alpha$. □

Another example are reflections and Coxeter functors.

Theorem 17. *1. If a vertex i of a quiver Γ is positive, the functor s_i^+ is left exact. So is also the positiv Coxeter functor C^+.*

2. If a vertex i of a quiver Γ is negative, the functor s_i^i is right exact. So is also the negative Coxeter functor C^-.

We prove (2); the proof of (1) is analogous. We leave it as an exercise. So, let $M \xrightarrow{\alpha} N \xrightarrow{\beta} L \to 0$ be an exact sequence, that is $M(j) \xrightarrow{\alpha(j)} N(j) \xrightarrow{\beta(j)} L(j) \to 0$ be an exact sequence for every vertex j. Then we have a commutative diagram with exact columns and exact first two rows and have to show that the last row is also exact:

$$\begin{tikzcd}
M(i) \arrow[r, "\alpha(i)"] \arrow[d, "\varepsilon_i^M"'] & N(i) \arrow[r, "\beta(i)"] \arrow[d, "\varepsilon_i^N"'] & L(i) \arrow[r] \arrow[d, "\varepsilon_i^L"'] & 0 \\
M^-(i) \arrow[r, "\alpha^-(i)"] \arrow[d, "\gamma^M"'] & N^-(i) \arrow[r, "\beta^-(i)"] \arrow[d, "\gamma^N"'] & L^-(i) \arrow[r] \arrow[d, "\gamma^L"'] & 0 \\
M'(i) \arrow[r, "\alpha'(i)"] \arrow[d] & N'(i) \arrow[r, "\beta'(i)"] \arrow[d] & L'(i) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

As $\beta'(i)\gamma^N = \gamma^L\beta^-(i)$ is surjective, so is $\beta'(i)$. Obviously, $\beta'(i)\alpha'(i) = 0$, hence $\operatorname{Im}\alpha'(i) \subseteq \operatorname{Ker}\beta'(i)$. Let $x \in \operatorname{Ker}\beta'(i)$, i.e. $\beta'(i)x = 0$, and $x = \gamma^N y$. Then $\gamma^L\beta^-(i)y = \beta'(i)\gamma^N y = 0$, hence $\beta^-(i)y = \varepsilon_i^L z$ for some z. Let $z = \beta(i)v$, then $\gamma^N(y - \varepsilon_i^N v) = x$ and $\beta^-(i)(y - \varepsilon_i^N v) = 0$. Hence $y - \varepsilon^N(i)v = \alpha^-(i)u$ and $x = \gamma^N\alpha^-(i)u = \alpha'(i)\gamma^M u \in \operatorname{Im}\alpha'(i)$. Thus $\operatorname{Ker}\beta'(i) = \operatorname{Im}\alpha'(i)$, so the last row is exact. □

Exercise 39. Let $0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} L \to 0$ be an exact sequence of representations of a quiver, and neither of the representations M, N, L has a direct summand E_i, where i is a positive or a negative vertex. Prove that the sequence $0 \to s_i^\pm M \xrightarrow{s_\alpha^\pm} s_i^\pm N \xrightarrow{s_\beta^\pm} s_i^\pm L \to 0$ is also exact. (s_i^+ if i is positive, s_i^- if it is negative.)

Hint: If M does not have E_i as a direct summand, then $\mathbf{dim}\, s_i^{\pm} M = s_i \,\mathbf{dim}\, M$. Hence under the given conditions $\mathbf{dim}\, s_i^{\pm} N = \mathbf{dim}\, s_i^{\pm} M + \mathbf{dim}\, s_i^{\pm} L$. Now use the preceding theorem.

Theorem 18. *Applying an exact functor F to an exact sequence, we obtain an exact sequence.*

Proof. Obviously, it is enough to prove that if the sequence $N \xrightarrow{\alpha} M \xrightarrow{\beta} L$ is exact, the sequence $FN \xrightarrow{F\alpha} FM \xrightarrow{F\beta} FL$ is also exact. Applying F to the epimorphism $\alpha' : M \twoheadrightarrow \operatorname{Im}\alpha$, we obtain an epimorphism $F\alpha' : FM \twoheadrightarrow F(\operatorname{Im}\alpha)$. Applying F to the monomorphism $\alpha'' : \operatorname{Im}\alpha \hookrightarrow N$, we obtain an monomorphism $F\alpha'' : F(\operatorname{Im}\alpha) \hookrightarrow FN$. Thus $F\alpha = F(\alpha'')F(\alpha')$, where $F(\alpha')$ is epimorphism and $F(\alpha'')$ is monomorphism, that is $F(\operatorname{Im}\alpha) = \operatorname{Im} F\alpha$. The same consideration shows that $F(\operatorname{Im}\beta) = \operatorname{Im} F\beta$. Applying F to the exact sequence $0 \to \operatorname{Im}\alpha \to N \to \operatorname{Im}\beta$, we obtain the exact sequence $0 \to \operatorname{Im} F\alpha \to FN \to \operatorname{Im} F\beta$, whence $\operatorname{Im} F\alpha = \operatorname{Ker} F\beta$. □

The definitions of projective and injective objects imply the following fact (explain, why).

Proposition.
- *A module P is projective if and only if the functor $h^P = \operatorname{Hom}_\Lambda(P, \_\,)$ is exact.*
- *A module Q is injective if and only if the functor $h_Q = \operatorname{Hom}_\Lambda(\_\,, Q)$ is exact.*

If $M_\bullet = (M_n, d_n)$ is a complex and F is a functor, then $FM_\bullet = (FM_n, Fd_n)$ is also a complex, since $Fd_n \cdot Fd_{n+1} = F(d_n d_{n+1}) = F0 = 0$.

If F is a contravariant functor, we use the upper notation, setting $(FM)^n = FM_n$ and $(Fd)^n = Fd_{n+1} : (FM)^n \to (FM)^{n+1}$. Then we obtain a complex $(FM)^\bullet$.

An important fact is that homotopy is preserved by any functors.

Proposition.
- *If φ and ψ are morphisms of complexes and $\varphi \sim \psi$, then $F\varphi \sim F\psi$.*
- *If the complexes $M_\bullet$ and $N_\bullet$ are homotopic, so are the complexes $FM_\bullet$ and $FN_\bullet$.*
- *If a complex $M_\bullet$ is contractible, so is the complex $FM_\bullet$.*

In particular any functor maps a split exact sequence to a split exact sequence.

Proof. (1) If $\{\,\sigma_n\,\}$ is a homotopy of φ and ψ, then $\{\,F\sigma_n\,\}$ is a homotopy of $F\varphi$ and $F\psi$.

(2) and (3) follows immediatly from (1). □

On the other hand, a fiunctor need not preserve the exactness.

Example. The sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is exact, but applying the functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \_\,)$, we obtain the sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

(since $2 \cdot \mathbb{Z}/2\mathbb{Z} = 0$), which is not exact.

. Let $F : \Lambda\text{-}\mathsf{Mod} \to \Lambda'\text{-}\mathsf{Mod}$ be a (covariant) functor, $M \in \Lambda\text{-}\mathsf{Mod}$ and $P_\bullet$ be a projective resolution of M.

Set $L_nFM = H_n(FP_\bullet)$.

If $\alpha : M \to N$, $P'_\bullet$ is a projective resolution of N and $\varphi : P_\bullet \to P'_\bullet$ is a lifting of α, we denote by $L_nF\alpha : L_nFM \to L_nFN$ the map $H_n(F\varphi) : H_n(FP_\bullet) \to H_n(FP'_\bullet)$.

Note that a projective resolution of M as well as a lifing of α to resolutions are uniquely determined up to homotopy, hence neither L_nFM, nor $L_nF\alpha$ depend on the choice of a resolution and of a lifting.

Moreover, if $\beta : N \to L$, $P''_\bullet$ is a projective resolution of L and $\psi : P'_\bullet \to P''_\bullet$ is a lifting of β then $\psi\varphi$ is a lifting of $\beta\alpha$, whence $L_nF(\beta\alpha) = L_nF\beta \cdot L_nF\alpha$.

Therefore, L_nF is a functor $\Lambda\text{-}\mathsf{Mod} \to \Lambda'\text{-}\mathsf{Mod}$ called the *n-th left derived functor of F*.

The whole set $\{\, L_nF \mid n \in \mathbb{N} \,\}$ is called the *(full) left derived functor of F*.

If F is a contravariant functor, then, using the same notations, we set $R^nFM = H^n((FP)^\bullet)$ and denote by $R^nF\alpha : R^nFN \to R^nFM$ the map $H^n(F\varphi) : H^n((FP')^\bullet) \to H^n((FP)^\bullet)$.

By the same reason, neither R^nFM, nor $R^nF\alpha$ depend on the choice of a resolution and of a lifting, $R^nF(\beta\alpha) = R^nF\alpha \cdot R^nF\beta$, so we obtain the *$n$-the right derived functor* (also contravariant) R^nF.

If we use in the same way injective resolutions, we obtain *right derived* of covariant and *left derived* of contravarint functors. We leave to the reader the details of the corresponding definitions.

Note that there is a projective resolution of a projective module P with $P_0 = P$ and $P_n = 0$ for $n > 0$.

Hence, $L_nF(P) = 0$ for $n > 0$ and every covariant functor F, as well as $R^nF(P) = 0$ for $n > 0$ and every contravariant functor F.

By the same reasons, if Q is injective, $R^nF(Q) = 0$ for every covariant functor F and $L_nF(Q) = 0$ for every contravariant functor F.

Proposition. *If a functor F is right exact, then $L_0F \simeq F$.*

Proof. If $(P_\bullet, \pi)$ is a projective resolution of a module M, there is an exact sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varphi} M \to 0$.

As F is left exact, $FP_1 \xrightarrow{Fd_1} FP_0 \xrightarrow{F\varphi} FM \to 0$ is also an exact sequence. Therefore, $L_0F(M) = H_0(FP_\bullet) = FP_0/\operatorname{Im} Fd_1 \simeq FM$. □

We suggest the readers to formulate analogous results for right derived functors, as well as for contravariant functors.

Now we are going to prove the LES-theorem for derived functors. First, we constract lifting of exact sequences to resolutions.

Lemma 19. *Let* $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ *be an exact sequence. There are projecive resolutions* $(P_\bullet, \pi)$ *of* M*,* $(P'_\bullet, \pi')$ *of* N *and* $(P''_\bullet, \pi'')$ *of* L *and liftings* φ *of* α *and* ψ *of* β *such that* $0 \to P'_\bullet \xrightarrow{\varphi} P_\bullet \xrightarrow{\psi} P''_\bullet \to 0$ *is also an exact sequence.*

We call the latter sequence the *lifting* of the exact sequence $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ to projective resolutions.

Again, we suggest the reader to formulate and prove the analogous proposition for injective resolutions.

Proof. Let $\pi' : P'_0 \twoheadrightarrow N$ and $\pi'' : P''_0 \twoheadrightarrow L$ be surjections with projective P'_0 and P''_0. Set $P_0 = P'_0 \oplus P''_0$ and let $\varphi_0 : P'_0 \to P_0$ be the canonical embedding and $\psi_0 : P_0 \to P''_0$ be the canonical projection. As P''_0 is projective and $\beta : M \to L$ is surjective, there is $\gamma : P''_0 \to M$ such that $\pi'' = \beta\gamma$. Consider the map $\pi : P_0 = P'_0 \oplus P''_0 \to M$ with the components $\alpha\pi'$ and γ. One can verify (do it) that π is surjective, $\pi\varphi_0 = \alpha\pi'$ and $\beta\pi = \pi''\psi_0$. Denote by $K' = \operatorname{Ker}\pi'$, $K = \operatorname{Ker}\pi$ and $K'' = \operatorname{Ker}\pi''$. Obviously, $\varphi_0(K') \subseteq K$ and $\psi_0(K) \subseteq K''$. Therefore, we obtain a commutative diagram with exact columns, as well as the exact second and third rows.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L'_0 \arrow[r, "\overline{\varphi_0}"] \arrow[d, "\lambda'"] & L_0 \arrow[r, "\overline{\psi_0}"] \arrow[d, "\lambda"] & L''_0 \arrow[r] \arrow[d, "\lambda''"] & 0 \\
0 \arrow[r] & P'_0 \arrow[r, "\varphi_0"] \arrow[d, "\pi'"] & P_0 \arrow[r, "\psi_0"] \arrow[d, "\pi"] & P''_0 \arrow[r] \arrow[d, "\pi''"] & 0 \\
0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d] & M \arrow[r, "\beta"] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By 3×3-Lemma, the first row is also exact. Hence, we can apply to the first row the same construction of projectives as for the sequence $0 \to N \to M \to L$, obtaining a commutative diagram with exact columns and rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L'_1 \arrow[r, "\overline{\varphi_1}"] \arrow[d, "\lambda'_1"] & L_1 \arrow[r, "\overline{\psi_1}"] \arrow[d, "\lambda_1"] & L''_1 \arrow[r] \arrow[d, "\lambda''_1"] & 0 \\
0 \arrow[r] & P'_1 \arrow[r, "\varphi_1"] \arrow[d, "\pi'_1"] & P_1 \arrow[r, "\psi_1"] \arrow[d, "\pi_1"] & P''_1 \arrow[r] \arrow[d, "\pi''_1"] & 0 \\
0 \arrow[r] & L'_0 \arrow[r, "\overline{\varphi_0}"] \arrow[d] & L_0 \arrow[r, "\overline{\psi_0}"] \arrow[d] & L''_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Taking products $d_1' = \lambda'\pi_1'$, $d_1 = \lambda\pi_1$ and $d_1'' = \lambda''\pi_1''$, we obtain the first two terms of the projective resolutions, namely, commutative diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & P_1' \arrow[r, "\varphi_1"] \arrow[d, "d_1'"] & P_1 \arrow[r, "\psi_1"] \arrow[d, "d_1"] & P_1'' \arrow[r] \arrow[d, "d_1''"] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "\varphi_0"] \arrow[d, "\pi'"] & P_0 \arrow[r, "\psi_0"] \arrow[d, "\pi"] & P_0'' \arrow[r] \arrow[d, "\pi''"] & 0 \\
0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d] & M \arrow[r, "\beta"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Repeating this procedure, we construct recursively the whole resolutions $P_\bullet'$, $P_\bullet$ and $P_\bullet''$ (restore the details yourself). $\square$

Theorem 20 (LES-theorem)**.** *For every exact sequence* $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ *and any (covariant) functor* F *there are morphisms* $\delta_n : L_nF(L) \to L_{n-1}F(N)$ *and* $\delta^n : R^nF(L) \to R^{n+1}F(N)$ $(n \in \mathbb{N})$ *such that the following sequences are exact:*

$$\begin{aligned} \ldots \to L_nF(N) &\xrightarrow{L_nF(\alpha)} L_nF(M) \xrightarrow{L_nF(\beta)} L_nF(L) \xrightarrow{\delta_n} \\ &\xrightarrow{\delta_n} L_{n-1}F(N) \xrightarrow{L_{n-1}F(\alpha)} L_{n-1}F(M) \xrightarrow{L_{n-1}F(\beta)} L_{n-1}F(L) \to \ldots \end{aligned}$$

and

$$\begin{aligned} \ldots \to R^nF(N) &\xrightarrow{R^nF(\alpha)} R^nF(M) \xrightarrow{R^nF(\beta)} R^nF(L) \xrightarrow{\delta^n} \\ &\xrightarrow{\delta^n} R^{n+1}F(N) \xrightarrow{R^{n+1}F(\alpha)} R^{n+1}F(M) \xrightarrow{R^{n+1}F(\beta)} R^{n+1}F(L) \to \ldots \end{aligned}$$

Certainly, for contravariant functors the LES-theorem must be changed as follows.

Theorem 21 (LES-theorem)**.** *For every exact sequence* $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ *and any contravariant functor* F *there are morphisms* $\delta_n : L_nF(N) \to L_{n-1}F(L)$ *and* $\delta^n : R^nF(L) \to R^{n+1}F(N)$ $(n \in \mathbb{N})$ *such that the following sequences are exact:*

$$\begin{aligned} \ldots \to L_nF(L) &\xrightarrow{L_nF(\beta)} L_nF(M) \xrightarrow{L_nF(\alpha)} L_nF(N) \xrightarrow{\delta_n} \\ &\xrightarrow{\delta_n} L_{n-1}F(L) \xrightarrow{L_{n-1}F(\beta)} L_{n-1}F(M) \xrightarrow{L_{n-1}F(\alpha)} L_{n-1}F(N) \to \ldots \end{aligned}$$

and

$$\begin{aligned} \ldots \to R^nF(L) &\xrightarrow{R^nF(\beta)} R^nF(M) \xrightarrow{R^nF(\alpha)} R^nF(N) \xrightarrow{\delta^n} \\ &\xrightarrow{\delta^n} R^{n+1}F(L) \xrightarrow{R^{n+1}F(\beta)} R^{n+1}F(M) \xrightarrow{R^{n+1}F(\alpha)} R^{n+1}F(N) \to \ldots \end{aligned}$$

We prove the theorem for contravariant and right derived functors, leaving the other cases for the reader.

Proof. By Lemma 19, there is a lifting $0 \to P'_\bullet \xrightarrow{\varphi} P_\bullet \xrightarrow{\psi} P''_\bullet \to 0$ of the exact sequence $0 \to N \to M \to L \to 0$ to projective resolutions. For every n the exact sequence $0 \to P'_n \xrightarrow{\varphi_n} P_n \xrightarrow{\psi_n} P''_n \to 0$ splits, since P''_n is projective. Therefore, applying F, we get the exact sequence of complexes $\mathbf{S}$: $0 \to (FP'')^\bullet \xrightarrow{F\psi} (FP)^\bullet \xrightarrow{F\varphi} (FP')^\bullet \to 0$. As, by definition, $H^n(FP^\bullet) = R^nF(M)$, the LES for the exact sequence $\mathbf{S}$ just coincides with the LES for the derived functor R^nF. □

Remark.

- As projective (injective) resolutions are right (left) bounded, these LES have zero at the beginning or end. For instance, in the case of left derived of a covariant functor they are

$$\begin{aligned} \ldots \to L_1F(N) &\xrightarrow{L_1F(\beta)} L_1F(M) \xrightarrow{L_1F(\alpha)} L_1F(L) \xrightarrow{\delta_1} \\ &\xrightarrow{\delta_1} L_0F(N) \xrightarrow{L_0F(\beta)} L_0F(M) \xrightarrow{L_0F(\alpha)} L_0F(L) \to 0 \end{aligned}$$

- If F is left (right) exact, then $R^0F = F$ (respectively, $L_0F = F$), so the first terms, in the case of right derived of a contravariant functor, are

$$\begin{aligned} 0 \to F(L) &\xrightarrow{F(\beta)} F(M) \xrightarrow{F(\alpha)} F(N) \xrightarrow{\delta^0} \\ &\xrightarrow{\delta^0} R^1F(L) \xrightarrow{R^1F(\beta)} R^1F(M) \xrightarrow{R^1F(\alpha)} R^1F(N) \to \ldots \end{aligned}$$

Corollary. *1. Let $0 \to N \xrightarrow{\alpha} P \xrightarrow{\beta} M \to 0$ be an exact sequence with projective P.*

- *For every covariant functor F $L_nF(M) \simeq L_{n-1}F(N)$ if $n > 1$ and $L_1F(M) \simeq \operatorname{Ker} L_0F(\alpha)$.*
- *For every contravariant functor F $R^nF(M) \simeq R^{n-1}F(N)$ if $n > 1$ and $R^1F(M) \simeq \operatorname{Coker} R^0F(\beta)$.*

2. Let $0 \to M \xrightarrow{\alpha} Q \xrightarrow{\beta} N \to 0$ be an exact sequence with projective Q.

- *For every covariant functor F $R^nF(M) \simeq R^{n-1}F(N)$ if $n > 1$ and $R^1F(N) \simeq \operatorname{Coker} R^0F(\alpha)$.*
- *For every contravariant functor F $L_nF(M) \simeq L_{n-1}F(N)$ if $n > 1$ and $L_1F(M) \simeq \operatorname{Ker} L_0F(\beta)$.*

Recall that if F is left (right) exact, then $R^0F = F$ (respectively, $L_0F = F$).

Proof. We consider the case of right derived of a contravariant functor; other cases are quite analogous.

Applying LES-theorem and taking into account that $R^nF(P) = 0$, we obtain, for $n > 1$, the exact sequence $0 \to R^{n-1}F(N) \xrightarrow{\delta^{n-1}} R^nF(M) \to 0$, whence δ^n is an isomorphism. For $n = 1$, we get $R^0F(P) \xrightarrow{R^0F(\beta)} R^0F(N) \xrightarrow{\delta^0} R^1F(M) \to 0$, which means that $R^1F(M) \simeq \operatorname{Coker} R^0F(\beta)$. □

Exercise 40. Prove that if

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \arrow[r] & M' \arrow[r] & L' \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram with exact rows and F is a functor, all induced diagrams

$$\begin{tikzcd}
L_nF(L) \arrow[r, "\delta_n"] \arrow[d] & L_{n-1}F(N) \arrow[d] \\
L_nF(L') \arrow[r, "\delta'_n"'] & L_{n-1}F(N')
\end{tikzcd}$$

are commutative.

Formulate and prove analogous results for right derived and for contravariant functors.

10.4 Ext and extensions

10.5 Ext and extensions

Ext and extensions

Now we will consider the most important for us example of derived functors — functors $\operatorname{Ext}^n_\Lambda$.

. Functor $\operatorname{Ext}^n_\Lambda(\_, A)$ is the n-th right derived of the functor R^nh_A, where $h_A = \operatorname{Hom}_\Lambda(\_, A)$.

Note that h_A is a contravariant left exact functor. Hence $\operatorname{Ext}^0_\Lambda(M, A) = \operatorname{Hom}_\Lambda(M, A)$ and for any exact sequence $0 \to N \to M \to L \to 0$ there is the LES

$$0 \to \operatorname{Hom}_\Lambda(L, A) \to \operatorname{Hom}_\Lambda(M, A) \to \operatorname{Hom}_\Lambda(N, A) \xrightarrow{\delta^0}$$
$$\xrightarrow{\delta^0} \operatorname{Ext}^1_\Lambda(L, A) \to \operatorname{Ext}^1_\Lambda(M, A) \to \operatorname{Ext}^1_\Lambda(N, A) \xrightarrow{\delta^1} \dots$$

Actually, there are also LES with respect to the second argument of this functor.

Theorem 22. *If* $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ *is an exact sequence, for any module* A *there are homomorphisms* $\delta^n : \mathrm{Ext}_\Lambda(A, L) \to \mathrm{Ext}_\Lambda^{n+1}(A, M)$ *such that the following sequence is exact:*

$$0 \to \mathrm{Hom}_\Lambda(A, N) \xrightarrow{\alpha\cdot} \mathrm{Hom}_\Lambda(A, M) \xrightarrow{\beta\cdot} \mathrm{Hom}_\Lambda(A, L) \xrightarrow{\delta^0}$$
$$\xrightarrow{\delta^0} \mathrm{Ext}_\Lambda^1(A, N) \xrightarrow{\alpha\cdot} \mathrm{Ext}_\Lambda^1(A, M) \xrightarrow{\beta\cdot} \mathrm{Ext}_\Lambda^1(A, L) \xrightarrow{\delta^1} \dots$$

Proof. Let $P_\bullet$ be a projective resolution of A. As all modules P_n are projective, the functors $\mathrm{Hom}_\Lambda(P_n, \_\,)$ map surjections to surjections, hence all sequences

$$0 \to \mathrm{Hom}_\Lambda(P_n, N) \xrightarrow{\alpha\cdot} \mathrm{Hom}_\Lambda(P_n, M) \xrightarrow{\beta\cdot} \mathrm{Hom}_\Lambda(P_n, L) \to 0$$

are exact. Therefore, we obtain an exact sequence of complexes

$$0 \to \mathrm{Hom}_\Lambda(P_\bullet, N) \xrightarrow{\alpha\cdot} \mathrm{Hom}_\Lambda(P_\bullet, M) \xrightarrow{\beta\cdot} \mathrm{Hom}_\Lambda(P_\bullet, L) \to 0.$$

As $H^n(\mathrm{Hom}_\Lambda(P_\bullet, M)) = \mathrm{Ext}_\Lambda^n(A, M)$, the LES for this sequence of complexes is just the LES from the theorem. □

Proposition. *If* Q *is injective,* $\mathrm{Ext}_\Lambda^n(M, Q) = 0$ *for* $n > 0$.

Proof. Let $P_\bullet$ be a projective resolution of M. It is exact at all terms, except P_0. As Q is injective, the functor h_Q maps exact sequences to exact sequences, hence the complex $Hom_\Lambda(P_\bullet, Q)$ is also exact at all terms except FP_0. Therefore, $\mathrm{Ext}_\Lambda^n(M, Q)) = H^n(\mathrm{Hom}_\Lambda(P_\bullet, Q)) = 0$ for $n > 0$. □

Calculations of Ext are often based on the following fact (cf. Corollary 226).

Corollary. *1. Let* $0 \to N \xrightarrow{\alpha} P \xrightarrow{\beta} M \to 0$ *be an exact sequence with projective* P. *Then* $\mathrm{Ext}_\Lambda^n(M, A) \simeq \mathrm{Ext}_\Lambda^{n-1}(N, A)$ *for* $n > 1$, *while* $\mathrm{Ext}_\Lambda^1(M, A) \simeq \mathrm{Coker}(\cdot\alpha)$*Note that here* $\cdot\alpha : \mathrm{Hom}_\Lambda(P, A) \to \mathrm{Hom}_\Lambda(N, A)$.

2. Let $0 \to M \xrightarrow{\alpha} Q \xrightarrow{\beta} N \to 0$ *be an exact sequence with injective* Q. *Then* $\mathrm{Ext}_\Lambda^n(A, M) \simeq \mathrm{Ext}_\Lambda^{n-1}(A, N)$ *for* $n > 1$, *while* $\mathrm{Ext}_\Lambda^1(A, M) \simeq \mathrm{Coker}(\beta\cdot)$. *Note that here* $\beta\cdot : \mathrm{Hom}_\Lambda(A, Q) \to \mathrm{Hom}_\Lambda(A, N)$.

Proof. (1) Just write the corresponding LES and use the fact that $\mathrm{Ext}_\Lambda^n(P, A) = 0$ for $n > 0$. (2) analogously. □

One can also consider the derived functors $R^n h^A$, where $h^A = \mathrm{Hom}_\Lambda(A, \_\,,)$. But it so happen that these functors actually coincide with Ext_Λ^n.

Theorem 23. $R^n h^A(M) \simeq \mathrm{Ext}_\Lambda^n(A, M)$.

Proof. For $n = 0$, both sides are $\mathrm{Hom}_\Lambda(A, M)$. Now embed M into an injective module Q obtaining an exact sequence $0 \to M \xrightarrow{\alpha} Q \xrightarrow{\beta} N \to 0$. Then both $\mathrm{Ext}^1_\Lambda(A, M)$ and $R^1h^A(M)$ are isomorphic to $\mathrm{Coker}(\beta\cdot)$. For $n > 1$ use the induction and isomorhism $\mathrm{Ext}^n_\Lambda(A, M) \simeq \mathrm{Ext}^{n-1}_\Lambda(A, N)$, as well as analogous isomorphism for R^nh^A. $\square$

In particular, to calculate Ext^n_Λ, one can use both projective and injective resolutions.

Example. We calculate $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z})$, where $a, b \in \mathbb{N}$, both > 1.

A projective resolution for $\mathbb{Z}/a\mathbb{Z}$ is $0 \to \mathbb{Z} \xrightarrow{a\cdot} \mathbb{Z} \to 0$. Applying $\mathrm{Hom}_{\mathbb{Z}}(\_, \mathbb{Z}/b\mathbb{Z})$ and knowing that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) = M$, we obtain the complex $0 \to \overbrace{\mathbb{Z}/b\mathbb{Z}}^{0} \xrightarrow{a\cdot} \overbrace{\mathbb{Z}/b\mathbb{Z}}^{1} \to 0$ (we have shown the numbers of places in this complex). The cohomology H^0 is just $\mathrm{Ker}\left(\mathbb{Z}/b\mathbb{Z} \xrightarrow{a\cdot} \mathbb{Z}/b\mathbb{Z}\right)$, that is $\{\, x + b\mathbb{Z} \mid b \mid ax \,\} = \{\, x + b\mathbb{Z} \mid (b/d) \mid x \,\} = (b/d)\mathbb{Z}/b\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(a, b)$. The cohomology H^1 is $(\mathbb{Z}/b\mathbb{Z})/\mathrm{Im}(a\cdot)$, that is $\mathbb{Z}/(a\mathbb{Z} + b\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$. Thus $\mathrm{Hom}_\Lambda(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}) \simeq \mathrm{Ext}^1_\Lambda(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}) \simeq \mathbb{Z}/d\mathbb{Z}$.

Exercise 41. Formulate and prove the analogous result for any commutative principal ideals domain.

The functor Ext^1 is closely connected to the *extensions* of modules.

1. An *extension of a module M with the kernel N* is an exact sequence $\mathbf{E}:\ 0 \to N \xrightarrow{\alpha} X \xrightarrow{\beta} M \to 0$.

2. They say that the extensions $\mathbf{E}$ and $\mathbf{E}':\ 0 \to N \xrightarrow{\alpha'} X' \xrightarrow{\beta'} M \to 0$ are *equivalent* and write $\mathbf{E} \sim \mathbf{E}'$ if there is a homomorphism $\gamma : X \to X'$ such that $\gamma\alpha = \alpha'$ and $\beta'\gamma = \beta$, that is the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, equal] & X \arrow[r, "\beta"] \arrow[d, "\gamma"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "\alpha'"] & X \arrow[r, "\beta'"] & M \arrow[r] & 0
\end{tikzcd}$$

   is commutative. Note that γ is an isomorhism by 5-Lemma.

Obviously, it is indeed an equivalence relation. We denote by $\mathsf{Ex}(M, N)$ the set of equivalence classes of extensions of M with the kernel N under this relation. One easily see that if the sequence $\mathbf{E}$ splits, any equivalent extension splits too. Our aim is the next result.

Theorem 24. *For any extension* $\mathbf{E}:\ 0\to N\xrightarrow{\alpha}X\xrightarrow{\beta}M\to 0$ *denote by* $\varepsilon(\mathbf{E})$ *the image* $\delta^0_{\mathbf{E}}(1_N)$*, where* $\delta^0_{\mathbf{E}}:\mathrm{Hom}_\Lambda(N,N)\to\mathrm{Ext}^1_\Lambda(M,N)$ *is the connecting homomorphism in the LES defined by the exact sequence* $\mathbf{E}$*. The map* $\boldsymbol{\varepsilon}:\mathbf{E}\mapsto\varepsilon(\mathbf{E})$ *is a bijection* $\mathsf{Ex}(M,N)\xrightarrow{\sim}\mathrm{Ext}^1_\Lambda(M,N)$*. The exact sequence* $\mathbf{E}$ *splits if and only if* $\varepsilon(\mathbf{E})=0$*.*

During the proof of this theorem we write $\delta_{\mathbf{E}}$ instead of $\delta^0_{\mathbf{E}}$

First, we show that this definition is *consistent*, that is does not depend on the choice of $\mathbf{E}$ in the equivalence class. Indeed, if $\mathbf{E}\sim\mathbf{E}'$, that is there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, equal] & X \arrow[r, "\beta"] \arrow[d, "\gamma"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "\alpha'"] & X \arrow[r, "\beta'"] & M \arrow[r] & 0
\end{tikzcd}$$

it gives the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(N,N) \arrow[r, "\delta_{\mathbf{E}}"] \arrow[d, equal] & \mathrm{Ext}^1_\Lambda(M,N) \arrow[d, equal] \\
\mathrm{Hom}_\Lambda(N,N) \arrow[r, "\delta'_{\mathbf{E}}"] & \mathrm{Ext}^1_\Lambda(M,N)
\end{tikzcd}$$

whence $\varepsilon(\mathbf{E})=\delta_{\mathbf{E}}(1_N)=\delta_{\mathbf{E}'}(1_N)=\varepsilon(\mathbf{E}')$.

Now we fix an exact sequence $\mathbf{R}:\ 0\to K\xrightarrow{\psi}P\xrightarrow{\varphi}M\to 0$ with projective P. The corresponding LES gives the exact sequence

$$\mathrm{Hom}_\Lambda(P,N)\xrightarrow{\cdot\psi}\mathrm{Hom}_\Lambda(K,N)\xrightarrow{\delta_{\mathbf{R}}}\mathrm{Ext}^1_\Lambda(M,N)\to 0.$$

Hence, for every element $\varepsilon\in\mathrm{Ext}^1_\Lambda(M,N)$ there is $\eta:K\to N$ such that $\varepsilon=\delta_{\mathbf{R}}(\eta)$. Consider the quotient $Y=(P\oplus N)/\{\,(\psi(u),-\eta(u))\mid u\in K\,\}$ and the maps $\xi:P\to Y$, mapping p the class $[p,0]$ of $(p,0)$, $\lambda:N\to Y$ mapping v to the class $[0,v]$ and $\mu:Y\to M$ mapping the class $[p,v]$ to $\varphi(p)$.One can easily check that the following diagram is commutative (do it):

$$\begin{tikzcd}
 & 0 \arrow[r] & K \arrow[r, "\psi"] \arrow[d, "\eta"'] & P \arrow[r, "\varphi"] \arrow[d, "\xi"'] & M \arrow[r] \arrow[d, equal] & 0 \\
\mathbf{E}: & 0 \arrow[r] & N \arrow[r, "\lambda"] & Y \arrow[r, "\mu"] & M \arrow[r] & 0
\end{tikzcd} \qquad (*)$$

It gives an extension $\mathbf{E}\in\mathsf{Ex}(M,N)$ and the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(Y,N) \arrow[r, "\cdot\lambda"] \arrow[d, "\cdot\xi"] & \mathrm{Hom}_\Lambda(N,N) \arrow[r, "\delta_{\mathbf{E}}"] \arrow[d, "\cdot\eta"] & \mathrm{Ext}^1_\Lambda(M,N) \arrow[d, equal] \\
\mathrm{Hom}_\Lambda(P,N) \arrow[r, "\cdot\psi"] & \mathrm{Hom}_\Lambda(K,N) \arrow[r, "\delta_{\mathbf{R}}"] & \mathrm{Ext}^1_\Lambda(M,N)
\end{tikzcd}$$

Therefore, $\varepsilon=\delta_{\mathbf{R}}(\eta)=(\delta_{\mathbf{R}})(\cdot\eta)(1_N)=\delta_{\mathbf{E}}(1_N)=\varepsilon(\mathbf{E})$, so the map $\boldsymbol{\varepsilon}$ is surjective. In what follows, we denote the extension $\mathbf{E}$ constructed in this way from the homomorphism $\eta : K \to N$ by $\mathbf{E}(\eta)$. We see that $\varepsilon(\mathbf{E}(\eta)) = \delta_{\mathbf{R}}(\eta)$.

If $\delta_{\mathbf{R}}(\eta) = \delta_{\mathbf{R}}(\eta')$, then $\eta' = \eta + \theta\psi$ for some $\theta : P \to N$. Consider $\mathbf{E}(\eta')$: $0 \to N \xrightarrow{\lambda'} Y' \xrightarrow{\mu'} M \to 0$, where $Y' = P \oplus N / \{ (\psi(u), -\eta'(u)) \mid u \in K \}$, $\lambda' : v \mapsto [0, v]$, $\mu' : [p, v] \mapsto \varphi(p)$. Define $\gamma : Y \to Y'$ setting $\gamma[p, v] = [p, v - \theta(p)]$. One easily verifies (do it) that γ is correctly defined, $\gamma\lambda = \lambda'$ and $\mu'\gamma = \mu$.It means that $\mathbf{E}(\eta) \sim \mathbf{E}(\eta')$. Therefore, $\mathbf{E}(\eta)$ actually depends only on the image $\varepsilon = \delta_{\mathbf{R}}(\eta)$, so we denote it by $\mathbf{E}(\varepsilon)$.As we have already seen, $\varepsilon(E(\varepsilon)) = \varepsilon$.

On the other hand, if $\mathbf{E} :\ 0 \to N \xrightarrow{\alpha} X \xrightarrow{\beta} M \to 0$ is an extension, there is $\zeta : P \to X$ such that $\beta\zeta = \varphi$. Then $\beta\zeta\psi = \varphi\psi = 0$, so $\operatorname{Im}\zeta\psi \subseteq \operatorname{Ker}\beta = \operatorname{Im}\alpha$. Hence $\zeta\psi(u) = \alpha(v)$ for a unique v and $\zeta\psi = \alpha\eta$, where we define $\eta(u) = v$. Thus we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\psi"] \arrow[d, "\eta"'] & P \arrow[r, "\varphi"] \arrow[d, "\zeta"'] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "\alpha"] & X \arrow[r, "\beta"] & M \arrow[r] & 0
\end{tikzcd}$$

It gives a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_\Lambda(N, N) \arrow[r, "\delta_{\mathbf{E}}"] \arrow[d, "\cdot\eta"'] & \operatorname{Ext}^1_\Lambda(M, N) \arrow[d, equal] \\
\operatorname{Hom}_\Lambda(K, N) \arrow[r, "\delta_{\mathbf{R}}"] & \operatorname{Ext}^1_\Lambda(M, N)
\end{tikzcd}$$

It implies that $\varepsilon(\mathbf{E}) = \delta_{\mathbf{R}}(\eta)$.

Returning to the diagram (*) defining the extension $\mathbf{E}(\varepsilon)$, where $\varepsilon = \delta_{\mathbf{R}}(\eta)$, we define a map $\tau : P \oplus N \to X$ setting $\tau(p, v) = \zeta(p) + \alpha(v)$. If $\tau(p, v) = 0$, then $\zeta(p) = -\alpha(v)$, whence $\varphi(p) = \beta\zeta(p) = -\beta\alpha(v) = 0$. Hence $p = \psi(u)$ for some u. Therefore, $\alpha\eta(u) = \zeta\psi(u) = \zeta(p) = -\alpha(v)$ and $v = -\eta(u)$, that is $\operatorname{Ker}\tau = \{ (\psi(u), -\eta(u)) \}$ and τ defines a homomorphism $\gamma : Y \to X$: $\gamma[p, u] = \tau(p, u)$. $\gamma\lambda(v) = \gamma[0, v] = \alpha(v)$ and $\beta\gamma[p, u] = \beta(\zeta(p) + \alpha(v)) = \beta\zeta(p) = \varphi(p) = \mu[p, u]$. Therefore, γ defines an equivalence $\mathbf{E}(\varepsilon) \sim \mathbf{E}$, so the maps $\mathbf{E} \mapsto \varepsilon(\mathbf{E})$ and $\varepsilon \mapsto \mathbf{E}(\varepsilon)$ are mutually inverse. $\square$

In the same way, using injective modules, one can prove

Theorem 25. *For any extension* $\mathbf{E} :\ 0 \to N \xrightarrow{\alpha} X \xrightarrow{\beta} M \to 0$ *denote by* $\varepsilon'(\mathbf{E})$ *the image* $\delta_0^{\mathbf{E}}(1_M)$*, where* $\delta_0^{\mathbf{E}} : \operatorname{Hom}_\Lambda(M, M) \to \operatorname{Ext}^1_\Lambda(M, N)$ *is the connecting homomorphism in the LES defined by the exact sequence* $\mathbf{E}$*. The map* $\boldsymbol{\varepsilon}' : \mathbf{E} \mapsto \varepsilon'(\mathbf{E})$ *is a bijection* $\mathsf{Ex}(M, N) \xrightarrow{\sim} \operatorname{Ext}^1_\Lambda(M, N)$*. The exact sequence* $\mathbf{E}$ *splits if and only if* $\varepsilon'(\mathbf{E}) = 0$*.*

‘

Actually, one can prove that $\varepsilon'(\mathbf{E}) = -\varepsilon(\mathbf{E})$, though we will not use this fact.

Every homomorphism $\xi : M' \to M$ induces maps $\cdot\xi : \mathrm{Hom}_\Lambda(M,N) \to \mathrm{Hom}_\Lambda(M',N)$, hence maps $\mathrm{Ext}^n_\Lambda(M,N) \to \mathrm{Ext}^n_\Lambda(M',N)$, which we denote by $\overset{n}{\cdot}\xi$ and write $\varepsilon\xi$ for $(\overset{n}{\cdot}\xi)(\varepsilon)$. In the same way, every homomorphism $\eta : N \to N'$ induces maps $\eta\cdot : \mathrm{Hom}_\Lambda(M,N) \to \mathrm{Hom}_\Lambda(M,N')$, hence maps $\mathrm{Ext}^n_\Lambda(M,N) \to \mathrm{Ext}^n_\Lambda(M,N')$, which we denote by $\eta\overset{n}{\cdot}$ and write $\eta\varepsilon$ for $(\eta\overset{n}{\cdot})(\varepsilon)$. Thus we can consider $\mathrm{Ext}^n_\Lambda(M,N)$ as a right module over $\mathrm{End}_\Lambda M$ as well as a left module over $\mathrm{End}_\Lambda N$.

For $\mathrm{Ext}^1_\Lambda(M,N)$ one can realize these actions on the corresponding extensions.

Let $\mathbf{E} :\ 0 \to N \xrightarrow{\alpha} X \xrightarrow{\beta} M \to$ be an extension and $\xi : M' \to M$. Consider the submodule $X' \subseteq X \oplus M'$, $M' \{\, (x,v) \mid \beta(x) = \xi(v) \,\}$ and the maps $\beta' : X' \to M'$, $\beta'(x,v) = v$, $\xi' : X' \to X$, $\xi'(x,v) = x$ and $\alpha' : N \to X'$, $\alpha'(u) = (\alpha(u), 0)$. One can verify (do it) that we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
\mathbf{E}' : & 0 \arrow[r] & N \arrow[r, "\alpha'"] \arrow[d, equal] & X' \arrow[r, "\beta'"] \arrow[d, "\xi'"'] & M' \arrow[r] \arrow[d, "xi"'] & 0 \\
\mathbf{E} : & 0 \arrow[r] & N \arrow[r, "\alpha"] & X \arrow[r, "\beta"] & M \arrow[r] & 0
\end{tikzcd}$$

It gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(N,N) \arrow[r, "\delta_0^{\mathbf{E}'}"] \arrow[d, equal] & \mathrm{Ext}^1_\Lambda(M',N) \arrow[d, "\overset{1}{\cdot}\xi"'] \\
\mathrm{Hom}_\Lambda(N,N) \arrow[r, "\delta_0^{\mathbf{E}}"] & \mathrm{Ext}^1_\Lambda(M,N)
\end{tikzcd}$$

which implies that $\varepsilon(\mathbf{E}') = \varepsilon(\mathbf{E})\xi$. Further we write $\mathbf{E}' = \mathbf{E}\xi$ and call $\mathbf{E}'$ the *pullback* of $\mathbf{E}$ along ξ.

Just in the same way, if $\eta : N \to N'$, consider the quotient $X' = (X \oplus N')/\{\, (\alpha(v), -\eta(v)) \mid v \in N \,\}$. We denote by $[x,u]$ the coset of (x,u). There are maps $\alpha' : N' \to X'$, $\alpha'(u) = [0,u]$, $\eta' : X \to X'$, $\eta'(x) = [x,0]$ and $\beta' : X' \to M$, $\beta'[x,u] = \beta(x)$. We obtain a commutative diagram with exact rows

$$\begin{tikzcd}
\mathbf{E} :\ 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "\eta"'] & X \arrow[r, "\beta"] \arrow[d, "\eta'"'] & M \arrow[r] \arrow[d, equal] & 0 \\
\mathbf{E}' :\ 0 \arrow[r] & N' \arrow[r, "\alpha'"] & X' \arrow[r, "\beta'"] & M \arrow[r] & 0
\end{tikzcd}$$

It gives the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(M,M) \arrow[r, "\delta^0_{\mathbf{E}}"] \arrow[d, equal] & \mathrm{Ext}^1_\Lambda(M,N) \arrow[d, "\eta\overset{1}{\cdot}"'] \\
\mathrm{Hom}_\Lambda(M,M) \arrow[r, "\delta^0_{\mathbf{E}'}"] & \mathrm{Ext}^1_\Lambda(M,N')
\end{tikzcd}$$

whence $\varepsilon'(\mathbf{E}') = \eta\varepsilon'(\mathbf{E})$. We write $\mathbf{E}' = \eta\mathbf{E}$ and call $\mathbf{E}'$ the *pushdown* of $\mathbf{E}$ along η.

Note that if ξ (or η) in these considerations is an isomorphism, then, by 5-Lemma, the maps ξ' (or η') is also isomorphism. Thus, though extensions $\mathbf{E}$ and $\mathbf{E}\xi$ (or $\eta\mathbf{E}$) are, as a rule, not equivalent, their middle terms are isomorphic. Therefore, it is important to consider the action of the group $\mathsf{G}(M,N) = \operatorname{Aut} M \times \operatorname{Aut} N$ on the set $\operatorname{Ext}^1_\Lambda(M,N)$: $(g,h)\varepsilon = h\varepsilon g^{-1}$. The elements of $\operatorname{Ext}^1_\Lambda(M,N)$ belonging to one orbit of this action give isomorphic middle terms in the corresponding extensions. In particular, if $\dim_{\Bbbk} \operatorname{Ext}^1_\Lambda(M,N) = 1$, all non-split extenstions of M with the kernel N have isomorphic middle terms.

Here are some simple corollaries from the correspondence between Ext and extensions.

Corollary. *Let $M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_n = 0$ be a chain of submodules, $L_i = M_{i-1}/M_i$ $(1 \leqslant i \leqslant n)$. Suppose that $\{1,2,\ldots,n\} = I \cup J$ with $I \cup J = \emptyset$ and $\operatorname{Ext}^1_\Lambda(L_j, L_i) = 0$ for $i \in I$, $j \in J$. There is a chain of submodules $M = N_0 \supset N_1 \supset N_2 \supset \ldots \supset N_n = 0$ such that all quotients N_i/N_{i-1} are from $\{L_j \mid j \in I\}$ for $i \leqslant \#(I)$ and from $\{L_j \mid j \in J\}$ for $i > \#(I)$.*

If, morover, $\operatorname{Ext}^1_\Lambda(L_i, L_j) = 0$ for $i \in I$, $j \in J$, then $M \simeq N_m \oplus (M/N_m)$, where $m = \#(I)$.

Proof. Let $k = \min(I)$, $M' = M/M_k$ and $M'_i = M_i/M_k$. There is an exact sequence $0 \to L_k \to M' \to M'/L_k \to 0$ and all quotients $M'_{i-1}/M'_1 \simeq M_{i-1}/M_i$ $(1 \leqslant i < k)$ are from $\{L_j \mid j \in J\}$. Using the exact sequence for Ext, we obtain that $\operatorname{Ext}^1_\Lambda(M'/L_k.L_k) = 0$, so this exact sequence splits and $M' \simeq L_k \oplus M'/L_k$. If N_1 is the preimage of M'/L_k in M, then $M/N_1 \simeq L_k$ and there is a chain of submodules in N_1 with quotients L_{i-1}/L_i $(i \neq k)$.

Since there are less quotients in N_1 than in M, an easy induction accomplish the proof of the first claim.

The second claim follows from the fact that in this case $\operatorname{Ext}^1_\Lambda(M/N_m, N_m) = 0$ which also follows from the LES for Ext. $\square$

10.6 Hereditary rings

Hereditary rings

Quiver algebras belong to a special class of rings having rather specific homological properties.

. A ring is called *hereditary* if any submodule of any projective module is projective.

As every projective module is a direct summand of a free one, it is enough to know that any submodule of a free module is projective. Actually, the situation is even simpler.

Proposition. *A ring Λ is hereditary if and only if every left ideal of Λ is projective.*

We prove that in this case every submodule of a free module Λ^n of finite rank is projective. The case of modules of infinite rank requires some set-theoretical technique, like Zorn lemma or transfinite induction.

Actually, we have even a more general result.

Proposition. *If $M = \bigoplus_{i=1}^n M_i$ and for every i every submodule of M_i is projective, every submodule of M is projective too.*

Proof. We use induction. For $n = 1$ it is given. Let all submodules of $M' = \bigoplus_{i=1}^{n-1} M_i$ are projective and $N \subseteq M$. Set $N \cap M' = N'$, then $N/N' \subseteq M/M' = M_n$.

Then N' is projective by the inductive supposition and so is N/N'.

The projectivity of N/N' implies that $N \simeq N/N' \oplus N'$, hence N is also projective. □

This proposition is also true for infinite direct sums.

The following fact is obvious.

Proposition. *Let Λ be hereditary, $\pi : P \twoheadrightarrow M$ be an epimorphism with projective P, $K = \operatorname{Ker} \pi$ and $\varepsilon : K \hookrightarrow P$ be the embedding. Then $0 \to K \xrightarrow{\varepsilon} P \to 0$ is a projective resolution of M. Therefore, $L_n F(M) = 0$ for $n > 1$ and $L_1 F(M) = \operatorname{Coker} F\varepsilon$ for any covariant functor F, as well as $R^n F M = 0$ for $n > 1$ and $R^1 F(M) = \operatorname{Coker} F\varepsilon$ for any contravariant functor F. In particular, $\operatorname{Ext}^n_\Lambda(M, N) = 0$ for $n > 1$ and $\operatorname{Ext}^1_\Lambda(M, N) = \operatorname{Hom}_\Lambda(K, N)/\{\, \alpha\varepsilon \mid \alpha : P \to N \,\}$*

Corollary. *Let Λ be hereditary. Every exact sequence $\mathbf{E} :\ 0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ produces, for every module A, exact sequences*

$$\begin{aligned} 0 \to \operatorname{Hom}_\Lambda(L, A) &\xrightarrow{\cdot\beta} \operatorname{Hom}_\Lambda(M, A) \xrightarrow{\cdot\alpha} \operatorname{Hom}_\Lambda(N, A) \xrightarrow{\delta^{\mathbf{E}}} \\ \to \operatorname{Ext}^1_\Lambda(L, A) &\xrightarrow{\cdot^1\beta} \operatorname{Ext}^1_\Lambda(M, A) \xrightarrow{\cdot^1\alpha} \operatorname{Ext}^1_\Lambda(N, A) \to 0 \end{aligned}$$

and

$$\begin{aligned} 0 \to \operatorname{Hom}_\Lambda(A, N) &\xrightarrow{\alpha\cdot} \operatorname{Hom}_\Lambda(A, M) \xrightarrow{\beta\cdot} \operatorname{Hom}_\Lambda(A, L) \xrightarrow{\delta_{\mathbf{E}}} \\ \to \operatorname{Ext}^1_\Lambda(A, N) &\xrightarrow{\alpha\cdot^1} \operatorname{Ext}^1_\Lambda(A, M) \xrightarrow{\beta\cdot^1} \operatorname{Ext}^1_\Lambda(A, L) \to 0. \end{aligned}$$

Recall that the symbols $\cdot^1\alpha$ and $\alpha\cdot^1$ denote the maps $\operatorname{Ext}^1_\Lambda(\alpha, A)$ and $\operatorname{Ext}^1_\Lambda(A, \alpha)$. In particular, the functors $\operatorname{Ext}^1_\Lambda(\_\,, A)$ and $\operatorname{Ext}^1_\Lambda(A, \_\,)$ are right exact.

Example. 1. Obviously, every skewfield is hereditary.

2. If Λ is a (left) principale ideals domain, then every left ideal is of the form Λa and the map $x \mapsto xa$ is injective, so $\Lambda a \simeq \Lambda$, so is projective. Therefore, Λ is hereditary.

   In particular, $\mathbb{Z}$ and $\Bbbk[t]$ are hereditary.

3. The most important for us in the example of the *quiver algebra* $\Lambda = \Bbbk\Gamma$. We'll prove that it is hereditary if Γ is finite and has no oriented cycles, though this fact is true for any quiver. Actually, if $\Bbbk$ is algebraically closed, such quiver algebras are, in some sense, unique finite dimensional hereditary algebras [DK, Sec. 8.5].

Indeed, if $\operatorname{Ver}\Lambda = \{1, 2, \dots, n\}$, then $1 = \sum_{i=1}^{n} \emptyset_i$ and $\Lambda = \bigoplus_{i=1}^{n} \Lambda\emptyset_i$. We suppose that $1, 2, \dots, n$ is a positive numeration of vertices. Obviousely, the elements of $\Lambda\emptyset_i$ are linear combinations of paths starting it the vertex i. So, as a representation of the quiver Γ, it coincides with P_i defined above, so we denote it by P_i too. As direct summands of Λ, P_i are projective. According to the preceding proposition, we only have to prove that every submodule $M \subseteq P_i$ is projective. We prove it using induction. $P_1 = \langle \emptyset_i \rangle \simeq$ is simple, so has no notrivial submodules. Suppose the claim is true for P_i with $i < k$.

If $M \subset P_k$ is proper, it cannot contain 0_i. Hence it is contained in P'_k, with the basis $\mathbf{P}'_k$ consisting of non-trivial paths starting at k. If $a_1, a_2, \dots, a_r$ are all arrows starting at k, $\mathbf{P}'_k = \bigoplus_{j=1}^{r} \mathbf{A}_j$, where $\mathbf{A}_j$ is the set of paths with the frist arrow a_j. Therefore, $P'_k = \bigoplus_{j=1}^{r} \Lambda a_j$. One easily sees that if $a_j : k \to i_j$, then $\Lambda a_j \simeq P_{i_j}$ (just send $pa_j \mapsto p$). Moreover, since this numeration if positive, $i_j < k$ for all j. By inductive supposition, every submodule in P_{i_j} is projective. Therefore, every submodule in P_k is projective and $\Bbbk\Gamma$ is hereditary. □

As $P_k/P'_k \simeq E_k$ (the simple representation at the vertex k), the calculations above give the following result.

Corollary (of the proof)**.** *A projective resolution of the simple representation E_k of a quiver Γ is*

$$0 \to \bigoplus_{a:k\to j} P_j \xrightarrow{d} P_k \to E_k \to 0, \tag{2}$$

where the a-th component of d maps p to pa, so we wrire $d = (\cdot a)$

An important fact about representations of quivers is

Lemma 26. *For every vertex k and any representation M of a quiver Γ the map $\alpha \mapsto \alpha(\emptyset_k)$ is a bijection* $\operatorname{Hom}_\Lambda(P_k, M) \xrightarrow{\sim} M(k)$.

Proof. $= p\emptyset_k$ for every $p \in P_k$, hence $\alpha(p) = p\alpha(0_k)$. It implies that if $\alpha(\emptyset_k) = \beta(\emptyset_k)$, then $\alpha = \beta$, so this map is injective. On the other hand, given $x \in M(k)$, we define $\alpha : P_k \to M$ setting $\alpha(p) = px$. Evidently, α is a homomorphism and $\alpha(\emptyset_k) = x$. Hence this map is surjective. □

Together with the resolution

$$0 \to \bigoplus_{a:k\to j} P_j \xrightarrow{(\cdot a)} P_k \to E_k \to 0,$$

this lemma gives

Corollary. $\mathrm{Ext}^1_\Gamma(E_k, M) \simeq \bigoplus_{a:j\to k} M(j)/\{\,(ax) \mid x \in M(k)\,\}$. *In particular,* $\mathrm{Ext}^1_\Gamma(E_k, E_j) \simeq \mathrm{Ar}_\Gamma(k, j)$, *where* $\mathrm{Ar}_\Gamma(k, j)$ *is the vector space with the basis consisting of the arrows* $k \to j$.

The details of the proof we leave to the reader as a simple exercise.

From these calculations we obtain an important results relating Hom_Γ and Ext^1_Γ to the form χ_Γ.

Theorem 27. *Let* M, N *be representations of an acyclic quiver* Γ. *Then*

$$\chi_\Gamma(\mathbf{dim}\, M, \mathbf{dim}\, N) = \dim_{\Bbbk} \mathrm{Hom}_\Gamma(M, N) - \dim_{\Bbbk} \mathrm{Ext}^1_\Gamma(M, N).$$

During the proof, we denote

$$\xi_\Gamma(M, N) = \dim_{\Bbbk} \mathrm{Hom}_\Gamma(M, N) - \dim_{\Bbbk} \mathrm{Ext}^1_\Lambda(M, N).$$

So, we have to prove that $\xi_\Gamma(M, N) = \chi_\Gamma(\mathbf{dim}\, M, \mathbf{dim}\, N)$.

First, a simple result about vector spaces.

Lemma 28. *If* $0 \to V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{m-1}} V_m \to 0$ *is an exact sequence of vector spaces, then*

$$\sum_{i=1}^{n} (-1)^n \dim_{\Bbbk} V_i = 0.$$

Proof. If $U_i = \mathrm{Ker}\,\alpha_i = \mathrm{Im}\,\alpha_{i-1}$, there are exact sequences $0 \to U_i \to V_i \to U_{i+1} \to 0$ $(1 \leqslant i < m)$, $U_0 = 0$ and $U_m = V_m$. As $\dim V_i = \dim U_i + \dim U_{i+1}$, an easy calculation gives the result. □

Now we deduce a lemma about the fumction χ.

Lemma 29. *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence, then, for every* N,

$$\xi(M, N) = \xi(M', N) + \xi(M'', N),$$
$$\xi(N, M) = \xi(N, M') + \xi(N, M'').$$

Proof. Applying the functor $\mathrm{Hom}_\Lambda(\_, N)$ to this exact sequence, we obtain the LES

$$0 \to \mathrm{Hom}_\Lambda(M'', N) \xrightarrow{\cdot\beta} \mathrm{Hom}_\Lambda(M, N) \xrightarrow{\cdot\alpha} \mathrm{Hom}_\Lambda(M', N) \xrightarrow{\delta^{\mathbf{E}}}$$

$$\to \mathrm{Ext}^1_\Lambda(M'', N) \xrightarrow{\cdot\beta} \mathrm{Ext}^1_\Lambda(M, N) \xrightarrow{\cdot\alpha} \mathrm{Ext}^1_\Lambda(M', N) \to 0.$$

Counting dimensions, we obtain the first formula. To obtain the second one, just apply $\mathrm{Hom}_\Lambda(N, \_)$. □

Now we can prove the theorem.

Proof. First, note that Corollary 261 implies that $\xi(E_k, E_k) = 1 = \chi(\mathbf{e}_k, \mathbf{e}_k)$, and if $k \neq j$, then $\xi(E_k, E_j) = \chi(\mathbf{e}_k, \mathbf{e}_i) = -c_{jk}$ (the number of arrows $k \to j$). Hence the theorem is true if both M and N are simple modules.

As all representations are finite dimensional, there is a chain of subrepresentations $N = N_0 \supset N_1 \supset N_2 \supset \ldots \supset N_m = 0$ with simple quotients $L_i = N_{i-1}/N_i$ $(1 \leqslant i \leqslant m)$. Lemma 29 implies that

$$\xi(E_k, N) = \sum_{i=1}^{m} \xi(E_k, L_i) = \sum_{i=1}^{m} \chi(\mathbf{e}_k, \mathbf{dim}\, L_i) = \chi(\mathbf{e}_k, \mathbf{dim}\, N),$$

that is the theorem is true if M is simple. The same observation with a chain of subrepresentations of M prove the general case. □

11 Regular representations

11.1 Homogeneous representations

Regular representations

We study now regular representations of a Euclidean quiver Γ using the information about Ext and extensions.

Recall the main definitions and results concerning such representations.

- An indecomposable representation M is called *regular* if $\kappa(\mathbf{dim}\, M) = \chi(\boldsymbol{\omega}, \mathbf{dim}\, M) = 0$. Equivalently, $\partial(\mathbf{dim}\, M) = 0$, that is $C^m\, \mathbf{dim}\, M = \mathbf{dim}\, M$, where m is the order of the restriction of the Coxeter map C onto the quotient $\mathbb{R}^\Gamma/\mathbf{N}$ and $\mathbf{N} = \langle \boldsymbol{\omega} \rangle = \{\, \mathbf{x} \mid \mathsf{Q}_\Gamma(\mathbf{x}) = 0 \,\} = \{\, \mathbf{x} \mid \mathsf{B}_\Gamma(\mathbf{x}, \mathbf{y}) = 0 \text{ for all } \mathbf{y} \,\}$.
- A representation M is called *regular* if all its indecomposable direct summands are regular. We denote by $\mathcal{R}(\Gamma, \Bbbk)$ the category of regular representations.
- $M \in \mathcal{R}(\Gamma, \Bbbk)$ if and only if $\kappa(M) = 0$ and $\kappa(M') \leqslant 0$ for any $M' \subseteq M$.
- If $\alpha : M \to N$, where M and N are regular, then $\mathrm{Ker}\,\alpha$, $\mathrm{Im}\,\alpha$, $\mathrm{Coker}\,\alpha$ are regular.
- If $N \subseteq M$, $L = M/N$ and two of the modules M, N, L are regular, so is the third one.
- A regular representation is called *R-simple* (*regularly simple*) if it has no proper regular subrepresentations.

- One easily deduce that for any regular representation M there is a chain of subrepresentations

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_l = 0$$

such that all quotients $L_i = M_{i-1}/M_i$ $(1 \leqslant i \leqslant l)$ are R-simple. We call L_i *R-simple factors*, l the *R-length* of M and denote $l = \mathrm{rl}(M)$. One can prove that they are uniquely defined up to a permutation: just repeat the usual proof of the Jordan–Gölder theorem [DK, 1.5.1] using the fact that if N, N' are regular submodules of a reguar module M, then $N + N'$ and $N \cap N'$ are also regular as the image and the kernel of the natural map $N \oplus N' \to M$.

- An R-simple representation M is called *homogeneous* if $\mathbf{dim}\, M \in \mathbf{N}$, i.e. $\mathbf{dim}\, M = k\boldsymbol{\omega}$.

- A regular representation is called *homogeneous* if all its R-simple factors are homogeneous. We denote by $\mathcal{H}(\Gamma, \Bbbk)$ the category of homogeneous regular representations.

Example. 1. If Γ is the Kronecker quiver $1 \rightrightarrows 2$, then $\kappa(x_1, x_2) = x_1 - x_2$, hence regular representations are those of dimensions (n, n). In this case all of them are homogeneous.

2. On the other hand, if $\Gamma =$
$$\begin{tikzcd} & 1 \arrow[d] & \\ 2 \arrow[r] & 5 & 4 \arrow[l] \\ & 3 \arrow[u] & \end{tikzcd}$$
the dimension $\mathbf{d} = (1, 1, 0, 0, 1)$ is regular and there is an indecomposable representation M with $M(1) = M(2) = M(5) = \Bbbk$, $M(4) = M(5) = 0$, the maps $M(1) \to M(5)$ and $M(2) \to M(5)$ are identities. It is regular, R-simple and non-homogeneous.

There is a variant of the Schur lemma for R-simple representations.

Lemma 30. *Let M, L be regular representations and L be R-simple.*

1. *If $\alpha : L \to M$, either $\alpha = 0$ or α is injective.*

2. *If $\beta : M \to L$, either $\beta = 0$ or β is surjective.*

3. *If M is also R-simple, every $\alpha : L \to M$ is either 0 or an isomorphism.*

4. *In particular, $\mathrm{End}_\Gamma L$ is a skewfield.*

Proof. It evidently follows from the fact that $\mathrm{Ker}\, \alpha$ and $\mathrm{Im}\, \beta$ are regular. □

Corollary. *Let L, L' be R-simple representations.*

1. *If L is non-homogeneous and $\mathbf{dim}\, L' \equiv \mathbf{dim}\, L \,(\mathrm{mod}\, \boldsymbol{\omega})$, then $L' \simeq L$ and $\mathbf{dim}\, L = \mathbf{dim}\, L'$.*

2. *If L is homogeneous and $L' \not\simeq L$, then $\mathrm{Hom}_\Gamma(L, L') = \mathrm{Hom}_\Gamma(L', L) = \mathrm{Ext}^1_\Gamma(L, L') = \mathrm{Ext}^1_\Gamma(L', L) = 0$.*

Proof. (1) $\mathsf{Q}(\mathbf{dim}\, L) = \chi(\mathbf{dim}\, L, \mathbf{dim}\, L) = \chi(\mathbf{dim}\, L', \mathbf{dim}\, L) > 0$, hence $\mathrm{Hom}_\Gamma(L', L) \neq 0$ and $L' \simeq L$ by the Schur lemma.

(2) $\mathrm{Hom}_\Gamma(L, L') = \mathrm{Hom}_\Gamma(L', L) = 0$ by the Schur lemma. As L is homogeneous and L' is regular, $\chi(\mathbf{dim}\, L, \mathbf{dim}\, L') = 0$. Then also $\chi(\mathbf{dim}\, L', \mathbf{dim}\, L) = 0$, since $\chi(\mathbf{dim}\, L, \mathbf{dim}\, L') + \chi(\mathbf{dim}\, L', \mathbf{dim}\, L) = 2\mathsf{B}(\mathbf{dim}\, L, \mathbf{dim}\, L') = 0$. As $\chi_\Gamma(L, L') = \dim_{\Bbbk} \mathrm{Hom}_\Gamma(L, L') - \dim_{\Bbbk} \mathrm{Ext}^1_\Gamma(L, L'))$, we have that $\mathrm{Ext}^1_\Gamma(L, L') = 0$. In the same way, $\mathrm{Ext}^1_\Gamma(L', L) = 0$. ☐

For a homogeneous R-simple representation L we denote by $\mathcal{F}(L)$ the category of homogeneous representations such that all their R-simple factors are isomorphic to L.

Corollary. *Let L be an R-simple homogeneous representation, M, M' be indecomposable regular representations.*

1. *If L is an R-simple factor of M, then $M \in \mathcal{F}(L)$.*
2. *If $M \in \mathcal{F}(L)$, $M' \notin \mathcal{F}(L)$, then* $\mathrm{Hom}_\Gamma(M', M) = \mathrm{Hom}_\Gamma(M, M') = \mathrm{Ext}^1_\Gamma(M', M) = \mathrm{Ext}^1_\Gamma(M, M') = 0$.

Proof. Let $M = M_0 \supset M_1 \supset \ldots \supset M_n = 0$ be such that $L_i = M_{i-1}/M_i$ are R-simple, $I = \{\, i \mid L_i \simeq L \,\}$ and $J = \{\, i \mid L_i \not\simeq L \,\}$. By Corollary 250, $M \simeq M_I \oplus M_J$, where $M_I \in \mathcal{F}(L)$ and M_J has no R-simple factors isomorphic to L. It proves (1). If $M \in \mathcal{F}(L)$, $M' \notin \mathcal{F}(L)$, then (1) implies that neither R-simple factor L' of M' is isomorphic to L. By Corollary 270, for any such L' $\mathrm{Hom}_\Gamma(L, L') = \mathrm{Hom}_\Gamma(L', L) = \mathrm{Ext}^1_\Gamma(L, L') = \mathrm{Ext}^1_\Gamma(L', L) = 0$. Then (2) follows from the LES for Ext. ☐

Exercise 42. Verify that R-simple representations of the Kronecker quiver are $K(\varphi)$, where φ is an irreducible polynomial (in particular, $M_1(\lambda) \simeq M(t - \lambda)$) and $M_1(\infty)$.

Indecomposable representation of length l in $\mathcal{F}(K(\varphi))$ is $K(\varphi^l)$ ($K_l(\lambda)$ if $\varphi = t - \lambda$), and in $\mathcal{F}(K_1(\infty))$ it is $K_l(\infty)$.

In pafticular, if $\Bbbk$ is algebraically closed, all R-simple representations are $K_1(\lambda)$ ($\lambda \in \Bbbk \cup \{\infty\}$) and all indecomposable representations in $\mathcal{F}(K_1(\lambda))$ are $K_l(\lambda)$.

Now we describe the structure of the category $\mathcal{F}(L)$, where L is an R-simple homogeneous representation. We denote by $\mathbb{F}$ the skewfield $\mathrm{End}_\Gamma L$. (Further we will see that it is actually commutative, i.e. a field.) Note that $\chi(L, L) = 0$, hence $\dim \mathrm{Hom}_\Gamma(L, L) = \dim \mathrm{Ext}^1_\Lambda(L, L)$. Therefore, if we consider $\mathrm{Ext}^1_\Gamma(L, L)$ as a vector space over $\mathbb{F}$, it is 1-dimensional.

Theorem 31. *Let L be a homogeneous R-simple representation.*

1. *For every l there is a unique (up to isomorphism) indecomposable representation $M_l \in \mathcal{F}(L)$ with* $\mathrm{rl}(M) = l$.

2. $\mathrm{Hom}_\Gamma(L, M_l) \simeq \mathrm{Hom}_\Gamma(M_l, L) \simeq \mathrm{Ext}^1_\Gamma(L, M_l) \simeq \mathrm{Ext}^1_\Gamma(M_l, L) \simeq \mathbb{F}$ *for every* l.

3. *For every* l *and* $k \leqslant l$ *there is a unique regular submodule* $M_{l,k} \subseteq M_l$ *such that* $\mathrm{rl}(M_l/M_{l,k}) = k$. *(Such modules are called* serial.*)*

4. $M_{l,k} \simeq M_{l-k}$ *and* $M/M_{l,k} \simeq M_k$.

5. *Every homomorphism* $M_l \to M_r$ *arises from an isomorphism* $M_l/M_{l,k} \xrightarrow{\sim} M_{r,r-k}$ $(k \leqslant \min(l,r))$.

6. *If* $r < l$, *then* $\mathrm{Hom}_\Gamma(M_l, M_r) \simeq \mathrm{Hom}_\Gamma(M_r, M_l) \simeq \mathrm{Hom}_\Gamma(M_r, M_r)$.

Proof. We construct recursively indecomposable modules $M_l \in \mathcal{F}(L)$ of regular length l satisfying conditions (1-6). Certainly, the property (5) at each stage is checked for already constructed modules.

We set $M_1 = L$. It is a unique R-simple module in $\mathcal{F}(L)$, so satisfies (1). As $\chi(L, L) = 0$, $\mathrm{Hom}_\Gamma(L, L) \simeq \mathrm{Ext}^1_\Gamma(L, L)$, so M_1 satisfies (2). The properties (3-6) are obvious, since L is R-simple.

Now we suppose that the modules M_k satisfying (2-6) have been constructed for $k \leqslant l$ and consrtuct the module M_{l+1}.

Note that, since $\mathrm{Hom}_\Gamma(L, M_l) \simeq \mathrm{Hom}_\Gamma(M_l, L) \simeq \mathrm{Ext}^1_\Gamma(L, M_l) \simeq \mathrm{Ext}^1_\Gamma(M_l, L) \simeq \mathbb{F}$, the group $\mathrm{Aut}_\Gamma L = \mathbb{F}^\times$ acts transitively on nonzero elements of these spaces, hence all non-split extensions are isomorphic and any two nonzero homomorphisms differs by a multiple which is an automorphism of L, thus have the same kernels and images.

Let $\mathbf{E}:\ 0 \to L \xrightarrow{\alpha} M_{l+1} \xrightarrow{\beta} M_l \to 0$ be a non-split extension. Obviously, $M_{l+1} \in \mathcal{F}(L)$ and $\mathrm{rl}(M_{l+1}) = l + 1$. The LES for Ext gives the exact sequence

$$0 \to \mathrm{Hom}_\Gamma(M_l, L) \xrightarrow{\cdot\beta} \mathrm{Hom}_\Gamma(M_{l+1}, L) \xrightarrow{\cdot\alpha} \mathrm{Hom}_\Gamma(L, L) \xrightarrow{\delta_{\mathbf{E}}}$$

$$\xrightarrow{\delta} \mathrm{Ext}^1_\Gamma(M_l, L) \xrightarrow{\overset{1}{\cdot}\beta} \mathrm{Ext}^1_\Gamma(M_{l+1}, L) \xrightarrow{\overset{1}{\cdot}\alpha} \mathrm{Ext}^1_\Gamma(L, L) \to 0.$$

As the extension $\mathbf{E}$ was non-split, $\delta_{\mathbf{E}} \neq 0$. As $\mathrm{Hom}_\Gamma(M_l, L)$ and $\mathrm{Ext}^1_\Gamma(L, L)$ are 1-dimensional vector spaces over the skewfield $\mathbb{F}$, δ is an isomorphism, i.e. $\mathrm{Ker}\,\delta = \mathrm{Im}(\cdot\alpha) = 0$, $\mathrm{Im}\,\delta = \mathrm{Ker}(\overset{1}{\cdot}\beta) = \mathrm{Ext}^1_\Gamma(L, L)$. Therefore, $\cdot\alpha = 0 = \overset{1}{\cdot}\beta$, whence $\cdot\beta$ and $\overset{1}{\cdot}\alpha$ are isomorphisms and $\mathrm{Hom}_\Gamma(M_{l+1}, L) \simeq \mathrm{Ext}^1_\Gamma(M_{l+1}, L) \simeq \mathbb{F}$. In particular, it implies that M_{l+1} is indecomposable.

Every maximal regular submodule $M' \subset M_{l+1}$ defines a surjection $\varphi : M_{l+1} \twoheadrightarrow L \simeq M_{l+1}/M'$. If $\varphi' : M_{l+1} \to L$ is another surjetion, $\varphi' = \theta\varphi$ for some $\theta \in \mathrm{Aut}_\Gamma L$, whence $\mathrm{Ker}\,\varphi' = \mathrm{Ker}\,\varphi$. Thus M' is a unique maximal regular submodule in M_{l+1}.

The exact sequence $0 \to M' \xrightarrow{\alpha'} M_{l+1} \xrightarrow{\beta'} L \to 0$ gives a LES

$$0 \to \mathrm{Hom}_\Gamma(L, L) \xrightarrow{\cdot\beta'} \mathrm{Hom}_\Gamma(M_{l+1}, L) \xrightarrow{\cdot\alpha'} \mathrm{Hom}_\Gamma(M', L) \xrightarrow{\delta'}$$

$$\xrightarrow{\delta} \mathrm{Ext}^1_\Gamma(L, L) \xrightarrow{\overset{1}{\cdot}\beta'} \mathrm{Ext}^1_\Gamma(M_{l+1}, L) \xrightarrow{\overset{1}{\cdot}\alpha'} \mathrm{Ext}^1_\Gamma(L, L) \to 0.$$

As $\mathrm{Hom}_\Gamma(L,L) \simeq \mathrm{Hom}_\Gamma(M_{l+1},L) \simeq \mathrm{Ext}^1_\Gamma(M_{l+1},L) \simeq \mathrm{Ext}^1_\Gamma(L,,L) \simeq \mathbb{F}$, the embedding $\cdot\beta'$ and the surjection $\overset{1}{\cdot}\alpha'$ are isomorphisms. Therefore, $\cdot\alpha'$ and $\overset{1}{\cdot}\beta'$ are zero, thus δ' is also an isomoprhism and $\mathrm{Hom}_\Gamma(M',L) \simeq \mathrm{Ext}^1_\Gamma(L,L) \simeq \mathbb{F}$. It implies that M' is indecomposable, hence $M' \simeq M_l$, since $\mathrm{rl}(M') = l$. So we write M_l for M'.

Since M_l is a unique maximal regular submodule of M_{l+1}, it contains all proper regular submodules, in particular, the images of morphisms $L \to M_{l+1}$. Therefore, $\mathrm{Hom}_\Gamma(L,M_{l+1}) \simeq \mathrm{Hom}_\Gamma(L,M_l) \simeq \mathbb{F}$, hence also $\mathrm{Ext}^1_\Gamma(L,M_{l+1}) \simeq \mathbb{F}$, which accomplishes the proof of (2).

M_l contains a unique submodule $M_{l,k}$ such that $\mathrm{rl}(M_l/M_{l,k}) = k$, and $M_{l,k} \simeq M_{l-k}$. If we consider it as a submodule of M_{l+1} and denote it by $M_{l+1,k+1}$, we obtain a unique submodule of M_{l+1} such that $\mathrm{rl}(M_{l+1}/M_{l+1,k+1}) = k+1$ and $M_{l+1,k+1} \simeq M_{(l+1)-(k+1)}$.

Moreover, since $M_l/M_{l+1,k+1}$ is a unique maximal submodule of $M_{l+1}/M_{l+1,k+1}$, the latter module is indecomposable, hence isomorphic to M_{k+1}, which accomplishes the proof of (3) and (4). Let $k \leqslant l$. The image of every homomorphism $M_k \to M_{l+1}$ is a submodule of M_l and the kernel of every homomorphism $M_{l+1} \to M_k$ contains L, so it is actually a homomorphism $M_{l+1}/L \simeq M_l \to M_k$. Using induction, we obtain (5) and (6).

So, it only remains to prove that M_{l+1} is a unique indecomposable representation in $\mathcal{F}(L)$ of R-length $l+1$. To prove it, we need a lemma.

Lemma 32. *Consider a non-split exact sequence* $\mathbf{E}:\ 0 \to L \xrightarrow{\alpha} M_{l+1} \xrightarrow{\beta} M_l \to 0$. *For every* l, $\mathrm{Ext}^1_\Gamma(M_l,L) \simeq \mathrm{End}_\Gamma M_l/\mathfrak{r}$, *where* $\mathfrak{r} = \mathrm{Im}(\beta\cdot)$, *Moreover,* $\mathfrak{r}$ *is the radical of* $\mathrm{End}_\Gamma M_l$, *all endomorphisms* $\gamma \notin \mathfrak{r}$ *are invertible and* $\mathrm{Aut}_\Gamma M_l$ *acts transitively on nonzero elements of* $\mathrm{Ext}^1_\Gamma(M_l,L)$.

Recall that $\mathrm{rad}\,\Lambda$, where Λ is a finite dimensional algebra, is the biggest nilpotent ideal in Λ (see [DK, Sec. 3.1]). If M is indecomposable, $\mathrm{rad}\,\Lambda$ is the subset of all non-invertible elements and $\mathrm{End}_\Gamma M_l/\mathrm{rad}\,\mathrm{End}_\Gamma M_l$ is a skewfield [DK, Th. 3.2.2 & Cor. 3.2.3]. In particular, it is the case for $M = M_l$.

Proof. Applying $\mathrm{Hom}_\Gamma(M_l, \_\,)$ to this exact sequence, we obtain the LES

$$0 \to \mathrm{Hom}_\Gamma(M_l,L) \xrightarrow{\alpha\cdot} \mathrm{Hom}_\Gamma(M_l,M_{l+1}) \xrightarrow{\beta\cdot} \mathrm{Hom}_\Gamma(M_l,M_l) \xrightarrow{\delta}$$
$$\xrightarrow{\delta} \mathrm{Ext}^1_\Gamma(M_l,L) \xrightarrow{\overset{1}{\alpha\cdot}} \mathrm{Ext}^1_\Gamma(M_{l+1},M_l) \xrightarrow{\overset{1}{\beta\cdot}} \mathrm{Ext}^1_\Gamma(M_l,M_l) \to 0.$$

As $\chi(M_l,M_k) = 0$ and $\mathrm{Hom}_\Gamma(M_{l+1},M_l) \simeq \mathrm{Hom}_\Gamma(M_l,M_l)$, also $\dim \mathrm{Ext}^1_\Gamma(M_{l+1},M_l) = \dim \mathrm{Ext}^1_\Gamma(M_l,M_{l+1})$. Therefore, as $\overset{1}{\beta\cdot}$ is a surjection, it is a bijection, hence $\overset{1}{\alpha\cdot} = 0$, δ is a surjection and $\mathrm{Ext}^1_\Gamma(M_l,L) \simeq \mathrm{End}_\Gamma M_l/\mathfrak{r}$. Note that neither element from $\mathfrak{r}$ is invertible: otherwise we obtain a morphism $\beta' : M_l \to M_{l+1}$ such that $\beta\beta' = 1_{M_l}$ and the sequence $\mathbf{E}$ splits. On the other hand, $\mathrm{End}_\Gamma M_l$ acts on $\mathrm{Ext}^1_\Gamma(M_l,L) \simeq \mathbb{F} = \mathrm{End}_\Gamma L$. Let $\varepsilon = \delta(1_{M_l})$. For every $\gamma \in \mathrm{End}_\Gamma M_l$, $\varepsilon\gamma = \gamma'\varepsilon$ for some $\gamma' \in \mathbb{F}$. One easily sees that $\gamma \mapsto \gamma'$ is a homomorphism of rings

$\operatorname{End}_\Gamma M_l \to \mathbb{F}$ and its kernel is just $\mathfrak{r}$ (explain it). Therefore, $\operatorname{End}_\Gamma M_l/\mathfrak{r} \simeq \mathbb{F}$ (they are of the same dimension). It implies that $\mathfrak{r} = \operatorname{rad}\operatorname{End}_\Gamma M_l$. As M_l is indecomposable, all elements $\gamma \notin \mathfrak{r}$ are invertible, that accomplishes the proof. $\square$

We will also use the following fact.

Remark. Since $\operatorname{Ext}^1_\Gamma$ is right exact, the injection $\mu_{lk} : M_k \to M_l$, where $k < l$, induces a surjection $\overset{1}{\cdot}\mu_{lk} : \operatorname{Ext}^1_\Gamma(M_l, L) \to \operatorname{Ext}^1(M_k, L)$, As both these spaces are isomorphis to $\mathbb{F}$, $\overset{1}{\cdot}\mu_{lk}$ is an isomorphism.

Now we prove (1) for the regular length $l+1$.

Let M be an indecomposable module from $\mathcal{F}(L)$ of regular length $l+1$. Then L embeds into M. Let $M' = M/L$. It is regular of regular length l and there is a nonsplit exact sequence $\mathbf{E}:\ 0 \to L \to M \to M' \to 0$. We denote by $\varepsilon = \varepsilon(\mathbf{E})$ the corresponding element from $\operatorname{Ext}^1_\Gamma(M', L)$. If M' is indecomposable, $M' \simeq M_l$ and $M \simeq M_{l+1}$. We will prove that if M' decomposes, M decomposes too.

If M' is decomposable, there are numbers $l_i < l$ such that $M' \simeq \bigoplus_{i=1}^m M_{l_i}$ and we can suppose that $l_1 \geqslant l_2 \geqslant \ldots \geqslant l_m$. Then $\operatorname{Ext}^1_\Gamma(M', L) = \bigoplus_{i=1}^m \operatorname{Ext}^1_\Gamma(M_{l_i}, L)$. Thus ε can be considered as a vector $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$, where $\varepsilon_i \in \operatorname{Ext}^1_\Gamma(M_{l_i}, L)$. If some $\varepsilon_i = 0$, then M decomposes as $M_{l_i} \oplus N$, where $N/L \simeq \bigoplus_{j\neq i} M_{l_j}$. Actually, N arises as an extension corresponding to the element $\varepsilon' \in \operatorname{Ext}^1_\Gamma(\bigoplus_{j\neq i} M_{l_j}, L)$ given as the vector with the same coordinates as ε except ε_i (explain why). Let $\varepsilon_1 \neq 0$. Then $\varepsilon_1\mu_{l_1l_2} \neq 0$, hence, by Lemma, $\varepsilon_2 = \varepsilon_1\mu_{l_1l_2}\gamma$ for some $\gamma \in \operatorname{Aut}_\Gamma M_{l_2}$. Consider the automorphism θ of M' given by the matrix

$$\begin{pmatrix} 1 & -\mu_{l_1l_2}\gamma & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Then $\varepsilon\theta = (\varepsilon_1, 0, \varepsilon_3, \ldots, \varepsilon_m)$ and M decomposes.

It accomplishes the proof of the theorem. $\square$

11.2 Non-homogeneous representations

Non-homogeneous representations

Now we consider non-homogeneous representations. Note first of all the following fact.

Proposition. *1. If M is an irreducible regular representation, then $\overrightarrow{\sigma}^+_k M \neq 0$ and $\overleftarrow{\sigma}^-_k M \neq 0$ for any k (see 138 for the definition of $\overrightarrow{\sigma}$ and $\overleftarrow{\sigma}$).*

2. *If M and N are regular, then* $\mathrm{Hom}_{\overrightarrow{\sigma}\Gamma}(\overrightarrow{\sigma}_k^+ M, \overrightarrow{\sigma}_k^+ N) \simeq \mathrm{Hom}_\Gamma(M,N)$ *and* $\mathrm{Ext}^1_{\overrightarrow{\sigma}\Gamma}(\overrightarrow{\sigma}_k^+ M, \overrightarrow{\sigma}_k^+ N) \simeq \mathrm{Ext}^1_\Gamma(M,N)$, *as well as* $\mathrm{Hom}_{\overleftarrow{\sigma}\Gamma}(\overleftarrow{\sigma}_k^- M, \overleftarrow{\sigma}_k^- N) \simeq \mathrm{Hom}_\Gamma(M,N)$ *and* $\mathrm{Ext}^1_{\overleftarrow{\sigma}\Gamma}(\overleftarrow{\sigma}_k^- M, \overleftarrow{\sigma}_k^- N) \simeq \mathrm{Ext}^1_\Gamma(M,N)$, *In particular,* $\mathrm{Hom}_\Gamma(C^\pm M, C^\pm N) \simeq \mathrm{Hom}_\Gamma(M,N)$ *and* $\mathrm{Ext}^1_\Gamma(C^\pm M, C^\pm N) \simeq \mathrm{Ext}^1_\Gamma(M,N)$.

Proof. Obviously, it is enough to prove the claims for one reflection $s_i^\pm$. (1) If $s_i^\pm M = 0$, then $M \simeq E_i$. But if i is positive, E_i is preprojective and if i is negative, it is preinjective,

(2) follows now from Theorem 17 and Exercise 39, which show that the functor $s_i^\pm$ maps an extension of M with the kernel N to an extension of $s_i^\pm M$ with the kernel $s_i^\pm N$. Obviously, equivalent representations are mapped to equivalent and $s_i^\mp$ gives the inverse map. □

Therefore, $\overrightarrow{\sigma}_k^+$ is an equivalence of the categories of regular representations $\mathcal{R}(\Gamma, \Bbbk)$ and $\mathcal{R}(\overrightarrow{\sigma}_k\Gamma, \Bbbk)$, while $\overleftarrow{\sigma}_k^-$ is an equivalence of the categories $\mathcal{R}(\Gamma, \Bbbk)$ and $\mathcal{R}(\overrightarrow{\sigma}_k\Gamma, \Bbbk)$. In particular, these functors map R-simple representations to R-simple and preserve regular lengths. The following theorems show that, when studying regular representations, we can consider, for each Euclidean graph, one, arbitrary chosen orientation of arrows.

Theorem 33. *Let $|\Gamma| = |\Gamma'|$ be a tree. There is a sequence of vertices $(i_1, i_2, \dots, i_r)$ such that every i_k becomes positive in Γ after reflections at $i_1, i_2, \dots, i_{k-1}$ (in particular, i_1 is positive) and $s_{i_r} \dots s_{i_2} s_{i_1}\Gamma = \Gamma'$.*

Proof. We use induction by the number of vertices. If there is only 1 (or 2) vertices, the claim is obvious. So, we suppose that the theorem is true for quivers with less vertices than Γ.

As $|\Gamma|$ is a tree, there is a vertex i such that there is only one edge a incident to i. We denote by j the second end of a. Let $\Gamma_1 = \Gamma \setminus \{i\}$ and $\Gamma_1' = \Gamma' \setminus \{i\}$. By induction, there is a sequence of positive reflections that transform Γ_1 to Γ_1'. If at some stage we have to do the reflection s_j and $a : j \to i$, we do the positive reflection s_i and then s_j. Note that after all these reflections $a : j \to i$. If $a : i \to j$ in Γ', it remains to make the reflection s_i once more.

Analogously, one can prove the following result about quivers Γ of type $\tilde{A}_n$, i.e. such that $|\Gamma|$ is a cycle. Recall that we always suppose that there are no cycles in Γ. If p arrows in Γ go clockwise and q arrows go anticlockwise, we call (p,q) the *clock type* of Γ and say that Γ is of type $\tilde{A}_{p,q}$.

Theorem 34. *Let Γ and Γ' be acyclic quivers of type $\tilde{A}_n$. There is a sequence of positive reflections transforming Γ to Γ' if and only if these quivers have the same clock type.*

We propose the reader to prove this theorem.

Certainly, the same results holds true if we replace positive reflections by negative.

In what follows we write C instead of C^+.

. A non-homogeneous R-simple representation L of dimension $\mathbf{d}$ is called *generating* if $\chi(\mathbf{d},\mathbf{x}) = -\chi(\mathbf{x}, C\mathbf{d})$ for all $\mathbf{x}$.

As $s_i^{\pm}$ preserves χ and $\boldsymbol{\omega}$ and C is invertible, we have

Corollary. *If L is generating, so is $C^k L$, and vice versa.*

Since $\chi(M,N) = \dim \operatorname{Hom}_\Gamma(M,N) - \dim \operatorname{Ext}^1_\Gamma(M,N)$, we have the following result.

Proposition. *If L is generating and M regular, then*

1. *If $\chi(L,M) > 0$, L embeds into M.*
2. *If $\chi(L,M) < 0$, there is a surjection $M \twoheadrightarrow CL$.*
3. *If M is R-simple, non-isomorphic to L and to CL, then $\chi(L,M) = 0$ and* $\operatorname{Hom}_\Gamma(L,M) = \operatorname{Ext}^1_\Gamma(L,M) = 0$.

Proof. (1) In this case $\operatorname{Hom}_\Gamma(L,M) \neq 0$ and any nonzero morphism $L \to M$ is an embedding. (2) In this case $\chi(M,CL) > 0$, hence $\operatorname{Hom}_\Gamma(M,CL) \neq 0$ and any nonzero morphism $M \to CL$ is a surjection. (3) follows from (1),(2) and Schur lemma. □

Let L be generating and $r = r(L)$ be the smallest positive integer such that $C^r L \simeq L$. Note that $r \geqslant 2$ since L is non-homogeneous. Set $L_i = C^i L$ $(0 \leqslant i < r)$, $\mathbb{F} = \operatorname{End}_\Gamma L$, $f = \dim_{\Bbbk} \mathbb{F}$ (further we will see that $\mathbb{F} = \Bbbk$, so $f = 1$). We also set $L_j = L_i$ if $j \equiv i \pmod{r}$.

Proposition. *If M is an R-simple representation, then*

1. $\operatorname{Hom}_\Gamma(L_i,M) \simeq \operatorname{Hom}_\Gamma(M,L_i) \simeq \begin{cases} \mathbb{F} & \text{if } M \simeq L_i \\ 0 & \text{otherwise .} \end{cases}$

2. $\operatorname{Ext}^1_\Gamma(L_i,M) \simeq \begin{cases} \mathbb{F} & \text{if } M \simeq L_{i+1}, \\ 0 & \text{othewise .} \end{cases}$

3. $\operatorname{Ext}^1_\Gamma(M,L_i) \simeq \begin{cases} \mathbb{F} & \text{if } M \simeq L_{i-1}, \\ 0 & \text{otherwise .} \end{cases}$

Proof. Since C is bijective on $\mathcal{R}(\Gamma, \Bbbk)$ and preserves Hom_Γ and Ext^1_Γ, we only have to consider the case $L_i = L$, so $L_{i\pm 1} = C^{\pm} L$. Note that the spaces $\mathrm{Ext}^1_\Gamma(M, L)$ and $\mathrm{Ext}^1_\Gamma(L, M)$ are $\mathbb{F}$-vector spaces, hence their dimensions are multiples of f. (1) is just the Schur lemma. (2) $\dim \mathrm{Hom}_\Gamma(L, L) - \dim \mathrm{Ext}^1_\Gamma(L, L) = \chi(L, L) = \mathsf{Q}(L) > 0$, hence $\dim \mathrm{Ext}^1_\Gamma(L, L) < \dim \mathrm{End}_\Gamma L = f$ and $\mathrm{Ext}^1_\Gamma(L, L) = 0$. On the other hand, $\dim \mathrm{Ext}^1_\Gamma(L, CL) = -\chi(L, CL) = \chi(L, L) = f$, so $\mathrm{Ext}^1_\Gamma(L, CL) \simeq \mathbb{F}$. If $M \not\simeq L$ and $M \not\simeq CL$, then $\chi(L, M) = 0$ and $\mathrm{Hom}_\Gamma(L, M) = 0$, whence $\mathrm{Ext}^1_\Gamma(L, M) = 0$. (3) $\mathrm{Ext}^1_\Gamma(C^- L, L) \simeq \mathrm{Ext}^1_\Gamma(L, CL) \simeq \mathbb{F}$. If $M \not\simeq L$ and $M \not\simeq C^- L$, then $\chi(M, L) = \chi(CM, CL) = -\chi(L, CM) = 0$ and $\mathrm{Hom}_\Gamma(M, L) = 0$, whence $\mathrm{Ext}^1_\Gamma(M, L) = 0$. □

For a generating representation L we denote by $\mathcal{F}(L)$ the subcategory of regular representations M such that every R-simple factor of M is isomorphic to some $C^k L$ $(0 \leqslant k < r(L))$. Of course, $\mathcal{F}(L) = \mathcal{F}(C^k L)$, so this category only depends on the orbit of L under C.

Corollary. *If M is an indecomposable regular representation and one of its R-simple factors is isomorphic to $C^k L$ for some k, then $M \in \mathcal{F}(L)$.*

Proof. It follows from the preceding proposition and Corollary 250. □

Now we are going to describe all non-homogeneous R-simple representations. In particular, we will see that they only form finitely many C-orbits, hence finitely many subcategories of the sort $\mathcal{F}(L)$, and $\mathrm{End}_\Gamma(L) \simeq \Bbbk$ for every such representation. For every regular dimension $\mathbf{d}$ set $\mathbf{N}(\mathbf{d}) = \{\, \mathbf{x} \mid \chi(C^i \mathbf{d}, \mathbf{x}) = 0 \text{ for all } i \,\}$.

Theorem 35. *There is a finite set $\mathfrak{G} = \{\, L^1, L^2, \dots, L^t \,\}$ of non-homogeneous R-simple generating representations such that*

1. *They belong to different orbits of C.*
2. $\bigcap_{i,k} \mathbf{N}(\mathbf{dim}\, C^k L^i) = \langle \boldsymbol{\omega} \rangle$.
3. *Every indecomposable non-homogeneous regular representation belongs to one of $\mathcal{F}(L^i)$ and $\mathcal{F}(L^i) \cap \mathcal{F}(L^j) = \{0\}$ if $i \neq j$.*

The set $\mathfrak{G}$ is called a *generating set* for the quiver Γ.

Proof. First, we prove that (1) and (2) imply (3).

Indeed, let M be an indecomposable regular representation, L be its R-simple factor. If $L \simeq C^k L^i$, then $M \in \mathcal{F}(L^i)$, otherwise, $\chi(C^k L^i, L) = 0$. As it holds for any i and k, $\mathbf{dim}\, L \in \langle \boldsymbol{\omega} \rangle$, that is L is homogeneous and $M \in \mathcal{F}(L)$ is also homogeneous.

If $i \neq j$, then $\mathcal{F}(L^i) \cap \mathcal{F}(L^j) = \{0\}$ by Corollary 292, since $C^k L^i \not\simeq C^q L^j$ for all k, q.

Assertions (1) and (2) are proved by direct construction of a generating set. Due to Theorems 33 and 34, one can do it for one, arbitrary chosen orientation of the graph. We will present generating sets for a couple of examples, asking the readers to check the properties (1) and (2) in these cases, which is an easy exercise. In all cases the numeration of vertices is positive. We hope that the readers easily check that all representations that we present are indeed indecomposable and generating.

The remaining cases are left to the interested readers. □

Let Γ be if type $\tilde{A}_3$ (more precisely, $\tilde{A}_{2,2}$): $\begin{tikzcd}[row sep=small] & 1 & \\ 3 \arrow[ur] \arrow[dr] & & 4 \arrow[ul] \arrow[dl] \\ & 2 & \end{tikzcd}$. Then $\boldsymbol{\omega} = (1,1,1,1)$

Set $L^1 = \begin{tikzcd}[row sep=small] & \Bbbk & \\ \Bbbk \arrow[ur, "1"] \arrow[dr] & & 0 \arrow[ul] \arrow[dl] \\ & 0 & \end{tikzcd}$. Then $CL^1 = \begin{tikzcd}[row sep=small] & 0 & \\ 0 \arrow[ur] \arrow[dr] & & \Bbbk \arrow[ul] \arrow[dl, "1"] \\ & \Bbbk & \end{tikzcd}$, $C^2L^1 = L^1$ Set

$L^2 = \begin{tikzcd}[row sep=small] & \Bbbk & \\ 0 \arrow[ur] \arrow[dr] & & \Bbbk \arrow[ul, "1"'] \arrow[dl] \\ & 0 & \end{tikzcd}$. Then $CL^2 = \begin{tikzcd}[row sep=small] & 0 & \\ \Bbbk \arrow[ur] \arrow[dr, "1"'] & & 0 \arrow[ul] \arrow[dl] \\ & \Bbbk & \end{tikzcd}$, $C^2L^2 = L^2$

$\chi(\mathbf{dim}\, L^1, \mathbf{x}) = x_1 + x_3 - x_1 - x_2 = x_3 - x_2$. In the same way, $\chi(\mathbf{dim}\, CL^1, \mathbf{x}) = x_4 - x_1$, $\chi(\mathbf{dim}\, L^2, \mathbf{x}) = x_4 - x_2$ and $\chi(\mathbf{dim}\, CL^2, \mathbf{x}) = x_3 - x_1$. If all these forms are 0, $x_1 = x_2 = x_3 = x_4$, that is $\mathbf{x} = x_1\boldsymbol{\omega}$.

Thus, (1) and (2) hold, so $\mathfrak{G} = \{ L^1, L^2 \}$ is a generating set.

Let Γ be of type $\tilde{D}_4$: $\begin{tikzcd}[row sep=small] 2 \arrow[dr] & & 4 \arrow[dl] \\ & 1 & \\ 3 \arrow[ur] & & 5 \arrow[ul] \end{tikzcd}$ (it is the "4 subspaces problem"). Then $\boldsymbol{\omega} = (2,1,1,1,1)$

There are three representations in a generating set $\mathfrak{G}$: $L^1 = \begin{tikzcd}[row sep=small] \Bbbk \arrow[dr, "1"] & & 0 \arrow[dl] \\ & \Bbbk & \\ \Bbbk \arrow[ur, "1"'] & & 0 \arrow[ul] \end{tikzcd}$

, $L^2 = \begin{tikzcd}[row sep=small] \Bbbk \arrow[dr, "1"] & & \Bbbk \arrow[dl, "1"'] \\ & \Bbbk & \\ 0 \arrow[ur] & & 0 \arrow[ul] \end{tikzcd}$, $L^3 = \begin{tikzcd}[row sep=small] 0 \arrow[dr] & & \Bbbk \arrow[dl, "1"'] \\ & \Bbbk & \\ \Bbbk \arrow[ur, "1"'] & & 0 \arrow[ul] \end{tikzcd}$.

Find their C-orbits and check the properties (1) and (2).

Let Γ be of type $\tilde{A}_{3,2}$:

$\begin{tikzcd}[row sep=small] & 3 \arrow[rr] & & 2 \arrow[dr] & \\ 5 \arrow[ur] \arrow[drr] & & & & 1 \\ & & 4 \arrow[urr] & & \end{tikzcd}$ Then $\boldsymbol{\omega} = (1,1,1,1,1)$.

Check that in this case the set $\mathfrak{G} = \{ E_3, E_4 \}$, consisting of two simple representations, is generating and the lengths of C-orbits of these representations are, respectively, 3 and 2.

How will it be for the quiver of type $\tilde{A}_{p,q}$? (The case $q = 1$ is special!)

Let Γ be of type $\tilde{D}_6$:
$$\begin{tikzcd}[row sep=small] 4 \arrow[dr] & & & & 6 \arrow[dl] \\ & 3 \arrow[r] & 2 \arrow[r] & 1 & \\ 5 \arrow[ur] & & & & 7 \arrow[ul] \end{tikzcd}$$
Here $\boldsymbol{\omega} = (2, 2, 2, 1, 1, 1, 1)$

Check that $\mathfrak{G}$ consists of 3 representations: $L^1 = E_3$ with the orbit of length 4,
$$L^2 = \begin{tikzcd}[row sep=small] \Bbbk \arrow[dr, "1"] & & & & \Bbbk \arrow[dl, "1"'] \\ & \Bbbk \arrow[r, "1"] & \Bbbk \arrow[r, "1"] & \Bbbk & \\ 0 \arrow[ur] & & & & 0 \arrow[ul] \end{tikzcd}, \quad L^3 = \begin{tikzcd}[row sep=small] \Bbbk \arrow[dr, "1"] & & & & 0 \arrow[dl] \\ & \Bbbk \arrow[r, "1"] & \Bbbk \arrow[r, "1"] & \Bbbk & \\ 0 \arrow[ur] & & & & \Bbbk \arrow[ul, "1"] \end{tikzcd},$$

L^2 and L^3 with the orbits of length 2.

How will it be for the quiver of type $\tilde{D}_n$?

Let Γ be of type $\tilde{E}_6$:
$$\begin{tikzcd}[row sep=small] 5 \arrow[r] & 2 \arrow[dr] & & & \\ & & 1 & 4 \arrow[l] & 7 \arrow[l] \\ 6 \arrow[r] & 3 \arrow[ur] & & & \end{tikzcd}$$
. Then $\boldsymbol{\omega} = (3, 2, 2, 2, 1, 1, 1)$.

$\mathfrak{G}$ consists of 3 representations: $L^1 = \begin{tikzcd}[row sep=small] \Bbbk \arrow[r, "1"] & \Bbbk \arrow[dr, "1"] & & & \\ & & \Bbbk & \Bbbk \arrow[l, "1"'] & 0 \arrow[l] \\ 0 \arrow[r] & 0 \arrow[ur] & & & \end{tikzcd}$, $L^2 =$

$$\begin{tikzcd}[row sep=small] 0 \arrow[r] & 0 \arrow[dr] & & & \\ & & \Bbbk & \Bbbk \arrow[l, "1"'] & 0 \arrow[l] \\ \Bbbk \arrow[r, "1"'] & \Bbbk \arrow[ur, "1"'] & & & \end{tikzcd}, \quad L^1 = \begin{tikzcd}[row sep=small] 0 \arrow[r] & \Bbbk \arrow[dr, "1"] & & & \\ & & \Bbbk & \Bbbk \arrow[l, "1"'] & 0 \arrow[l] \\ 0 \arrow[r] & \Bbbk \arrow[ur, "1"'] & & & \end{tikzcd}.$$

Find their orbits and check (1) and (2).

For $\tilde{E}_7 = \begin{tikzcd}[row sep=small] & & & 4 \arrow[d] & & & \\ 7 \arrow[r] & 5 \arrow[r] & 2 \arrow[r] & 1 & 3 \arrow[l] & 6 \arrow[l] & 8 \arrow[l] \end{tikzcd}$ $\quad \boldsymbol{\omega} = (4, 3, 3, 2, 2, 2, 1, 1).$

There are 3 orbits of length 2, 3, 4 generated by: $\begin{tikzcd}[row sep=small] & & & \Bbbk \arrow[d, "\binom{1}{1}"] & & & \\ \Bbbk \arrow[r, "\binom{1}{0}"] & \Bbbk \arrow[r, "1"] & \Bbbk^2 \arrow[r, "1"] & \Bbbk^2 & \Bbbk \arrow[l, "\binom{0}{1}"'] & \Bbbk \arrow[l, "1"'] & 0 \arrow[l] \end{tikzcd}$

$$\begin{tikzcd}[row sep=small] & & & \Bbbk \arrow[d, "1"] & & & \\ 0 \arrow[r] & 0 \arrow[r] & \Bbbk \arrow[r, "1"] & \Bbbk & \Bbbk \arrow[l, "1"'] & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & & & \Bbbk \arrow[d, "1"] & & & \\ 0 \arrow[r] & \Bbbk \arrow[r, "1"] & \Bbbk \arrow[r, "1"] & \Bbbk & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

For $\tilde{E}_8 = \begin{tikzcd}[row sep=small] & & 4 \arrow[d] & & & & & \\ 5 \arrow[r] & 2 \arrow[r] & 1 & 3 \arrow[l] & 6 \arrow[l] & 7 \arrow[l] & 8 \arrow[l] & 9 \arrow[l] \end{tikzcd}$ $\quad \boldsymbol{\omega} = (6, 4, 5, 3, 2, 4, 3, 2, 1).$

There are 3 orbits of length $2, 3, 5$ generated by:

$$\begin{tikzcd}
 & & \Bbbk^2 \arrow[d, "\left(\begin{smallmatrix}1&0\\1&1\\0&1\end{smallmatrix}\right)"] & & & & & \\
\Bbbk \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & \Bbbk^2 \arrow[r, "\left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right)"] & \Bbbk^3 & \Bbbk^2 \arrow[l, "\left(\begin{smallmatrix}0&0\\1&0\\0&1\end{smallmatrix}\right)"'] & \Bbbk^2 \arrow[l, "1"'] & \Bbbk \arrow[l, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] & \Bbbk \arrow[l, "1"'] & {} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \Bbbk \arrow[d, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"] & & & & & \\
\Bbbk \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & \Bbbk^2 \arrow[r, "1"] & \Bbbk^2 & \Bbbk \arrow[l, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] & \Bbbk \arrow[l, "1"'] & \Bbbk \arrow[l, "1"'] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \Bbbk \arrow[d, "1"] & & & & \\
0 \arrow[r] & \Bbbk \arrow[r, "1"] & \Bbbk & \Bbbk \arrow[l, "1"'] & 0 \arrow[l] & 0 \arrow[l] & {} \arrow[l]
\end{tikzcd}$$

Remark. Note that all representations L from the constructed generating sets have at least one coordinate in $\mathbf{dim}\, L$ equal 0. Therefore, they are actually representations of a Dynkin graph. Hence, $\mathsf{Q}(\mathbf{dim}\, L) = 1$ and $\operatorname{End}_\Gamma L = \Bbbk$. Therefore, the same is true for every R-simple non-homogeneous representation. The latter is not true for homogeneous R-regular representations if $\Bbbk$ is not algebraically closed. For instance, if $p(t)$ is an irreducible polynomial over $\Bbbk$, the representation $K(p)$ of the Kronecker quiver $\mathbf{K}$ is R-simple, but $\operatorname{End}_{\mathbf{K}} K(p) \simeq \Bbbk[t]/p(t)\Bbbk[t]$ (check it). From some posterior results it follows that analogous examples exist for all Euclidean quivers.

Moreover, one can check that $\sum_{i=0}^{r-1} \mathbf{dim}\, C^i L = \boldsymbol{\omega}$, where $r = r(L)$.

Finally, we give a description of $\mathcal{F}(L)$ analogous to the homogeneous case. In this theorem $r = r(L)$, $a \equiv b$ means $a \equiv b \pmod r$, $L_i = C^i L$.

Theorem 36. *1. For every l and $0 < k \leqslant r$ there is a unique (up to isomorphism) indecomposable representation $M_l^k \in \mathcal{F}(L)$ of regular length l such that L_k is its quotient. Thus $\mathcal{F}(L)$ contains exactly $r(L)$ indecomposable representations of each regular length l.* If it is necessary to precise the generating representation L, we denote this module by $M_l^k(L)$.

2. *For every $0 < m < l$ there is a unique regular submodule $M_{l,m}^k \subset M_l^k$ such that $\operatorname{rl}(M_l^k/M_{l,m}^k) = m$.*

3. *$M_{l,m}^k \simeq M_{l-m}^{k+m}$ and $M_l^k/M_{l,m}^k \simeq M_m^k$.*

4. *Every homomorphism $M_l^k \to M_s^q$ arises from an isomorphism $M_l^k/M_{l,m}^k \xrightarrow{\sim} M_{s,s-m}^q$ for some $m \leqslant \min(l,s)$ such that $m \equiv s+q-k$.*

(continued on the next frame)

. 5.

$$\operatorname{Hom}_\Gamma(M_l^k, L_i) \simeq \begin{cases} \Bbbk & \text{if } k = i, \\ 0 & \text{if } k \neq i; \end{cases}$$

$$\operatorname{Hom}_\Gamma(L_i, M_l^k) \simeq \begin{cases} \Bbbk & \text{if } k + l \equiv i + 1, \\ 0 & \text{if } k + l \not\equiv i + 1; \end{cases}$$

$$\operatorname{Ext}^1_\Gamma(M_l^k, L_i) \simeq \begin{cases} \Bbbk & \text{if } k + l \equiv i, \\ 0 & \text{if } k + l \not\equiv i; \end{cases}$$

$$\operatorname{Ext}^1_\Gamma(L_i, M_l^k) \simeq \begin{cases} \Bbbk & \text{if } k \equiv i - 1, \\ 0 & \text{if } k \not\equiv i - 1. \end{cases}$$

The proof is very much alike that of Theorem 31, though with more technical details. Again, we construct the representations M_l^k recursively, starting from $M_1^k = L_k$. So, suppose that they have been constructed for all length $\leqslant l$.

Consider a non-split extension $0 \to L_{k+l} \xrightarrow{\alpha} M \xrightarrow{\beta} M_l^k \to 0$. Note that, as $\operatorname{Ext}^1_\Gamma(M_l^k, L_{k+l}) \simeq \Bbbk$, all such extensions have isomorphic middle terms. Obviously, $\operatorname{rl}(M) = l + 1$. The LES for this extension gives exact sequences

$$0 \to \operatorname{Hom}_\Gamma(M_l^k, L_i) \xrightarrow{\cdot\beta} \operatorname{Hom}_\Gamma(M, L_i) \xrightarrow{\cdot\alpha} \operatorname{Hom}_\Gamma(L_{k+l}, L_i) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} \operatorname{Ext}^1_\Gamma(M_l^k, L_i) \xrightarrow{\overset{1}{\cdot}\beta} \operatorname{Ext}^1_\Gamma(M, L_i) \xrightarrow{\overset{1}{\cdot}\alpha} \operatorname{Ext}^1_\Gamma(L_{k+l}, L_i) \to 0.$$

If $i \neq k$, $\operatorname{Hom}_\Gamma(M_l^k, L_i) = 0$. If $i \not\equiv k + l$, $\operatorname{Hom}_\Gamma(L_{k+l}, L_i) = \operatorname{Ext}^1_\Gamma(M_l^k, L_i) = 0$. If $i \equiv k + l$, both $\operatorname{Hom}_\Gamma(L_{k+l}, L_i) = \operatorname{Ext}^1_\Gamma(M_l^k, L_i) \simeq \Bbbk$ and $\delta \neq 0$, hence δ is an isomorphism and both $\cdot\alpha$ and $\overset{1}{\cdot}\beta$ are zero, while $\cdot\beta$ and $\overset{1}{\cdot}\alpha$ are isomorphisms. So, if $i \neq k$, $\operatorname{Hom}_\Gamma(M, L_i) = 0$, and $\operatorname{Hom}_\Gamma(M, L_k) \simeq \Bbbk$. It implies that M is indecomposable, has a unique maximal regular submodule M' and $M/M' \simeq L_k$. Moreover, as $\operatorname{Ext}^1_\Gamma(L_{k+l}, L_i) = \Bbbk$ if $i \equiv k+l+1$ and 0 otherwise, the same holds true for $\operatorname{Ext}^1_\Gamma(M, L_i)$.

Consider now the exact sequence $0 \to M' \xrightarrow{\xi} M \xrightarrow{\eta} L_k \to 0$. It gives LES

$$0 \to \operatorname{Hom}_\Gamma(L_k, L_i) \xrightarrow{\cdot\eta} \operatorname{Hom}_\Gamma(M, L_i) \xrightarrow{\cdot\xi} \operatorname{Hom}_\Gamma(M', L_i) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} \operatorname{Ext}^1_\Gamma(L_k, L_i) \xrightarrow{\overset{1}{\cdot}\eta} \operatorname{Ext}^1_\Gamma(M, L_i) \xrightarrow{\overset{1}{\cdot}\xi} \operatorname{Ext}^1_\Gamma(M', L_i) \to 0$$

If $i \neq k$, the first two terms are 0. If $i = k$ they both are $\Bbbk$, so $\cdot\eta$ is an isomorphism. In both cases $\cdot\xi = 0$ and δ is injective. If $i \not\equiv k + 1$, $\operatorname{Ext}^1_\Gamma(L_k, L_i) = 0$, so $\operatorname{Hom}_\Gamma(M', L_i) = 0$. If $i \equiv k + 1$, $\operatorname{Ext}^1_\Gamma(L_k, L_i) = \Bbbk$, whence also $\operatorname{Hom}_\Gamma(M', L_i) = \Bbbk$. It implies that M' is indecomposable and has a quotient L_{k+1}. Therefore $M' \simeq M_l^{k+1}$.

As M' is a unique maximal submodule in M, all other submodules are contained in it, which gives the property (2) for M. The property (3) is deduced just as in Theorem 31, taking into account that all quotients of M have a quotient L_k, so are of the form M_m^k.

The same short exact sequence gives also LES

$$0 \to \mathrm{Hom}_\Gamma(L_i, M_l^{k+1}) \xrightarrow{\xi\cdot} \mathrm{Hom}_\Gamma(L_i, M) \xrightarrow{\eta\cdot} \mathrm{Hom}_\Gamma(L_i, L_k) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} \mathrm{Ext}^1_\Gamma(L_i, M_l^{k+1}) \xrightarrow{\xi\cdot^1} \mathrm{Ext}^1_\Gamma(L_i, M) \xrightarrow{\eta\cdot^1} \mathrm{Ext}^1_\Gamma(L_i, L_k) \to 0.$$

Just as above, it implies that $\mathrm{Hom}_\Gamma(L_i, M) = \Bbbk$ if $i \equiv k + l$ and 0 otherwise, while $\mathrm{Ext}^1_\Gamma(L_i, M) = \Bbbk$ if $k \equiv i + 1$ and 0 otherwise. So M satisfies conditions (5) for M_{l+1}^k.Therefore, we have constructed M_l^k for all k, l.

Condition (4) follows from the description of submodules and quotients of M_l^k, taking into account that $\mathrm{Im}\,\varphi \simeq M/ker\varphi$ for any morphism $\varphi : M \to N$.

The uniqueness of M_l^k can be proved in the same way as that of M_l in Theorem 31, so we only sketch it, remaining the details to the reader. □

Lemma 37. *Let $\mathcal{E}_{l,k} = \mathrm{End}_\Gamma M_l^k$, $\mathfrak{r}_{l,k} = \mathrm{rad}\,\mathcal{E}_{n,k}$ and $\mathbf{A}_{l,k} = \mathrm{Aut}_\Gamma M_l^k$.*

1. *$\mathcal{E}_{l,k}/\mathfrak{r}_{l,k} \simeq \Bbbk$.*
2. *The group $\mathbf{A}_{l,k}$ acts transitively on $\mathrm{Ext}^1_\Gamma(M_l^k, L_i)$, where $i \equiv l + k$.*
3. *If $k + l \equiv s + q \equiv i$ and $q \leqslant l$, the embedding $\mu_{lq}^{ks} : M_q^s \simeq M_{l,l-q}^k \hookrightarrow M_l^k$ induces a surjection $\mathrm{Ext}^1_\Gamma(M_q^s, L_i) \twoheadrightarrow \mathrm{Ext}^1_\Gamma(M_l^k, L_i)$*

Proof. (1) If $\varphi \in \mathrm{End}_\Gamma M_l^k$ is surjective, it is an automorphism. Otherwise, $\mathrm{Im}\,\varphi \in M'$, where M' is the unique maximal submodule in M_l^k. Thus $\mathfrak{r}_{l,k} = \mathrm{Hom}_\Gamma(M_l^k, M')$. Therefore, the quotient $\mathcal{E}_{l,k}/\mathfrak{r}_{l,k}$ acts on $M_l^k/M' \simeq L_k$, which gives a homomorphism $\mathcal{E}_{l,k}/\mathfrak{r}_{l,k} \to \mathrm{End}_\Gamma L_k$. As $\mathrm{End}_\Gamma L_k \simeq \Bbbk$, also $\mathcal{E}_{l,k}/\mathfrak{r}_{l,k} \simeq \Bbbk$.

(2) follows from (1) and the isomorphism $\mathrm{Ext}^1_\Gamma(M_l^k, L_i) \simeq \Bbbk$.

(3) It follows from the fact that in our case Ext^1_Γ is right exact. □

Now, let M be indecomposable of regular length $l + 1$ and has a submodule L_i, $M' = M/L_i$. If M' is indecomposable, it must be M_l^k, where $k + l \equiv i$. In this case $M \simeq M_{l+1}^k$. Suppose that M' decomposes: $M' = \bigoplus_{j=1}^m M_{l_j}^{k_j}$. Then M is given by an element of $\mathrm{Ext}^1_\Gamma(M', L_i)$, which can be considered as a vector $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$, where $\varepsilon_j \in \mathrm{Ext}^1_\Gamma(M_{l_j}, L_i)$. As M is indecomposable, $\varepsilon_j \neq 0$, whence $l_j + k_j \equiv i$. Let $l_1 \geqslant l_2$. Then there is $\lambda \in \Bbbk^\times$ such that $\varepsilon_2 = \varepsilon_1 \mu_{l_1 l_2}^{k_1 k_2} \lambda$. If θ is the automorphism of M' given by the matrix

$$\theta = \begin{pmatrix} 1 & -\mu_{l_1 l_2}^{k_1 k_2}\lambda & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

then $\varepsilon\theta = (\varepsilon_1, 0, \dots, \varepsilon_m)$ and M decomposes. This contradiction accomplishes the proof of the uniqueness and of the whole theorem. □

Exercise 43. Using Remark 302 and the description of $\mathcal{F}(L)$, calculate the dimensions of indecomposable representations in $\mathcal{F}(L^i)$ and deduce that:

1. $\mathsf{Q}(\mathbf{dim}\, M) = 1$ for every indecomposable representation $M \in \mathcal{F}(L)$, provided that $\mathbf{dim}\, M \notin \langle \boldsymbol{\omega} \rangle$, that is it is a real root of Q.

2. On the contrary, every regular real root is indeed a dimension of a non-homogeneous indecomposable regular representation.

3. If $M \in \mathcal{F}(L)$ is indecomposable, then $\mathbf{dim}\, M \in \langle \boldsymbol{\omega} \rangle$ if and only if $\mathrm{rl}(M) = kr$, where $r = r(L)$, and there are exactly r such representations (up to isomorphism).

4. If $M, N \in \mathcal{F}(L)$ are indecomposable and both $\mathbf{dim}\, M, \mathbf{dim}\, N \in \langle \boldsymbol{\omega} \rangle$, then $\mathrm{Hom}_\Gamma(M, N) \neq 0$.

11.3 Parametrization

Parametrization

We have proved that the category $\mathcal{R}(\Gamma, \Bbbk)$ can be considered as a product of the categories $\mathcal{F}(L)$, where L runs through representatives of the orbits of R-simple modules under the action of C. It means that every module from $\mathcal{R}(\Gamma, \Bbbk)$ is a direct sum of modules from different $\mathcal{F}(L)$ and $\mathrm{Hom}_\Gamma(M, N) = 0$ if M and N belong to different categories $\mathcal{F}(L)$.

Now we are going to show that the categories $\mathcal{F}(L)$ can be parametrized by the set

$$\mathbb{P}^1_\Bbbk = \{\, \varphi \mid \varphi(t) \text{ a unital irreducible polynomial from } \Bbbk[t]\,\} \cup \{\infty\}.$$

In the mordern algebraic geometry this set is considered as the projective line over the field $\Bbbk$.

Certainly, if $\Bbbk$ is algebraically closed, we can replace the irreducible polynomial $t - \lambda$ by λ and set $\mathbb{P}^1_\Bbbk = \Bbbk \cup \{\infty\}$, which is more usual.

. A functor T is called *fully faithful* if all induced maps $\mathrm{Hom}(M, N) \to \mathrm{Hom}(TM, TN)$ $(\alpha \mapsto T\alpha)$ is bijective.

Proposition. *Let T be fully faithful.*

1. *If $TM \simeq TN$, then $M \simeq N$.*

2. *$T(M)$ is indecomposable if and only if so is M*

Proof. (1) Let $\varphi : TM \xrightarrow{\sim} TN$, $\varphi^{-1} : TN \to TM$. Since T is fully faithful, $\varphi = T\alpha$, $\varphi^{-1} = T\beta$ for some $\alpha : M \to N$, $\beta : N \to M$. Moreover, $T(\alpha\beta) = \varphi\varphi^{-1} = 1_{TN} = T1_N$, hence $\alpha\beta = 1_N$. In the same way $\beta\alpha = 1_M$, so $\beta - \alpha^{-1}$.

(2) M is decomposable if and only if there is an idempotent $e \in \operatorname{End}_\Gamma M$ ($e^2 = e$) which is neither 1 nor 0. Namely, then $M = \operatorname{Im} e \oplus \operatorname{Ker} e$ (prove it). On the contrary, if $M = M_1 \oplus M_2$, take for e the projection onto M_1. As $\operatorname{Hom}(M, M) \to \operatorname{Hom}(TM, TM)$ is bijective, $\operatorname{End} M$ has nontrivial idempotent if and only if $\operatorname{End} TM$ does. □

Theorem 38. *For every Euclidean quiver Γ there is a generating set $\mathfrak{G}$ of non-homogeneous R-simple representations, an exact fully faithfully functor $T : \mathcal{R}(\mathbf{K}, \Bbbk) \to \mathcal{R}(\Gamma, \Bbbk)$ and a set $\mathfrak{g} \subseteq \Bbbk \cup \{\infty\}$ with a bijection $\tau : \mathfrak{g} \to \mathfrak{G}$ such that*

1. *If $\bar{L}$ is an R-simple representation of $\mathbf{K}$ and $\bar{L} \not\simeq K_1(\lambda)$ for all $\lambda \in \mathfrak{g}$, then $T\bar{L}$ is R-simple and homogeneous.*

2. *Every R-simple homogeneous representation $L \in \mathcal{H}(\Gamma, \Bbbk)$ is isomorphic to $T\bar{L}$ for some $\bar{L} \in \mathcal{R}(\mathbf{K}, \Bbbk)$.*

3. *If $\lambda \in \mathfrak{g}$, then $TK_1(\lambda) \simeq M_r^k(\tau(\lambda))$ for $r = r(\tau(L))$ and some k.*

Note that *exact* in this context means that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of regular representations, the induced sequence $0 \to M' \to M \to M'' \to 0$ is also exact. Note also that the representation $\bar{L}$ in item (2) must also be R-simple and non-isomorphic to $K_1(\lambda)$ for $\lambda \in \mathfrak{g}$.

As R-simple regular representations of the Kronecker quiver $\mathbf{K}$ are $K(\varphi)$, where φ runs through the unital irreducible polynomials from $\Bbbk[t]$ and $K_1(\infty)$, we obtain a parametrization of the components $\mathcal{F}(L)$ of the category $\mathcal{R}(\Gamma, \Bbbk)$ of regular representations of every Euclidean quiver Γ by the "projective line" $\mathbb{P}^1_\Bbbk$. That is why we call a functor T with these properties a $\mathbb{P}^1$-*parametrization* of $\mathcal{R}(\Gamma, \Bbbk)$. The components corresponding to the "points" φ from $\mathbb{P}^1_\Bbbk \setminus \mathfrak{g}$ consist of homogeneous representations. Namely, for every $l \in \mathbb{N}$ such component has a unique indecomposable representation of dimension $ld\boldsymbol{\omega}$, where $d = \deg \varphi$, which is just $TK(\varphi^l)$. The component corresponding to $\lambda \in \mathfrak{g}$ consists of non-homogeneous representations. If $r = \operatorname{rl}(L)$, where $L = \tau(\lambda)$, it has r representations in each dimension $l\boldsymbol{\omega} + \mathbf{d}$, where either $\mathbf{d} = 0$ or $\mathbf{d} = \sum_{i=k}^{k+j} \operatorname{\mathbf{dim}} C^i L$, where $0 \leqslant j < r$ and $k \leqslant r$ (explain this claim).

Again, we construct the functor T individually for each Euclidean diagram. We will check the properties (1-3) in a couple of cases and propose the reader to do it in some other cases.

For a representation K of the Kronecker quiver $\mathbf{K} = \; 1 \underset{b}{\overset{a}{\rightrightarrows}} 2$ we denote $A = K(a)$, $B = K(b)$ and write (A, B) instead of K.

In what follows we denote by I the identity matrices of the appropriate dimensions.

$$\Gamma = \begin{tikzcd}[row sep=small] 2 \arrow[dr] & & 4 \arrow[dl] \\ & 1 & \\ 3 \arrow[ur] & & 5 \arrow[ul] \end{tikzcd}.$$ For a representation $K(1) \overset{A}{\underset{B}{\rightrightarrows}} K(2)$ of the Kronecker quiver we set $$T(A,B) = \begin{tikzcd}[row sep=small] K(2) \arrow[dr, "\left(\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right)"] & & K(2) \arrow[dl, "\left(\begin{smallmatrix} I \\ I \end{smallmatrix}\right)"'] \\ & K(2)\oplus K(2) & \\ K(2) \arrow[ur, "\left(\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right)"'] & & K(1) \arrow[ul, "\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)"] \end{tikzcd}$$ If a set of matrices $(S_1, S_2, S_3, S_4, S_5)$ gives a morphism $\varphi : T(A,B) \to T(A',B')$, then, for every matrices X, X' corresponding to the arrow $i \to 1$, it must be $S_1 X = X' S_i$. If we present S as the block matrix $S = \left(\begin{smallmatrix} C_1 & C_2 \\ C_3 & C_4 \end{smallmatrix}\right)$, these equations give for $i = 2$: $C_1 = S_2$, $C_3 = 0$; for $i = 3$: $C_4 = S_3$, $C_2 = 0$; for $i = 4$: $C_1 = C_4 = S_4$; finally, for $i = 5$: $C_1 A = A' S_5$, $C_1 B = B' S_5$, that is $\psi = (C_1, S_5)$ is a morphism $(A,B) \to (A',B')$ and φ is completely defined by ψ, so we can set $\varphi = T\psi$ obtaining a fully faithful functor $\mathrm{rep}(\mathbf{K}) \to \mathrm{rep}(\Gamma)$.

Note that if the representation $K = (A,B)$ has a nontrivial subrepresentation K', choosing the bases in the corresponding subspaces of $K(1)$ and $K(2)$, we see that A and B can be presented in the form $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, $B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}$. Then $K' = (A_1, B_1)$ and $K/K' = (A_3, B_3)$ (explain it). It immediately implies that the functor T maps the submodule $K' \subset K$ to a submodule $T(K') \subset T(K)$ and $T(K/K') \simeq T(K)/T(K')$ (expalin it!). It means that T is exact. These considerations remain valid in all examples considered below and we will not repeat them.

Note that $\mathbf{dim}\, T(K) \in \langle \boldsymbol{\omega} \rangle$ if and only if $\mathbf{dim}\, K = (n,n)$. If, moreover, K is indecomposable, $T(K)$ is also indecomposable, hence regular. Set $\mathfrak{g} = \{0, 1, \infty\}$. If $K = K_1(\lambda)$, then $\mathbf{dim}\, K = (1,1)$, $(A,B) \in \{\, (1,0), (1,1), (0,1) \,\}$ and $\mathbf{dim}\, TK = (2,1,1,1,1)$. If $(A,B) = (1,0)$, then $$T(A,B) = \begin{tikzcd}[row sep=small] \Bbbk \arrow[dr, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & & \Bbbk \arrow[dl, "\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)"'] \\ & \Bbbk^2 & \\ \Bbbk \arrow[ur, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"'] & & \Bbbk \arrow[ul, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \end{tikzcd}$$ contains a subrepresentation $$\begin{tikzcd}[row sep=small] \Bbbk \arrow[dr, "1"] & & 0 \arrow[dl] \\ & \Bbbk & \\ 0 \arrow[ur] & & \Bbbk \arrow[ul, "1"'] \end{tikzcd}$$,which is CL^3 from the list on the page 296. In the same way, if $(A,B) = (0,1)$, $T(A,B)$ contains CL^2, and if $(A,B) = (1,1)$, it contains CL^1 (verify it). Hence, this choice satisfies the condition (3) of the theorem if we set $\lambda(0) = L^3$, $\lambda(1) = L^1$, $\lambda(\infty) = L^2$.

We have now to prove that all indecomposable homogeneous representations arise as $T(I, F(\varphi))$ for non-exceptional φ.

Consider the subquiver $\Gamma' = \Gamma \setminus \{5\}$. It is the Dynkin quiver of type D_4. Its representations correspond to the real roots, which are $\mathbf{e}_i$ ($1 \leqslant i \leqslant 4$), $\mathbf{e}_1 + \mathbf{e}_i$ ($2 \leqslant i \leqslant 4$), $\mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_j$ ($2 \leqslant i < j \leqslant 4$), $(1,1,1,1)$ and $(2,1,1,1)$. The corresponding representations are E_i, N_i, N_{ij}, $N =$
$$\begin{tikzcd} \Bbbk \arrow[dr, "1"] & & \Bbbk \arrow[dl, "1"'] \\ & \Bbbk & \\ \Bbbk \arrow[ur, "1"'] & & \end{tikzcd}$$
and
$$N^* = \begin{tikzcd} \Bbbk \arrow[dr, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & & \Bbbk \arrow[dl, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"'] \\ & \Bbbk^2 & \\ \Bbbk \arrow[ur, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] & & \end{tikzcd}$$
N_i (N_{ij}) have $\Bbbk$ at the places $1, i$ (respectively, $1, i, j$) and 0 elsewhere. Actually, N_{23}, N_{24} and N_{34} are just the representations L^1, L^2 and L^3 from page 296. Note that $N_i \subset N_{ij} \subset N$.

Let M' be a restriction onto Γ' of a homogeneous indecomposable representation M of the quiver Γ. Obviously, every subrepresentation of M' can be considered as a subrepresentation of M. In particular, N_{ij} gives a subrepresentation isomorphic to one of L^k from page 296. Therefore, since M is homogeneous, neither N_{ij} nor N can be submodules of M'. If E_i is a direct summand of M', it is also a direct summand of M. Hence, $M' \simeq xN^* \oplus y_2N_2 \oplus y_3N_3 \oplus y_4N_4 \oplus zE_1$, whence $\mathbf{dim}\, M = (2x + y_2 + y_3 + y_4 + z, x + y_2, x + y_3, x + y_4)$. But, as M is homogeneous, $\mathbf{dim}\, M' = (2n, n, n, n)$ for some n. It immediately implies that $y_2 = y_3 = y_4 = z = 0$ (check it).

Therefore, $M' \simeq nN^*$ and $M \simeq$
$$\begin{tikzcd} \Bbbk^n \arrow[dr, "\left(\begin{smallmatrix}I\\0\end{smallmatrix}\right)"] & & \Bbbk^n \arrow[dl, "\left(\begin{smallmatrix}I\\I\end{smallmatrix}\right)"'] \\ & \Bbbk^{2n} & \\ \Bbbk^n \arrow[ur, "\left(\begin{smallmatrix}0\\I\end{smallmatrix}\right)"'] & & \Bbbk^n \arrow[ul, "\left(\begin{smallmatrix}A\\B\end{smallmatrix}\right)"] \end{tikzcd}$$
for some $n \times n$ matrices A, B, that is $M \simeq T(A, B)$ for a regular indecomposable (A, B).

Note that, as we have excluded $\lambda \in \{0, 1, \infty\}$, it must be some of $K(\varphi) = (I, F(\varphi))$, where $\varphi \neq t^n$ and $\varphi \neq (t-1)^n$. Such representation is homogeneous if if contains no submodules from the orbits of L^k ($1 \leqslant k \leqslant 3$), which are just the submodules of the sort N_{ij} ($2 \leqslant i < j \leqslant 5$). We check that $T(I, F(\varphi))$ does not contain $N_{45} = CL^1$ and propose the reader to check the other cases.

So, let $\alpha : N_{45} \hookrightarrow T(I, M(\varphi))$, where $N_{45} =$
$$\begin{tikzcd} 0 \arrow[dr] & & \Bbbk \arrow[dl, "1"'] \\ & \Bbbk & \\ 0 \arrow[ur] & & \Bbbk \arrow[ul, "1"] \end{tikzcd}$$
and $T(I, F(\varphi)) =$
$$\begin{tikzcd} \Bbbk^n \arrow[dr, "\left(\begin{smallmatrix}I\\0\end{smallmatrix}\right)"] & & \Bbbk^n \arrow[dl, "\left(\begin{smallmatrix}I\\I\end{smallmatrix}\right)"'] \\ & \Bbbk^{2n} & \\ \Bbbk^n \arrow[ur, "\left(\begin{smallmatrix}0\\I\end{smallmatrix}\right)"'] & & \Bbbk^n \arrow[ul, "\left(\begin{smallmatrix}I\\F\end{smallmatrix}\right)"] \end{tikzcd}$$
, where $F = F(\varphi)$. Let $\alpha(1) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\alpha(4) = y$, $\alpha(5) = z$, where x_1, x_2, y, z are $n \times 1$ matrices. Then $\begin{pmatrix} I \\ I \end{pmatrix} y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot 1$, i.e. $x_1 = x_2 = y$. Also

$\begin{pmatrix} I \\ F \end{pmatrix} z = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot 1$, i.e. $z = x_1 = y$ and $Fz = x_2 = y$, whence $(F - I)z = 0$. It is impossible, since $\varphi \neq (t-1)^n$, hence $\varphi(1) \neq 0$ and F does not have eigenvalue 1.

Therefore, T is indeed a $\mathbb{P}^1$-parametrization of $\mathcal{R}(\Gamma, \Bbbk)$.

For other quivers of type $\tilde{D}_n$ the construction is similar. For instance, for the $\tilde{D}_6$

$$\Gamma = \begin{tikzcd}[row sep=small] 4 \arrow[dr] & & & & 6 \arrow[dl] \\ & 3 \arrow[r] & 2 \arrow[r] & 1 & \\ 5 \arrow[ur] & & & & 7 \arrow[ul] \end{tikzcd}$$

, we set

$$T(A,B) = \begin{tikzcd}[row sep=small] M(2) \arrow[dr, "\binom{I}{0}"] & & & & M(2) \arrow[dl, "\binom{I}{I}"'] \\ & M(2)^2 \arrow[r, "1"] & M(2)^2 \arrow[r, "1"] & M(2)^2 & \\ M(2) \arrow[ur, "\binom{0}{I}"'] & & & & M(1) \arrow[ul, "\binom{A}{B}"] \end{tikzcd}$$

The proof that it is indeed a $\mathbb{P}^1$-parametrization with $\mathfrak{g} = \{0, 1, \infty\}$ is almost the same as for $\tilde{D}_4$. The additional problem is to prove that the maps $3 \to 2$ and $2 \to 1$ in a homogeneous regular representations must be isomorphisms. But if they have kernels, M has a subrepresentation E_3 or E_2, which are R-simple non-homogeneous (they are L^1 and CL^1 from page 298). So they are monomorphisms. As these spaces are of the same dimension, they are indeed isomorphisms.

Quite analogous is the calculation for any quiver of type $\tilde{D}_m$. We remain the details for the readers.

Let $\Gamma = \begin{tikzcd}[row sep=small] & 3 \arrow[r] & 2 \arrow[dr] & \\ 5 \arrow[ur] \arrow[dr] & & & 1 \\ & 4 \arrow[urr] & & \end{tikzcd}$ (of type $\tilde{A}_{3,2}$). Set $\mathfrak{g} = \{0, \infty\}$

and $T(A,B) = \begin{tikzcd}[row sep=small] & M(1) \arrow[r, "1"] & M(1) \arrow[dr, "A"] & \\ M(1) \arrow[ur, "1"] \arrow[dr, "1"'] & & & M(2) \\ & M(1) \arrow[urr, "B"'] & & \end{tikzcd}$ One easily verifies that T is an exact and fully faithful functor (check it!).

$TK_1(0) = T(1,0)$ has an R-simple non-homogeneous submodule E_4 and $TK_1(\infty) = T(0,1)$ has an R-simple non-homogeneous submodule E_2.

Therefore, it remains to prove that every indecomposable homogeneous representation M is isomorphic to $TK(\varphi)$, where $\varphi \notin \{\, t^k, (t-1)^k \mid k \in \mathbb{N} \,\}$.

If the map $M(3) \to M(2)$ has a kernel, M has an R-simple non-homogeneous submodule E_3. If the map $M(2) \to M(1)$ has a kernel, M has an R-simple non-homogeneous submodule E_2. If the map $M(4) \to M(1)$ has a kernel, M has an R-simple non-homogeneous submodule E_4. So we suupose now that these maps are monomorphisms, hence isomorphisms, since all spaces $M(i)$ are of the same dimension. If the map $M(5) \to M(3)$ has a kernel, M has an R-simple non-homogeneous submodule $CE_2 = \begin{tikzcd}[row sep=small] & 0 \arrow[r] & 0 \arrow[dr] & \\ \Bbbk \arrow[ur] \arrow[dr, "1"'] & & & \Bbbk \\ & \Bbbk \arrow[urr, "1"'] & & \end{tikzcd}$. Just take a vector $v \in M(5)$ which goes to 0 under this map and all vectors obtained from v by the maps $M(5) \to M(4) \to M(1)$. If the map $M(5) \to M(4)$ has a kernel, M has an R-simple non-homogeneous submodule CE_4 (check it!).

Therefore, all maps in the representation M are isomorphisms. Then we can suppose that all of them except $M(4) \to M(1)$ are identities and the latter is conjugate to an indecomposable Frobenius matrix $F(\varphi)$, so $M \simeq TK(\varphi)$, where $\varphi \neq t^k$ (explain it).

On the contrary, the representation $TK(\varphi)$, $\varphi \neq t^k$, contains no non-homogeneous R-simple representations, which are E_2, E_3, E_4, CE_2 and CE_4 (why?). Thus T is indeed a $\mathbb{P}^1$-parametrization of $\mathcal{R}(\Gamma, \Bbbk)$.

Similarly, one constructs a $\mathbb{P}^1$-paramerization of $\mathcal{R}(\Gamma, \Bbbk)$ for every quiver of type $\tilde{A}_{p,q}$. Again the case $q = 1$ is special. Namely, in this case there is only one exceptional value: $\mathfrak{g} = \{\infty\}$ (verify it).

In the remaining cases we only present a $\mathbb{P}^1$-parametrization, remaining the proofs to an interested reader (see also Tables in [DR], page 38).

For the type $\tilde{E}_6$, when $\Gamma =$
$$\begin{tikzcd}[row sep=small] 5 \arrow[r] & 2 \arrow[dr] & & & \\ & & 1 & 4 \arrow[l] & 7 \arrow[l] \\ 6 \arrow[r] & 3 \arrow[ur] & & & \end{tikzcd},$$

a $\mathbb{P}^1$-parametrization is

$$\begin{tikzcd}[row sep=small] K(2) \arrow[r, "\binom{1}{0}"] & K(2) \oplus K(1) \arrow[dr, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] & & & \\ & & K(2) \oplus K(1)^2 & K(1)^2 \arrow[l, "\left(\begin{smallmatrix} B & A \\ 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)"'] & K(1) \arrow[l, "\binom{1}{0}"'] \\ K(1) \arrow[r, "\binom{0}{1}"'] & K(1)^2 \arrow[ur, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & & & \end{tikzcd}$$

For the type $\tilde{E}_7$, when $\Gamma =$
$$\begin{tikzcd} & & & 4 \arrow[d] & & & \\ 7 \arrow[r] & 5 \arrow[r] & 2 \arrow[r] & 1 & 3 \arrow[l] & 6 \arrow[l] & 8 \arrow[l] \end{tikzcd},$$

a $\mathbb{P}^1$-parametrization is

$$\begin{tikzcd} & & & U^2 \arrow[d, "\left(\begin{smallmatrix} A & B \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)"] & & & \\ V \arrow[r, "\binom{1}{0}"'] & V \oplus U \arrow[r, "\binom{I}{0}"'] & V \oplus U^2 \arrow[r, "\binom{I}{0}"'] & V \oplus U^3 & U^3 \arrow[l, "\binom{0}{I}"] & U^2 \arrow[l, "\binom{0}{I}"] & U \arrow[l, "\binom{0}{1}"] \end{tikzcd},$$

where $U = K(1)$, $V = K(2)$.

For the type $\tilde{E}_8$, when $\Gamma =$
$$\begin{tikzcd} & & & & & 4 \arrow[d] & & \\ 9 \arrow[r] & 8 \arrow[r] & 7 \arrow[r] & 5 \arrow[r] & 2 \arrow[r] & 1 & 3 \arrow[l] & 6 \arrow[l] \end{tikzcd},$$

a $\mathbb{P}^1$-parametrization is

$$
\begin{tikzcd}
 & & & & & U^3 \arrow[d, "\begin{pmatrix} A & B & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}"] & & \\
V \arrow[r, "\binom{1}{0}"] & V \oplus U \arrow[r, "\binom{I}{0}"] & V \oplus U^2 \arrow[r, "\binom{I}{0}"] & V \oplus U^3 \arrow[r, "\binom{I}{0}"] & V \oplus U^4 \arrow[r, "\binom{I}{0}"] & V \oplus U^5 & U^4 \arrow[l, "\binom{0}{I}"'] & U^2 \arrow[l, "\binom{0}{I}"']
\end{tikzcd}
,
$$

where $U = K(1)$, $V = K(2)$.

12 Bibliography

‘

References

[Bass] H. Bass. Algebraic K-Theory. W.A. Benjamin, Inc., 1968. Русский перевод: Х. Басс. Алгебраическая K-теория. Мир, 1973.

[DR] V. Dlab, C.M. Ringel. Indecomposable Representations of Graphs and Algebras. Memoirs Amer. Math. Soc. 173, 1976.

[DK] Yu.A. Drozd, V.V. Kirichenko. Finite Dimensional Algebras. Springer-Verlag, 1994. https://www.imath.kiev.ua/~drozd/drozd-kirichenko_en.pdf

[Ringel] C.M. Ringel. Tame Algebras and Integral Quadratic Forms. Lecture Notes in Math. 1099, Springer-Verlag, 1984.

[Stekolshchik] R. Stekolshchik. Notes on Coxeter Transformations and the McKay Correspondence. Springer, 2008.

[Weibel] Ch.A. Weibel. An Introduction to Homological Algebra. Cambridge University Press, 1994.

References

[BGP] И.Н. Бернштейн, И.М. Гельфанд, В.А. Пономарев. Функторы Кокстера и теорема Габриеля. Успехи мат. наук, 28, №2 (1973) 19–33. http://www.mathnet.ru/links/8bc0e41d4986540971ce4edf6026c7ef/rm4860.pdf

[Gantmacher] Ф.Р. Гантмахер. Теория матриц. Наука, 1988 (та інші видання).

[AG] Ю.А. Дрозд. Вступ до алгебричної геометрії. ВНТЛ–Класика, 2004 https://www.imath.kiev.ua/~drozd/agukr.pdf

[Shafarevich] И.Р. Шафаревич. Основы алгебраической геометрии. МЦНМО, 2007 (та інші видання).