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Research article

On Hom-Spaces of Tame Algebras*

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Abstract: Let Λ be a finite dimensional algebra over an algebraically closed field k and Λ has tame representation type. In this paper, the structure of Hom-spaces of all pairs of indecomposable Λ-modules having dimension smaller than or equal to a fixed natural number is described, and their dimensions are calculated in terms of a finite number of finitely generated Λ-modules and generic Λ-modules. In particular, such spaces are essentially controlled by those of the corresponding generic modules.

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1 Introduction

Let Λ be a finite-dimensional k-algebra of tame representation type, k an algebraically closed field. We recall that Λ is of tame representation type if for all natural numbers d, there is a finite number of Λ-$k[x]$-bimodules $M_1, ..., M_n$ which are free of finite rank as right $k[x]$-modules and such that if M is an indecomposable Λ-module of k-dimension equal to d, then $M \cong M_i \otimes_{k[x]} k[x]/(x-\lambda)$ for some $1 \leq i \leq n$ and $\lambda \in k$.

It is known from [6] that for each dimension d, almost all Λ-modules of dimension at most d are controlled by finitely many isomorphism classes of generic modules in the sense of (i) of Theorem 1.2. A question arises naturally: are Hom-spaces of Λ-modules also controlled by those of generic modules? In this paper, we will give a positive answer.

If G is a left Λ-module then G can be regarded as a left $\mathrm{End}_\Lambda(G)$-module, and we call its length as $\mathrm{End}_\Lambda(G)$-module, the endolength of G. We say that G is a generic module if it is indecomposable, of infinite dimension over k but finite endolength. We recall that if G is a generic Λ-module and R a commutative principal ideal domain which is finitely generated over k, then a realization of G over R is a finitely generated Λ-R-bimodule T such that if K is the quotient field of R, then $G \cong T \otimes_R K$ and $\dim_K(T \otimes_R K)$ is equal to the endolength of G.

As an example consider, Λ=kQ, the Kronecker algebra defined by quiver Q, then G is a generic module, and T is a realization of G over $R = k[x]$.

$$Q =: \cdot \underset{b}{\overset{a}{\rightrightarrows}} \cdot, \qquad G =: k(x) \underset{id}{\overset{x}{\rightrightarrows}} k(x), \qquad T =: k[x] \underset{id}{\overset{x}{\rightrightarrows}} k[x].$$

We denote by Λ-Mod the category of left Λ-modules, by Λ-mod the full subcategory of Λ-Mod consisting of the finite-dimensional Λ-modules, and by Λ-ind the full subcategory of Λ-mod consisting of the indecomposable Λ-modules.

We recall from Theorem 5.4 of [6] that if Λ is of tame representation type then given any generic Λ-module there is a *good realization* of G over some R in the sense of the following definition:

Definition 1.1. Let T be a realization of a generic module G over some R, then T is called a good realization if:

(i) T is free as right R-module;

(ii) the functor $T \otimes_R - : R\text{-Mod} \to \Lambda\text{-Mod}$ preserves isomorphism classes and indecomposability;

(iii) if $p \in R$ is a prime, $n \geq 1$ and $S_{p,n}$ denotes the exact sequence

$$0 \to R/(p^n) \overset{(p,\pi)}{\to} R/(p^{n+1}) \oplus R/(p^{n-1}) \overset{\begin{pmatrix} \pi \\ -p \end{pmatrix}}{\to} R/(p^n) \to 0$$

where π is the canonical projection, then $T \otimes_R S_{p,n}$ is an almost split sequence in Λ-mod.

We know from Theorem 4.6 of [6] that if G is a generic Λ-module then there is a splitting $\mathrm{End}_\Lambda(G) = k(x) \oplus \mathrm{radEnd}_\Lambda(G)$. This splitting induces a structure of left $\Lambda^{k(x)} = \Lambda \otimes_k k(x)$-module for G and such structure is called an *admissible structure*. The main aim of this paper is to prove of the following theorem:

Theorem 1.2. *Let Λ be a finite-dimensional k-algebra of tame representation type, k an algebraically closed field. Let d be an integer greater than the dimension of Λ over k. Then there are generic Λ-modules $G_1, ..., G_s$ with admissible structures of left $\Lambda^{k(x)}$-modules and good realizations T_i over some R_i, finitely generated localization of $k[x]$, of each G_i and indecomposable Λ-modules $L_1, ..., L_t$ with $\dim_k L_j \leq d$ for $j = 1, ..., t$ with the following properties:*

(i) If M is an indecomposable left Λ-module with $\dim_k M \leq d$, then either $M \cong L_j$ for some $j \in \{1, ..., t\}$ or $M \cong T_i \otimes_{R_i} R_i/(p^m)$ for some $i \in \{1, ..., s\}$ some prime element $p \in R_i$ and some natural number m. If M is an indecomposable which is simple, projective or injective left Λ-module, then $M \cong L_j$ for some $j \in \{1, ..., t\}$.

(ii) If $M = T_i \otimes_{R_i} R_i/(p^m), N = T_j \otimes_{R_j} R_j/(q^n), L_u^{k(x)} = L_u \otimes_k k(x)$ with $i, j \in \{1, ..., s\}, u \in \{1, ..., t\}$, p a prime in R_i, q a prime in R_j, then

$$\dim_k \mathrm{rad}^\infty_\Lambda(M, N) = mn \dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(G_i, G_j),$$

$$\dim_k \mathrm{rad}^\infty_\Lambda(L_u, M) = m \dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(L_u^{k(x)}, G_i),$$

$$\dim_k \mathrm{rad}^\infty_\Lambda(M, L_u) = m \dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(G_i, L_u^{k(x)}).$$

(iii) Suppose $M = T_i \otimes_{R_i} R_i/(p^m), N = T_j \otimes_{R_j} R_j/(q^n)$, then if $i = j$, $p = q$,

$$\mathrm{Hom}_\Lambda(M, N) \cong \mathrm{Hom}_{R_i}(R_i/(p^m), R_i/(p^n)) \oplus \mathrm{rad}^\infty_\Lambda(M, N).$$

And if $i \neq j$ or $(p) \neq (q)$:

$$\mathrm{Hom}_\Lambda(M, N) = \mathrm{rad}^\infty_\Lambda(M, N).$$

Moreover, $\mathrm{Hom}_\Lambda(L_u, M) = \mathrm{rad}^\infty_\Lambda(L_u, M)$, $\mathrm{Hom}_\Lambda(M, L_u) = \mathrm{rad}^\infty_\Lambda(M, L_u)$.

For the proof of our main result we first study layered bocses of tame representation type (see Theorem 9.2). For this we use the method of reduction functors $F : \mathcal{B}_1$-Mod $\to \mathcal{B}_2$-Mod between the representation categories of two layered bocses $\mathcal{B}_1$ and $\mathcal{B}_2$ (see [5], [7] and section 7 of this paper). We prove that given a layered bocs $\mathcal{A}$ of tame representation type and a dimension vector $\mathbf{d}$ of $\mathcal{A}$ there is a composition of reduction functors $F : \mathcal{B}$-Mod $\to \mathcal{A}$-Mod with $\mathcal{B}$ a minimal bocs such that if $M \in \mathcal{A}$-Mod with $\mathbf{dim} M \leq \mathbf{d}$, then there is a $N \in \mathcal{B}$-Mod with $F(N) \cong M$. Observe that in Theorem A of [5] several minimal bocses are needed. In section 6 we study the Hom-spaces for minimal bocses. Consider now the category $P^1(\Lambda)$ of morphisms $f : P \to Q$ with P,Q projective Λ-modules and $f(P) \subset \mathrm{rad} Q$. There is a layered bocs $\mathcal{D}(\Lambda)$, the Drozd's bocs, such that $\mathcal{D}(\Lambda)$-Mod is equivalent to $P^1(\Lambda)$. Using our results on Hom-spaces for minimal layered bocses we study the Hom-spaces in $P^1(\Lambda)$ obtaining a version of Theorem 1.2 for $P^1(\Lambda)$ (see Theorem 9.5). Finally, we use the relations between Hom-spaces in $P^1(\Lambda)$ and Λ-Mod collected in the results of sections 2 and 3.

2 Generalities

Here we state the general results needed in our work. We recall that an additive k-category $\mathcal{R}$ is a Krull-Schmidt category if each object is a finite direct sum of indecomposable objects with local endomorphism rings. In this case, the indecomposable objects coincide with those having local endomorphism rings.

Let $\mathcal{R}$ be a Krull-Schmidt category. A morphism $f : E \to M$ in $\mathcal{R}$ is called irreducible if it is neither a retraction nor a section and for any factorization $f = vu$, either u is a section or v is a retraction.

A morphism $f : E \to M$ in $\mathcal{R}$ is called right almost split if

(i)f is not a retraction ;

(ii) if $g : X \to M$ is not a retraction, there is a $s : X \to E$ with $fs = g$.

Moreover, $f : E \to M$ a right almost split morphism is said to be minimal if $fu = f$ with $u \in \mathrm{End}_{\mathcal{R}}(E)$ implies u is an isomorphism.

One has the dual concepts for left almost split morphisms and minimal left almost split morphisms.

Remark. Any minimal right almost split morphism $f : E \to M$ is an irreducible morphism. Moreover if $X \neq 0$, $g : X \to M$ is an irreducible morphism iff there is a section $\sigma : X \to E$ with $f\sigma = g$.

In particular if $h : F \to M$ is also a minimal right almost split morphism there is an isomorphism $u : F \to E$ with $fu = h$.

Similar properties hold for minimal left almost split morphisms.

Definition 2.1. A pair of composable morphisms in $\mathcal{R}$,

$$M \xrightarrow{f} E \xrightarrow{g} N$$

is said to be almost split if

(i) g is a minimal right almost split morphism;

(ii) f is a minimal left almost split morphism, and;

(iii) $gf = 0$

In the following, we use the following notation. If $f : E \to M$ and $f' : E' \to M'$ are morphisms in $\mathcal{R}$, a morphism from f to f' is a pair (u, v) where $u : E \to E'$ and $v : M \to M'$ are morphisms such that $f'u = vf$. If u, v are isomorphisms, we say that f and g are isomorphic. Similarly if $M \xrightarrow{f} E \xrightarrow{g} N$, $M' \xrightarrow{f'} E' \xrightarrow{g'} N'$ are pairs of composable morphisms, a morphism from (f, g) into (f', g') is a triple (u_1, u_2, u_3) where $u_1 : M \to M'$,

$u_2 : E \to E'$, $u_3 : N \to N'$ are morphisms such that $u_2 f = f' u_1, u_3 g = g' u_2$. If u_1, u_2, u_3 are isomorphisms we say that the pair (f, g) is isomorphic to the pair (f', g'). The pairs (f, g) and (f', g') are equivalent if $M = M'$, $N = N'$ and there is an isomorphism from the first pair into the second one of the form $(1_M, u, 1_N)$.

If $\mathcal{A}$ is an additive category with split idempotents a pair (i, d) of composable morphisms $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{A}$ is said to be exact if i is a kernel of d, and d is a cokernel of i. Let $\mathcal{E}$ be a class of exact pairs closed under isomorphisms. The morphisms i and d appearing in a pair of $\mathcal{E}$ are called an inflation and a deflation of $\mathcal{E}$, respectively.

We recall from [9] that the class $\mathcal{E}$ is an exact structure for $\mathcal{E}$ if the following axioms are satisfied:

E.1 The composition of two deflations is a deflation.

E.2 If $f : Z' \to Z$ is a morphism in $\mathcal{A}$ for each deflation $d : Y \to Z$ there is a morphism $f' : Y' \to Y$ and a deflation $d' : Y' \to Z'$ such that $df' = fd'$.

E.3 Identities are deflations. If de is deflation, then so is d.

E.3^{op} Identities are inflations. If ji is a inflation, then so is i.

If $\mathcal{E}$ is an exact structure for $\mathcal{A}$ then we denote by $\mathrm{Ext}_{\mathcal{A}}(X, Y)$ the equivalence class of the pairs $Y \xrightarrow{i} E \xrightarrow{d} X$ in $\mathcal{E}$. If $\mathcal{A}$ is a k-category, $\mathrm{Ext}_{\mathcal{A}}(?, -)$ is a bifunctor from $\mathcal{A}$ into the category of k-vector spaces, contravariant in the first variable and covariant in the second variable.

An object $X \in \mathcal{A}$ is called $\mathcal{E}$-projective if $\mathrm{Ext}_{\mathcal{A}}(X, -) = 0$, and it is called $\mathcal{E}$-injective if $\mathrm{Ext}_{\mathcal{A}}(-, X) = 0$.

Definition 2.2. An almost split pair $X \to Y \to Z$ in $\mathcal{A}$ which is in $\mathcal{E}$ is called an almost split $\mathcal{E}$-sequence.

As in the case of modules, one can prove that in the above definition, X and Z are indecomposables.

Now, consider $(\mathcal{A}, \mathcal{E})$ an exact category with $\mathcal{A}$ a Krull-Schmidt k-category such that for $X, Y \in \mathcal{A}, \dim_k \mathrm{Hom}_{\mathcal{A}}(X, Y)$ is finite. Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$ having the following property:

(A) If X is an indecomposable object in $\mathcal{C}$ there is a minimal left almost split morphism in $\mathcal{A}$, $f : X \to Y_1 \oplus ... \oplus Y_t$ with $Y_i \in \mathcal{C}$.

We recall that a morphism $f : M \to N$ with M, N indecomposable objects in $\mathcal{A}$ is called a radical morphism if f is not an isomorphism.

Proposition 2.3. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$ with condition (A).*

Suppose $h : M \to N$ is a morphism in $\mathcal{A}$ with M, N indecomposable objects in $\mathcal{C}$ such that $h = \sum h_i$, where each h_i is a composition of m radical morphisms between indecomposables in $\mathcal{A}$, then $h = \sum g_j$ with each g_j composition of m radical morphisms between indecomposables in $\mathcal{C}$.

Proof. By induction on m. If $m = 1$ our assertion is trivial. Assume our assertion is true for $m-1$. We may assume $h = s_m \cdots s_1$ with $s_i : M_i \to M_{i+1}$, M_j indecomposable object of $\mathcal{A}$ for $j = 1, ..., m+1$, $M_1 = M, M_{m+1} = N$. By (A), there is a left almost split morphism $M = M_1 \xrightarrow{u} Y_1 \oplus ... \oplus Y_t$ with $Y_1, ..., Y_t \in \mathcal{C}$. We have $u = \begin{pmatrix} u_1 \\ \vdots \\ u_t \end{pmatrix}$. Then there is $v = (v_1, ..., v_t) : Y_1 \oplus ... \oplus Y_t \to M_2$ with $vu = s_1 = \sum_{i=1}^t v_i u_i$. Therefore,

$$h = s_m \cdots s_2 s_1 = \sum_{i=1}^{t} s_m \cdots s_2 v_i u_i.$$

Now, consider $g_i = s_m \cdots s_2 v_i : Y_i \to N$ which is a composition of $m-1$ radical morphisms. Then, by induction hypothesis, each g_i is a sum of $m-1$ radical morphisms between indecomposables in $\mathcal{C}$. Consequently, h is a sum of compositions of m radical morphisms between objects in $\mathcal{C}$. This proves our claim. □

We recall that an ideal of a k-category $\mathcal{R}$ is a subfunctor of $\mathrm{Hom}_{\mathcal{R}}(-, ?)$. If I, J are ideals of $\mathcal{R}$, IJ is the ideal such that for $X, Y \in \mathcal{R}$, $IJ(X, Y)$ consists of sums of compositions gf with $f \in J(X, Z), g \in I(Z, Y)$ for some $Z \in \mathcal{R}$. We denote by I^2 the ideal II and, by induction, $I^n = I^{n-1}I$. For $\mathcal{R}$ a Krull-Schmidt k-category we define the ideal $\mathrm{rad}_{\mathcal{R}}$ such that for X and Y indecomposable objects of $\mathcal{R}$, $\mathrm{rad}_{\mathcal{R}}(X, Y) =$ the morphisms which are not isomorphisms. The infinity radical is defined by

$$\mathrm{rad}_{\mathcal{R}}^{\infty} = \bigcap_n \mathrm{rad}_{\mathcal{R}}^n.$$

Corollary 2.4. *With the hypothesis of proposition 2.3, for $X, Y \in \mathcal{C}$,*

$$\mathrm{rad}_{\mathcal{C}}^{\infty}(X, Y) = \mathrm{rad}_{\mathcal{A}}^{\infty}(X, Y).$$

Proof. We may assume X and Y are indecomposables. It follows from Proposition 2.3 that $\mathrm{rad}_{\mathcal{C}}^m(X, Y) = \mathrm{rad}_{\mathcal{C}}^m(X, Y)$ for all m. Hence,

$$\mathrm{rad}_{\mathcal{C}}^{\infty}(X, Y) = \bigcap_m \mathrm{rad}_{\mathcal{C}}^m(X, Y) = \bigcap_m \mathrm{rad}_{\mathcal{A}}^m(X, Y) = \mathrm{rad}_{\mathcal{A}}^{\infty}(X, Y).$$

□

Now, we recall the following definition of [5], section 2:

Definition 2.5. If $(\mathcal{A}, \mathcal{E})$ is an exact category with $\mathcal{A}$ a Krull-Schmidt category, we say that it has almost split sequences if

i) for any indecomposable Z in $\mathcal{A}$ there is a right almost split morphism $Y \to Z$ and a left almost split morphism $Z \to X$;

ii) for each indecomposable Z in $\mathcal{A}$ which is not $\mathcal{E}$-projective, there is an almost split $\mathcal{E}$-sequence ending in Z, and for each indecomposable Z in $\mathcal{A}$ which is not $\mathcal{E}$-injective, there is an almost split $\mathcal{E}$-sequence starting in Z.

Remark. If the exact category $(\mathcal{A}, \mathcal{E})$ has almost split sequences one can consider the valued Auslander-Reiten quiver of $\mathcal{A}$ as in the case of the category of finitely generated modules over an artin algebra.

Proposition 2.6. *Suppose $(\mathcal{A}, \mathcal{E}_{\mathcal{A}})$ and $(\mathcal{B}, \mathcal{E}_{\mathcal{B}})$ are two exact categories such that the first category has almost split sequences and $F : \mathcal{B} \to \mathcal{A}$ is a full and faithful functor sending $\mathcal{E}_{\mathcal{B}}$-sequences into $\mathcal{E}_{\mathcal{A}}$-sequences. Let $\{E_i\}_{i\in\mathbb{N}}$ be a set of pairwise non-isomorphic objects in $\mathcal{B}$ which are not $\mathcal{E}_{\mathcal{B}}$-projectives, and almost split $\mathcal{E}_{\mathcal{B}}$-sequences:*

$$(e_1) : \ E_1 \xrightarrow{f_1} E_2 \xrightarrow{g_1} E_1$$

$$(e_i) : \ E_i \xrightarrow{\begin{pmatrix} f_i \\ g_{i-1} \end{pmatrix}} E_{i+1} \oplus E_i \xrightarrow{(g_i, f_{i-1})} E_i,$$

for $i > 1$. Then, if there is an almost split $\mathcal{E}_{\mathcal{A}}$- sequence ending in $F(E_1)$ which is the image under F of a sequence in $\mathcal{E}_{\mathcal{B}}$, then the image $F(e_i)$ of the sequence e_i is an $\mathcal{E}_{\mathcal{A}}$-almost split sequence for all $i \in \mathbb{N}$.

Proof. There is a sequence in $\mathcal{E}_{\mathcal{B}}$, $(a) : M \xrightarrow{u} E \xrightarrow{v} E_1$ whose image under F is an almost split $\mathcal{E}_{\mathcal{A}}$-sequence. Since F is a full and faithful functor, then (a) is an almost split sequence. This implies that (a) is isomorphic to (e_1). Therefore, the image under F of (e_1) is isomorphic to the image under F of (a) which is an almost split sequence, and so, the image of (e_1) under F is an almost split sequence.

Suppose that $F(e_l)$ is an almost split sequence for all $l \leq i$. By hypothesis, (e_{i+1}) is a non-trivial $\mathcal{E}_{\mathcal{B}}$-sequence, since F is a full and faithful functor. Then $F(e_{i+1})$ is a non-trivial $\mathcal{E}_{\mathcal{A}}$-sequence. Thus, $F(E_{i+1})$ is not $\mathcal{E}_{\mathcal{A}}$-projective. Then there is an almost split sequence

$$L_{i+1} \to M_{i+1} \to F(E_{i+1}).$$

Here $F(e_i)$ is an almost split sequence. Then we have an almost split sequence:

$$F(E_i) \to F(E_{i+1}) \oplus F(E_{i-1}) \to F(E_i),$$

and so, we have an irreducible morphism $F(E_i) \to F(E_{i+1})$. Therefore, $M_{i+1} \cong F(E_i) \oplus Y$. Thus, we have an irreducible morphism $L_{i+1} \to F(E_i)$. This implies that $L_{i+1} \cong F(E_{i+1})$ or $L_{i+1} \cong F(E_{i-1})$. But we have an almost split sequence starting and ending in $F(E_{i-1})$. Therefore, if $L_{i+1} \cong F(E_{i-1})$, then $F(E_{i+1}) \cong F(E_{i-1})$ implies $E_{i+1} \cong E_{i-1}$, which is not the case, therefore $L_{i+1} \cong F(E_{i+1})$. Then the socle of $\mathrm{Ext}_{\mathcal{A}}(F(E_{i+1}), F(E_{i+1}))$ as $\mathrm{End}_{\mathcal{A}}(F(E_{i+1}))$-module is simple. As previously stated, $F(e_{i+1})$ is a non-zero element of the above socle, and; therefore, $F(e_{i+1})$ is an almost split sequence. □

3 The categories $P(\Lambda)$ and $P^1(\Lambda)$

Let Λ be a finite-dimensional algebra over an arbitrary field k. We denote by Λ-Proj the full subcategory of Λ-Mod whose objects are projective Λ-modules, and by Λ-proj, the full subcategory of Λ-mod whose objects are projective Λ-modules.

Here Λ-proj has only a finite number of isoclasses of indecomposable objects, then for any indecomposable projective Λ-module P there are morphisms

$$\rho(P) : r(P) \to P, \quad \lambda(P) : P \to l(P)$$

such that they are a minimal right almost split in Λ-proj and a minimal left almost split in Λ-proj, respectively. Observe that $\rho(P)$ and $\lambda(P)$ are also a minimal right almost split and a minimal left almost split morphism, respectively, in the category Λ-Proj.

Denote by $P(\Lambda)$ the category whose objects are morphisms $X = f_X : P_X \to Q_X$, with $P_X, Q_X \in \Lambda$-Proj. The morphisms from X to Y, objects of $P(\Lambda)$, are pairs $u = (u_1, u_2)$ with $u_1 : P_X \to P_Y$, $u_2 : Q_X \to Q_Y$ such that $u_2 f_X = f_Y u_1$. If $u = (u_1, u_2) : X \to Y$ and $v = (v_1, v_2) : Y \to Z$ are morphisms, its composition is defined by $vu = (v_1 u_1, v_2 u_2)$.

We denote by $\mathcal{E}$ the class of pairs of composable morphisms $X \xrightarrow{u} Y \xrightarrow{v} Z$ such that the sequences of Λ-modules:

$$0 \to P_X \xrightarrow{u_1} P_Y \xrightarrow{v_1} P_Z \to 0$$

$$0 \to Q_X \xrightarrow{u_2} Q_Y \xrightarrow{v_2} Q_Z \to 0$$

are exact and then split exact.

Proposition 3.1. *The pair $(P(\Lambda), \mathcal{E})$ is an exact category.*

Proof. See [1]. □

For P any projective Λ-module consider $J(P) = (P \xrightarrow{id_P} P)$, $Z(P) = (P \xrightarrow{0} 0)$, $T(P) = (0 \xrightarrow{0} P)$. It is easy to see that the objects $J(P)$ and $T(P)$ are $\mathcal{E}$-projectives and the objects $J(P), Z(P)$ are $\mathcal{E}$-injectives. One can see without difficulty that the exact category $(P(\Lambda), \mathcal{E})$ has enough projectives and enough injectives.

Proposition 3.2. *The indecomposable $\mathcal{E}$-projectives in $P(\Lambda)$ are the objects $J(P)$ and $T(P)$ for P indecomposable projective Λ-module.*

The indecomposable $\mathcal{E}$-injectives in $P(\Lambda)$, are the objects $J(P)$ and $Z(P)$ for P indecomposable projective Λ-module.

We denote by $\overline{P(\Lambda)}$ the category having the same objects as $P(\Lambda)$ and morphisms those of $P(\Lambda)$ modulo the morphisms which factorizes through $\mathcal{E}$-injective objects.

We have a full and dense functor $Cok : P(\Lambda) \to \Lambda\text{-Mod}$ which in objects is given by $Cok(f_X : P_X \to Q_X) = \mathrm{Coker} f_X$.

Proposition 3.3. *The functor $Cok : P(\Lambda) \to \Lambda\text{-Mod}$ induces an equivalence $\overline{Cok} : \overline{P(\Lambda)} \to \Lambda\text{-Mod}$.*

Proof. One can prove (see [1]) that if $f : X \to Y$ is a morphism in $P(\Lambda)$ then $Cok(f) = 0$ iff f factorizes through some $\mathcal{E}$-injective object in $P(\Lambda)$. □

We consider now $p(\Lambda)$, the full subcategory of $P(\Lambda)$ whose objects are morphisms between finitely generated Λ-modules.

Proposition 3.4. *The exact category $(p(\Lambda), \mathcal{E})$ has almost split $\mathcal{E}$- sequences.*

Proof. See [1]. □

Now consider $P^1(\Lambda)$ the full subcategory of $P(\Lambda)$ whose objects are those $X = f_X : P_X \to Q_X$ with $\mathrm{Im}(f_X) \subset \mathrm{rad}(Q_X)$. We denote by $\mathcal{E}_1$ the class of composable morphisms in $P^1(\Lambda)$ which are in $\mathcal{E}$. By $p^1(\Lambda)$ we denote the full subcategory of $P^1(\Lambda)$, whose objects are morphisms between finitely generated projective Λ-modules.

Proposition 3.5. *The pair $(P^1(\Lambda), \mathcal{E}_1)$ is an exact category.*

Proof. See [1]. □

For an indecomposable projective Λ-module P denote by $R(P)$ the object $\rho(P) : r(P) \to P$ and by $L(P)$ the object $\lambda(P) : P \to l(P)$. Observe that P a left Λ-module is in Λ-proj if P is indecomposable and projective.

Lemma 3.6. *The morphism*

$$\sigma(P) = (\rho(P), id_P) : R(P) \to J(P)$$

is a minimal right almost split morphism in $P(\Lambda)$, the morphism

$$\tau(P) = (id_P, \lambda(P)) : J(P) \to L(P)$$

is a minimal left almost split morphism in $P(\Lambda)$.

Proposition 3.7. *Suppose $u : X \to Y$ is a morphism in $P^1(\Lambda)$ such that $Cok(u) = 0$, then $u = gh$ with $h : X \to W$, $g : W \to Y$ and W a sum of objects of the form $Z(P)$ and $R(Q)$.*

Proof. It follows from Proposition 3.3 and Lemma 3.6. □

Proposition 3.8. *The indecomposable $\mathcal{E}_1$-projectives in $P^1(\Lambda)$ are the objects $T(P)$ and $L(P)$ with P indecomposable projective Λ-module. The indecomposable $\mathcal{E}_1$-injectives are the objects $Z(P)$ and $R(P)$ with P an indecomposable projective Λ-module.*

Proof. It follows from Proposition 3.2 and Lemma 3.6. □

Proposition 3.9. *For $X, Y \in P^1(\Lambda)$, there is an exact sequence*

$$0 \to \mathrm{Hom}_{P^1(\Lambda)}(X, Y) \xrightarrow{i} \mathrm{Hom}_\Lambda(P_X, P_Y) \oplus \mathrm{Hom}_\Lambda(Q_X, Q_Y)$$

$$\xrightarrow{\delta} \mathrm{rad}_\Lambda(P_X, Q_Y) \xrightarrow{\eta} \mathrm{Ext}_{P^1(\Lambda)}(X, Y) \to 0$$

Proof. See Proposition 5.1 of [1]. □

Now, if $X = (P_X \xrightarrow{f_X} Q_X) \in P(\Lambda)$ choose some minimal projective cover $P_2 \xrightarrow{g} P_1 \xrightarrow{\eta} \mathrm{Ker}h \to 0$ with $h = D(\Lambda) \otimes f_X : D(\Lambda) \otimes_\Lambda P_X \to D(\Lambda) \otimes_\Lambda Q_X$. We put $\tau X = (P_2 \xrightarrow{g} P_1)$.

Proposition 3.10. *If X is an indecomposable which is not $\mathcal{E}_1$-projective in $\mathrm{p}^1(\Lambda)$, then there is an almost split $\mathcal{E}_1$-sequence:*

$$(1) \quad Y \to E \to X$$

with $Y \cong \tau X$. Dually if Y is indecomposable non $\mathcal{E}_1$-injective, then there is an almost split $\mathcal{E}_1$-sequence (1).

Proof. See [10] for k a perfect field and [1] for the general case. □

Proposition 3.11. *For $X, Y \in p^1(\Lambda)$, there is an isomorphism of k-modules*

$$\mathrm{Ext}_{P^1(\Lambda)}(X, Y) \cong D\overline{\mathrm{Hom}}_{P^1(\Lambda)}(Y, \tau(X)).$$

Here $\overline{\mathrm{Hom}}_{p^1(\Lambda)}(Z, W)$ stands for the morphisms from Z to W modulo those morphisms which are factorized through $\mathcal{E}_1$-injectives objects.

Proof. It follows from Corollary 9.4 of [9]. □

As a consequence we obtain:

Proposition 3.12. *(See [3] and [1]) For $X, Y \in \mathrm{p}^1(\Lambda)$, there is an isomorphism of k-modules:*

$$\mathrm{Ext}_{P^1(\Lambda)}(X, Y) \cong D(\mathrm{Hom}_\Lambda(Cok(Y), DtrCok(X))/\mathcal{S}(Cok(Y), Dtr(Cok(X)))$$

where $\mathcal{S}(M, N)$ are the morphisms which factorizes through semisimple Λ-modules.

Proposition 3.13. *If $Y \xrightarrow{v} E \xrightarrow{u} X$ is an almost split sequence in $p(\Lambda)$ with $Cok(Y) \neq 0$ and $Cok(X) \neq 0$, then*

$$0 \to Cok(Y) \xrightarrow{Cok(v)} Cok(E) \xrightarrow{Cok(u)} Cok(X) \to 0$$

is an almost split sequence in Λ-mod. Moreover, if $Cok(Y)$ is not a simple Λ-module, then the sequence $Y \xrightarrow{v} E \xrightarrow{u} X$ lies in $\mathrm{p}^1(\Lambda)$.

Proof. For the first part of our statement see Proposition 5.6 of [1], for the second part see Theorem 2.6 of [10] and Proposition 5.7 of [1]. □

Suppose now that Λ is a basic finite-dimensional k-algebra, and $1_\Lambda = \sum_{i=1}^n e_i$ is a decomposition into pairwise orthogonal primitive idempotents. Moreover, assume that $\dim_k(\Lambda/\mathrm{rad}\Lambda)e_i = 1$ for all $i = 1, ..., n$. For $M \in \Lambda$-mod we put

$$\mathbf{dim}M = (\dim_k e_1 M, ..., \dim_k e_n M).$$

For $X = f_X : P_X \to Q_X$ an object in $p^1(\Lambda)$ we put

$$\mathbf{dim}X = (\mathbf{dim}(P_X/\mathrm{rad}P_X), \mathbf{dim}(Q_X/\mathrm{rad}Q_X)) \in \mathbb{Z}^{2n}.$$

In the following, we consider three bilinear forms defined on $\mathbb{Z}^{2n}$:
For $\mathbf{x} = (x_1, ..., x_n; x'_1, ..., x'_n), \mathbf{y} = (y_1, ..., y_n; y'_1, ..., y'_n)$, we put

$$h_\Lambda(\mathbf{x}, \mathbf{y}) = \sum_{i,j}(x_i y_j + x'_i y'_j)\dim_k(e_i \Lambda e_j) - \sum_{i,j} x_i y'_j \dim_k(e_i \mathrm{rad}\Lambda e_j),$$

$$s_\Lambda(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y'_i, \quad g_\Lambda(x, y) = \sum_{i,j}(x_i y_j + x'_i y'_j - x_i y'_j)(\dim_k e_i \Lambda e_j).$$

Clearly $g_\Lambda(\mathbf{x}, \mathbf{y}) = h_\Lambda(\mathbf{x}, \mathbf{y}) - s_\Lambda(\mathbf{x}, \mathbf{y})$.

Proposition 3.14. *For $X, Y \in p^1(\Lambda)$ we have:*

$$(1) \dim_k \mathrm{Hom}_{p^1(\Lambda)}(X, Y) - \dim_k \mathrm{Ext}_{p^1(\Lambda)}(X, Y) = h_\Lambda(\boldsymbol{dim}X, \boldsymbol{dim}Y);$$

$$(2) \dim_k \mathrm{Ext}_{p^1(\Lambda)}(X, Y) = \dim_k \mathrm{Hom}_\Lambda(Cok(Y), DtrCok(X)) - s_\Lambda(\boldsymbol{dim}X, \boldsymbol{dim}Y);$$

$$(3) \dim_k \mathrm{Hom}_\Lambda(Cok(Y), DtrCok(X)) = \dim_k \mathrm{Hom}_{p^1(\Lambda)}(X, Y) - g_\Lambda(\boldsymbol{dim}X, \boldsymbol{dim}Y).$$

Proof. The part (1) follows from Proposition 3.9, part (2) follows from Proposition 3.12 and from the equalities:

$$\dim_k \mathcal{S}(Cok(Y), DtrCok(X)) = \dim_k \mathrm{Hom}_\Lambda(\mathrm{top}Cok(Y), \mathrm{soc}DtrCok(X))$$

$$= s_\Lambda(\mathbf{dim}X, \mathbf{dim}Y).$$

Finally, (3) follows from (1) and (2). □

4 Bocses

We recall that a coalgebra over a k-category A is an A-bimodule V endowed with two bimodule homomorphisms, a comultiplication $\mu : V \to V \otimes_A V$ and a counit $\epsilon : V \to A$, subject to the conditions

$$(\mu \otimes 1)\mu = (1 \otimes \mu)\mu$$

$$(\epsilon \otimes 1)\mu = i_l, \quad (1 \otimes \epsilon)\mu = i_r$$

with $i_l : V \cong A \otimes_A V$ and $i_r : V \cong V \otimes_A A$ the natural isomorphisms. Observe that A is a coalgebra over A with comultiplication $A \cong A \otimes_A A$ the natural isomorphism and the counit the identity morphism $id_A : A \to A$.

A bocs is a pair $\mathcal{A} = (A, V)$ with A a skeletally small k-category and V a coalgebra over A.

The bocs (A, A) is called the principal bocs.

The category $\mathcal{A}$-Mod has the same objects as A-Mod, the covariant functors $A \to k$-Mod. Then, if M, N are in $\mathcal{A}$-Mod, a morphism in $\mathcal{A}$-Mod is given by an A-module morphism from $V \otimes_A M$ to N. The composition of $f : V \otimes_A M \to N$ and $g : V \otimes_A N \to L$ is given by the composition

$$V \otimes_A M \overset{\mu\otimes 1}{\to} V \otimes_A V \otimes_A M \overset{1\otimes f}{\to} V \otimes_A N \overset{g}{\to} L,$$

the identity morphism for M in $\mathcal{A}$-Mod is given by the composition:

$$V \otimes_A M \overset{\epsilon\otimes 1}{\to} A \otimes_A M \overset{\sigma}{\to} M,$$

where σ is given by $\sigma(a \otimes m) = am$ for $a \in A$, $m \in M$. We identify A-Mod with (A, A)-Mod.

Suppose now $\mathcal{A} = (A, V)$ and $\mathcal{B} = (B, W)$ are two bocses, denote by $\epsilon_V, \mu_V, \epsilon_W$, μ_W the corresponding counits and comultiplications. A morphism of bocses $\theta : \mathcal{A} \to \mathcal{B}$ is a pair (θ_0, θ_1) where $\theta_0 : A \to B$ is a functor and $\theta_1 : V \to {}_{\theta_0}W_{\theta_0}$ is a morphism of A-A bimodules such that

$$\epsilon_W \theta_1 = \theta_0 \epsilon_V, \text{and} \quad \pi(\theta_1 \otimes \theta_1)\mu_V = \mu_W \theta_1,$$

where π is the natural map $W \otimes_A W \to W \otimes_B W$. A morphism of bocses $\theta : \mathcal{A} \to \mathcal{B}$ induces a functor $\theta^* : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$. For $M \in \mathcal{B}$-Mod we put $\theta^* M = {}_{\theta_0}M$ and if $f : W \otimes_B M \to N$ is a morphism in $\mathcal{B}$-Mod then $\theta^*(f)$ is the composition:

$$V \otimes_A ({}_{\theta_0}M) \overset{\theta_1\otimes 1}{\to} W \otimes_A ({}_{\theta_0}M) \overset{\pi}{\to} W \otimes_B M \overset{f}{\to} N$$

where π is the natural morphism.

Observe that if

$$\mathcal{A} \overset{(\theta_0,\theta_1)}{\to} \mathcal{B} \overset{(\phi_0,\phi_1)}{\to} \mathcal{C}$$

are morphisms of bocses then $(\phi_0\theta_0, \phi_1\theta_1) = \phi\theta : \mathcal{A} \to \mathcal{C}$ is a morphism of bocses. Clearly $(\phi\theta)^* = (\theta)^*(\phi)^*$.

Lemma 4.1. *If $\theta = (\theta_0, \theta_1) : \mathcal{A} = (A, V) \to \mathcal{B} = (B, W)$ is a morphism of bocses then*

$$(\theta)^*(1, \epsilon_W)^* = (1, \epsilon_V)^*(\theta_0, \theta_0)^*.$$

Proof. It follows from the definition of morphism of bocses and the above. □

Let $\mathcal{A} = (A, V)$ be a bocs and A' a subcategory of A with the same objects as A. A morphism $\omega : A' \to {}_{A'}V_{A'}$ of A'-A' bimodules is said to be a grouplike of $\mathcal{A}$ relative to

A' if $(i,\omega):(A',A')\to\mathcal{A}$ is a morphism of bocses, where $i:A'\to A$ is the inclusion. If the induced functor $(i,\omega)^*:\mathcal{A}\text{-Mod}\to A'\text{-Mod}$ reflects isomorphisms we say that ω is a reflector. If $\omega:{}_{A'}A'_{A'}\to -_{A'}V_{A'}$ is a grouplike we have that ω is completely determined by the elements $\omega_X=\omega(id_X)$ for all $X\in \text{ind}A'$ such that $\mu(\omega_X)=\omega_X\otimes\omega_X$.

If $\mathcal{A}=(A,V)$ is a bocs $\overline{V}=\text{Ker}\epsilon$ is called the kernel of $\mathcal{A}$. Then there is the following exact sequence of A-A bimodules:

$$0\to\overline{V}\xrightarrow{\sigma}V\xrightarrow{\epsilon}A\to 0$$

where σ is the inclusion.

We recall that if $\omega:A'\to {}_{A'}V_{A'}$ is a grouplike, it determines two morphisms $\delta_1:{}_{A'}A_{A'}\to {}_{A'}\overline{V}_{A'}$ and $\delta_2:{}_{A'}\overline{V}_{A'}\to {}_{A'}\overline{V}\otimes_A\overline{V}_{A'}$, given for $a\in\text{Hom}_A(X,Y)$ and $v\in V(X,Y)$ by :

$$\delta_1(a)=a\omega_X-\omega_Y a,\quad \delta_2(v)=\mu(v)-\omega_Y\otimes v-v\otimes\omega_X.$$

Observe that $(id_A,\epsilon):\mathcal{A}\to(A,A)$ is a morphism of bocses. Therefore, it induces a functor $(id_A,\epsilon)^*:A\text{-Mod}\to\mathcal{A}\text{-Mod}$. For $M\in A\text{-Mod}$, $(id_A,\epsilon)^*(M)=M$, and for $h:M\to N$ a morphism of A-modules $(id_A,\epsilon)^*h:V\otimes{}_AM\to N$ is given by $(id_A,\epsilon)^*(h)(v\otimes m)=h(\epsilon(v)m)$ for $m\in M, v\in V$.

For $M\in\mathcal{A}\text{-Mod}$, $(i,\omega)^*(M)={}_{A'}M$ and if $f:V\otimes_A M\to N$ is a morphism in $\mathcal{A}$-Mod, $f^0=(i,\omega)^*f:{}_{A'}M\to{}_{A'}N$ is given by $f^0(m)=f(\omega_X\otimes m)$ for $m\in M(X)$.

Given $\mathcal{A}=(A,V)$ a bocs with a grouplike ω relative to some A' subcategory of A, for any morphism, $f:V\otimes_A M\to N$ we have the morphisms $f^0=(i,\omega)^*f\in\text{Hom}_{A'}(M,N)$, $f^1=f(\sigma\otimes 1):\overline{V}\otimes_A M\to N$. The pair of morphisms (f^0,f^1) satisfies the following property:

$$(A)\quad f^0(am)=af^0(m)+f^1(\delta_1(a)\otimes m).$$

Now, for any object $Y\in A$ we have :

$$(V\otimes_A M)(Y)=V(-,Y)\otimes_A M=\omega_Y\otimes M(Y)\oplus(\overline{V}\otimes_A M)(Y),$$

therefore, a pair of morphisms (f^0,f^1) with

$$f^0\in\text{Hom}_{A'}(M,N)\quad\text{and}\quad f^1\in\text{Hom}_A(\overline{V}\otimes_A M,N)$$

which satisfies the condition (A) determines a morphism of A-modules $f:V\otimes_A M\to N$. Thus, any morphism $f:V\otimes_A M\to N$ is completely determined by the pair (f^0,f^1) satisfying property (A). In the rest of the paper, we put $f=(f^0,f^1)$.

Proposition 4.2. *If $f=(f^0,f^1):M\to N$, $g=(g^0,g^1):N\to L$ are morphisms in $\mathcal{A}$-Mod then $gf=(g^0f^0,(gf)^1)$ with*

$$(gf)^1(v\otimes m)=g^1(v\otimes f^0(m))+g^0(f^1(v\otimes m))+\sum_i g^1(v_i^1\otimes f^1(v_i^2\otimes m)),$$

where $v\in V, m\in M$ and $\delta_2(v)=\sum_i v_i^1\otimes v_i^2$.

Proof. It follows from the fact that $(i,\omega)^*$ is a functor and from the definitions. □

Following [5], if A is a k-category a morphism $a \in A(X,Y)$ is called indecomposable if both X and Y are indecomposable objects of A. Similarly, if W is an A-A bimodule an element of W is an element $w \in W(X,Y)$ for some X, Y. In case both X and Y are indecomposable, w will be called indecomposable. If X and Y are objects of A , then we denote by $F_{X,Y}$ the A-A bimodule given by

$$F_{X,Y} = \mathrm{Hom}_A(-,X) \otimes_k \mathrm{Hom}_A(Y,-).$$

We say that the A-A bimodule W is freely generated by the elements $w_i \in W(X_i, Y_i), i = 1, ..., n$ if there is an isomorphism of A-A bimodules

$$\psi : F_{X_1,Y_1} \oplus ... \oplus F_{X_n,Y_n} \to W$$

such that $\psi(id_{X_i} \otimes id_{Y_i}) = w_i$, for $i = 1, ..., n$.

Now, suppose that A' has the same objects as A, and T is an A'-A' -subimodule of ${}_{A'}A_{A'}$, denote by $T^{\otimes n}$ the tensor product $T \otimes_{A'} T \otimes_{A'} ... \otimes_{A'} T$ of n copies of T and set $T^0 = A'$. Then the direct sum of A'-A'-bimodules:

$$T^{\otimes} = \bigoplus_{n=0}^{\infty} T^{\otimes n}$$

can be regarded as a category with the same objects as A and product given by the natural isomorphisms $T^{\otimes n} \otimes_A T^{\otimes m} \to T^{\otimes m+n}$.

We recall from Definition 2.5 of [5] that if A' has the same objects as A, we say that A is freely generated over A by morphisms $a_1, ..., a_n$ in A if the a_i freely generate an A'-A' subimodule T of ${}_{A'}A_{A'}$ such that the functor $T^{\otimes} \to A$ induced by the inclusion of A' and T in A is an isomorphism.

Definition 4.3. A k-category A is called minimal if it is skeletal and is equivalent to

$$\mathrm{mod}(k) \times ... \times \mathrm{mod}(k) \times P(R_1) \times ... \times P(R_n)$$

where $R_i = k[x, f_i(x)^{-1}]$ with $f_i(x)$ is a nonzero element of $k[x]$ and $P(R)$ denotes the category of finitely generated projective left R-modules. We denote by ind$A$ the set of indecomposable objects of a minimal category A.

Definition 4.4. Let $\mathcal{A} = (A, V)$ be a bocs with kernel $\overline{V}$. A collection $L = (A'; \omega; a_1, ..., a_n; v_1, ...v_m)$, is a layer for $\mathcal{A}$, if

(L1) A' is a minimal category;

(L2) A is freely generated over A' by indecomposable elements $a_1, ..., a_n$;

(L3) ω is a reflector for $\mathcal{A}$ relative to A';

(L4) $\overline{V}$ is freely generated as an A-A bimodule by indecomposable elements $v_1, ...v_m$;

(L5) let $\delta_1 : A \to \overline{V}$ be the morphism induced by ω, $A_0 = A'$ and for $i \in \{1, ..., n-1\}$, A_i the subcategory of A generated by A' and $a_1, ...a_i$, then for any $0 \leq i < n$, $\delta_1(a_{i+1})$ is contained in the A_i-A_i subimodule of $\overline{V}$ generated by $v_1, ...v_m$.

A bocs having a layer will be called layered.

Suppose $\mathcal{A} = (A, V)$ is a bocs with layer $L = (A'; \omega; a_1, ..., a_n; v_1, ...v_m)$. Throughout this paper, we denote by $\mathcal{A}$-mod the full subcategory of $\mathcal{A}$-Mod whose objects are representations M such that $\sum_{X \in \mathrm{ind} A'} \dim_k M(X) < \infty$.

For $\mathcal{A}$ as before we have

$$\overline{V} \otimes_A M \cong \bigoplus_{v_i} A(-, Y_i) \otimes_k M(X_i)$$

for $M \in A$-Mod. Thus, for $M, N \in A$-Mod we have an isomorphism:

$$\phi_{M,N} : \bigoplus_{v_i} \mathrm{Hom}_k(M(X_i), N(Y_i)) \to \mathrm{Hom}_A(\overline{V} \otimes_A M, N).$$

Therefore, in this case a morphism $f : M \to N$ in $\mathcal{A}$-Mod is given by a pair of morphisms

$$(f^0, \phi_{M,N}(f_1^1, ..., f_m^1)), f^0 \in \mathrm{Hom}_{A'}(M, N), f_i^1 \in \mathrm{Hom}_k(M(X_i), N(Y_i)),$$

$i = 1, ..., m$ such that for all $a_j : X_j \to Y_j, j = 1, ..., n$ and $u \in M(X_j)$

$$f_{Y_j}^0(a_j u) = a_j f_{X_j}^0(u) + \phi_{M,N}(f_1^1, ..., f_m^1)(\delta_1(a_j) \otimes u).$$

Observe that $\phi_{M,N}(f_1^1, ..., f_m^1)(v_i \otimes u) = f_i^1(u)$ for $u \in M(X_i), i = 1, ..., m$.

Lemma 4.5. *With the above notations, if $(f, 0) : M \to N$ and $(h^0, \phi_{N,L}(h_1, ..., h_m)) : N \to L$ are morphisms in $\mathcal{A}$-*Mod *then:*

$$(h^0, \phi_{N,L}(h_1, ..., h_m))(f, 0) = (h^0 f, \phi_{M,L}(g_1, ..., g_m)) \quad \text{with} \quad g_i = h_i f_{X_i}.$$

*Similarly, if $(h^0, \phi_{M,N}(h_1, ..., h_m)) : M \to N$, $(f, 0) : N \to L$ are morphisms in $\mathcal{A}$-*Mod*, then:*

$$(f, 0)(h^0, \phi_{M,N}(h_1, ..., h_m)) = (f h^0, \phi_{M,N}(g_1, ..., g_m)), \quad \text{with} \quad g_i = f_{Y_i} h_i.$$

In later sections we need the following.

Definition 4.6. Let $\mathcal{A} = (A, V)$ be a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$. Then a sequence of morphisms in $\mathcal{A}$-Mod,

$$M \xrightarrow{f} E \xrightarrow{g} N$$

is called proper exact if $gf = 0$ and the sequence of morphisms

$$0 \to M \overset{(i,\omega)^* f}{\to} E \overset{(i,\omega)^* g}{\to} N \to 0$$

in A'-Mod is exact. An almost split sequence in $\mathcal{A}$-mod which is also a proper exact sequence is called a proper almost split sequence.

Definition 4.7. With the notation of Definition 4.6 an indecomposable object $X \in A'$ is called marked if $A'(X, X) \neq k id_X$.

5 Hom-spaces of Minimal Bocses

We recall from [5] that a minimal bocs is a bocs $\mathcal{A} = (A, V)$ with layer

$$L = (A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$$

such that $A' = A$. Therefore in this case the $a_1, ..., a_n$ do not appear.

Throughout this section, $\mathcal{B} = (B, W)$ is a minimal bocs with layer

$$L = (B; \omega; w_1, ..., w_m), \quad \text{where} \quad w_i \in \overline{W}(X_i, Y_i).$$

For $M, N \in \mathcal{B}$-Mod we put $\text{Hom}_{\mathcal{B}}(M, N)^1 = \{f : M \to N | (1, \omega)^*(f) = 0\}$.

Proposition 5.1. *Let $\mathcal{B} = (B, W)$ be a minimal bocs and $\epsilon : W \to B$ the counit of W. Then for $M, N \in \mathcal{B}$-Mod we have*

$$\text{Hom}_{\mathcal{B}}(M, N) = (1, \epsilon)^*(\text{Hom}_B(M, N)) \oplus \text{Hom}_{\mathcal{B}}(M, N)^1.$$

Proof. We have $(1, \epsilon)^*(1, \omega)^* \cong id_{B\text{-Mod}}$. □

Observe that if we have any pair of morphisms $(f, \phi_{M,N}(h_1, ..., h_m))$ with $f \in \text{Hom}_B(M, N)$, $h_i \in \text{Hom}_k(M(X_i), N(Y_i))$ where $w_i : X_i \to Y_i$, this pair is a morphism from M to N in $\mathcal{B}$-Mod, because in a minimal bocs $\delta_1 = 0$ and condition (A) before Proposition 4.2 is trivially satisfied. Then we have:

Corollary 5.2. *For $M, N \in \mathcal{B}$-mod :*

$$\dim_k \text{Hom}^1_{\mathcal{B}}(M, N) = \sum_{w_i} \dim_k \text{Hom}_k(M(X_i), N(Y_i)).$$

The morphisms in the image of $(1, \epsilon)^*$ have the form $(f, 0)$ where the morphism f is in $\text{Hom}_B(M, N)$.

Lemma 5.3. *(Compare Definition 3.8 in [5]) Let M, N be two objects in $\mathcal{B}$-Mod, then $M \cong N$ in $\mathcal{B}$-Mod iff $M \cong N$ in B-Mod.*

Proof. If $h : M \to N$ is an isomorphism in $\mathcal{B}$-Mod then $(1, \omega)^*(h)$ is an isomorphism in B-Mod. Conversely, if $g : M \to N$ is an isomorphism in B-Mod then $(1, \epsilon)^*(g)$ is an isomorphism in $\mathcal{B}$-Mod. □

Clearly, Lemma 5.3 implies that indecomposable objects in B-Mod and $\mathcal{B}$-Mod coincide.

We have $B(Z, Z') = 0$ for $Z \neq Z' \in \text{ind}B$ and for $Z \in \text{ind}B$, $B(Z, Z) = R_Z = k[x, h(x)^{-1}]id_Z$ with $h(x) \in k[x]$ or $B(Z, Z) = kid_Z$. Take M an indecomposable object in B-mod, then there is only one $Z \in \text{ind}B$ such that $M(Z) \neq 0$. Here M is a covariant functor of B into k-Mod, $M(Z)$ is a left R_Z- module. Therefore if $B(Z,Z) = R_Z \neq kid_Z$, $M(Z) \cong R_Z/(p^n)$ with $p = x - \lambda$ a prime element in R_Z, if $B(Z,Z) = kid_Z$, $M(Z) = k$.

For $Z \in \mathrm{ind}B$ with $B(Z,Z) = R_Z \neq kid_Z$ and $p = x - \lambda$, a prime element in R_Z we define $M(Z,p,n) \in B$-Mod by

$$M(Z,p,n)(W) = 0 \quad \text{for} \quad W \neq Z, W \in \mathrm{ind}B, \quad M(Z,p,n)(Z) = R_Z/(p^n).$$

If $B(Z,Z) = kid_Z$ we define $S_Z \in B$-mod by

$$S_Z(W) = 0 \quad \text{for} \quad W \neq Z, W \in \mathrm{ind}B, \quad S_Z(Z) = k.$$

Lemma 5.4. *If M is an indecomposable object in B-mod then $M \cong M(Z,p,n)$ or $M \cong S_Z$ for some $Z \in \mathrm{ind}B$.*

Lemma 5.5. *Let $(f,0) : M \to N$ be a morphism in $\mathcal{B}$-Mod such that for all $Z \in \mathrm{ind}B$, $f_Z : M(Z) \to N(Z)$ is surjective. Then if $h : L \to N$ is a morphism in $\mathcal{B}$-Mod with $(1,\omega)^*(h) = 0$, there is a morphism $g : L \to M$ in $\mathcal{B}$-Mod with $(f,0)g = h$.*

Proof. Take $h : L \to N$ with $(1,\omega)^*(h) = 0$, then $h = (0, \phi_{L,N}(h_1, ..., h_m))$. We may assume that there is a j with $0 \neq h_j \in \mathrm{Hom}_k(M(X_j), N(Y_j))$ and $h_i = 0$ for $i \neq j$.

We have that $f_{Y_j} : M(Y_j) \to N(Y_j)$ is an epimorphism. Consequently, there is a k-linear map $\sigma : N(Y_j) \to M(Y_j)$ with $f_{Y_j}\sigma = id_{N(Y_j)}$. Take now $g_j = \sigma h_j \in \mathrm{Hom}_k(L(X_j), M(Y_j))$, and $0 = g_i \in \mathrm{Hom}_k(L(X_i), M(Y_i))$, for $i \neq j$. Take now the morphism

$$g = (0, \phi_{L,M}(g_1, ..., g_m)) : L \to M$$

then by Lemma 4.5 $(f,0)g = (0, \phi_{L,N}(\lambda_1, ..., \lambda_m))$ with $\lambda_i = f_{Y_i} g_i$. Therefore, $\lambda_i = 0$ for $i \neq j$ and $\lambda_j = f_{Y_j} g_j = f_{Y_j}\sigma h_j = h_j$. Consequently, $(f,0)g = (0, \phi_{L,N}(\lambda_1, ..., \lambda_m)) = (0, \phi_{L,N}(h_1, ..., h_m)) = h$. □

Similarly, we have the dual version of the above result.

Lemma 5.6. *Let $(f,0) : M \to N$ be a morphism in $\mathcal{B}$-Mod such that for all $Z \in \mathrm{ind}B$, $f_Z : M(Z) \to N(Z)$ is an injection. Then if $u : M \to L$ is a morphism with $(1,\omega)^*(u) = 0$ there is a morphism $v : N \to L$ with $v(f,0) = u$.*

For $Z, Z' \in \mathrm{ind}B$ we denote by $t(Z,Z')$ the number of $w_i \in \overline{W}(Z,Z')$.

Lemma 5.7. *Suppose M, N are indecomposable objects in $\mathcal{B}$-mod with $M(Z) \neq 0, N(Z') \neq 0$, $Z, Z' \in \mathrm{ind}B$. Then*

$$\dim_k \mathrm{Hom}_{\mathcal{B}}(M,N)^1 = t(Z,Z') \dim_k M(Z) \dim_k N(Z').$$

Proof. It follows from Corollary 5.2. □

Lemma 5.8. *If M, N are indecomposable objects in $\mathcal{B}$-mod, then*

$$\mathrm{rad}^{\infty}_{\mathcal{B}}(M,N) \subset \mathrm{Hom}_{\mathcal{B}}(M,N)^1.$$

Proof. Suppose there is a $h \in \mathrm{rad}^{\infty}_{\mathcal{B}}(M,N)$ with $(1,\omega)^*(h) \neq 0$. Then there is a $Z \in \mathrm{ind}B$ with $M(Z) \neq 0, N(Z) \neq 0$. Since $(1,\omega)^*$ reflects isomorphisms, then $(1,\omega)^*(h)$ is not an isomorphism. Consequently, $B(Z,Z) = R_Z \neq kid_Z$ and $M \cong M(Z,p,m), N \cong M(Z,p,n)$.

Here $\mathrm{rad}^{\infty}_{B}(M,N) \cong \mathrm{rad}^{\infty}_{R_Z}(R_Z/(p^m), R_Z/(p^n)) = 0$. Then there is a s with $\mathrm{rad}^{s}_{B}(M,N) = 0$.

On the other hand, there is a chain of non-isomorphisms between indecomposables:

$$M \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \ldots \to X_{s-1} \xrightarrow{f_s} N$$

with $g = (1,\omega)^*(f_s \cdots f_2 f_1) \neq 0$.

But $g = (1,\omega)^*(f_s) \cdots (1,\omega)^*(f_1) \in \mathrm{rad}^{s}_{B}(M,N) = 0$, a contradiction. This proves our claim. □

Consider $M = M(Z,p,m), N = M(Z,p,n)$ indecomposables in B-mod. If $f : R_Z/(p^m) \to R_Z/(p^n)$ is a morphism of R_Z-modules, we put $u(f) : M \to N$ given by $u(f)_Z = f$ and $u(f)_W = 0$ for $W \neq Z$.

Proposition 5.9. *Let M, N be indecomposables in $\mathcal{B}$-mod with $M(Z) \neq 0$ or $N(Z) \neq 0$ for some $Z \in \mathrm{ind}B$ with $B(Z,Z) \neq kid_Z$, then*

$$\mathrm{rad}^{\infty}_{\mathcal{B}}(M,N) = \mathrm{Hom}_{\mathcal{B}}(M,N)^1.$$

Proof. By Lemma 5.8, it is enough to prove that if $f : M \to N$ is a morphism in $\mathcal{B}$-mod with $(1,\omega)^*(f) = 0$ then $f \in \mathrm{rad}^{\infty}_{\mathcal{B}}(M,N)$. Suppose $M(Z) \neq 0$ with $B(Z,Z) = R_Z \neq id_Z k$. Then we may assume $M = M(Z,p,m)$. Take any natural number n. Consider the monomorphism $i_l : R_Z/(p^l) \to R_Z/(p^{l+1})$ given by $i_l(\eta_l(a)) = \eta_{l+1}(pa)$ for $a \in R_Z$ and $\eta_j : R_Z \to R_Z/(p^j)$ the quotient map. Take $(u,0) = (u(i_{n+m-1}),0)\ldots(u(i_{m+1}),0)(u(i_m),0) : M(Z,p,m) \to M(Z,p,m+n)$. Here $u_Z : M(Z,p,m)(Z) \to M(Z,p,m+n)(Z)$ is a monomorphism . By Lemma 5.6, there is a morphism $t : M(Z,p,m+n) \to N$ in $\mathcal{B}$-Mod such that $t(u,0) = f$.

Now, $(u,0) \in \mathrm{rad}^{n}_{\mathcal{B}}(M, M(Z,p,m+n))$, and, therefore, $f = t(u,0) \in \mathrm{rad}^{n}_{\mathcal{B}}(M,N)$ for all n, then $f \in \mathrm{rad}^{\infty}_{\mathcal{B}}(M,N)$.

For the case in which $N(Z) \neq 0$ with $B(Z,Z) \neq kid_Z$ one proceeds in a similar way. □

Corollary 5.10. *If M, N are indecomposable objects in $\mathcal{B}$-mod, and $Z, Z' \in \mathrm{ind}B$ with $M(Z) \neq 0$, $N(Z') \neq 0$, and $B(Z,Z) \neq kid_Z$ or $B(Z',Z') \neq kid_{Z'}$, then*

$$\dim_k \mathrm{rad}^{\infty}_{\mathcal{B}}(M,N) = \dim_k M(Z) \dim_k N(Z') t(Z,Z').$$

Corollary 5.11. *Let $M = M(Z, p, m), N = M(Z', q, n), S = S_W$ be indecomposables in $\mathcal{B}$-mod, with $B(Z, Z) \neq kid_Z$, $B(Z', Z') \neq kid_{Z'}$, $B(W, W) = kid_W$. Then if $Z = Z'$, $p = q$,*

$$\mathrm{Hom}_{\mathcal{B}}(M, N) \cong \mathrm{Hom}_B(M, N) \oplus \mathrm{rad}^{\infty}_{\mathcal{B}}(M, N),$$

with $\dim_k(\mathrm{Hom}_B(M, N)) = min\{m, n\}$.
And if $Z \neq Z'$ or $Z = Z'$, and $(p) \neq (q)$

$$\mathrm{Hom}_{\mathcal{B}}(M, N) = \mathrm{rad}^{\infty}_{\mathcal{B}}(M, N).$$

Moreover,

$$\mathrm{Hom}_{\mathcal{B}}(M, S) = \mathrm{rad}^{\infty}_{\mathcal{B}}(M, S) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{B}}(S, M) = \mathrm{rad}^{\infty}_{\mathcal{B}}(S, M).$$

Lemma 5.12. *If $0 \to M \xrightarrow{f^0} E \xrightarrow{g^0} N \to 0$ is a short exact sequence in B-Mod, then the pair of morphisms in $\mathcal{B}$-Mod, $M \xrightarrow{(f^0,0)} E \xrightarrow{(g^0,0)} N$ is an exact pair of morphisms.*

Proof. We claim that $f = (f^0, 0)$ is a kernel of $(g^0, 0)$. Assume there is a morphism $u = (u^0, u^1) = (u^0, 0) + (0, u^1) : L \to E$ such that $gu = (g^0u^0, (gu)^1) = 0$. Here $g^0u^0 = 0$, then there is a unique morphism in B-Mod, $v^0 : L \to M$ with $f^0v^0 = u^0$. Now, $u^1 = \phi_{L,E}(u_1, ..., u_m)$, with $u_i : L(X_i) \to E(Y_i)$ where $w_i \in \overline{W}(X_i, Y_i)$. Then $(gu)^1 = \phi_{L,N}(g^0_{Y_1}u_1, ..., g^0_{Y_m}u_m)$. Therefore, for $i = 1, ..., m$, $g^0_{Y_i}u_i = 0$. Thus, there are linear maps $v_i : L(X_i) \to M(Y_i)$ with $f^0_{Y_i}v_i = u_i$ for $i = 1, ..., m$. Then taking $v = (v^0, \phi_{L,M}(v_1, ..., v_m))$ we have $fv = u$. Clearly v is unique with this property. This proves our claim. In a similar way one can prove that g is a cokernel of f. □

Lemma 5.13. *Suppose $(a) : M \xrightarrow{f} E \xrightarrow{g} N$ is a proper exact sequence in $\mathcal{B}$-Mod. Then (a) is isomorphic to the sequence: $M \xrightarrow{(f^0,0)} E \xrightarrow{(g^0,0)} N$.*

Proof. By Lemma 5.5 and its proof, there is a morphism $u = (0, u^1) : E \to E$ such that $(g^0, 0)u = (0, g^1)$. Then $(g^0, 0)(1_E, u^1) = g$, with $\sigma = (1_E, u^1)$ an isomorphism. Thus, $(g^0, 0)\sigma f = gf = 0$. But by the above Lemma, $(f^0, 0)$ is a kernel of $(g^0, 0)$, then there is a morphism $\lambda = (\lambda^0, \lambda^1) : M \to M$ with $(f^0, 0)\lambda = \sigma f$. Here $f^0\lambda^0 = f^0$, since f^0 is a monomorphism then $\lambda^0 = 1_M$. Therefore, $\lambda : M \to M$ is an isomorphism. This proves our claim. □

From Lemma 5.12 and Lemma 5.13, we deduce that proper exact sequences are exact pairs of morphisms. Denote by $\mathcal{E}_p$ the class of proper exact sequences in $\mathcal{B}$-Mod, then we have the following.

Proposition 5.14. *The pair $(\mathcal{B}\text{-Mod}, \mathcal{E}_p)$ is an exact category.*

Proof. Observe first that $g = (g^0, g^1) : E \to M$ is a deflation if and only if g^0 is an epimorphism. In fact, if g is a deflation, by definition of proper exact sequence g^0 is an epimorphism. Conversely, suppose g^0 is an epimorphism, then as in the proof of Lemma 5.5 there is an isomorphism $\tau : E \to E$ such that $(g^0, 0) = g\tau$. Taking $f^0 : N \to E$ the kernel of g^0 in B-Mod, we see that $(g^0, 0)$ is a deflation, thus g is a deflation too. Similarly, one can prove that $f : N \to E$ is an inflation if and only if f^0 is a monomorphism. From this, it is clear that conditions E.1, E.3 and E.3^{op} hold. For proving E.2, assume $g : E \to N$ is a deflation and $h : L \to N$ is an arbitrary morphism. Then we have the morphism $(g, h) : E \oplus L \to N$. Now, $(g, h) = ((g^0, h^0), (g^1, h^1))$, here g^0 is an epimorphism, then (g^0, h^0) is also an epimorphism, thus (g, h) is a deflation, therefore it has a kernel, $M \xrightarrow{u} E \oplus L$. Take $u_1 : M \to E$ equal to u composed with the projection on E and $-u_2 : M \to L$, the composition of u with the projection on L. Now, one can see that u_2 is a deflation and $gu_1 = hu_2$. Therefore, E.2 holds. □

Let $Z_1, ..., Z_s$ be all marked objects in ind$B$. For $i = 1, ..., s$ take $R_i = B(Z_i, Z_i)$ and the B-R_i-bimodule $B_i = B(Z_i, -)$. Then if p is a prime element of R_i and n a positive integer, $M(Z_i, p, n) \cong B_i \otimes_{R_i} R_i/(p^n)$. We denote by $S^i_{p,n}$ the exact sequence in R_i-mod:

$$0 \to R_i/(p^n) \xrightarrow{(p,\pi)} ((R_i/(p^{n+1}) \oplus R_i/(p^{n-1})) \xrightarrow{\begin{pmatrix} \pi \\ -p \end{pmatrix}} R/(p^n) \to 0.$$

Proposition 5.15. *The sequence $B_i \otimes_{R_i} S^i_{p,n}$:*

$$B_i \otimes_{R_i} R_i/(p^n) \xrightarrow{id\otimes(p,\pi)} B_i \otimes_{R_i} ((R_i/(p^{n+1}) \oplus R_i/(p^{n-1}))$$

$$\xrightarrow{id\otimes\begin{pmatrix} \pi \\ -p \end{pmatrix}} B_i \otimes_{R_i} R_i/(p^n)$$

*is a proper almost split sequence in $\mathcal{B}$-*mod.

Proof. The sequence $S^i_{p,n}$ is an almost split sequence in R_i-mod. Now, using Lemma 5.5 and Lemma 5.6 one can prove that $B_i \otimes_{R_i} S^i_{p,n}$ is a proper almost split sequence. □

6 Hom-spaces between $\mathcal{A}$-$k(x)$-bimodules

Let $\mathcal{A} = (A, V)$ be a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$. We recall from [6] that an $\mathcal{A}$-$k(x)$-bimodule is an object $M \in \mathcal{A}$-Mod with a morphism $\alpha_M : k(x) \to \text{End}_{\mathcal{A}}(M)$. If M and N are $\mathcal{A}$-$k(x)$-bimodules, a morphism $f : M \to N$ in $\mathcal{A}$-Mod is a morphism of $\mathcal{A}$-$k(x)$-bimodules if for all $q \in k(x)$, $f\alpha_M(q) = \alpha_N(q)f$.

We denote by $\mathcal{A}$-$k(x)$-Mod the category whose objects are the $\mathcal{A}$-$k(x)$-bimodules and the morphisms are morphisms of $\mathcal{A}$-$k(x)$-bimodules. If $F : \mathcal{B}$-Mod $\to \mathcal{A}$-Mod is a functor with $\mathcal{A}, \mathcal{B}$ layered bocses, then F induces a functor $F^{k(x)} : \mathcal{B}$-$k(x)$-Mod $\to \mathcal{A}$-$k(x)$-Mod. If

M is a $\mathcal{B}$-$k(x)$-bimodule, with $\alpha_M : k(x) \to \mathrm{End}_{\mathcal{B}}(M)$ then $F(M)$ is an $\mathcal{A}$-$k(x)$-bimodule with $\alpha_{F(M)} = F\alpha_M : k(x) \to \mathrm{End}_{\mathcal{A}}(F(M))$. Observe that if $f : M \to N$ is a morphism of $\mathcal{B}$-$k(x)$-bimodules, then $F(f)$ is a morphism of $\mathcal{A}$-$k(x)$-bimodules. Now, if F is full and faithful then $F(f) : F(M) \to F(N)$ is a morphism of $\mathcal{A}$-$k(x)$-bimodules if and only if for all $q \in k(x)$, $F(f)F(\alpha_M(q)) = F(\alpha_N(q))F(f)$ and this is true if and only if $f\alpha_M(q) = \alpha_N(q)f$ for all $q \in k(x)$. Thus, F induces a full and faithful functor

$$F^{k(x)} : \mathcal{B}\text{-}k(x)\text{-Mod} \to \mathcal{A}\text{-}k(x)\text{-Mod}.$$

The $\mathcal{A}$-$k(x)$-bimodule M is called proper if there is a $\beta_M : k(x) \to \mathrm{End}_A(M)$ such that $\alpha_M = (1, \epsilon)^* \beta_M$, thus $\alpha_M(q) = (\beta_M(q), 0)$ for all $q \in k(x)$. Observe that if M is a proper $\mathcal{A}$-$k(x)$-bimodule then M is an A-$k(x)$-bimodule. We denote by $\mathcal{A}$-$k(x)$-Mod^p, the full subcategory of $\mathcal{A}$-$k(x)$-Mod whose objects are the proper bimodules. Suppose $\theta : \mathcal{A} \to \mathcal{B}$ is a morphism of bocses with $\epsilon_{\mathcal{B}}$ the counit of $\mathcal{B}$ and $\epsilon_{\mathcal{A}}$ the counit of $\mathcal{A}$, then $\theta^* : \mathcal{B}$-Mod $\to \mathcal{A}$-Mod is a full and faithful functor. Observe that if M is a proper $\mathcal{B}$-$k(x)$-bimodule then $\alpha_M = (1, \epsilon_{\mathcal{B}})^* \beta_M$ with $\beta_M : k(x) \to \mathrm{End}_B(M)$. Then $\theta^*(M)$ is a $\mathcal{A}$-$k(x)$-bimodule, using Lemma 4.1 we have

$$\alpha_{\theta^*(M)} = (\theta_0, \theta_1)^*(1, \epsilon_{\mathcal{B}})^* \beta_M = (1, \epsilon_{\mathcal{A}})^*(\theta_0, \theta_0)^* \beta_M,$$

thus $\theta^*(M)$ is a proper $\mathcal{B}$-$k(x)$-bimodule, consequently θ^* induces a full and faithful functor $(\theta^*)^{k(x)} : \mathcal{B}$-$k(x)$-Mod$^p \to \mathcal{A}$-$k(x)$-Mod^p.

Proposition 6.1. *Let M, N be proper $\mathcal{A}$-$k(x)$-bimodules. Then $f = (f^0, \phi_{M,N}(f_1, ..., f_m)) : M \to N$ is a morphism of $\mathcal{A}$-$k(x)$-bimodules if and only if f^0 is a morphism of A'-$k(x)$-bimodules and $f_i \in \mathrm{Hom}_{k(x)}(M(X_i), N(Y_i))$ for all $v_i \in \overline{V}(X_i, Y_i)$.*

Proof. We have that M and N are proper bimodules so, $\alpha_M(q) = (\beta_M(q), 0)$ and $\alpha_N(q) = (\beta_N(q), 0)$ with morphisms of k-algebras $\beta_M : k(x) \to \mathrm{End}_A(M)$ and $\beta_N : k(x) \to \mathrm{End}_A(N)$. Then a morphism $f : M \to N$ in $\mathcal{A}$-Mod is a morphism of $\mathcal{A}$-$k(x)$-bimodules if and only if $f\alpha_M(q) = \alpha_N(q)f$ for all $q \in k(x)$. Then, by Proposition 4.2, the above holds if and only if $f^0\beta_M(q) = \beta_N(q)f^0$ for all $q \in k(x)$, and for all v_i and all $q \in k(x)$, $u \in M(X_i)$: $\beta_N(q)\phi_{M,N}(f_1, ..., f_m)(v_i \otimes u) = \phi_{M,N}(f_1, ..., f_m)(v_i \otimes \beta_M(q)(u))$. Using the relations given in Lemma 4.5, we obtain that the latter equality is equivalent to $\beta_N(q)f_i(u) = f_i(\beta_M(q)(u))$. From here we obtain our result. □

Corollary 6.2. *Let $\mathcal{B} = (B, W)$ be a minimal bocs with layer $(B; \omega_B; w_1, ..., w_m)$, with $w_i \in \overline{W}(X_i, Y_i)$. Then if M and N are proper $\mathcal{B}$-$k(x)$-bimodules we have:*

$$\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(M, N) \cong \mathrm{Hom}_{B\text{-}k(x)}(M, N) \oplus \bigoplus_i \mathrm{Hom}_{k(x)}(M(X_i), N(Y_i)).$$

Let $\mathcal{B} = (B, W)$ be a minimal bocs with layer $(B; \omega; w_1, ..., w_m)$, for Z a marked object in ind$B$ we define $Q_Z \in \mathcal{B}$-Mod as follows: $Q_Z(Z) = k(x)$ where $B(Z, Z) = k[x, f(x)^{-1}]id_Z$ and the action of x on $Q_Z(Z)$ is the multiplication by x, $Q_Z(W) = 0$ for

$Z \neq W$. The action of $k(x)$ is the multiplication on the right by the elements of $k(x)$. Here Q_Z is a proper $\mathcal{B}$-$k(x)$-bimodule. Using the notation of section 5, we have as a consequence of the above corollary:

Corollary 6.3. *If Z, Z' are marked objects and W is a non-marked object in* $\mathrm{ind}B$*, write* $S_W^{k(x)} = S_W \otimes_k k(x)$. *We have:*

$$\dim_{k(x)}\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) = \delta(Z, Z') + t(Z, Z')$$

where $\delta(Z, Z') = 1$ if $Z = Z'$ and zero otherwise. Moreover

$$\dim_{k(x)}(\mathrm{rad}_{\mathcal{B}\text{-}k(x)}(Q_Z, S_W^{k(x)})) = t(Z, W),$$

$$\dim_{k(x)}(\mathrm{rad}_{\mathcal{B}\text{-}k(x)}(S_W^{k(x)}, Q_Z)) = t(W, Z).$$

Corollary 6.4. *With the notations in Corollary 6.3 we have :*

$$\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) = k(x) \oplus \mathrm{rad}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) \quad \textit{when} \quad Z = Z',$$

$$\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) = \mathrm{rad}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) \quad \textit{when} \quad Z \neq Z'.$$

Moreover:

$$\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(Q_Z, S_W^{k(x)}) = \mathrm{rad}_{\mathcal{B}\text{-}k(x)}(Q_Z, S_W^{k(x)}),$$

$$\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(S_W^{k(x)}, Q_Z) = \mathrm{rad}_{\mathcal{B}\text{-}k(x)}(S_W^{k(x)}, Q_Z).$$

From the above corollaries, we obtain the next proposition.

Proposition 6.5. *Let $\mathcal{B} = (B, W)$ be a minimal bocs with layer $(B; \omega; w_1, ..., w_m)$. Suppose Z, Z', and W are objects in* $\mathrm{ind}\,B$ *with $B(W, W) = id_W k$, $B(Z, Z) \neq id_Z k$, $B(Z', Z') \neq id_{Z'} k$. Take $M = M(Z, p, m)$, $N = M(Z', q, n)$, $L = S_W$ with p, q prime elements in $B(Z, Z)$ and $B(Z', Z')$, respectively. Then*

$$\dim_k \mathrm{rad}_{\mathcal{B}}^{\infty}(M, N) = mn(\dim_{k(x)}\mathrm{Hom}_{\mathcal{B}\text{-}k(x)}(Q_Z, Q_{Z'}) - \delta(Z, Z'));$$

$$\dim_k \mathrm{rad}_{\mathcal{B}}^{\infty}(M, L) = m\dim_{k(x)}\mathrm{rad}_{\mathcal{B}\text{-}k(x)}(Q_Z, L^{k(x)});$$

$$\dim_k \mathrm{rad}_{\mathcal{B}}^{\infty}(L, M) = m\dim_{k(x)}\mathrm{rad}_{\mathcal{B}\text{-}k(x)}(L^{k(x)}, Q_Z).$$

7 $\mathcal{D}$-isolated Objects

Let $\mathcal{A} = (A, V)$ be a bocs with layer $L = (A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$. We recall that an object $X \in \mathrm{ind}A'$ is called marked if $A'(X, X) \neq kid_X$, we denote by $m(A')$, the set of marked objects of A'. For $M \in \mathcal{A}$-mod we define its dimension vector

$$\mathbf{dim}M : \mathrm{ind}A' \to \mathbb{N} \quad \text{by} \quad \mathbf{dim}M(X) = \dim_k M(X).$$

By Dim$\mathcal{A}$ we denote the set of functions $\mathbf{d} : \text{ind}A' \to \mathbb{N}$. If $\mathbf{d}, \mathbf{d}' \in$ Dim$\mathcal{A}$ we have $\mathbf{d}+\mathbf{d}'$, defined by $(\mathbf{d}+\mathbf{d}')(X) = \mathbf{d}(X) + \mathbf{d}'(X)$ for all $X \in \text{ind}A'$. The norm of $\mathbf{d} \in$ Dim$\mathcal{A}$ is defined by $||\mathbf{d}|| = \sum_{i=1}^{n} \mathbf{d}(X_i)\mathbf{d}(Y_i) + \sum_{X\in m(A')} \mathbf{d}(X)^2$, where $a_i : X_i \to Y_i$. For $M \in \mathcal{A}$-mod we define the norm of M, $||M|| = ||\mathbf{dim}M||$.

If $\mathbf{d} \in \text{Dim}(\mathcal{A})$ we define $|\mathbf{d}| = \sum_{X\in \text{ind}A'} \mathbf{d}(X)$. For $M \in \mathcal{A}$-mod, we put $|M| = |\mathbf{dim}M|$ which is called the dimension of M.

Take $\theta : A \to B$ a functor with B a skeletally small category, the induced bocs $\mathcal{A}^B = (B, W)$ is given as follows: $W = B \otimes_A V \otimes_A B$ with counit

$$\epsilon_B : W \to B$$

given by $\epsilon_B(b_1 \otimes v \otimes b_2) = b_1\theta(\epsilon(v))b_2$ for b_1, b_2 morphisms in B, $v \in V$. The coproduct

$$\mu_B : W \to W \otimes_B W$$

is given by $\mu_B(b_1 \otimes v \otimes b_2) = \sum_i b_1 \otimes v_i^1 \otimes 1 \otimes 1 \otimes v_i^2 \otimes b_2$, where b_1, b_2 are morphisms in B and $v \in V$ with $\delta(v) = \sum_i v_i^1 \otimes v_i^2$.

There is a morphism of A-A-bimodules

$$\theta_1 : V \to W$$

given by $\theta_1(v) = 1 \otimes v \otimes 1$, for $v \in V$. Then we obtain a morphism of bocses $(\theta, \theta_1) : \mathcal{A} \to \mathcal{A}^B$ which induces a full and faithful functor $\theta^* : \mathcal{A}^B\text{-Mod} \to \mathcal{A}\text{-Mod}$.

Assume $\mathcal{A}^B$ has layer

$$L^\theta = (B'; \omega'; b_1, ..., b_{n'}; w_1, ..., w_{m'}).$$

There is an additive function $t^\theta : \text{Dim}(\mathcal{A}^B) \to \text{Dim}(\mathcal{A})$, given by $t^\theta(\mathbf{d})(X) = \sum_j \mathbf{d}(Y_j)$ with $\theta(X) = \bigoplus_j Y_j$, $Y_j \in \text{ind}B'$. We have $\mathbf{dim}\theta^*(M) = t^\theta(\mathbf{dim}M)$, for $M \in \mathcal{A}^B$-mod.

Following [6], we say that that the bocs $\mathcal{A} = (A, V)$ with counit $\epsilon : V \to A$ and layer $L = (A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$ is of wild representation type or simply wild if there is a functor $F : A \to \Sigma$, where Σ are the finitely generated free $k\langle x, y\rangle$-modules such that the induced functor:

$$(F, F\epsilon)^* : \Sigma\text{-Mod} \to \mathcal{A}\text{-Mod}$$

preserves isomorphism classes and indecomposables.

From [7], we know that a layered bocs $\mathcal{A} = (A, V)$ which is not of wild representation type is of tame representation type. This is, for each natural number d, there are a finite number of A-$k[x]$-bimodules $M_1, ..., M_s$ free of finite rank as right $k[x]$-modules, and such that every indecomposable M in $\mathcal{A}$-Mod with $|\mathbf{dim}M| \le d$ is isomorphic to $M_i \otimes_{k[x]} k[x]/(x - \lambda)$ for some $1 \le i \le s$ and $\lambda \in k$.

This section is devoted to find some subset $\mathcal{D}$ of Dim$\mathcal{A}$ with $\mathcal{A}$ a bocs of tame representation type such that the marked indecomposable objects of A become $\mathcal{D}$-isolated objects in the sense of Definition 7.4. For this we need the following specific functors (see section 4 of [5]):

1. **Regularization**. Suppose $a_1 : X_1 \to Y_1$ with $\delta(a_1) = v_1$. Then B is freely generated by A' and $a_2, ..., a_n$. The functor $\theta : A \to B$ is the identity on A', $\theta(a_1) = 0$, $\theta(a_i) = a_i$ for $i = 2, ..., n$. The bocs $\mathcal{A}^B = (B, W)$ has layer $(A'; \omega_B; a_2, ..., a_n; \theta_1(v_2), ..., \theta_1(v_m))$.

The functor $\theta^* : \mathcal{A}^B\text{-Mod} \to \mathcal{A}\text{-Mod}$ is an equivalence of categories, $\text{Dim}(\mathcal{A}^B) = \text{Dim}(\mathcal{A})$ and $t^\theta = id$. In this case $||t^\theta(\mathbf{d})|| \geq ||\mathbf{d}||$, and one has the equality if and only if $\mathbf{d}(X_1)\mathbf{d}(Y_1) = 0$.

2. **Deletion of objects** . Let C be a subcategory of A. Let B' be the full subcategory of A' whose objects have no non-zero direct summand isomorphic to a direct summand of an object of C. Take I_0 the set of $i \in \{1, ..., n\}$ such that $a_i \in A(X_i, Y_i)$ with X_i, Y_i in B', and I_1 the set of $j \in \{1, ..., m\}$ such that $v_j \in V(X_j, Y_j)$ with X_j, Y_j in B'. Then B is freely generated by B' and the a_i with $i \in I_0$. The functor $\theta : A \to B$ is the identity on B' and $\theta(X) = 0$ for all $X \in C$. The bocs $\mathcal{A}^B$ has layer $(B'; \omega_B; (a_i)_{i\in I_0}; (\theta_1(v_j))_{j\in I_1})$. Here $M \in \mathcal{A}-\text{Mod}$ is isomorphic to some $\theta^*(N)$ if and only if $M(X) = 0$ for all X indecomposable objects of C. The function $t^\theta : \text{Dim}(\mathcal{A}^B) \to \text{Dim}(\mathcal{A})$ is an inclusion, $\mathbf{d} \in \text{Dim}(\mathcal{A})$ is in the image of t^θ if and only if $\mathbf{d}(X) = 0$ for all X indecomposable objects of C. In this case $||t^\theta(\mathbf{d})|| = ||\mathbf{d}||$.

3. **Edge reduction** . Suppose $a_1 : X_1 \to Y_1$ with $X_1 \neq Y_1$ is such that $\delta(a_1) = 0$, and $A'(X_1, X_1) = kid_{X_1}$, $A'(Y_1, Y_1) = kid_{Y_1}$. Let C be the full subcategory of A' whose objects have no direct summands isomorphic to X_1 or Y_1. Now denote by D a minimal category with three indecomposable objects Z_1, Z_2, Z_3, $D(Z_i, Z_i) = kid_{Z_i}$ for $i = 1, 2, 3$. Take $B' = C \times D$. The category B is freely generated by B' and elements $b_1, ..., b_s$. The number of arrows $b_j : W_j \to W'_j$ with W_j and W'_j different from Z_2 is $n - 1$, where n is the number of a_i.

The functor $\theta : A \to B$ is the identity on C and $\theta(X_1) = Z_1 \oplus Z_2$, $\theta(Y_1) = Z_2 \oplus Z_3$.

The bocs $\mathcal{A}^B = (B, W)$ has a layer of the form $(B', \omega_B; b_1, ..., b_s; w_1, ...w_u)$. Moreover, if $M \in \mathcal{A}^B\text{-Mod}$, $\theta^*(M)(a_i) = 0$ for all $i \in \{1, ..., n\}$ if and only if $M(b_j) = 0$ for all $j \in \{1, ..., s\}$ and $M(Z_2) = 0$. The functor θ^* is an equivalence of categories. Moreover $||t^\theta(\mathbf{d})|| > ||\mathbf{d}||$ if and only if $(t^\theta(\mathbf{d}))(X_1)(t^\theta(\mathbf{d})(Y_1)) \neq 0$. If $||t^\theta(\mathbf{d})|| = ||\mathbf{d}||$ and $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'||$, then $t^\theta(\mathbf{d}) = t^\theta(\mathbf{d}')$ implies $\mathbf{d} = \mathbf{d}'$.

4. **Unraveling** . Let X be an indecomposable object in A' with $A'(X, X) = k[x, f(x)^{-1}]id_X$. Suppose $S = \{\lambda_1, ..., \lambda_t\}$ is a set of elements of k which are not roots of $f(x)$. For r a positive integer there is a functor $\theta : A \to B$, where B is freely generated by B' and elements $b_1, ..., b_s$, $B' = C \times D$, where C is the full subcategory of A' whose objects have no direct summands isomorphic to X. The category D is the minimal category with indecomposable objects $Y, Z_{i,j}$ with $i \in \{1, ..., r\}, j \in \{1, ..., t\}$, $D(Z_{i,j}, Z_{i,j}) = kid_{Z_{i,j}}$, $D(Y, Y) = k[x, f(x)^{-1}, g(x)^{-1}]id_Y$, where $g(x) = (x-\lambda_1)...(x-\lambda_t)$. The functor $\theta : A \to B$ acts as the identity on C and $\theta(X) = Y \oplus \bigoplus_{j=1}^{t} \bigoplus_{i=1}^{r} Z_{i,j}^i$, where $Z_{i,j}^i$ is the direct sum of i copies of $Z_{i,j}$.

The bocs $\mathcal{A}^B = (B, W)$ has a layer of the form $(B'; \omega_B; b_1, ..., b_s; w_1, ..., w_u)$.

Moreover for $N \in \mathcal{A}^B\text{-mod}$ we have the following:

(a) $||N|| \leq ||\theta^*(N)||$, with strict inequality if $\theta^*(N)(g(x))$ is not invertible.

(b) If $M \in \mathcal{A}\text{-mod}$ and for all $Z \in \text{ind}A'$, $\dim_k M(Z) \leq r$ then there is a $N \in \mathcal{A}^B\text{-mod}$

such that $\theta^*(N) \cong M$.

(c) $\theta^*(N)(x) = N(x) \oplus \bigoplus_{j=1}^{s} \bigoplus_{i=1}^{r} N(Z_{i,j}^i)(x)$ with eigenvalues of $N(x)$ not in S, and $N(Z_{i,j}^i)(x) = J_i(\lambda_j)$, the Jordan block of size i and eigenvalue λ_j.

(d) Suppose $M \in \mathcal{A}$-mod is an indecomposable with $M(X) \neq 0$ and $M(W) = 0$ for all $W \neq X$, $W \in \text{ind}A'$, $M(a_i) = 0$ for $i \in \{1, ..., n\}$. Then if the unique eigenvalue of $M(x)$ is not in the set S, there is a $N \in \mathcal{A}^B$-mod with $N(W) = 0$ for all $W \in \text{ind}B'$, with $W \neq Y$, $N(b_j) = 0$ for all $j \in \{1, ..., s\}$ and $\theta^*(N) \cong M$.

(e) The number of $b_j : Y_1 \to Y_2$ with Y_1, Y_2 non isomorphic to $Z_{i,j}$ is equal to n, the number of a_i.

Definition 7.1. Let $\mathcal{A} = (A, V)$ be a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ...v_m)$. We say that $M \in \mathcal{A}$-Mod is concentrated in the indecomposable $X \in A'$ if $M(X) \neq 0$, $M(Y) = 0$ for Y indecomposable in A', $Y \neq X$ and $M(a_i) = 0$ for all $i \in \{1, ..., n\}$.

Proposition 7.2. *Let $\mathcal{A} = (A, V)$ be a bocs which is not wild, with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$. Let X be an indecomposable object in A' with $A'(X, X) = k[x, f(x)^{-1}]$. Then given a fixed dimension vector $\mathbf{d}$ with $\mathbf{d}(X) \neq 0$, there is a finite subset $S(X, \mathbf{d})$ of k such that if M is indecomposable in $\mathcal{A}$-mod with $\mathbf{dim}M = \mathbf{d}$ and λ in k but not in $S(X, \mathbf{d})$ is an eigenvalue of $M(x)$, then $M \cong M'$, with M' concentrated in X.*

Proof. We may assume $\mathbf{d}$ is sincere. We prove our assertion by induction on $||\mathbf{d}||$. If $||\mathbf{d}|| = 1$, take $S(X, \mathbf{d})$ the set of roots of $f(x)$. Then if M is an indecomposable in $\mathcal{A}$-mod, $M(X) \neq 0$, $\mathbf{dim}M = \mathbf{d}$, clearly M is concentrated in X.

Suppose our result proved for all non-wild layered bocses and dimension vectors with norm smaller than r. We may assume that for all $a_i : X_i \to Y_i$ with $\delta(a_i) = 0$, Y_i is not equal to X_i, since if $X_i = Y_i$, then because $\mathcal{A}$ is not wild and by Proposition 9 of [7] we have $A'(X_i, X_i) = kid_{X_i}$, so we may move a_i into A', such that $A'(X_i, X_i) = k[z]$, with $z = a_i$.

Take $a_1 : X_1 \to Y_1$ the first arrow. By condition $L.5$ of a layered bocs we have

$$\delta(a_1) = \sum_{j \in T} c_j v_j d_j,$$

where $c_j \in A'(Y_1, Y_1), d_j \in A'(X_1, X_1)$ and T is the set of all $j \in \{1, ..., m\}$ such that $v_j : \overline{V}(X_1, Y_1)$. We have then the following possibilities: $\delta(a_1) = 0$ or $\delta(a_1) = \sum_j c_j v_j d_j$ with some $c_j v_j d_j \neq 0$. If all $c_i, d_i \in k$, we may assume $d_i = 1$ for all $i \in T$. In this case we put $v'_i = v_i$ for $i \neq j$ and $v'_j = \sum_j c_j v_j$. Taking $\{v'_j, v'_1, ..., v'_m\}$ instead of $\{v_1, ..., v_m\}$ we have again a layer for $\mathcal{A}$, thus in this case we may assume $\delta(a_1) = v_1$. In case that for some $j \in T$, c_j is not in k or d_j is not in k, we have $A'(Y_1, Y_1) \neq kid_{Y_1}$ or $A'(X_1, X_1) \neq kid_{X_1}$.

Case 1. $\delta(a_1) = v_1$. Take $\theta^* : \mathcal{A}^B$-Mod $\to \mathcal{A}$-Mod the regularization of a_1. Here θ^* is an equivalence and the norm of $\mathbf{d}$ in $\mathcal{A}^B$ is smaller than r. Our claim is true for X and the norm r' of $\mathbf{d}$ in $\mathcal{A}^B$. Take $S(X, \mathbf{d}) = S'(X, \mathbf{d})$, with $S'(X, \mathbf{d})$ the subset of k for which our claim is true in $\mathcal{A}^B$.

Then if $M \in \mathcal{A}$-mod is indecomposable with $\mathbf{dim}M = \mathbf{d}$ and λ is an eigenvalue of $M(x)$ which is not in $S(X, \mathbf{d})$, we may assume $M = \theta^*(N)$. Here $M(x) = N(x)$, thus $N \cong N'$, with N' concentrated in X, but this implies that $\theta^*(N')$ is concentrated in X, thus $\theta^*(N') \cong \theta^*(N) = M$, proving our claim.

Case 2. $\delta(a_1) = 0$. Since $\mathcal{A}$ is not wild, by Proposition 9 of [7], $A'(X_1, X_1) = kid_{X_1}$ and $A'(Y_1, Y_1) = kid_{Y_1}$. Here X_1 is not equal to Y_1. We have the edge reduction of a_1, $\theta^* : \mathcal{A}^B$-Mod $\to \mathcal{A}$-Mod, with $\mathcal{A}^B = (B, W)$. Consider the dimension vectors $\mathbf{d}_1, ..., \mathbf{d}_l$ of those $N \in \mathcal{A}^B$-mod such that $\mathbf{dim}\theta^*(N) = \mathbf{d}$.

The norms of the $\mathbf{d}_i$ are smaller than r. Here X is not equal to X_1 and to Y_1. Therefore X is an indecomposable object of B'. We may consider the subsets $S(X, \mathbf{d}_1), ..., S(X, \mathbf{d}_l)$. Take $S(X, \mathbf{d}) = S(X, \mathbf{d}_1) \cup ... \cup S(X, \mathbf{d}_l)$.

Let M be an indecomposable in $\mathcal{A}$-mod with $\mathbf{dim}M = \mathbf{d}$. Suppose λ is an eigenvalue of $M(x)$ which is not in $S(X, \mathbf{d})$. Since θ^* is an equivalence there is a $N \in \mathcal{A}^B$-mod such that $\theta^*(N) \cong M$. We may assume $\theta^*(N) = M$, then $M(X) = N(X)$ and $M(x) = N(x)$. Here $\mathbf{dim}N = \mathbf{d_i}$ for some $i \in [1, l]$. Therefore, since λ is an eigenvalue of $N(x)$ which is not in $S(X, \mathbf{d}_i)$, $N \cong N'$, with N' concentrated in X, consequently $\theta^*(N')$ is concentrated in X and $\theta^*(N') \cong M$.

Case 3. $a_1 : X_1 \to Y_1$ with $A'(X_1, X_1) \neq kid_{X_1}$ or $A'(Y_1, Y_1) \neq kid_{Y_1}$.

Using the notation of [5], we have an unraveling in X_1 or in Y_1, for r and some elements of k, $\lambda_1, ..., \lambda_s$ followed by regularization of $b : Y \to Y_1$ or of $b : X_1 \to Y$, with b the generator corresponding to a_1. Let $\theta^* : \mathcal{A}^B$-Mod $\to \mathcal{A}$-Mod be the unraveling functor followed by the corresponding regularization, with $\mathcal{A}^B = (B, W)$ and layer $(B', \omega_B; b_1, ..., b_v; w_1, ..., w_u)$.

In case X is not equal to X_1 and to Y_1 we proceed as in Case 2.

Suppose now that the unraveling is in X with $X = X_1$ or $X = Y_1$, such that $\theta(X) = Y \oplus (\bigoplus_{i,j} Z^i_{i,j})$. Take all dimension vectors $\mathbf{d}_1, ..., \mathbf{d}_l$ of those $N \in \mathcal{A}^B$-mod with $\mathbf{dim}\theta^*(N) = \mathbf{d}$.

The norms of all $\mathbf{d_i}$ are smaller than r. Then we may take $S(Y, \mathbf{d}_i)$. We put $S(X, \mathbf{d}) = S(Y, \mathbf{d}_1) \cup ... \cup S(Y, \mathbf{d}_l) \cup \{\lambda_1, ..., \lambda_s\}$.

Let M be an indecomposable in $\mathcal{A}$-mod with $\mathbf{dim}M = \mathbf{d}$, $M(X) \neq 0$ and λ an eigenvalue of $M(x)$ which is not in $S(X, \mathbf{d})$.

There is a $N \in \mathcal{A}^B$ with $\theta^*(N) \cong M$. We may assume $\theta^*(N) = M$. There is a $\mathbf{d}_i$ with $i \in [1, l]$ such that $\mathbf{dim}N = \mathbf{d}_i$.

Here $M(x) = N(x) \oplus M'(x)$ with eigenvalues of $M'(x)$ contained in $\{\lambda_1, ..., \lambda_s\}$. The eigenvalue λ of $M(x)$ is not in $S(X, \mathbf{d})$, therefore, λ is an eigenvalue of $N(x)$. But λ is not in $S(Y, \mathbf{d}_i)$, then $N \cong N'$, with N' concentrated in Y. This implies that $\theta^*(N')$ is concentrated in X and $M \cong \theta^*(N')$. □

Notation 7.3. We recall that if $\mathbf{d}$ and $\mathbf{d}'$ are dimension vectors of the bocs $\mathcal{A} = (A, V)$ we say that $\mathbf{d} \leq \mathbf{d}'$ if for all indecomposable objects X of A', $\mathbf{d}(X) \leq \mathbf{d}'(X)$. Then if $\mathcal{D}$ is a finite set of dimension vectors of $\mathcal{A}$, we denote by $s(\mathcal{D})$ the set consisting of all vectors in $\mathcal{D}$, all sums $\mathbf{d} + \mathbf{d}'$ with $\mathbf{d}, \mathbf{d}' \in \mathcal{D}$, and all vectors $\mathbf{e}$ with $\mathbf{e} \leq \mathbf{f}$ with $\mathbf{f}$ one of the above dimension vectors. Clearly $s(\mathcal{D})$ is also a finite set.

Definition 7.4. Let $\mathcal{A} = (A, V)$ be a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$ and $\mathcal{D}$ be a finite set of dimension vectors of $\mathcal{A}$. We say that X, an indecomposable object in A', with $A'(X, X) = k[x, f(x)^{-1}]id_X$ is $\mathcal{D}$-isolated if for any indecomposable $M \in \mathcal{A}$-mod with $\mathbf{dim}M \in s(\mathcal{D})$ and $M(X) \neq 0$, there is a $M' \in \mathcal{A}$-mod, concentrated in X with $M \cong M'$.

Lemma 7.5. *Let $\mathcal{A} = (A, V)$ be a layered bocs as above, which is not of wild representation type, and $\mathcal{D}$ be a finite set of dimension vectors of $\mathcal{A}$ such that for all indecomposable $X \in A'$ there is a $\mathbf{d} \in \mathcal{D}$ with $\mathbf{d}(X) \neq 0$, and $a_1 : X_1 \to Y_1$. Then*

(1) if X_1 and Y_1 are both $\mathcal{D}$-isolated and $\delta(a_1) \in \mathcal{I}_2\overline{V} + \overline{V}\mathcal{I}_1$ with $\mathcal{I}_1$ an ideal of $A'(X_1, X_1)$, $\mathcal{I}_2$ an ideal of $A'(Y_1, Y_1)$, then $\mathcal{I}_1 = A'(X_1, X_1)$ or $\mathcal{I}_2 = A'(Y_1, Y_1)$;

(2) if X_1 is $\mathcal{D}$-isolated, $A'(Y_1, Y_1) = kid_{Y_1}$, $\delta(a_1) \in \overline{V}\mathcal{I}_1$ with $\mathcal{I}_1$ an ideal of $A'(X_1, X_1)$, then $\mathcal{I}_1 = A'(X_1, X_1)$;

(3) if Y_1 is $\mathcal{D}$-isolated, $A'(X_1, X_1) = kid_{X_1}$, $\delta(a_1) \in \mathcal{I}_2\overline{V}$ with $\mathcal{I}_2$ an ideal of $A'(Y_1, Y_1)$, then $\mathcal{I}_2 = A'(Y_1, Y_1)$.

Proof. We have

$$(*) \quad \delta(a_1) = \sum_{s \in T_1} h_s v_s + \sum_{s \in T_2} v_s g_s$$

with $h_s \in \mathcal{I}_2, g_s \in \mathcal{I}_1$.

(1) Suppose our claim is not true, then we may assume $\mathcal{I}_1$ and $\mathcal{I}_2$ are maximal ideals. Then $A'(X_1, X_1)/\mathcal{I}_1 \cong k$ and $A'(Y_1, Y_1)/\mathcal{I}_2 \cong k$. First assume $X_1 = Y_1$. Take the representation M of A such that $M(X_1) = M_1 \oplus M_2$ with $M_i = A'(X_1, X_1)/\mathcal{I}_i$ for $i = 1, 2$, $M(W) = 0$ for $W \neq X_1$. Take $M(a_1)$ such that $0 \neq M(a_1)(M_1) \subset M_2$, $M(a_1)(M_2) = 0$ and $M(a_j) = 0$ for $j > 1$. Here $\mathbf{dim}M \in s(\mathcal{D})$, then if M is indecomposable, $M \cong M'$ with M' concentrated in X_1, but this implies that M' is indecomposable as A'-module, which is not the case because as A'-modules, we have $M' \cong M \cong M_1 \oplus M_2$. Therefore, $M \cong L_1 \oplus L_2$, with L_1, L_2 indecomposables, and $\mathbf{dim}L_1, \mathbf{dim}L_2$ are in $s(\mathcal{D})$. Then $L_1 \cong L_1', L_2 \cong L_2'$, with L_1', L_2' concentrated in X_1, thus $M \cong L = L_1' \oplus L_2'$, and $L(a_1) = 0$. There is an isomorphism $f = (f^0, f^1) : M \to L$. Then from $(*)$ we obtain

$$L(a_1) f^0_{X_1} - f^0_{Y_1} M(a_1) = \sum_{s \in T_1} L(h_s) f^1(v_s) + \sum_{s \in T_2} f^1(v_s) M(g_s),$$

then, since $L(a_1) = 0$ and $\mathcal{I}_1 M_1 = 0$, from the above formula we obtain

$$f^0_{Y_1} M(a_1)(M) = f^0_{Y_1} M(a_1)(M_1) \subset \mathcal{I}_2 L,$$

then if $\mathcal{I}_1 = \mathcal{I}_2$, $\mathcal{I}_2 L = 0$, so $f^0_{Y_1} M(a_1)(M) = 0$. If $\mathcal{I}_1 \neq \mathcal{I}_2$, $A'(X_1, X_1) = \mathcal{I}_1 + \mathcal{I}_2$. We have

$$\mathcal{I}_1 f^0_{Y_1} M(a_1)(M) \subset \mathcal{I}_1 \mathcal{I}_2 L = 0,$$

$$\mathcal{I}_2 f^0_{Y_1} M(a_1)(M) \subset f^0_{Y_1}(\mathcal{I}_2 M_2) = 0.$$

Consequently, $f^0_{Y_1}M(a_1) = 0$, a contradiction to $M(a_1) \neq 0$. Thus we obtain our statement in this case.

Now, assume $X_1 \neq Y_1$, take M the representation of A such that $M(X_1) = A'(X_1, X_1)/\mathcal{I}_1$, $M(Y_1) = A'(Y_1, Y_1)/\mathcal{I}_2$, $M(Z) = 0$ for Z indecomposable non-isomorphic to X_1 or Y_1; $M(a_1) \neq 0$ and $M(a_j) = 0$ for all $j > 1$. Clearly $\mathbf{dim} M \in s(\mathcal{D})$. We claim that $M \cong L$ with $L(a_1) = 0$. In fact if M is indecomposable then $M \cong M'$ with M' concentrated in X_1 since $M(X_1) \neq 0$, and $M \cong M''$ with M'' concentrated in Y_1, since $M(Y_1) \neq 0$. Thus $X_1 = Y_1$ a contradiction, therefore M is decomposable $M \cong L = L_1 \oplus L_2$ with $L_1(X_1) \cong M(X_1)$, $L_1(Y_1) = 0$ and $L_2(X_1) = 0, L_2(Y_1) \cong M(Y_1)$, consequently, $L_1(a_1) = 0$ and $L_2(a_1) = 0$, and, therefore $L(a_1) = 0$, proving our claim.

Then there is an isomorphism $(f^0, f^1) : M \to L$. Here $f^0_{X_1} : M(X_1) \to L(X_1)$ and $f^0_{Y_1} : M(Y_1) \to L(Y_1)$ are isomorphisms. From $(*)$ we obtain

$$L(a_1)f^0_{X_1} - f^0_{Y_1}M(a_1) = \sum_{s\in T_1} L(h_s)f^1(v_s) + \sum_{s\in T_2} f^1(v_s)M(g_s) = 0,$$

consequently, $f^0_{Y_1}M(a_1) = 0$, so $M(a_1) = 0$, a contradiction.

(2) We are assuming that X_1 is $\mathcal{D}$-isolated, by Definition 7.4, $A'(X_1, X_1) \neq kid_{X_1}$. Here we suppose $A'(Y_1, Y_1) = kid_{Y_1}$, then $X_1 \neq Y_1$. If our claim is not true, we may assume that $\mathcal{I}_1$ is a maximal ideal and $A'(X_1, X_1)/\mathcal{I}_1 = k$. Consider now M, the representation of A, such that $M(X_1) = A'(X_1, X_1)/\mathcal{I}_1, M(Y_1) = k$, $M(Z) = 0$ for Z indecomposable non-isomorphic to X_1 and to Y_1, $M(a_1) \neq 0, M(a_j) = 0$ for all $j \geq 2$. If M is indecomposable, then $M \cong M'$ with M' concentrated in X_1, since $M(X_1) \neq 0$, a contradiction to $M(Y_1) \neq 0$. If M is decomposable, we may construct a module $L = L_1 \oplus L_2$ and lead to a contradiction similar to (1).

(3) The proof is similar to (2). □

Remark 7.6. Let $\mathcal{A}$ be a non wild bocs and $\theta : A \to B$ any of our reduction functors such that it does not delete marked indecomposable objects. If $\mathcal{A}$ has layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$ and $\mathcal{A}^B$ has layer $(B'; \omega_B; b_1, ..., b_{n'}; w_1, ...w_{m'})$, then to each marked $X \in \text{ind}A'$ corresponds a marked $X^m \in B'$ such that $\theta(X) = X^m \oplus Y$ with Y either 0 or a sum of non-marked indecomposables. Conversely each marked object in B' is equal to some X^m. Moreover,

i) if $N \in \mathcal{A}^B$-Mod is concentrated in X^m then $\theta^*(N)$ is concentrated in X.

ii) Suppose $N \in \mathcal{A}^B$-Mod is indecomposable with $N(X^m) \neq 0$ and $\theta^*(N) \cong M$ with M concentrated in X, then there exists $N' \in \mathcal{A}^B$-Mod concentrated in X^m such that $N' \cong N$.

Lemma 7.7. *If $\theta : A \to B$ is a reduction functor and $(e) : M \xrightarrow{f} E \xrightarrow{g} N$ is a proper exact sequence in $\mathcal{A}^B$-mod, then $\theta^*(e) : \theta^*(M) \xrightarrow{\theta^*(f)} \theta^*(E) \xrightarrow{\theta^*(g)} \theta^*(N)$ is a proper exact sequence in $\mathcal{A}$-mod (see Definition 4.6).*

Proof. Let $f : L \to H$ be a morphism in $\mathcal{A}^B$-Mod. From the explicit description of θ^*

for each of the reduction functors given in section 4 of [5] one can see that if $(i, \omega_B)^*(f)$ is a monomorphism (respectively an epimorphism), then $(i, \omega)^*\theta^*(f)$ is a monomorphism (respectively an epimorphism). We have $\mathbf{dim}E = \mathbf{dim}M + \mathbf{dim}N$, then $\mathbf{dim}\theta^*(E) = t^\theta(\mathbf{dim}E) = \mathbf{dim}\theta^*(M) + \mathbf{dim}\theta^*(N)$. Therefore, $\dim_k\theta^*(E)(X) = \dim_k\theta^*(M)(X) + \dim_k\theta^*(N)(X)$, for each $X \in \mathrm{ind}A'$. From this and our first observation we may conclude that $\theta^*(e)$ is a proper exact sequence, proving our claim. $\square$

8 An improvement of the Tame Theorem

In this section, we prove in Theorem 8.5 that given a tame layered bocs $\mathcal{A}$ and a positive integer r, then there is a minimal layered bocs $\mathcal{B}$ and a functor $F : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$, which is a composition of the reduction functors of section 7, such that for any M representation of $\mathcal{A}$, with dimension smaller than or equal to r there is a representation N of $\mathcal{B}$ with $F(N) \cong M$. This is an improvement of Theorem A in [5] which needs several minimal bocses.

We recall that if $\mathcal{A} = (A, V)$ is a bocs, then a family $\mathcal{F}$ of non-isomorphic indecomposable objects in $\mathcal{A}$-mod is called a one-parameter family if there is T an A-$k[x, f(x)^{-1}]$-bimodule free of finite rank as right $k[x, f(x)^{-1}]$-module, such that for all $\lambda \in k$ which is not a root of $f(x)$, there is a $N \in \mathcal{F}$ with $T \otimes_{k[x,f(x)^{-1}]} k[x]/(x-\lambda) \cong N$ and for each $N \in \mathcal{F}$ there is an unique $\lambda \in k$ which is not a root of $f(x)$ with $N \cong T \otimes_{k[x,f(x)^{-1}]} k[x]/(x-\lambda)$.

Two one-parameter families $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be equivalent if there is only a finite number of elements in $\mathcal{F}_1$ which are not isomorphic to objects in $\mathcal{F}_2$. It follows from Theorem 5.6 of [6] that if $\mathcal{A}$ is not of wild representation type and $\mathcal{D}$ is a finite set of dimension vectors there is only a finite number $m(\mathcal{A}, \mathcal{D})$ of non-equivalent one-parameter families of objects in $\mathcal{A}$-mod having dimension vectors in $s(\mathcal{D})$. Observe that the number of $\mathcal{D}$-isolated objects X in A' is smaller than or equal to $m(\mathcal{A}, \mathcal{D})$.

In the following, $\mathcal{A}_0 = (A_0, V_0)$ is a fixed layered bocs which is not of wild representation type and $\mathcal{D}_0$ a fixed finite set of dimension vectors of $\mathcal{A}_0$. Consider the family $\mathcal{P}$ of pairs $(\mathcal{A}, \mathcal{D})$ with $\mathcal{A}$ a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$, $\mathcal{D}$ a finite set of dimension vectors of $\mathcal{A}$ such that there exists $\theta : A_0 \to A$ a composition of reduction functors with $\mathcal{A}_0^A = \mathcal{A}$ and $t^\theta(\mathcal{D}) \subset \mathcal{D}_0$. We denote by m_0 the number $m(\mathcal{A}_0, s(\mathcal{D}_0))$. Observe that since θ^* is a full and faithful functor and $\mathcal{A}_0$ is not of wild representation type, then $\mathcal{A}$ is not of wild representation type.

If $(\mathcal{A}, \mathcal{D}) \in \mathcal{P}$, for each $X \in \mathrm{ind}A'$ which is $\mathcal{D}$-isolated we have a one-parameter family of representations of $\mathcal{A}$. To different $\mathcal{D}$-isolated indecomposables in $\mathrm{ind}A'$ correspond non-equivalent one-parameter families of representations of $\mathcal{A}$. By the definition of $\mathcal{P}$, there exists a composition of reduction functors $\theta : A_0 \to A$ with $t^\theta(\mathcal{D}) \subset \mathcal{D}_0$. Therefore, the image under θ^* of the one-parametric family corresponding to a $\mathcal{D}$-isolated indecomposable in A' is a one-parametric family of $\mathcal{A}_0$ with dimension vector in $s(\mathcal{D}_0)$. Therefore, the number of $\mathcal{D}$-isolated indecomposables in A' is smaller or equal to m_0.

Notation. Suppose $\mathcal{A}$ is a layered bocs which is not of wild representation type and $\mathcal{D}$ is a finite set of dimension vectors of $\mathcal{A}$. For j a non-negative integer, we denote by

$\mathcal{S}(\mathcal{A},\mathcal{D})(j)$ the subset of $\mathcal{D}$ consisting of the $\mathbf{d}$ in $\mathcal{D}$ with $||\mathbf{d}|| = j$.
Take $(\mathcal{A},\mathcal{D})$ a pair in $\mathcal{P}$, we define a function $c(\mathcal{A},\mathcal{D}) : \{-1, 0, 1, 2, ..., \infty\} \to \{0, 1, 2, ...\}$ in the following way:

$$c(\mathcal{A},\mathcal{D})(\infty) = m_0 - i(\mathcal{A},\mathcal{D})$$

with $i(\mathcal{A},\mathcal{D})$ the number of indecomposables in A' which are $\mathcal{D}$-isolated.

$$c(\mathcal{A},\mathcal{D})(-1) = n$$

where n is the number of a_i in the layer of $\mathcal{A}$. For j a non-negative integer we put

$$c(\mathcal{A},\mathcal{D})(j) = \mathrm{Card}\mathcal{S}(\mathcal{A},\mathcal{D})(j).$$

The functions $c(\mathcal{A},\mathcal{D})$ belong to $\mathcal{H}$, the set of functions

$$f : \{-1, 0, 1, ..., \infty\} \to \{0, 1, ..., \}$$

with $f(x) = 0$ for almost all $x \in \{-1, 0, 1, ..., \infty\}$.

If f, g are elements in $\mathcal{H}$ we put $f < g$ if there is a s in $\{-1, 0, 1, ..., \infty\}$ such that $f(s) < g(s)$ and $f(u) = g(u)$ for $u \in \{-1, 0, 1, ..., \infty\}, u > s$. Clearly if we have an infinite sequence of elements in $\mathcal{H}$ with:

$$f_1 \geq f_2 \geq .... \geq f_m \geq f_{m+1} \geq ....$$

then there exists l such that for all $m > l$, $f_m = f_l$.

Notation. If $\theta : A \to B$ is any of our reduction functors and $\mathcal{D}$ is a finite set of dimension vectors of $\mathcal{A}$, we say that θ^* is $\mathcal{D}$-covering if for each $M \in \mathcal{A}$-mod with $\mathbf{dim}M \in \mathcal{D}$ there exists a $N \in \mathcal{A}^B$-mod with $\theta^*(N) \cong M$. If $\theta : A \to B$ is a composition of our reduction functors, we denote by $\mathcal{D}^B$ the set of $\mathbf{d}' \in \mathrm{Dim}(\mathcal{A}^B)$ such that $t^\theta(\mathbf{d}') \in \mathcal{D}$.

In the statement of the following Lemma, we use the notation of Remark 7.6.

Lemma 8.1. *Let $\theta : A \to B$ be any of our reduction functors such that it does not delete marked objects. Then if X is $\mathcal{D}$-isolated, one has that X^m is $\mathcal{D}^B$-isolated. Conversely if θ is a regularization or the deletion of an object W such that $\mathbf{d}(W) = 0$ for all $\mathbf{d} \in \mathcal{D}$ and X^m is $\mathcal{D}^B$-isolated then X is $\mathcal{D}$-isolated.*

Proof. Suppose X is $\mathcal{D}$-isolated in $\mathcal{A}$. We shall prove that X^m is $\mathcal{D}^B$-isolated in $\mathcal{A}^B$. For this take an indecomposable $N \in \mathcal{A}^B$-mod, with $\mathbf{dim}N \in s(\mathcal{D}^B)$ and $N(X^m) \neq 0$. Consider $M = \theta^*(N)$, then following the notation of Remark 7.6, $M(X) = N(X^m) \oplus N(Y)$, thus $M(X) \neq 0$, moreover $\mathbf{dim}M \in s(\mathcal{D})$. Since X is $\mathcal{D}$-isolated, then there exists $M' \in \mathcal{A}$-mod, with $M \cong M'$ and M' concentrated in X. Therefore, by Remark 7.6 there is a N' concentrated in X^m such that $N \cong N'$. From here we conclude that X^m is $\mathcal{D}^B$-isolated. This proves the first part of our claim.

Suppose now that θ is a regularization. In this case $t^\theta = id$ and $\mathcal{D}^B = \mathcal{D}$. Suppose X^m is $\mathcal{D}^B$-isolated, let us prove that X is $\mathcal{D}$-isolated. Let M be an indecomposable in

$\mathcal{A}$-mod, with $\mathbf{dim}M \in s(\mathcal{D})$ and $M(X) \neq 0$. Since θ^* is an equivalence of categories, there is a $N \in \mathcal{A}^B$-mod with $\theta^*(N) \cong M$. We have $N(X^m) = M(X)$, and, therefore, $N(X^m) \neq 0$. Moreover, $\mathbf{dim}N \in s(\mathcal{D}^B)$. Since X^m is $\mathcal{D}^B$-isolated, there is a $N' \in \mathcal{A}^B$-mod, concentrated in X^m such that $N' \cong N$. We have $M' = \theta^*(N')$ is concentrated in X, clearly $M \cong M'$, proving our claim.

A similar proof is done for the case θ is the deletion of an indecomposable W with $\mathbf{d}(W) = 0$ for all $\mathbf{d} \in \mathcal{D}$. □

Lemma 8.2. *Let $\theta : A \to B$ be a reduction functor which is not an unraveling or the deletion of some X for which there is a $\mathbf{d} \in \mathcal{D}$ with $\mathbf{d}(X) \neq 0$. Suppose there is a $\mathbf{d}'$ with $t^\theta(\mathbf{d}') \in \mathcal{D}$ and $||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||$. Let*

$$r = \max\{||t^\theta(\mathbf{d}')|| \mid t^\theta(\mathbf{d}') \in \mathcal{D}, \text{ and } ||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||\}.$$

Then for $j > r$,

$$c(\mathcal{A}^B, \mathcal{D}^B)(j) = c(\mathcal{A}, \mathcal{D})(j) \quad \text{and} \quad c(\mathcal{A}^B, \mathcal{D}^B)(r) < c(\mathcal{A}, \mathcal{D})(r).$$

Proof. Let us prove first that for $j \geq r$, t^θ induces an injective function

$$t^\theta_j : \mathcal{S}(\mathcal{A}^B, \mathcal{D}^B)(j) \to \mathcal{S}(\mathcal{A}, \mathcal{D})(j).$$

Take $\mathbf{d}' \in \mathcal{S}(\mathcal{A}^B, \mathcal{D}^B)(j)$, then $||t^\theta(\mathbf{d}')|| \geq ||\mathbf{d}'|| = j \geq r$. By definition of r, $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'|| = j$. Thus, t^θ induces a function t^θ_j. If $t^\theta_j(\mathbf{d}') = t^\theta_j(\mathbf{d}'')$, we have $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'||$ and $||t^\theta(\mathbf{d}'')|| = ||\mathbf{d}''||$, therefore $\mathbf{d}' = \mathbf{d}''$. Consequently, t^θ_j is an injective function.

Suppose $j > r$. Take $\mathbf{d} \in \mathcal{S}(\mathcal{A}, \mathcal{D})(j)$, since θ^* does not delete indecomposable objects $X \in \text{ind}A'$ for which there is a $\mathbf{f} \in \mathcal{D}$ with $\mathbf{f}(X) \neq 0$ then there is a $\mathbf{d}' \in \mathcal{S}(\mathcal{A}^B, \mathcal{D}^B)$ with $t^\theta(\mathbf{d}') = \mathbf{d}$. We have $r < ||\mathbf{d}|| = ||t^\theta(\mathbf{d}')|| \geq ||\mathbf{d}'||$. By definition of r, $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'|| = j$. Thus $\mathbf{d}' \in \mathcal{S}(\mathcal{A}^B, \mathcal{D}^B)(j)$. Consequently, t^θ_j is a bijective function and we have proved the first part of our claim.

For the second part of our claim, take $\mathbf{d}' \in \mathcal{D}^B$ such that $r = ||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||$. We have $\mathbf{d} = t^\theta(\mathbf{d}')$ in $\mathcal{S}(\mathcal{A}, \mathcal{D})(r)$. Let us prove that $\mathbf{d}$ is not in the image of $t^\theta_r : \mathcal{S}(\mathcal{A}^B, \mathcal{D}^B)(r) \to \mathcal{S}(\mathcal{A}, \mathcal{D})(r)$. If θ is a regularization or deletion of objects, t^θ is an injective function and if $\mathbf{d} = t^\theta_r(\mathbf{d}'')$, with $||\mathbf{d}''|| = r$, since t^θ is injective we have $\mathbf{d}' = \mathbf{d}''$, a contradiction. We only need consider the case in which θ is an edge reduction of $a_1 : X_1 \to Y_1$. Since $||\mathbf{d}|| = ||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||$, $\mathbf{d}(X_1)\mathbf{d}(Y_1) \neq 0$ and if $\mathbf{d} = t^\theta(\mathbf{d}'')$ then $r = ||t^\theta(\mathbf{d}'')|| > ||\mathbf{d}''||$, proving our claim. □

Lemma 8.3. *Suppose $(A, \mathcal{D})$ is a pair in $\mathcal{P}$. Let $\theta : A \to B$ be the deletion of a non-marked indecomposable $X \in A'$, such that for all $\mathbf{d} \in \mathcal{D}$, $\mathbf{d}(X) = 0$, then $c(\mathcal{A}^B, \mathcal{D}^B)(u) = c(\mathcal{A}, \mathcal{D})(u)$ for all $u \in \{0, 1, .., \infty\}$.*

Proof. By Lemma 8.1 $c(\mathcal{A}^B, \mathcal{D}^B)(\infty) = c(\mathcal{A}, \mathcal{D})(\infty)$. On the other hand, by our hypothesis, t^θ induces a bijective function $t^\theta : \mathcal{D}^B \to \mathcal{D}$ and $||t^\theta(\mathbf{d})|| = ||\mathbf{d}||$, for all $\mathbf{d} \in \mathcal{D}^\theta$.

Therefore, $c(\mathcal{A}^B,\mathcal{D}^B)(j) = c(\mathcal{A},\mathcal{D})(j)$ for all non-negative integers j. This proves our claim. □

Lemma 8.4. *Let $(\mathcal{A},\mathcal{D})$ be a pair in $\mathcal{P}$. Suppose that for each $X \in \mathrm{ind}A'$ there exists $\mathbf{d} \in \mathcal{D}$ with $\mathbf{d}(X) \neq 0$. Then, if $\mathcal{A}$ is not a minimal bocs, there is a composition of reduction functors $\theta : A \to B$, with θ^* a $s(\mathcal{D})$-covering functor, such that $c(\mathcal{A}^B,\mathcal{D}^B) < c(\mathcal{A},\mathcal{D})$, or there is a change of layer of $\mathcal{A}$ such that if $c'(\mathcal{A},\mathcal{D})$ is the corresponding function we have $c'(\mathcal{A},\mathcal{D}) < c(\mathcal{A},\mathcal{D})$.*

Proof. (1) Suppose $a_1 : X_1 \to X_1$ and $\delta(a_1) = 0$. Since $\mathcal{A}$ is not of wild representation type, then by Proposition 9 of [7] we have $A'(X_1,X_1) = kid_{X_1}$. Take $B' = A'(a_1)$ and change the layer $(A';\omega;a_1,...,a_n;v_1,...v_m)$ by the layer $(B';\omega;a_2,...,a_n;v_1,...,v_m)$. We have $B'(X_1,X_1) = k[a_1]id_{X_1}$. Clearly if W is an object non isomorphic to X_1 in $\mathrm{ind}A'$, this object is $\mathcal{D}$-isolated with respect to the original layer of $\mathcal{A}$ if and only if it is $\mathcal{D}$-isolated with respect to the new layer. Here it is possible that X_1, which is not marked with respect to the original layer of $\mathcal{A}$, becomes a $\mathcal{D}$ -isolated object with respect to the new layer. Therefore, if we denote by $c'(\mathcal{A},\mathcal{D})$ the corresponding function with respect to the new layer we have $c'(\mathcal{A},\mathcal{D})(\infty) \leq c(\mathcal{A},\mathcal{D})(\infty)$.

The norm of a dimension vector does not depend of the choice of the layer, therefore, $c'(\mathcal{A},\mathcal{D})(j) = c(\mathcal{A},\mathcal{D})(j)$ for all non-negative integers j. Moreover,

$$c'(\mathcal{A},\mathcal{D})(-1) = c(\mathcal{A},\mathcal{D})(-1) - 1.$$

Therefore, $c'(\mathcal{A},\mathcal{D}) < c(\mathcal{A},\mathcal{D})$.

(2) Suppose there is a marked $X \in \mathrm{ind}A'$ which is not $\mathcal{D}$-isolated. Take $S = \bigcup_{\mathbf{d}\in s(\mathcal{D})} S(X,\mathbf{d})$, with $S(X,\mathbf{d})$ the sets of Proposition 7.2. Take r the maximal of the numbers $\mathbf{d}(X)$ with $\mathbf{d} \in s(\mathcal{D})$. Consider now the unraveling $\theta : A \to B$ in X with respect to r and S. Clearly, the functor $\theta^* : \mathcal{A}^B\text{-Mod} \to \mathcal{A}\text{-Mod}$ is a $s(\mathcal{D})$-covering functor. We have $\theta(X) = X^m \oplus \bigoplus_{i,j} Z^i_{i,j}$. We shall see that X^m is $\mathcal{D}^B$-isolated. Take N an indecomposable in $\mathcal{A}^B$-mod with $N(X^m) \neq 0$ and $\mathbf{dim}N \in s(\mathcal{D}^B)$, then $\mathbf{dim}\theta^*(N) \in s(\mathcal{D})$. We have $\theta^*(N)(X) = N(X^m) \oplus \bigoplus_{i,j} N(Z_{i,j})^i \neq 0$. Take any eigenvalue of $N(x)$, this is an eigenvalue of $\theta^*(N)(x)$ which is not in S, therefore, it is not in $S(X,\mathbf{d})$ with $\mathbf{d} = \mathbf{dim}\theta^*(N)$. Therefore, by Proposition 7.2, $\theta^*(N) \cong M$, with M concentrated in X. But this implies that $M(x)$ has only one eigenvalue which is not in S. Therefore, $M \cong \theta^*(N')$ with N' concentrated in X^m. But $N \cong N'$, this proves that X^m is $\mathcal{D}^B$-isolated. We have

$$c(\mathcal{A}^B,\mathcal{D}^B)(\infty) \leq c(\mathcal{A},\mathcal{D})(\infty) - 1.$$

Therefore, $c(\mathcal{A}^B,\mathcal{D}^B) < c(\mathcal{A},\mathcal{D})$.

(3) Suppose $a_1 : X_1 \to Y_1$ with $\delta(a_1) = 0$ and $X_1 \neq Y_1$. Take $\theta : A \to B$ the reduction of a_1. By Lemma 8.1, $c(\mathcal{A}^B,\mathcal{D}^B)(\infty) \leq c(\mathcal{A},\mathcal{D})(\infty)$. If there is a $\mathbf{d}' \in \mathcal{D}^B$ such that $||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||$, by Lemma 8.2, $c(\mathcal{A}^B,\mathcal{D}^B) < c(\mathcal{A},\mathcal{D})$. On the other hand if for all $\mathbf{d}' \in \mathcal{D}^B$, $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'||$, then again by Lemma 8.2, $c(\mathcal{A}^B,\mathcal{D}^B)(j) = c(\mathcal{A},\mathcal{D})(j)$ for all non-negative integers j. We have that for all $\mathbf{d} \in \mathcal{D}$, $\mathbf{d}(X_1)\mathbf{d}(Y_1) = 0$. This implies

that for all $\mathbf{d}' \in \mathcal{D}^B$, $\mathbf{d}'(Z_2) = 0$. Take $\theta : B \to C$ the deletion of Z_2. By Lemma 8.3 we have $c(((\mathcal{A})^B)^C, (\mathcal{D}^B)^C)(u) = c(\mathcal{A}^B, \mathcal{D}^B)(u) = c(\mathcal{A}, \mathcal{D})(u)$ for all $u \neq -1$. Moreover, $c(((\mathcal{A})^B)^C, (\mathcal{D}^B)^C)(-1) = c(\mathcal{A}, \mathcal{D})(-1) - 1$, therefore, $c((\mathcal{A}^B)^C), (\mathcal{D}^B)^C) < c(\mathcal{A}, \mathcal{D})$.

(4) $\delta(a_1) = v_1$. In this case take $\theta : A \to B$ the regularization of a_1. As in the above case if there is a $\mathbf{d}' \in \mathcal{D}^B$ with $||t^\theta(\mathbf{d}')|| > ||\mathbf{d}'||$, then $c(\mathcal{A}^B, \mathcal{D}^B) < c(\mathcal{A}, \mathcal{D})$. On the other hand if for all $\mathbf{d}' \in \mathcal{D}^B$, $||t^\theta(\mathbf{d}')|| = ||\mathbf{d}'||$, by Lemma 8.1 $c(\mathcal{A}^B, \mathcal{D}^B)(\infty) = c(\mathcal{A}, \mathcal{D})(\infty)$. By Lemma 8.2, $c(\mathcal{A}^B, \mathcal{D}^B)(j) = c(\mathcal{A}, \mathcal{D})(j)$ for all non-negative integers j. Moreover, $c(\mathcal{A}^B, \mathcal{D}^B)(-1) = c(\mathcal{A}, \mathcal{D})(-1) - 1$. Therefore, $c(\mathcal{A}^B, \mathcal{D}^B) < c(\mathcal{A}, \mathcal{D})$.

(5) $\delta(a_1) = \sum_{s \in T} r_s v_s$ with $a_1 : X_1 \to Y_1$, T the set of s such that $v_s \in \overline{V}(X_1, Y_1)$ and $r_s \in A'(Y_1, Y_1) \otimes_k (A'(X_1, X_1))^{op} = H$. If there is a marked object in indA' which is not $\mathcal{D}$-isolated we may proceed as in (2). Therefore, we may assume that all marked objects in indA' are $\mathcal{D}$-isolated. The ring H is isomorphic either to k, or to $k[x, f(x)^{-1}]$, or to $k[x, y, f(x)^{-1}, g(y)^{-1}]$. Let $\mathcal{I}$ be the ideal of H generated by the elements $\{r_s\}_{s \in T}$. If $\mathcal{I} \neq H$, then $A'(X_1, X_1) \neq kid_{X_1}$ or $A'(Y_1, Y_1) \neq id_{Y_1}$. Moreover there are ideals $\mathcal{I}_2 \subset A'(Y_1, Y_1)$ and $\mathcal{I}_1 \subset A'(X_1, X_1)$ with $\mathcal{I} \subset \mathcal{I}_2 \otimes_k (A'(X_1, X_1))^{op} + A'(Y_1, Y_1) \otimes_k \mathcal{I}_1$, $\mathcal{I}_2 \neq A'(Y_1, Y_1)$ and $\mathcal{I}_1 \neq A'(X_1, X_1)$. Thus, $\delta(a_1) \in \mathcal{I}_2 \overline{V}(X_1, Y_1) + \overline{V}(X_1, Y_1)\mathcal{I}_1$ with $\mathcal{I}_2 \neq A'(Y_1, Y_1)$ and $\mathcal{I}_1 \neq A'(X_1, X_1)$.

Then if $A'(X_1, X_1) \neq kid_{X_1}$ and $A'(Y_1, Y_1) \neq kid_{Y_1}$, both X_1 and Y_1 are $\mathcal{D}$-isolated. But this contradicts (1) of Lemma 7.5 (recall that $\mathcal{A}$ is not of wild representation type).

If $A'(X_1, X_1) \neq kid_{X_1}$ and $A'(Y_1, Y_1) = kid_{Y_1}$, then X_1 is marked, so it is $\mathcal{D}$-isolated, we have $\mathcal{I}_1 \neq A'(X_1, X_1)$, and $\mathcal{I}_2 = 0$, but this contradicts (2) of Lemma 7.5. In case $A'(X_1, X_1) = kid_{X_1}$, then Y_1 is a marked object in indA', so it is $\mathcal{D}$-isolated and this contradicts (3) of Lemma 7.5.

Therefore, $\mathcal{I} = H$ and $1 = \sum_{s \in T} u_i r_i$. This implies that there is a free basis of $\overline{V}(X_1, Y_1)$, with one of their elements equal to $\delta(a_1)$, then we may apply case (4). □

Theorem 8.5. *Let $\mathcal{A}_0 = (A_0, V_0)$ be a layered bocs which is not of wild representation type. Then given a positive integer r there is a composition of reduction functors $\theta : A_0 \to B$ with $\mathcal{A}^B$ a minimal layered bocs such that for all $M \in \mathcal{A}_0$-*mod *with $|M| \leq r$ there exists $N \in \mathcal{B}$-*Mod *with $\theta^*(N) \cong M$.*

Proof. Take $\mathcal{D}_0$ the set of $\mathbf{d} \in \mathrm{Dim}(\mathcal{A}_0)$ such that $\sum_{X \in \mathrm{ind} A'_0} \mathbf{d}(X) \leq r$, $\mathcal{D}_0$ is a finite set. Denote by $\mathcal{P}$ the family of pairs $(\mathcal{A}, \mathcal{D})$, with $\mathcal{A}$ a layered bocs, $\mathcal{D}$ a finite subset of $\mathrm{Dim}(\mathcal{A})$ such that there is a functor, composition of reduction functors $\theta : A_0 \to B$ with $t^\theta(\mathcal{D}) \subset \mathcal{D}_0$ and θ^* a $s(\mathcal{D}_0)$-covering functor.

Let $\mathcal{A} = (A, V)$ be a bocs with layer $(A'; \omega; a_1, ..., a_n; v_1, ..., v_m)$ and $\mathcal{D}$ be a set of dimension vectors of $\mathcal{A}$, such that $(\mathcal{A}, \mathcal{D})$ is in $\mathcal{P}$.

For $X \in \mathrm{ind} A'$ we denote by $\mathbf{d}_X$ the dimension vector of $\mathcal{A}$ such that $\mathbf{d}_X(X) = 1$ and $\mathbf{d}_X(Z) = 0$ for $Z \in \mathrm{ind} A'$ with $Z \neq X$.

We will consider non-empty sets $\mathcal{D}$ of dimension vectors of $\mathcal{A}$ with the following two conditions:

(a) If $\mathbf{d} \in \mathcal{D}$ and $\mathbf{d}' < \mathbf{d}$, then $\mathbf{d}' \in \mathcal{D}$.

(b) If X is a marked object in indA' then $\mathbf{d}_X \in \mathcal{D}$.

Let $\theta : A \to B$ be a reduction functor which does not delete marked objects of indA' and such that $\theta^* :$ Mod-$\mathcal{A}^B \to$ Mod-$\mathcal{A}$ is a $s(\mathcal{D})$-covering functor, we claim that if $\mathcal{D}$ satisfies properties (a) and (b), then $\mathcal{D}^B$ also satisfies these properties. Let $(B'; \omega; b_1, ..., b_t; w_1, ..., w_s)$ be a layer for $\mathcal{A}^B$.

Here θ^* is a $s(\mathcal{D})$-covering functor, then $\mathcal{D}^B$ is a non-empty set. Suppose now that $\mathcal{D}$ satisfies properties (a) and (b). Property (a) for $\mathcal{D}^B$, follows from the fact that $\mathbf{d}' < \mathbf{d}$ in $\mathcal{D}$ implies $t^\theta(\mathbf{d}') \leq t^\theta(\mathbf{d})$.

For proving property (b) of $\mathcal{D}^B$, suppose W is a marked object in B'. Then following the notation of Lemma 7.6, $W = X^m$ for some marked object $X \in$ indA'. Consider $\mathbf{d}_{X^m}$, dimension vector of $\mathcal{A}^B$. Then for $Z \in$ indA', $Z \neq X$ we have $\theta(Z) = \bigoplus_i Z_i$ with $Z_i \in$ indB', $Z_i \neq X^m$. Then $t^\theta(\mathbf{d}_{X^m})(Z) = \sum_i \mathbf{d}_{X^m}(Z_i) = 0$. We have $\theta(X) = X^m \oplus \bigoplus_j Y_j$ with $Y_j \in$ indB', $Y_j \neq X^m$, then $t^\theta(\mathbf{d}_{X^m})(X) = \mathbf{d}_{X^m}(X^m) = 1$. Consequently, $t^\theta(\mathbf{d}_{X^m}) = \mathbf{d}_X \in \mathcal{D}$, thus $\mathbf{d}_{X^m} \in \mathcal{D}^B$, proving our claim.

Now, suppose $\mathcal{D}$ satisfies properties (a) and (b), and $\theta : A \to B$ is the deletion of all objects $Z \in$ indA' such that $\mathbf{d}(Z) = 0$ for all $\mathbf{d} \in \mathcal{D}$. Since $\mathcal{D}$ satisfies property (b), then θ does not delete marked objects. Therefore, $\mathcal{D}^B$ satisfies properties (a) and (b).

Now, if $\mathcal{A}^B$ is not a minimal bocs, by Lemma 8.4 there is a reduction functor $\rho : B \to A_1$ such that ρ^* is a $s(\mathcal{D}^B)$-covering functor with

$$c((\mathcal{A}^B)^{A_1}, (\mathcal{D}^B)^{A_1}) < c(\mathcal{A}^B, \mathcal{D}^B),$$

or there exists a new layer for $\mathcal{A}^B$ such that

$$c'(\mathcal{A}^B, \mathcal{D}^B) < c(\mathcal{A}^B, \mathcal{D}^B).$$

By the proof of Lemma 8.4, we know that ρ does not delete marked objects, then $(\mathcal{D}^B)^{A_1}$ satisfies properties (a) and (b). Now for any $Z \in$ indB' there exists some $\mathbf{d} \in \mathcal{D}^B$ with $\mathbf{d}(Z) \neq 0$, thus $\mathbf{d}_Z \leq \mathbf{d}$, so by property (a), $\mathbf{d}_Z \in \mathcal{D}^B$, then $\mathcal{D}^B$ also satisfies property (b) with respect to the new layer.

Then starting from $(\mathcal{A}_0, \mathcal{D}_0)$, we can construct a sequence of composition of reduction functors:

$$A_0 \stackrel{\theta_0}{\to} A_1 \stackrel{\theta_1}{\to} A_2 \to ... \stackrel{\theta_{l-1}}{\to} A_l,$$

with sets of dimension vectors $\mathcal{D}_i = (\mathcal{D}_{i-1})^{A_i}$ of $\mathcal{A}_i = (\mathcal{A}_{i-1})^{A_i}$ having conditions (a) and (b), such that all functors θ_i^* are $s(\mathcal{D}_i)$-covering functors. Moreover, we have a strictly decreasing sequence in $\mathcal{H}$,

$$c(\mathcal{A}_0, \mathcal{D}_0) > c(\mathcal{A}_1, \mathcal{D}_1) > ... > c(\mathcal{A}_l, \mathcal{D}_l).$$

In $\mathcal{H}$ we can not have infinite strictly decreasing sequences, so there is a sequence of reduction functors as before with $\mathcal{A}_l$ a minimal bocs, proving our result. $\square$

9 Hom-spaces in $\mathcal{D}(\Lambda)$-mod and in $P(\Lambda)$

We may observe that if Λ_1 and Λ_2 are two Morita-equivalent finite-dimensional k-algebras, then Theorem 1.2 is valid for Λ_1 if and only if it is valid for Λ_2. Therefore, without loss of generality, we assume in the rest of the paper that Λ is a basic algebra.

Assume k is an algebraically closed field and $1 = \sum_{i=1}^{n} e_i$ is a decomposition of the unit element of Λ as a sum of pairwise orthogonal primitive idempotents. Then we have ${}_\Lambda\Lambda = \bigoplus_{i=1}^{n} \Lambda e_i$ a decomposition as sum of indecomposable projective Λ -modules and $\Lambda = S \oplus J$ a decomposition as a direct sum of S-S-bimodules, with $J = \text{rad}(\Lambda)$, $S = ke_1 \oplus ... \oplus ke_n$ a basic semisimple algebra. We can construct a basis $T = \{\alpha_1, ..., \alpha_m\}$ of J with $\alpha_j \in e_{s(j)}\text{rad}\Lambda e_{t(j)}$, inductively extending a basis of J^i to J^{i-1} by adding elements each of which lies in $e_s J e_t$ for some s and t. In the following, if L is a right S-modulo we denote its dual with respect to S by $L^* = \text{Hom}_S(L, S)$. For each element $\alpha_j \in e_{s(j)} T e_{t(j)}$ we define the element $\alpha_j^* \in J^*$, by $\alpha_j^*(\alpha_i) = 0$ for $\alpha_i \neq \alpha_j$ and $\alpha_j^*(\alpha_j) = e_{t(j)}$, clearly $\alpha_j^* \in e_{t(j)} J^* e_{s(j)}$ the elements α_j^* form a basis for J^*.

In the following, if U_1, U_2, U_3 are k-vector spaces we denote by $\begin{pmatrix} U_1 & 0 \\ U_2 & U_3 \end{pmatrix}$, the set of matrices of the form $\begin{pmatrix} u_1 & 0 \\ u_2 & u_3 \end{pmatrix}$, with $u_i \in U_i$, $i = 1, 2, 3$. With the usual sum of matrices and multiplication of scalars in k by matrices, the above set is a k-vector space.

In order to define the Drozd's bocs of Λ we need to consider the following two matrix algebras $A = \begin{pmatrix} S & 0 \\ J^* & S \end{pmatrix}$, and $A' = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$. We are going to define a coalgebra V over A which is isomorphic to the coalgebra given in Proposition 6.1 of [5]. First consider the morphism of S-S-bimodules:

$$m : J^* \xrightarrow{\nu^*} (J \otimes_S J)^* \cong J^* \otimes_S J^*$$

where $\nu : J \otimes_S J \to J$ is the multiplication. We have the k-vector spaces $W_0 = \begin{pmatrix} 0 & 0 \\ J^* & 0 \end{pmatrix}$, and $W_1 = \begin{pmatrix} J^* & 0 \\ 0 & J^* \end{pmatrix}$, the elements of both vector spaces can be multiplied as matrices by the right and the left by elements of A', thus W_0 and W_1 are A'-A'-bimodules.

We have a morphism of A'-A'-bimodules,

$$\underline{m} : W_1 \to W_1 \otimes_{A'} W_1$$

such that its composition with the isomorphism

$$W_1 \otimes_{A'} W_1 \cong \begin{pmatrix} J^* \otimes_S J^* & 0 \\ 0 & J^* \otimes_S J^* \end{pmatrix},$$

is the map that sends $\begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix}$ to $\begin{pmatrix} m(h) & 0 \\ 0 & m(g) \end{pmatrix}$.

Now, consider the k-vector space $\overline{V} = \begin{pmatrix} J^* & 0 \\ M \oplus M & J^* \end{pmatrix}$, with $M = J^* \otimes_S J^*$, this is an A-A-bimodule with the following actions of A over $\overline{V}$:

$$\begin{pmatrix} s_1 & 0 \\ g & s_2 \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ (w_1, w_2) & h_2 \end{pmatrix} = \begin{pmatrix} s_1 h_1 & 0 \\ (s_2 w_1 + g \otimes h_1, s_2 w_2) & s_2 h_2 \end{pmatrix},$$

$$\begin{pmatrix} h_1 & 0 \\ (w_1, w_2) & h_2 \end{pmatrix} \begin{pmatrix} s_1 & 0 \\ g & s_2 \end{pmatrix} = \begin{pmatrix} h_1 s_1 & 0 \\ (w_1 s_1, w_2 s_1 + h_2 \otimes g) & h_2 s_2 \end{pmatrix}.$$

The k-linear map $\delta : A \to \overline{V}$ given by

$$\delta(\begin{pmatrix} s_1 & 0 \\ h & s_2 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ (m(h), -m(h)) & s_2 \end{pmatrix},$$

is a derivation, thus it gives an extension of A-A-bimodules:

$$0 \to \overline{V} \xrightarrow{i} V \xrightarrow{\epsilon} A \to 0$$

where $V = \overline{V} \oplus A$ as right A-modules, and putting $\omega = (0, 1)$, the left action of A over V is given by $a(v + \omega b) = av + \delta(a)b + \omega ab$, for $a, b \in A$, $v \in \overline{V}$. Here $\overline{V}$ is generated by W_1 as A'-A'-bimodule. We have:

(a) $A \cong W_0^{\otimes} = A' \oplus W_0$.

(b) The multiplication map $A \otimes_{A'} W_1 \otimes_{A'} A \to \overline{V}$ is an isomorphism.

We have a morphism of A-A-bimodules $\mu : V \to V \otimes_A V$, with $\mu(\omega) = \omega \otimes \omega$ and for $v \in W_1$, $\mu(v) = v \otimes \omega + \omega \otimes v + \lambda(v)$, where λ is the composition of morphisms:

$$W_1 \xrightarrow{m} W_1 \otimes_{A'} W_1 \to \overline{V} \otimes_A \overline{V} \to V \otimes_A V.$$

The A-A-bimodule V is a coalgebra over A with counit ϵ and comultiplication μ.

We have $1 = \sum_{i=1,j=1}^{n,2} f_{i,j}$ a decomposition of the unit of A as a sum of pairwise orthogonal primitive idempotents, where $f_{i,2} = \begin{pmatrix} e_i & 0 \\ 0 & 0 \end{pmatrix}$ and $f_{i,1} = \begin{pmatrix} 0 & 0 \\ 0 & e_i \end{pmatrix}$.

Denote by D the full subcategory of A-proj whose objects are all finite direct sums of objects $Af_{i,j}$. By D' we denote the subcategory of D with the same objects as D and such that $D'(X,X) = kid_X$ for all $X \in \mathrm{ind}D$ and $D'(X,Y) = 0$ for $X, Y \in \mathrm{ind}D$ with $X \neq Y$. If Af and Ag are in $\mathrm{ind}D$, and $x \in fAg$ we denote by $\nu_x : Af \to Ag$ the right multiplication by x.

Now, if W is an A-A-bimodule we denote by $\vartheta(W)$ the D-D bimodule given by $\vartheta(W)(Af, Ag) = fWg$ and if $\nu_x : Af' \to Af$, $\nu_y : Ag \to Ag'$ are morphisms then $\vartheta(W)(\nu_x, \nu_y) : \vartheta(W)(Af, Ag) \to \vartheta(W)(Af', Ag')$ is given by $\vartheta(W)(\nu_x, \nu_y)(w) = xwy$ for $w \in \vartheta(W)(Af, Ag)$. Similarly, for L a right A-module and M a left A-module we define functors, $\vartheta(L) : D \to \text{Mod-}k$ and $\vartheta(M) : D^{op} \to \text{Mod-}k$. If $f : W_1 \to W_2$ is a morphism of A-A-bimodules we have an induced morphism $\vartheta(f) : \vartheta(W_1) \to \vartheta(W_2)$. If $g : W_2 \to W_3$ is a morphism of A-A-bimodules then $\vartheta(f_2 f_1) = \vartheta(f_2)\vartheta(f_1)$. The morphisms between left A-modules and right A-modules induce also morphisms between the corresponding functors.

Fixed L a right A-module we have $F : A\text{-mod} \to \text{Mod-}k$, given in objects by $F(M) = \vartheta(L) \otimes_D \vartheta(M)$ and if $f : M_1 \to M_2$ is a morphism of left A-modules, then $F(f) = 1 \otimes \vartheta(f)$. The functor F is right exact and commutes with direct sums. Consequently, $F \cong W \otimes_A M$, with W the right A-module $\vartheta(L)(A) \cong L$, therefore $\vartheta(L) \otimes_D \vartheta(M) \cong L \otimes_A M$ an isomorphism natural in L and M.

Now, suppose V_1 and V_2 are A-A-bimodules then for $Af, Ag \in \mathrm{ind}D$ we have $(\vartheta(V_1) \otimes_D \vartheta(V_2))(Af, Ag) = \vartheta(V_1)(Af, -) \otimes_D \vartheta(V_2)(-, Ag) \cong \vartheta(fV_1) \otimes_D \vartheta(V_2 g) \cong fV \otimes_A Vg$. Now, it is easy to see that in fact we have :

$$(c) \qquad \vartheta(V_1) \otimes_D \vartheta(V_2) \cong \vartheta(V_1 \otimes_A V_2)$$

The morphism of A-bimodules $\mu : V \to V \otimes_A V$ induces a morphism of D-D-bimodules $\vartheta(\mu) : \vartheta(V) \to \vartheta(V) \otimes_D \vartheta(V)$. In a similar way the morphism of A-A bimodules $\epsilon : V \to A$ induces a morphism of D-D-bimodules $\vartheta(\epsilon) : \vartheta(V) \to \vartheta({}_A A_A) \cong D$. Now it is clear that $\mathcal{D}(\Lambda) = (D, V_D)$ with $V_D = \vartheta(V)$ is a bocs, the Drozd's bocs of Λ.

The bocs $\mathcal{D}(\Lambda)$ is isomorphic to the one given in Theorem 4.1 of [8] (see also the bocs given in the proof of Theorem 11 in [7]). We have now a grouplike ω_D relative to D', given by $\omega_{Af} = f\omega f \in \vartheta(V)(Af, Af)$. Observe that we have $\vartheta(\mu)(\omega_{Af}) = \omega_{Af} \otimes \omega_{Af}$. The set of elements ω_{Af} is called a *normal section* in [8].

We are now going to construct a layer for $\mathcal{D}(\Lambda)$, with this purpose for each $i = 1, ..., n$, consider the following elements of D and $V_D = \vartheta(V)$,

$b_i = \nu_{x(i)} \in D(Af_{t(i),1}, Af_{s(i),2}) = \mathrm{Hom}_A(Af_{t(i),1}, Af_{s(i),2})$, $x(i) = \begin{pmatrix} 0 & 0 \\ \alpha_i^* & 0 \end{pmatrix}$; $v_{i,1} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha_i^* \end{pmatrix} \in \vartheta(V)(Af_{t(i),1}, Af_{s(i),1}) = f_{t(i),1} V f_{s(i),1}$, $v_{i,2} = \begin{pmatrix} \alpha_i^* & 0 \\ 0 & 0 \end{pmatrix}$, an element in $\vartheta(V)(Af_{t(i),2}, Af_{s(i),2}) = f_{t(i),2} V f_{s(i),2}$.

Consider the set $L = (D'; \omega_D; b_1, ..., b_n; v_{1,1}, ..., v_{n,1}, v_{1,2}, ..., v_{n,2})$. We will see that L is a layer for $\mathcal{D}(\Lambda)$. Here D' is a minimal category, so $L.1$ is satisfied. Properties (a), (b) and (c) imply $L.2$ and $L.4$. By (1) of Proposition 3.1 of [8], we have $L.3$.

For proving $L.5$ observe that $m(\alpha_i^*) = \sum_{s,t} \alpha_i^*(\alpha_s\alpha_t)\alpha_t^* \otimes \alpha_s^*$, then

$$\delta_1(b_i) = V(1,b_i)\omega_{X_{t(i),1}} - V(b_i,1)\omega_{X_{s(i),2}} = -\delta(x_i) =$$

$$\sum_{s,t} \alpha_i^*(\alpha_s\alpha_t)(v_{t,1}x_s - x_t v_{s,2}) = \sum_{s,t} \alpha_i^*(\alpha_s\alpha_t)(b_s v_{t,1} - v_{s,2}b_t).$$

Then by our choice of the α_i, we have $\alpha_i^*(\alpha_s\alpha_t) = 0$ for $s \geq i$ or $t \geq i$. This proves $L.5$, therefore L is a layer for $\mathcal{D}(\Lambda)$.

In the following we put $\mathcal{D}(\Lambda) = \mathcal{D}$ and $X_{i,j} = Af_{i,j}$ for $i = 1, ..., n; j = 1, 2$.

There is an equivalence of categories $\Xi : \mathcal{D}\text{-Mod} \to P^1(\Lambda)$. If $M \in \mathcal{D}$- Mod then,

$$\Xi(M) : \bigoplus_{i=1}^{n} \Lambda e_i \otimes_k M(X_{1,i}) \to \bigoplus_{i=1}^{n} \Lambda e_i \otimes_k M(X_{2,i}),$$

such that for $m_i \in M(X_{1,i})$, and $c_i \in \Lambda e_i$,

$$\Xi(M)(\sum_{i=1}^{n} c_i \otimes m_i) = \sum_{j=1}^{n} c_{s(j)}\alpha_j \otimes M(b_j)(m_{s(j)}).$$

For a morphism of the form $f = (f^0, f^1) : M \to N$ in $\mathcal{D}$-Mod, $\Xi(f)$ is given by the pair of morphisms:

$$\Xi(f)_u : \bigoplus_{i=1}^{n} \Lambda e_i \otimes_k M(X_{u,i}) \to \bigoplus_{i=1}^{n} \Lambda e_i \otimes_k N(X_{u,i}), \quad u = 1, 2$$

such that for $m_i \in M(X_{i,u})$ and $c_i \in \Lambda e_i$ we have

$$\Xi(f)_u(\sum_{i=1}^{n} c_i \otimes m_i) = \sum_{i=1}^{n} c_i \otimes f^0_{X_{i,u}}(m_i) + \sum_{j=1}^{n} c_{s(j)}\alpha_j \otimes f^1(v_{j,u})(m_{s(j)}).$$

Observe that if M is a proper $\mathcal{D}$-$k(x)$-bimodule then $\Xi(M)$ is an object in $P^1(\Lambda^{k(x)})$, and if $f : M \to N$ is a morphism between proper $\mathcal{D}$-$k(x)$-bimodules then $\Xi(f)$ is a morphism in $P^1(\Lambda^{k(x)})$. Therefore Ξ induces an equivalence:

$$\Xi^{k(x)} : \mathcal{D}\text{-}k(x)\text{-Mod}^p \to P^1(\Lambda^{k(x)}).$$

Lemma 9.1. *There are constants l_1 and l_2 such that if we have an almost split sequence in $\mathcal{D}(\Lambda)$-mod starting in H' and ending in H such that ΞH is not $\mathcal{E}$-injective, then $|H'| \leq l_1|H|$ and $|H| \leq l_2|H'|$.*

Proof. We put $l = \dim_k\Lambda$. Suppose $\Xi H = f : P_1 \to P_2$, here ΞH is indecomposable and it is not $\mathcal{E}$-injective. Therefore, ΞH has not direct summands of the form $P \to 0$, this implies that $\ker f$ is contained in $\mathrm{rad}P_1$, then f induces a monomorphism $P_1/\mathrm{rad}P_1 \rightarrow \mathrm{Im}f/\mathrm{radIm}f$, consequently $\dim_k(P_1/\mathrm{rad}P_1) \leq \dim_k\mathrm{Im}f \leq \dim_k P_2$. Then we have:

$$\dim_k Cok(\Xi H) \leq \dim_k P_2 \leq \dim_k P_1 + \dim_k P_2 \leq |H|l.$$

Moreover:

$$\dim_k P_2 \leq l\dim_k(P_2/\mathrm{rad}P_2) \leq l\dim_k Cok(\Xi H)$$

and $|H| = \dim_k(P_1/\mathrm{rad}P_1) + \dim_k(P_2/\mathrm{rad}P_2) \leq \dim_k P_2 + \dim_k Cok(\Xi H)$
$\leq (1+l)\dim_k Cok(\Xi H)$.

On the other hand, there is a constant l_0 such that for all non projective indecomposable $M \in \Lambda$-mod, $\dim_k M \leq l_0 \dim_k DtrM$ (see proof of Theorem D in [5]). By Propositions 3.10 and 3.13, $Cok(\Xi H') \cong DtrCok(\Xi H)$. Then $\dim_k Cok(\Xi H') \leq l_0 \dim_k Cok(\Xi H)$. Therefore :

$$|H'| \leq \dim_k(Cok(\Xi H'))(1+l) \leq$$

$$l_0 \dim_k(Cok(\Xi H))(1+l) \leq l_0|H|l(1+l) = l_1|H|.$$

The second part of our statement is proved in a similar way. □

Theorem 9.2. *Let $\mathcal{D} = (D, V)$ be the Drozd's bocs of a tame algebra Λ. Then ($\mathcal{D}$-Mod, $\mathcal{E}_{\mathcal{D}}$) is an exact category, with $\mathcal{E}_{\mathcal{D}}$ the class of proper exact sequences. This exact category restricted to $\mathcal{D}$-mod has almost split sequences in the sense of Definition 2.5. Given a positive integer r, there is a composition of reduction functors $\theta : D \rightarrow B$ with $\mathcal{B} = (B, V_B) = \mathcal{D}^B$ a minimal layered bocs having the following properties.*

(i) For any indecomposable $M \in \mathcal{D}$-mod with $|M| \leq r$ there is a $N \in \mathcal{B}$-mod with $M \cong \theta^*(N)$. Moreover any proper almost split sequence in $\mathcal{D}$-mod starting or ending in an indecomposable M with $|M| \leq r$ is the image under θ^* of an almost split sequence (in the sense of Definition 2.1) in $\mathcal{B}$-mod.

(ii) The image under θ^* of a proper exact sequence in $\mathcal{B}$-mod is a proper exact sequence in $\mathcal{D}$-mod.

(iii) The image under θ^* of a proper almost split sequence in $\mathcal{B}$-mod is an almost split sequence in $\mathcal{D}$-mod.

(iv) Let $Z_1, ..., Z_s$ be all the marked objects of $\mathrm{ind}B$ with

$$R_i = B(Z_i, Z_i) = k[x, h_i(x)^{-1}], \quad h_i(x) \in k[x],$$

and $M(Z_i, p, m)$, Q_{Z_i}, the indecomposable objects in $\mathcal{B}$-Mod defined in section 5 and 6 respectively. Then $B_i = \mathrm{Hom}_B(Z_i, -)$ is a B-R_i-bimodule such that $Q_{Z_i} \cong B_i \otimes_{R_i} k(x)$ and $M(Z_i, p, m) \cong B_i \otimes_{R_i} R_i/(p^m)$.

Take the D-R_i-bimodule $D_i = \theta^*(B_i)$, then

$$\theta^*(Q_{Z_i}) \cong D_i \otimes_{R_i} k(x), \quad \text{and} \quad \theta^*(M(Z_i, p, n)) \cong D_i \otimes_{R_i} R_i/(p^m).$$

Moreover, $\mathbf{dim}(D_i \otimes_{R_i} R_i/(p^m)) = m\mathbf{dim}_{k(x)}(D_i \otimes_{R_i} k(x))$.

Proof. There is an equivalence $\Xi : \mathcal{D}$-Mod $\rightarrow P^1(\Lambda)$, observe that if (a) is a pair of composable morphisms $X \rightarrow E \rightarrow Y$ in $\mathcal{D}$-Mod, $\Xi(a)$ is a sequence in the class $\mathcal{E}$ in $P^1(\Lambda)$ if and only if (a) is a proper exact sequence. Therefore if $\mathcal{E}_1$ is the class of proper exact sequences in $\mathcal{D}$-mod, the pair $(\mathcal{D}\text{-mod}, \mathcal{E}_1)$ is an exact category with almost split sequences, moreover if (a) is a pair of composable morphisms in $\mathcal{D}$-mod, $\Xi(a)$ is an almost split $\mathcal{E}$-sequence if and only if (a) is an almost split $\mathcal{E}_1$-sequence.

Take the number $r(1+l)$, with $l = max\{l_1, l_2\}$, l_1, l_2 the constants of Lemma 9.1. Then by Theorem 8.5 there is a composition of reduction functors $\theta_1 : D \rightarrow C$ with $\mathcal{C} = (C, V_C) = \mathcal{D}^C$ a minimal bocs with layer $(C'; \omega; w_1, ..., w_s)$ such that the full and faithful functor $\theta_1^* : \mathcal{C}$-Mod $\rightarrow \mathcal{D}$-Mod has the property that for all $M \in \mathcal{D}$-Mod with $|M| \leq r$, there is a $N \in \mathcal{C}$-Mod with $(\theta_1)^*(N) \cong M$. Take now $\theta_2 : C \rightarrow B$ the deletion of all marked indecomposable objects $Z \in \text{ind}C$ with $|t^{\theta_1}(\mathbf{d}_Z)| > r$, where $\mathbf{d}_Z \in \text{Dim}(\mathcal{C})$ with $\mathbf{d}_Z(Z) = 1$, and $\mathbf{d}_Z(Z') = 0$ for $Z' \neq Z$, $Z' \in \text{ind}C$. Then we have $\theta = \theta_2\theta_1 : D \rightarrow B$ and $\mathcal{B} = (B, V_B) = ((\mathcal{D})^C)^B = \mathcal{D}^B$ is a minimal layered bocs.

(i) Take an indecomposable object $M \in \mathcal{D}$-mod with $|M| \leq r$, then there is a $N_1 \in \mathcal{C}$-mod with $(\theta_1)^*(N_1) \cong M$. Since N_1 is an indecomposable object in the minimal bocs $\mathcal{C}$, then either $M \cong M(Z, p, m)$ for some marked $Z \in \text{ind}C$ or $M \cong S_Z$ for some non-marked $Z \in \text{ind}C$. In the first case $|t^{\theta_1}(\mathbf{dim}N_1)| = m|t^{\theta_1}(\mathbf{d}_Z)| = |\mathbf{dim}M| \leq r$. Thus, $|t^{\theta_1}(\mathbf{d}_Z)| \leq r$. Consequently, in both cases $N_1(W) = 0$ for W a marked object in ind$C$ with $|t^{\theta_1}(\mathbf{d}_W)| > r$, then there is a $N \in \mathcal{B}$-mod with $N_1 \cong (\theta_2)^*(N)$. Therefore $M \cong \theta^*(N)$ proving the first part of (i). For the second part take $M \rightarrow E \rightarrow L$ a proper almost split sequence in $\mathcal{D}$-mod, then if either M or L have dimension equal or smaller than r, all indecomposable summands of the other terms of the sequence have dimension equal or smaller than $(l+1)r$, consequently our proper almost split sequence is isomorphic to the image under $(\theta_1)^*$ of an almost split sequence (in the sense of Definition 2.1) (a_1) : $M_1 \rightarrow E_1 \rightarrow L_1$ in $\mathcal{C}$-mod. Then if M_1 or L_1 is an object of the form $M(Z, p, m)$, with Z a marked object in ind$C$, we have $M_1 \cong L_1$ and $E_1 = M(Z, p, m-1) \oplus M(Z, p, m+1)$. Here $|M(Z, p, m)| \leq r$ implies $|t^{\theta_1}(\mathbf{d}_Z)| \leq r$, then the sequence (a_1) is the image under $(\theta_2)^*$ of an almost split sequence in $\mathcal{B}$-mod. In case that M_1 or L_1 is an object of the form S_Z for a non marked object in ind$C$, then all other terms of (a_1) are sums of objects of the form S_W with W a non-marked object in ind$C$. Therefore, again (a_1) is the image under $(\theta_2)^*$ of an almost split sequence in $\mathcal{B}$-mod. This proves the second part of (i).

(ii) Follows from Lemma 7.7.

(iii) Take now Z a marked indecomposable in B and $M(Z, p, 1) \in \mathcal{B}$-mod with p a fixed prime element in $R_Z = B(Z, Z)$. By definition of B we have $|t^{\theta}(\mathbf{d}_Z)| \leq r$ and $\theta_2(Z) = Z \in C$. There is a non-trivial proper sequence ending and starting in $M(Z, p, 1)$, since θ^* is a full and faithful functor, there is a non-trivial proper exact sequence ending and starting in $\theta^*(M(Z, p, 1))$. Then $H = \theta^*(M(Z, p, 1))$ is not $\mathcal{E}_1$-projective. Therefore, there is an almost split sequence $(a) : H' \rightarrow H_0 \rightarrow H$. By the second part of (i) the sequence (a) is the image under θ^* of an almost split sequence (b) in $\mathcal{B}$-mod. Then using Proposition 2.6 we obtain (iii).

(iv) The first part follows from the definition of θ^*. For proving the second part take

X an indecomposable object in D and assume $\theta(X) = \bigoplus_{j=1}^{t} n_j Z_j$, where $Z_1, ..., Z_j$ are all indecomposable objects of B. Then for each $i \in \{1, ..., s\}$:

$$\mathbf{dim}_{k(x)}(\theta^* B_i \otimes_{R_i} k(x))(X) = \dim_{k(x)}(B(Z_i, \theta(X)) \otimes_{R_i} k(x)) =$$

$$\dim_{k(x)}(R_i^{n_i} \otimes_{R_i} k(x)) = n_i.$$

On the other hand:

$$t^\theta(\mathbf{d}_{Z_i})(X) = \mathbf{d}_{Z_i}(\theta(X)) = n_i.$$

Therefore $t^\theta(\mathbf{d}_{Z_i}) = \mathbf{dim}(\theta^* B_i \otimes_{R_i} k(x))$. Then

$$\mathbf{dim}(D_i \otimes_{R_i} R_i/(p^m)) = \mathbf{dim}(\theta^*(M(Z_i, p, m)) = m t^\theta(\mathbf{d}_{Z_i}),$$

proving (iv). □

In the following we put $\Lambda^{k(x)} = \Lambda \otimes_k k(x)$.

Definition 9.3. If R is a k-algebra a $P(\Lambda)$-R-bimodule is a morphism $X = f_X : P_X \to Q_X$, where P_X and Q_X are Λ-R-bimodules which are projectives as left Λ-modules and f_X is a morphism of Λ-R-bimodules. If Z is a left R-module, $X \otimes_R Z = f \otimes 1 : P_X \otimes_R Z \to Q_X \otimes_R Z$.

We recall from section 3 that if $X : P_X \to Q_X$ is an object in $p^1(\Lambda)$, then $\mathbf{dim}X = (\mathbf{dim}(\text{top}P_X), \mathbf{dim}(\text{top}Q_X))$. Then if $H' \in \mathcal{D}$-mod, $\mathbf{dim}(\Xi H') = \mathbf{dim}H'$. In case $X \in p^1(\Lambda^{k(x)})$ we put $\mathbf{dim}_{k(x)}X = (\mathbf{dim}_{k(x)}(\text{top}P_X), \mathbf{dim}_{k(x)}(\text{top}Q_X))$, then if $H' \in \mathcal{D}$-$k(x)$-mod, we have $\mathbf{dim}_{k(x)}(\Xi H') = \mathbf{dim}_{k(x)}H'$.

An indecomposable object $H = f_H : P_H \to Q_H$ in $P(\Lambda)$ which is not in $p(\Lambda)$ is called generic if P_H and Q_H have finite length as $\text{End}_{P(\Lambda)}(H)$-modules. A structure of $P(\Lambda)$-$k(x)$-bimodule for H is called admissible in case $\text{End}_{P(\Lambda)}(H) = k(x)_m \oplus \mathcal{R}$, where $\mathcal{R} = \text{radEnd}_{P(\Lambda)}(H)$ and $k(x)_m$ denotes the set of morphisms $h : H \to H$ of the form $h = (m(x)id_{P_H}, m(x)id_{Q_H})$ with $m(x) \in k(x)$.

Definition 9.4. Suppose $\hat{T} = f_{\hat{T}} : P_{\hat{T}} \to Q_{\hat{T}}$ is a $P(\Lambda)$-R-bimodule with R a finitely generated localization of $k[x]$ and $P_{\hat{T}}, Q_{\hat{T}}$ finitely generated as right R-modules. We say that $\hat{T}$ is a realization of H if $\hat{T} \otimes_R k(x) \cong H$. The realization $\hat{T}$ of H over R is called good if:

(i) $P_{\hat{T}}$ and $Q_{\hat{T}}$ are free as right R-modules;

(ii) the functor $\hat{T} \otimes_R - : R\text{-Mod} \to P(\Lambda)$ preserves isomorphism classes and indecomposable objects;

(iii) for p a prime in R, and n a positive integer $\hat{T} \otimes_R S_{p,n}$ is an almost split sequence, where $S_{p,n}$ is the sequence given in (iii) of Definition 1.1.

We are now ready for giving a version of Theorem 1.2 for $P(\Lambda)$.

Theorem 9.5. *Let Λ be a finite-dimensional algebra over an algebraically closed field k of tame representation type. Let r be a positive integer. Then there are indecomposable objects in $p^1(\Lambda)$, $\hat{L}_1, ..., \hat{L}_t$ with $|\hat{L}_j| \leq r$ for $j = 1, ..., t$ and generic objects in $P^1(\Lambda)$ with admissible structure of $P(\Lambda)$-$k(x)$-bimodules, $H_1, ..., H_s$ such that for $j = 1, ..., s$, H_j has a good realization $\hat{T}_j$ over R_j, a finitely generated localization of $k[x]$, with the following properties:*

(i) If X is an indecomposable object in $p^1(\Lambda)$ with $|X| \leq r$, then either $X \cong \hat{L}_j$ for some $j \in \{1, ..., t\}$ or $X \cong \hat{T}_i \otimes_{R_i} R_i/(p^m)$ for some $i \in \{1, ..., s\}$, some prime element $p \in R_i$ and some natural number m.

(ii) If $X = \hat{T}_i \otimes_{R_i} R_i/(p^m), Y = \hat{T}_j \otimes_{R_j} R_j/(q^n)$, with $i, j \in \{1, ..., s\}$, p a prime in R_i, q a prime in R_j, and $\hat{L}_u$ with $u \in \{1, ..., t\}$, then

$$\dim_k \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, Y) = mn\dim_{k(x)}\mathrm{rad}_{p^1(\Lambda^{k(x)})}(H_i, H_j),$$

$$\dim_k \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, \hat{L}_u) = m\dim_{k(x)}\mathrm{rad}_{p^1(\Lambda^{k(x)})}(H_i, \hat{L}_u^{k(x)}),$$

$$\dim_k \mathrm{rad}^{\infty}_{p^1(\Lambda)}(\hat{L}_u, X) = m\dim_{k(x)}\mathrm{rad}_{p^1(\Lambda^{k(x)})}(\hat{L}_u^{k(x)}, H_i).$$

(iii) If $X = \hat{T}_i \otimes_{R_i} R_i/(p^m), Y = \hat{T}_j \otimes_{R_j} R_j/(q^n)$, then if $i = j$ and $p = q$,

$$\mathrm{Hom}_{p^1(\Lambda)}(X, Y) \cong \mathrm{Hom}_{R_i}(R_i/(p^n), R_i/(p^m)) \oplus \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, Y).$$

If $i \neq j$ or $i = j$ and $(p) \neq (q)$:

$$\mathrm{Hom}_{p^1(\Lambda)}(X, Y) = \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, Y).$$

Moreover:

$$\mathrm{Hom}_{p^1(\Lambda)}(\hat{L}_u, X) = \mathrm{rad}^{\infty}_{p^1(\Lambda)}(\hat{L}_u, X), \quad \mathrm{Hom}_{p^1(\Lambda)}(X, \hat{L}_u) = \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, \hat{L}_u).$$

Proof. We apply Theorem 8.5 for the Drozd's bocs $\mathcal{D} = (D, V_D)$ of Λ and the positive integer $r(l+1)$ with $l = max\{l_1, l_2\}$ where l_1, l_2 are the integers given in Lemma 9.1. Then we obtain a minimal layered bocs $\mathcal{B} = (B, V_B)$ having properties (i)-(iv) of Theorem 9.2. We have the reduction functor $\theta : D \to B$, suppose $\theta(X_{j,i}) = \bigoplus_l n^l_{j,i} Z_l$ with $j = 1, 2$ and $i = 1, ..., n$ given in the beginning of this section.

Let $Z_1, ..., Z_s$ be the marked objects of ind$B$ and $Z_{s+1}, ..., Z_{s+t}$ be the non-marked objects. We have B_i, R_i and D_i given in (iv) of Theorem 9.2.

Consider $\hat{T}_i = \Xi D_i$. $\hat{T}_i = g_i : P_i \to Q_i$, then:

$$P_i = \bigoplus_v \Lambda e_v \otimes D_i(X_{1,v}) = \bigoplus_v \Lambda e_v \otimes_k \mathrm{Hom}_B(Z_i, \theta(X_{1,v})) \cong \bigoplus_v \Lambda e_v \otimes_k n^i_{1,v} R_i.$$

Similarly $Q_i \cong \bigoplus_v \Lambda e_v \otimes_k n^i_{2,v} R_i$. If $\lambda \in \Lambda e_v$, and $m \in D_i(X_{1,v})$, then:

$$g_i(\lambda \otimes m) = \sum_{d_j : X_{1,s(j)} \to X_{2,t(j)}, s(j) = v} \lambda \alpha_j \otimes \mathrm{Hom}_B(1, \theta(b_j))(m)$$

We have

$$H_i = \Xi D_i \otimes_{R_i} k(x) = f_i : P_{H_i} \to Q_{H_i}, P_{H_i} = P_i \otimes_{R_i} k(x), Q_{H_i} = Q_i \otimes_{R_i} k(x),$$

with $f_i = g_i \otimes 1_{k(x)}$, therefore $H_i = \hat{T}_i \otimes_{R_i} k(x)$.

Moreover, $P_{H_i} \cong \bigoplus_v n^i_{1,v}\Lambda^{k(x)}(e_v \otimes 1)$ and $Q_{H_i} \cong \bigoplus_v n^i_{2,v}\Lambda^{k(x)}(e_v \otimes 1)$.

For $i = 1, ..., s$ consider the objects $H_i \in P^1(\Lambda)$. For all $i = 1, ..., s$ we have an isomorphism induced by the functor $\Xi\theta^*$:

$$\mathrm{End}_{\mathcal{B}}(Q_{Z_i}) = \mathrm{End}_{\mathcal{B}}(Q_{Z_i})^0 \oplus \mathrm{End}_{\mathcal{B}}(Q_{Z_i})^1 \to \mathrm{End}_{P^1(\Lambda)}(H_i),$$

where $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^0$ denotes the morphisms of the form $(f^0, 0)$ and $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^1$ denotes the morphisms of the form $(0, f^1)$. Here $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^0 \cong \mathrm{End}_{R_i}(k(x)) = k(x)_m$, where $k(x)_m$ denotes the right multiplication by elements of $k(x)$. Here $\mathcal{B}$ is a layered bocs, therefore a morphism (f^0, f^1) is an isomorphism if and only if f^0 is an isomorphism, thus the elements in $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^1$ are the non-units in $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})$. Thus since the sum of non-units is again non-unit, $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})$ is a local ring and its radical is $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^1$. The image under $\Xi\theta^*$ of an element in $\mathrm{End}_{\mathcal{B}}(Q_{Z_i})^0$ is of the form $(id_{P_{H_i}} m(x), id_{Q_{H_i}} m(x))$, with $m(x) \in k(x)$. From here we obtain that the $P(\Lambda)$-$k(x)$-structure of H_i is admissible. Clearly, $\hat{T}_i$ is a realization of H_i.

In order to prove that $\hat{T}_i$ is a good realization of H_i, we must prove conditions (i), (ii) and (iii) of Definition 9.4. Condition (i) is clear. For proving condition (ii) take $\epsilon_{\mathcal{B}} : V_{\mathcal{B}} \to B$ the counit of the bocs $\mathcal{B}$. By Lemma 5.3 the functor $(id_B, \epsilon_{\mathcal{B}})^* : B$-Mod $\to \mathcal{B}$-Mod preserves indecomposables and isomorphism classes. Consider $\hat{B}_i$ the full subcategory of B whose unique indecomposable object is Z_i, then we have the composition η_i of full and faithful functors:

$$R_i\text{-Mod} \to \hat{B}_i\text{-Mod} \to B\text{-Mod}.$$

The composition:

$$R_i\text{-Mod} \xrightarrow{\eta_i} B\text{-Mod} \xrightarrow{(id_B,\epsilon_{\mathcal{B}})^*} \mathcal{B}\text{-Mod} \xrightarrow{\theta^*} \mathcal{D}\text{-Mod} \xrightarrow{\Xi} P^1(\Lambda)$$

is isomorphic to $\hat{T}_i \otimes_{R_i} -$. Therefore the functor $\hat{T}_i \otimes_{R_i} -$ preserves isomorphism classes and indecomposable modules. The condition (iii) of Definition 9.4 is a consequence of (iii) of Theorem 9.2.

Now, we may assume that $\hat{L}_j = \Xi\theta^*(S_{Z_{s+j}})$ for $j = 1, ..., t$ is such that $|\hat{L}_j| \leq r$.

(i) Take X an indecomposable object in $p^1(\Lambda)$ with $|X| \leq r$, then by (i) of Theorem 9.2 there is an indecomposable object N in $\mathcal{B}$-mod with $\Xi\theta^*(N) \cong X$. Since N is indecomposable, then $N \cong S_{Z_{s+j}}$ for some $j = 1, ..., t$ and then either $X \cong \hat{L}_j$, or $N \cong M(Z_i, p, n)$ for some $i = 1, ..., s$, some prime element $p \in R_i$ and some positive integer n, in this case by (iv) of Theorem 9.2 we have $M(Z_i, p, n) \cong B_i \otimes_{R_i} R_i/(p^n)$. Then $X \cong \Xi\theta^* B_i \otimes_{R_i} R_i/(p^n) \cong \hat{T}_i \otimes_{R_i} R_i/(p^n)$. Thus we have proved i).

(ii) Consider $\mathcal{C}$ the full subcategory of $p^1(\Lambda)$ whose objects are the objects of the form $\hat{T}_i \otimes_{R_i} R_i/(p^m)$. We have already proved that $\hat{T}_i$ is a good realization of H_i, then by property (iii) of Definition 9.4 the category $\mathcal{C}$ consists of whole Auslander-Reiten components of $p^1(\Lambda)$, thus $\mathcal{C}$ has property (A) of section 2, then by Corollary 2.4 for $X = \hat{T}_i \otimes_{R_i} R_i/(p^m)$, $Y = \hat{T}_i \otimes_{R_i} R_i/(q^n)$, $\dim_k \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X,Y) = \dim_k \mathrm{rad}^{\infty}_{\mathcal{C}}(X,Y) = \dim_k \mathrm{rad}^{\infty}_{\mathcal{B}}(M(Z,p,m), M(Z',q,n))$.

We recall from the discussion at the beginning of section 6 that the full and faithful functor $\theta^* : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$ restricts to a full and faithful functor $(\theta^*)^{k(x)} : \mathcal{B}\text{-}k(x)\text{-Mod}^p \to \mathcal{D}\text{-}k(x)\text{-Mod}^{op}$. Then the first equality of (ii) follows from that of Proposition 6.5.

Observe that $\hat{L}_u^{k(x)} = \Xi\theta^*(S_{Z_{s+u}})^{k(x)} \cong \Xi\theta^*(S^{k(x)}_{Z_{s+u}})$. The second and third equality of (ii) follow from those of Proposition 6.5.

(iii) Follows from Corollary 5.11 and from Corollary 2.4. □

10 Hom-spaces in Λ-Mod

In this section we discuss the Hom-spaces in Λ-Mod for a tame algebra Λ and prove our main result, Theorem 1.2. For $X = f_X : P_X \to Q_X \in p(\Lambda)$ we define $|X| = |\mathbf{dim}X| = \dim_k(P_X/\mathrm{rad}P_X) + \dim_k(Q_X/\mathrm{rad}Q_X)$.

There is an integer l_0 such that for any indecomposable non-injective Λ-module M, $\dim_k trDM \leq l_0 \dim_k M$. Let d be any positive integer greater than $\dim_k\Lambda$, consider $d_0 = d(1 + l_0)$ take $s(d_0) = (\dim_k(\Lambda) + 1)d_0$. If $M \in \Lambda$-mod with $\dim_k M \leq d_0$ and $X = f_X : P_X \to Q_X$ is a minimal projective presentation of M, we have $\dim_k(Q_X/\mathrm{rad}Q_X) \leq d_0$ and $\dim_k(P_X/\mathrm{rad}P_X) \leq \dim_k(\mathrm{Im} f_X) \leq \dim_k Q_X \leq \dim_k(M/\mathrm{rad}M)\dim_k\Lambda \leq d_0\dim_k\Lambda$, so $|X| \leq s(d_0)$. Taking the number $r = s(d_0)(1 + l)$ in Theorem 9.5 with $l = max\{l_1, l_2\}$, where l_1 and l_2 are the constants of Lemma 9.1, we obtain the generic objects in $P(\Lambda)$, $H_1, ..., H_s$ with admissible Λ-$k(x)$ structures and the indecomposables in $p^1(\Lambda)$, $\hat{L}_1, ..., \hat{L}_t$. For each $i = 1, ..., s$ we have the realizations $\hat{T}_i$ over R_i of H_i. We have the generic Λ-modules $G_i = Cok(H_i)$ and the following isomorphism of Λ-$k(x)$-bimodules, $G_i = Cok(H_i) \cong Cok(\hat{T}_i \otimes_{R_i} k(x)) \cong Cok(\hat{T}_i) \otimes_{R_i} k(x)$, with $T_i = Cok(\hat{T}_i)$ a Λ-R_i-bimodule finitely generated as right R_i-module. The Λ-$k(x)$ structure of H_i is admissible, then $\mathrm{End}_{P(\Lambda)}(H_i) = k(x)_m \oplus \mathcal{R}_i$ with $\mathcal{R}_i$ a nilpotent ideal. Then, $\mathrm{End}_\Lambda(G_i) = k(x)id_{G_i} \oplus \mathrm{radEnd}_\Lambda(G_i)$, therefore, the endolength of G_i coincides with $\dim_{k(x)} G_i$. Consequently, T_i is a realization of G_i.

Lemma 10.1. *G_i and T_i satisfy the conditions (ii) and (iii) of Definition 1.1.*

Proof. Take $W \in R_i$-Mod, we claim that $\hat{T}_i \otimes_{R_i} W$ has not indecomposable direct summands of the form $Z(P) = P \to 0$. Suppose some indecomposable $Z(P)$ is a direct summand of $\hat{T}_i \otimes_{R_i} W = \Xi\theta^*(W')$, with $W' = (id_B, \epsilon_{\mathcal{B}})^*\eta_i(W)$. Here $Z(P)$ is injective in $P^1(\Lambda)$, then $Z(P) = \Xi\theta^*(S_{Z_u})$ for some non-marked indecomposable object $Z_u \in B$. Since the functor $\Xi\theta^*$ is full and faithful, we have that S_{Z_u} is direct summand of W', but this is impossible because $W'(Z_u) = 0$. The above proves that $\hat{T}_i \otimes_{R_i} W$ is in $P^2(\Lambda)$, the full subcategory of $P^1(\Lambda)$ whose objects have not direct summands of the form $Z(P)$.

Now the functor $Cok : P^2(\Lambda) \to \Lambda$-Mod preserves indecomposables and isomorphism classes (see (2) of Lemma 3.2 of [6]). Consequently, the functor $Cok(\hat{T}_i \otimes_{R_i} -) \cong T_i \otimes_{R_i} -$ preserves indecomposables and isomorphism classes. This proves that T_i has property (ii) of Definition 1.1.

For proving condition (iii) of Definition 1.1 take p a prime element in R_i. There is an almost split sequence in $p^1(\Lambda)$ starting in $\hat{T}_i \otimes_{R_i} R_i/(p^m)$, therefore this object is not injective in $p^1(\Lambda)$ and therefore its cokernel is not zero. By Proposition 3.13 the image under the functor Cok of the almost split sequence starting in $\hat{T}_i \otimes_{R_i} R_i/(p^m)$ is an almost split sequence in Λ-mod. This proves that the Λ-R_i-bimodule T_i satisfies condition iii) for all $i \in \{1, ..., s\}$. □

Lemma 10.2. *Let $L_j = Cok(\hat{L}_j)$ with $j = 1, ..., t$. If M is an indecomposable Λ-module with $\dim_k M \leq d$, then M has the form given in (i) of Theorem 1.2.*

Proof. There is an indecomposable object $X \in p^1(\Lambda)$ with $M \cong Cok(X)$, since $|X| \leq s(d) \leq r$, $X \cong \hat{T}_i \otimes_{R_i} R_i/(p^m)$ or $X \cong \hat{L}_j$. But then either $M \cong Cok(\hat{T}_i \otimes_{R_i} R_i/(p^m)) \cong T_i \otimes_{R_i} R_i/(p^m)$, or $M \cong L_j$. This proves the first part of (i). For the second part of (i), by Proposition 5.9 of [1] we have that if X is an indecomposable object in $p^1(\Lambda)$ with $Cok(X)$ non-simple injective, then there is an almost split sequence in $p(\Lambda)$ starting in X and ending in an injective object with all its terms in $p^1(\Lambda)$, so this is an almost split sequence in $p^1(\Lambda)$. If $Cok(X)$ is simple then X is injective in $p^1(\Lambda)$, if $Cok(X)$ is projective, then X is projective in $p^1(\Lambda)$. Now if $X \cong \hat{T}_i \otimes_{R_i} R_i/(p^m)$, since $\hat{T}_i$ is a good realization of H_i, there is an almost split sequence starting and ending in X. Therefore, if M is an injective, projective or simple Λ-module, then $M \cong L_j$ for some $j = 1, ..., t$. □

Lemma 10.3. *Let $X = \hat{T}_i \otimes_{R_i} R_i/(p^n), Y = \hat{T}_i \otimes_{R_i} R_i/(p^m)$, $M = CokX$, $N = CokY$, then the functor Cok induces an isomorphism:*

$$\underline{Cok} : \mathrm{Hom}_{P^1(\Lambda)}(X, Y)/\mathrm{rad}^\infty(X, Y) \to \mathrm{Hom}_\Lambda(M, N)/\mathrm{rad}^\infty(M, N).$$

Proof. In fact, take a morphism $u : X \to Y$ such that $Cok(u) = 0$. Then by Proposition 3.3, u is a morphism which is a sum of compositions of the form u_2u_1 with $u_1 : X \to W$, $u_2 : W \to Y$ and W an indecomposable injective in $P(\Lambda)$. Then either $W = Z(P) = (P \xrightarrow{0} 0)$ or $W = J(P) = (P \xrightarrow{id_P} P)$ for some indecomposable projective Λ-module P. In the first case W is also an injective object in $p^1(\Lambda)$, then W is not in the Auslander-Reiten component containing X, therefore $u_2u_1 \in \mathrm{rad}^\infty(X, Y)$. Now, if $W = J(P)$, we recall (see Lemma 3.6) that there is a right minimal almost split morphism $\sigma(P) : R(P) \to J(P)$, then $u_1 = \sigma(P)u_1'$, with $u_1' : X \to R(P)$. Here $R(P)$ is injective in $p^1(\Lambda)$, then $u_2u_1 = u_2\sigma(P)u_1'$ is in $\mathrm{rad}^\infty(X, Y)$, therefore, $u \in \mathrm{rad}^\infty(X, Y)$, proving our Lemma. □

Lemma 10.4. *If $M = T_i \otimes_{R_i} R_i/(p^m)$, $N = T_j \otimes_{R_j} R_j/(q^n)$, $L_u^{k(x)} = L_u^{k(x)}$ with $i, j \in \{1, ..., s\}$, $u \in \{1, ..., t\}$, p a prime element of R_i, q a prime element of R_j, then M, N, L_u satisfy (iii) of Theorem 1.2.*

Proof. Let $M = CokX, N = CokY$, $X, Y \in p^1(\Lambda)$. If $i = j$ and $p = q$ by the first formula in (iii) of Theorem 9.5 and Lemma 10.3 we obtain our result. If $i \neq j$ or $(p) \neq (q)$ we have $\mathrm{Hom}_{p^1(\Lambda)}(X, Y) = \mathrm{rad}^{\infty}_{p^1(\Lambda)}(X, Y)$, thus $\mathrm{Hom}_\Lambda(M, N) = \mathrm{rad}^{\infty}_\Lambda(M, N)$. Moreover, the third and fourth formula of (iii) of Theorem 9.5 gives $\mathrm{Hom}_\Lambda(L_u, M) = \mathrm{rad}^{\infty}_\Lambda(L_u, M)$ and $\mathrm{Hom}_\Lambda(M, L_u) = \mathrm{rad}^{\infty}_\Lambda(M, L_u)$ respectively. □

Lemma 10.5. *Let $M = T_i \otimes_{R_i} R_i/(p^m), N = T_j \otimes_{R_j} R_j/(q^n)$, for $i, j \in \{1, ..., s\}$, p a prime in R_i, q a prime in R_j. Then*

$$\dim_k \mathrm{rad}^{\infty}_\Lambda(M, N) = mn \dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(G_i, G_j).$$

Proof. Suppose $X = \hat{T}_i \otimes_{R_i} R_i/(p^m)$ and $Y = \hat{T}_j \otimes_{R_j} R_j/(q^n)$ are minimal projective presentations of M and N respectively. Then if $\mathbf{z}_u = \mathbf{dim}_{k(x)} H_u$ for $u = 1, ..., s$, by (iv) of Theorem 9.2 we have $\mathbf{dim}_k X = m\mathbf{z}_i$, $\mathbf{dim}_k Y = n\mathbf{z}_j$.

Suppose now $i \neq j$ or $i = j$ and $(p) \neq (q)$. In this case $\mathrm{Hom}_\Lambda(M, N) = \mathrm{rad}^{\infty}_\Lambda(M, N)$ and $\mathrm{Hom}_{p^1(\Lambda)}(Y, X) = \mathrm{rad}^{\infty}_{p^1(\Lambda)}(Y, X)$. Here $DtrN \cong N$, then by (3) of Proposition 3.14 and the first equality in (ii) of Theorem 9.5 we obtain

$$\dim_k \mathrm{Hom}_\Lambda(M, N) = mn(\dim_{k(x)} \mathrm{rad}_{p^1(\Lambda^{k(x)})}(H_j, H_i) - g_\Lambda(\mathbf{z}_j, \mathbf{z}_i)).$$

On the other hand, since $Dtr_{\Lambda^{k(x)}} G_j \cong G_j$ (see Proposition 6.5 of [2]) we have

$$\dim_{k(x)} \mathrm{Hom}_{\Lambda^{k(x)}}(G_i, G_j) = \dim_{k(x)} \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(H_j, H_i) - g_{\Lambda^{k(x)}}(\mathbf{z}_j, \mathbf{z}_i).$$

We know from Corollary 2.3 of [2], that the indecomposable projective $\Lambda^{k(x)}$-modules are of the form $P \otimes_k k(x)$, with P indecomposable projective Λ-module, then $g_\Lambda = g_{\Lambda^{k(x)}}$. Observe that if $i \neq j$, $\mathrm{rad}_{p^1(\Lambda^{k(x)})}(H_j, H_i) = \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(H_j, H_i)$ and $\mathrm{rad}_{\Lambda^{k(x)}}(G_i, G_j) = \mathrm{Hom}_{\Lambda^{k(x)}}(G_i, G_j)$, moreover for $i = j$,
$\dim_{k(x)} \mathrm{End}_{p^1(\Lambda^{k(x)})}(H_i) = 1 + \dim_{k(x)} \mathrm{radEnd}_{p^1(\Lambda^{k(x)})}(H_i)$ and
$\dim_{k(x)} \mathrm{End}_{\Lambda^{k(x)}}(G_i) = 1 + \dim_{k(x)} \mathrm{radEnd}_{\Lambda^{k(x)}}(G_i)$. Thus we obtain:

$$\dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(G_i, G_j) = \dim_{k(x)} \mathrm{rad}_{p^1(\Lambda^{k(x)})}(H_j, H_i) - g_\Lambda(\mathbf{z}_j, \mathbf{z}_i).$$

From here we obtain our equality for $i \neq j$ or $i = j$ and $(p) \neq (q)$.

For $i = j$ and $p = q$ and the first equality of (iii) of Theorem 9.5 we obtain

$$\dim_k \mathrm{Hom}_{p^1(\Lambda)}(X, Y) = min\{m, n\} + mn \dim_{k(x)} \mathrm{radHom}_{p^1(\Lambda^{k(x)})}(H_i, H_i),$$

therefore

$$\dim_k \mathrm{Hom}_\Lambda(M, N) = min\{m, n\} + mn \dim_{k(x)} \mathrm{radHom}_{\Lambda^{k(x)}}(G_i, G_i).$$

By Lemma 10.4 the first equality of (iii) Theorem 1.2 holds, then we have $\dim_k \mathrm{rad}^{\infty}_\Lambda(M, N) = mn \dim_{k(x)} \mathrm{radEnd}_{\Lambda^{k(x)}}(G_i)$, obtaining our result. □

Lemma 10.6. *Let $M = T_i \otimes_{R_i} R_i/(p^m)$ for $i \in \{1, ..., s\}$, p a prime element in R_i, $L_u = Cok(\hat{L}_u)$, for some $u \in \{1, ..., t\}$. Then*

$$\dim_k \mathrm{rad}^{\infty}_\Lambda(L_u, M) = m \dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(G_i, L_u^{k(x)}).$$

In particular for Λe an indecomposable projective Λ-module there is a $u \in \{1, ..., t\}$ such that $\Lambda e \cong L_u$, then $\dim_k eM = m\dim_{k(x)} eG_i$.

Proof. Consider $\mathbf{l}_u = \mathbf{dim}_k \hat{L}_u = \mathbf{dim}_{k(x)} \hat{L}_u^{k(x)}$. We have $DtrM \cong M$, then by (3) of Proposition 3.14 and the second equality of (ii) of Theorem 9.5 we have:

$$\dim_k \mathrm{Hom}_\Lambda(L_u, M) = m\dim_{k(x)} \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(H_i, \hat{L}_u^{k(x)}) - mg_\Lambda(\mathbf{z}_i, \mathbf{l}_u).$$

We have $Cok\hat{L}_u^{k(x)} \cong (Cok\hat{L}_u)^{k(x)} = L_u^{k(x)}$, thus again by 3) of Proposition 3.14, recalling that $Dtr_{\Lambda^{k(x)}} G_i \cong G_i$, we obtain:

$$\dim_{k(x)} \mathrm{Hom}_{\Lambda^{k(x)}}(L_u^{k(x)}, G_i) = \dim_{k(x)} \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(H_i, \hat{L}_u^{k(x)}) - g_\Lambda(\mathbf{z}_i, \mathbf{l}_u).$$

From here we obtain the first part of our Lemma. For the second part of the Lemma, observe that by assumption, $\dim_k \Lambda \le d$, then by Lemma 10.4 we obtain our result. □

Lemma 10.7. *Let $M = T_i \otimes_{R_i} R_i/(p^m)$ for $i \in \{1, ..., s\}$, p a prime in R_i, $L_u = Cok(\hat{L}_u)$ for $u \in \{1, ..., t\}$. Then*

$$\dim_k \mathrm{rad}^\infty_\Lambda(M, L_u) = m\dim_{k(x)} \mathrm{rad}_{\Lambda^{k(x)}}(G_i, L_u^{k(x)}).$$

Proof. Assume first L_u is injective, then we may suppose $L_u = D(e\Lambda)$. We have:

$$\dim_k \mathrm{Hom}_\Lambda(M, D(e\Lambda)) = \dim_k \mathrm{Hom}_{\Lambda^{op}}(e\Lambda, D(M)) = \dim_k D(M)e = \dim_k(eM)$$

$$= m\dim_{k(x)} \mathrm{Hom}_{\Lambda^{k(x)}}(G_i, D_x((e \otimes 1)\Lambda^{k(x)})) = m\dim_{k(x)} \mathrm{Hom}_{\Lambda^{k(x)}}(G_i, (D(e\Lambda)^{k(x)})).$$

Where $D_x(-) = \mathrm{Hom}_{k(x)}(-, k(x))$.

Now assume L is not injective. Consider an almost split sequence starting in L:

$$0 \to L \xrightarrow{f} \oplus_{s=1}^m E_s \xrightarrow{g} L' \to 0,$$

with E_s indecomposable for $s = 1, ..., m$.

By the choice of the integer d_0, the objects E_s and L' are isomorphic to objects L_v or $T_j \otimes_{R_j} R_j/(p^m)$, but in this latter case L is in the component of an object of the form $T_j \otimes_{R_j} R_j/(p^m)$, which implies that $L \cong T_j \otimes_{R_j} R_j/(p^n)$ for some n, which is not the case therefore $L' \cong L_v$ for some $v = 1, ..., t$. Then $L' \cong Cok\hat{L}_v$. Take $\mathbf{l}_v = \mathbf{dim}\hat{L}_v = \mathbf{dim}_{k(x)} \hat{L}_v^{k(x)}$.

By (3) of Proposition 3.14 and the third equality of (iii) of Theorem 9.5 we obtain

$$\dim_k \mathrm{Hom}_\Lambda(M, L) = m(\dim_{k(x)} \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(\hat{L}_v^{k(x)}, H_i) - g_\Lambda(\mathbf{l}_v, \mathbf{z}_i)).$$

On the other hand, by Corollary 2.2 of [2] we have

$$Dtr_{\Lambda^{k(x)}}(L_v^{k(x)}) \cong (DtrL_v)^{k(x)} \cong L^{k(x)}.$$

Then:

$$\dim_{k(x)} \mathrm{Hom}_{\Lambda^{k(x)}}(G_i, L^{k(x)}) = \dim_{k(x)} \mathrm{Hom}_{p^1(\Lambda^{k(x)})}(\hat{L}_v^{k(x)}, H_i) - g_\Lambda(\mathbf{l}_v, \mathbf{z}_i).$$

From here we obtain our Lemma. □

Lemma 10.8. *T_i is a free right R_i-module, for $i = 1, ...s$.*

Proof. Since T_i is a finitely generated right R_i-module if it is not a free right R_i-module there is a primitive idempotent e of Λ such that $eT_i = C_0 \oplus C_1$ with C_0 free and C_1 a torsion R_i-module, then we may assume $C_1 = (\oplus_{j=1}^{a} R_i/(p^{m_j})) \oplus C_2$ with a prime element $p \in R_i$, positive integers m_j, and $C_2 \cong \oplus_b R_i/(q_b^{n_b})$, where p, q_b are coprime in R_i. Suppose $m = min\{m_1, ..., m_a\}$, $C_0 \cong R_i^l$. Take $M = T_i \otimes_{R_i} R_i/(p^m)$, then by the second part of Lemma 10.6, $\dim_k eM = m\dim_{k(x)} eG_i = m\dim_{k(x)} eT_i \otimes_{k(x)} k(x) = m\dim C_0 \otimes_{k(x)} k(x) = ml$. But $\dim_k eM = \dim_k eT_i \otimes_{R_i} R_i/(p^m) = \dim_k C_0 \otimes_{R_i} R_i/(p^m) + \dim_k (R_i/(p^m))^a = ml + am$, a contradiction. Therefore, T_i is free as right R_i-module proving our result. □

Proof (of Theorem 1.2)**.** The Λ-R_i-bimodule T_i is a good realization of G_i over R_i for $i = 1, ..., s$ by Lemma 10.8 and Lemma 10.1.

(i) of Theorem 1.2 follows from Lemma 10.2, (ii) follows from Lemma 10.5, Lemma 10.6 and Lemma 10.7. Finally (iii) follows from Lemma 10.4. □

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