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Tame-wild dichotomy for Cohen-Macaulay modules

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As it was conjectured in [DF] and proved in [D1], finite-dimensional algebras of infinite type (i.e. having infinitely many indecomposable representations) split into two classes. For the first one, called tame, indecomposable representations of any fixed dimension form a finite set of at most 1-parameter families, while for the second one, called wild, there exist arbitrarily large families of non-isomorphic indecomposable representations. Moreover, in some sense, knowing representations of one wild algebra, one would know those of any other algebras.

A lot of examples showed that the same should hold for Cohen-Macaulay modules over Cohen-Macaulay algebras of Krull dimension 1. In this paper we give a proof of it based on the same method of "matrix problems" or so called representations of bocses (cf. Sect. 1). But we had to consider a new situation, namely that of "open subcategories" (Sect. 2) and first reprove the results of [D1] for it. This new shape seems to be unavoidable in the case of Cohen-Macaulay modules but it should be also of use for other questions in representation theory. In Sect. 3 we propose a method to reduce the calculation of Cohen-Macaulay modules to some open subcategory and use the results of Sect. 2 to prove the tame-wild dichotomy.

The method we use is rather well-known in the theory of integral representations (cf. [GR] or [RR]). In principle, it almost coincides with that used in [J] for representations of commutative orders. We hope that it will be possible to spread both the method and the main theorem on tame-wild dichotomy to any orders over a complete discrete valuation ring, although at the moment we lack some technics to do it.

1 Preliminaries

As the notions of bocses and their representations are not well-known, remind the main definitions (cf. [Roi, D1]). All considered categories will be linear over some base field K which will always be supposed algebraically closed. Respectively, all functors are K-linear (bifunctors bilinear). We write Hom, $\otimes$ instead of Hom_K, $\otimes_K$. A *module* over a category A is a functor M: $A \to \mathrm{Vect}$ (the category of K-vector spaces); an *A-B-bimodule* (where A, B are categories) is a bifunctor V: $A^{op} \times B \to \mathrm{Vect}$; if $A = B$, we call V an A-bimodule. For $v \in V(X, Y)$, $a \in A(X', X)$, $b \in B(Y, Y')$ we write bva instead of $V(a, b)(v)$. A *bocs* is a pair $\mathbf{a} = (A, V)$ where A is some category and V an A-coalgebra, i.e. an A-bimodule V supplied with a comultiplication μ: $V \to V \otimes_A V$ and a counit ε: $V \to A$ satisfying the usual conditions.

A *representation* of $\mathbf{a}$ over some algebra R is defined as a functor M: $A \to pr - R$, the category of finitely generated projective R-modules. If N is another representation, define

$$\mathrm{Hom}_{\mathbf{a}}(M, N) = \mathrm{Hom}_{A-A}(V, (M, N))$$

where (M, N) is an A-bimodule defined by the rules:

$$(M, N)(X, Y) = \mathrm{Hom}_R(M(X), N(Y)) \quad \text{for } X, Y \in \mathbf{ob} A\ ;$$

$$afb = N(a) f M(b) \quad \text{for } f \in (M, N)(X, Y)\ ,$$

$$a\colon Y \to Y', \quad b\colon X' \to X \quad \text{in } A\ .$$

The product of $\varphi \in \mathrm{Hom}_{\mathbf{a}}(M, N)$ and $\psi \in \mathrm{Hom}_{\mathbf{a}}(L, M)$ is defined as the composition

$$V \xrightarrow{\mu} V \otimes_A V \xrightarrow{\varphi \otimes \psi} (M, N) \otimes_A (L, M) \xrightarrow{m} (L, N)$$

where m is the multiplication of R-homomorphisms. Thus the *category of representations* $\mathrm{Rep}(\mathbf{a}, R)$ is defined. We write $\mathrm{Rep}(\mathbf{a})$ instead of $\mathrm{Rep}(\mathbf{a}, K)$.

Any algebra R can be considered as a bocs ("principal bocs") if we put $A = V = R$. Of course, representations of such bocses are just representations of R. Remark that if $M \in \mathrm{Rep}(\mathbf{a}, R)$ and $L \in \mathrm{Rep}(R, R')$, then their tensor product $M(L) = M \otimes_R L$ lies in $\mathrm{Rep}(\mathbf{a}, R')$; so M can be viewed as "a family of representations of $\mathbf{a}$ parametrized by R".

As a rule, the category A will be finitely generated over K, i.e. with finite object set and a finite set of morphisms (generators) whose products span all spaces of morphisms $A(X, Y)$. A *dimension* of a representation of $\mathbf{a}$ is defined as a function $\underline{d}$: $\mathbf{ob} A \to \mathbf{N}$. In cases when there is a notion of rank for finitely generated projective R-modules, we can associate to $M \in \mathrm{Rep}(\mathbf{a}, R)$ its dimension $\underline{\dim}\, M$: $\mathbf{ob} A \to \mathbf{N}$, namely, $(\underline{\dim}\, M)(X) = \mathrm{rank}\, M(X)$ and denote by $\mathrm{Rep}_{\underline{d}}(\mathbf{a}, R)$ the set of representations having dimension $\underline{d}$. For instance, this is the case if $R = K$ (hence rank = dim), so $\mathrm{Rep}_{\underline{d}}(\mathbf{a})$ is defined. If $\mathbf{S}$ is a system of generators for A, then each representation $M \in \mathrm{Rep}(\mathbf{a})$ determines (and is determined by) linear mappings $M(a)$: $M(X) \to M(Y)$, $a \in S$, a: $X \to Y$. Hence, treating all linear mappings $M(a)$ as matrices, we can consider $\mathrm{Rep}_{\underline{d}}(\mathbf{a})$ as an algebraic variety lying in affine space $\mathbf{A}^{\|\underline{d}\|}$, carrying the Zariski topology, where

$$\|\underline{d}\| = \sum_{\substack{a \in S,\\ a: X \to Y}} \underline{d}(X)\underline{d}(Y)\ .$$

All considered bocses are supposed *normal* – which means that for any $X \in \mathbf{ob} A$ an element $\omega_X \in V(X, X)$ exists such that $\varepsilon(\omega_X) = 1_X$, $\mu(\omega_X) = \omega_X \otimes \omega_X$. In this case the bimodule structure on V is completely determined if we know the *kernel* of the bocs $\mathbf{a}$, $\bar{V} = \operatorname{Ker} \varepsilon$ and for each $a \in A(X, Y)$ its *differential* $\partial a = a\omega_X - \omega_Y a \in \bar{V}$. Moreover, the coalgebra structure is determined if we know the *differentials* $\partial v = \mu(v) - v \otimes \omega_X - \omega_Y \otimes v \in \bar{V} \otimes_A V$ for all $v \in \bar{V}(X, Y)$.

In main applications *free bocses* arise, i.e. such that A is a free category (that of paths $K\Gamma$ of an oriented graph Γ) and the kernel $\bar{V}$ is a free A-bimodule. A free bocs is completely determined if we know the set S_0 of free generators of A, the set S_1 of free generators of $\bar{V}$ and their differentials. The set $S = S_0 \cup S_1$ is called a *set of free generators* of the bocs $\mathbf{a}$.

For technical purposes, *semi-free bocses* are needed. A *semi-free category* is, by definition, a category of the form $K\Gamma[g_a(a)^{-1}]$ where a ranges through the set of loops (i.e. elements of S_0 such that a: $X \to X$) and $g_a(t) \in K[t]$ is a non-zero polynomial (depending on a). If $g_a \neq$ const, call the loop a *marked*. A bocs is called *semi-free* if A is a semi-free category, $\bar{V}$ a free A-bimodule and $\partial a = 0$ for all marked loops. In this case call S a set of *semi-free generators* of $\mathbf{a}$.

If $\mathbf{a}$ is free, then, of course, $\operatorname{Rep}_{\underline{d}}(\mathbf{a}) \simeq \mathbf{A}^{\|\underline{d}\|}$; if $\mathbf{a}$ is semi-free, then $\operatorname{Rep}_{\underline{d}}(\mathbf{a})$ is an open subset in $\mathbf{A}^{\|\underline{d}\|}$.

A semi-free category is called *triangular* if there exists a system S of semi-free generators and a function h: $S \to \mathbf{N}$ such that for any $a \in S$ ∂a belongs to the subbocs generated by $b \in S$ with $h(b) < h(a)$.

A representation $M \in \operatorname{Rep}(\mathbf{a}, R)$ is called *strict* if it satisfies the following two conditions:

(1) If $L \in \operatorname{Rep}(R, R')$ is indecomposable, then $M(L) \in \operatorname{Rep}(\mathbf{a}, R')$ is also indecomposable.

(2) If $L, L' \in \operatorname{Rep}(R, R')$ are non-isomorphic, then $M(L) \not\simeq M(L')$, too.

One can say that if such M exists, the representation theory of $\mathbf{a}$ is at least as complicated as that of R.

If a set $F = \{M_i | M_i \in \operatorname{Rep}(\mathbf{a}, R_i)\}$ is given (each M_i can be a representation over its own R_i), we call F *strict* provided each M_i is strict and if $i \neq j$, then $M_i(L) \not\simeq M_j(L')$ for any $L \in \operatorname{Rep}(R_i, R)$, $L' \in \operatorname{Rep}(R_j, R)$.

We need also "*bimodule categories*" defined as follows. Let U be an R_1-R_2-bimodule where R_1, R_2 are some algebras. For each algebra R let $P_i = P_i(R)$ be the category of finitely generated projective $R_i \otimes R^{op}$-modules. Consider a P_1-P_2-bimodule U_R such that $U_R(P_1, P_2) = \operatorname{Hom}_{R_1 \otimes R^{op}}(P_1, U \otimes_{R_2} P_2)$.

Take the elements of all $U_R(P_1, P_2)$ as objects of a new category $U(R)$ and as morphisms from $u \in U_R(P_1, P_2)$ to $u' \in U_R(P'_1, P'_2)$ take all pairs (f_1, f_2) with $f_i \in \operatorname{Hom}_{R_i \otimes R^{op}}(P_i, P'_i)$ such that $u' f_1 = f_2 u$.

If $L \in \operatorname{Rep}(R, R')$, then $P_i \otimes_R L \in P_i(R')$, so L defines a natural mapping

$$\otimes L\colon\ U_R(P_1, P_2) \to U_{R'}(P_1 \otimes_R L, P_2 \otimes_R L)\,.$$

Hence, one can reproduce for bimodule categories the above notion of strictness.

Note that this definition is formally distinct from that of [D1] though they provide equivalent categories.

Usually the algebras R_i are finite-dimensional and in this case the following theorem is valid [D1]:

Theorem 1. *If R_1, R_2 are finite-dimensional algebras and U is a finite-dimensional R_1-R_2-bimodule, then there exists a free triangular bocs $\mathbf{a} = \mathbf{a}_U$ and for each algebra*

R an equivalence of categories T_R: $\mathrm{Rep}(\mathbf{a}, R) \to U(R)$ *commuting with tensor products, i.e.*

$$T_{R'}(M \otimes_R L) \simeq T_R(M) \otimes_R L \quad \text{for any } L \in \mathrm{Rep}(R, R') .$$

2 Tame and wild open subcategories

Let $\mathbf{a}$ be a finitely generated bocs and $\mathbf{X} \subset \mathrm{Rep}(\mathbf{a})$ a full subcategory. Call $\mathbf{X}$ an *open subcategory* if it satisfies the following conditions:

(1) If $M \in \mathbf{X}$ and $N \simeq M$, then $N \in \mathbf{X}$;

(2) $M \oplus N \in \mathbf{X}$ if and only if $M \in \mathbf{X}$ and $N \in \mathbf{X}$;

(3) for each dimension $\underline{\mathrm{d}}$ the subset $\mathbf{X}_{\underline{\mathrm{d}}} = \mathbf{X} \cap \mathrm{Rep}_{\underline{\mathrm{d}}}(\mathbf{a})$ is open in $\mathrm{Rep}_{\underline{\mathrm{d}}}(\mathbf{a})$.

For any algebra R put $\mathbf{X}(R) = \{M \in \mathrm{Rep}(\mathbf{a}, R) | M(L) \in \mathbf{X} \text{ for any } L \in \mathrm{Rep}(R)\}$. It is clear that if $M \in \mathbf{X}(R)$ and $L \in \mathrm{Rep}(R, R')$, then $M(L) \in \mathbf{X}(R')$.

Call $\mathbf{X}$ *wild* if for any finitely generated algebra R there exists a strict representation $M \in \mathbf{X}(R)$. Non-formally this means that to know the representations of $\mathbf{X}$ we have to know the representations for all finitely generated algebras.

It is well-known (and easy to check) that to prove wildness it is sufficient to find a strict representation $M \in \mathbf{X}(K\langle x, y\rangle)$ (free non-commutative algebra with 2 generators), as the latter has a strict representation over any other one. A little more complicated but also known (cf. [GP] or [D2]) is that here we can replace $K\langle x, y\rangle$ by the polynomial ring $K[x, y]$ or even the power series ring $K[[x, y]]$.

Call a *rational algebra* any algebra of the form $K[x, f(x)^{-1}]$ for a non-zero polynomial $f(x)$, i.e. the affine algebra of a smooth rational affine curve.

Theorem 2. *Let* $\mathbf{a} = (A, V)$ *be a finitely generated semi-free bocs,* $\mathbf{X} \subset \mathrm{Rep}(\mathbf{a})$ *an open subcategory. Then the following conditions are equivalent*:

(1) $\mathbf{X}$ *is non-wild;*

(2) *for each dimension* $\underline{\mathrm{d}}$ *there exists a subvariety* $X_{\underline{\mathrm{d}}} \subset \mathbf{X}_{\underline{\mathrm{d}}}$ such that

$$\dim X_{\underline{\mathrm{d}}} \leqq |\underline{\mathrm{d}}| = \sum_{T \in \mathrm{ob}\mathbf{A}} \underline{\mathrm{d}}(T)$$

and any representation from $\mathbf{X}_{\underline{\mathrm{d}}}$ *is isomorphic to one belonging to* $X_{\underline{\mathrm{d}}}$;

(3) *for each dimension* $\underline{\mathrm{d}}$ *there exists a subvariety* $Y_{\underline{\mathrm{d}}} \subset \mathbf{X}_{\underline{\mathrm{d}}}$ *such that* $\dim Y_{\underline{\mathrm{d}}} \leqq 1$ *and any indecomposable representation from* $\mathbf{X}_{\underline{\mathrm{d}}}$ *is isomorphic to one belonging to* $Y_{\underline{\mathrm{d}}}$;

(4) *there exists a strict set* $\{M_i | i \in I, M_i \in \mathbf{X}(R_i)\}$ *with rational algebras* R_i *such that for each dimension* $\underline{\mathrm{d}}$ *all indecomposable representations from* $\mathbf{X}_{\underline{\mathrm{d}}}$ *except a finite number (up to isomorphism) are isomorphic to* $M_i(L)$ *for some* $i \in I_{\underline{\mathrm{d}}}$ *and some* $L \in \mathrm{Rep}(R_i)$ *where* $I_{\underline{\mathrm{d}}}$ *is a finite subset of* I *(depending on* $\underline{\mathrm{d}}$*).*

(If these conditions are satisfied, call $\mathbf{X}$ *tame*).

Proof. (4) $\Rightarrow$ (3) as any indecomposable n-dimensional representation L of a rational algebra $K[x, f(x)^{-1}]$ maps x to a Jordan cell $J(\lambda)$ with eigenvalue λ such that $f(\lambda) \neq 0$. Hence representations $M_i(L)$ for such L produce a 1-dimensional subvariety of $\mathbf{X}_{\underline{\mathrm{d}}}$ and as $\underline{\mathrm{d}}$ is fixed, n is also fixed.

(3) $\Rightarrow$ (2) is quite evident as $|\underline{\mathrm{d}}|$ is an upper bound for the maximal number of indecomposable direct summands of any representation of dimension $\underline{\mathrm{d}}$.

(2) ⇒ (1) if $M \in \mathbf{X}_{\underline{d}}(K\langle x, y\rangle)$ is strict, then $M(L)$ for $L \in \mathrm{Rep}_n(K\langle x, y\rangle)$ form in $\mathbf{X}_{n\underline{d}}$ a subset of dimension at least n^2 consisting of pairwise non-isomorphic representations and $n^2 > |n\underline{d}|$ if $n > |\underline{d}|$.

At last, (1) ⇒ (4) can be proved just by repeating the proof of the above Theorem 1 given in [D1] if we make the following simple observation. Let $a \in A(X, Y)$ with $\partial a = 0$. Then if $M \simeq N$ in Rep(**a**), we have $N(a) = DM(a)C^{-1}$ for some isomorphisms $C\colon M(X) \to M(Y)$ and $D\colon N(X) \to N(Y)$ ($C = D$ if $X = Y$). Denote $\mathbf{X}(a) = \{M(a) | M \in \mathbf{X}\}$. *As* $\mathbf{X}$ is an open subcategory, $\mathbf{X}(a)$ form an open subset in the space of all linear mapping $L \to L'$ for any fixed $L = M(X)$ and $L' = M(Y)$. Then the only possibilities for $\mathbf{X}(a)$ are:

– if $X \neq Y$, either all linear mappings, or those $F\colon L \to L'$ with rk $F = \dim L$, or those with rk $F = \dim L'$ or isomorphisms only;

– if $X = Y$ there exists a finite subset $E(a) \subset K$ such that $\mathbf{X}(a) = \{F\colon L \to L | F$ has no eigenvalue from $E(a)\}$.

Of course, the proof of [D1], based on algorithms of reduction of matrices, is rather complicated. Unfortunately, till now the only known way to obtain the equivalences (1) ⇔ (2) ⇔ (3) is to prove that (1) ⇒ (4).

3 Cohen-Macaulay algebras

In this paragraph we consider algebras Λ over K satisfying the following conditions:

(A1) The centre Z of Λ is a complete local noetherian Cohen-Macaulay ring of Krull dimension 1 with residue field K;

(A2) Λ is a (finitely generated) Cohen-Macaulay module over Z;

(A3) Λ is semi-prime, i.e. has no nilpotent ideals.

We call such algebras CM-*Algebras.* Denote by CM(Λ) the category of Λ-modules which are maximal Cohen-Macaulay modules over Z, i.e., in our case, finitely generated and torsion free. Call them CM-*Λ-modules.*

If Λ is a CM-algebra, its full quotient ring Q is a semi-simple artinian ring and there exists a (not necessarily unique) *maximal overring* $\tilde{\Lambda}$, i.e. a CM-algebra such that $\Lambda \subset \tilde{\Lambda} \subset Q$ and there are no CM-algebras $\Lambda' \neq \tilde{\Lambda}$ with $\tilde{\Lambda} \subset \Lambda' \subset Q$ (cf. [D3]). It follows from [Rog] that $\tilde{\Lambda}$ is always hereditary, i.e. any CM-$\tilde{\Lambda}$-module is projective over $\tilde{\Lambda}$.

If R is any K-algebra, denote by CM(Λ, R) the category of R-Λ-bimodules M satisfying the following conditions:

(M1) M is finitely generated as bimodule;

(M2) ${}_Z M$ is torsion free;

(M3) M_R is flat;

(M4) $M(L) = M \otimes_R L$ is a CM-Λ-module for any $L \in \mathrm{Rep}(R)$.

If R/m is finite-dimensional over K for any maximal left ideal $m \subset R$, then (M4) is equivalent to

(M4′) for any non-zero divisor $\lambda \in Z$ the R-module $M/\lambda M$ is also flat.

Surely, if $M \in \mathrm{CM}(\Lambda, R)$ and $L \in \mathrm{Rep}(R, R')$, then $M(L) \in \mathrm{CM}(\Lambda, R')$. So we are able to define strict modules $M \in \mathrm{CM}(\Lambda, R)$ and strict sets of such modules just as in Sect. 1. If R is a finitely generated commutative K-algebra of Krull dimension d, call any bimodule $M \in \mathrm{CM}(\Lambda, R)$ a *d-parameter family* of CM-Λ-modules (with base R).

Call Λ CM-*wild* if for every finitely generated algebra R there exists a strict module $M \in \mathrm{CM}(\Lambda, R)$. Again we have to check the existence of M only for $R = K\langle x, y\rangle$, or $R = K[x, y]$, or $R = K[|x, y|]$.

If a Λ-module M is torsion free (over Z) it can be embedded into the Q-module $Q \otimes_\Lambda M$, so if Λ' is an overring of Λ, i.e. a CM-algebra such that $\Lambda \subset \Lambda' \subset Q$, we can consider the Λ'-module $\Lambda' M$, which is the image of $\Lambda' \otimes_\Lambda M$ in $Q \otimes_\Lambda M$. If M was a CM-module, then so is $\Lambda' M$. In this case $Q \otimes_\Lambda M$ is finitely generated over Q, thus $Q \otimes_\Lambda M \simeq r_1 Q_1 \oplus \cdots \oplus r_t Q_t$ where $Q_1, \ldots, Q_t$ are all pairwise non-isomorphic simple Q-modules. Call the vector $\underline{r}(M) = (r_1, \ldots, r_t)$ the (*vector*) *rank* of M and denote $\mathrm{CM}_{\underline{r}}(\Lambda)$ the set of all CM-Λ-modules of rank $\underline{r}$.

Theorem 3. *For a* CM-*algebra Λ the following conditions are equivalent*:

(1) *Λ is not* CM-*wild*;

(2) *for any rank $\underline{r} = (r_1, \ldots, r_t)$ there exists a d-parameter family M of* CM-*Λ-modules with $d \leqq |\mathrm{r}| = \sum_{i=1}^{t} r_i$ such that any* CM-*Λ-module of rank $\underline{r}$ is isomorphic to some $M(L)$*;

(3) *for any rank $\underline{r}$ there exists a 1-parameter family M of* CM-*Λ-modules such that any indecomposable* CM-*Λ-module of rank $\underline{r}$ is isomorphic to some $M(L)$*;

(4) *there exists a strict set $\{M_i | i \in I, M_i \in \mathrm{CM}(\Lambda, R_i)\}$ with rational algebras R_i such that for each rank $\underline{r}$ all indecomposable* CM-*Λ-modules of rank $\underline{r}$ except a finite number (up to isomorphism) are isomorphic to $M_i(L)$ for some $i \in I_{\underline{r}}$ and $L \in \mathrm{Rep}(R_i)$ where $I_{\underline{r}}$ is a finite subset of I (depending on $\underline{r}$).*

If these conditions are satisfied, call Λ CM-*tame*.

Proof. Again (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is clear, so we have only to prove (1) $\Rightarrow$ (4).

Fix an overring $\Lambda' \supset \Lambda$ and denote by $\mathrm{CM}(\Lambda|\Lambda')$ the full subcategory in $\mathrm{CM}(\Lambda)$ consisting of all modules M such that $\Lambda' M$ is Λ'-projective. Of course, if Λ' is hereditary (e.g. maximal), then $\mathrm{CM}(\Lambda|\Lambda') = \mathrm{CM}(\Lambda)$. Let $I \subset \mathrm{rad}\,\Lambda$ be a two-sided Λ'-ideal such that $\dim_K \Lambda'/I < \infty$ (it exists as Λ'/Λ is a finitely generated torsion Z-module). Then $IM \subset M \subset \Lambda' M$ for any CM-module M and any homomorphism φ: $M \to N$ can be uniquely prolonged to φ': $\Lambda' M \to \Lambda' N$. Put

$$\Lambda_1 = \Lambda/I, \quad \Lambda_2 = \Lambda'/I$$

and consider a new category $C = C(\Lambda|\Lambda')$ whose objects are pairs (P, X) with P a (finitely generated) projective Λ_2-module, $X \subset P$ a Λ_1-submodule, and morphisms $(P, X) \to (P_1, X_1)$ are Λ_2-homomorphisms φ: $P \to P_1$ such that $\varphi(X) \subset X_1$. Define a functor T: $\mathrm{CM}(\Lambda|\Lambda') \to C$ putting $T(M) = (\Lambda' M/IM, M/IM)$ and let C_0 be the full subcategory of C consisting of all such pairs (P, X) that $\Lambda_2 X = P$. Then the following lemma is evident (cf. [GR] or [RR]):

Lemma 1. *$T(M) \in C_0$ for any $M \in \mathrm{CM}(\Lambda|\Lambda')$ and the functor T: $\mathrm{CM}(\Lambda|\Lambda') \to C_0$ is full, dense and reflects isomorphisms and indecomposability.*

Now consider the Λ_1-Λ_2-bimodule $U = \Lambda_2$ and define a functor $\underline{\mathrm{Im}}$: $U(K) \to C$ putting, for φ: $P_1 \to P_2$, $\underline{\mathrm{Im}}\,\varphi = (P_2, \mathrm{Im}\,\varphi)$. Denote $\mathbf{X}$ the full subcategory of $U(K)$ consisting of all such φ that $\mathrm{Ker}\,\varphi \subset \mathrm{rad}\,P_1$ and $\Lambda_2 \cdot \mathrm{Im}\,\varphi = P_2$. Certainly, these conditions define an open subset in $\mathrm{Hom}_\Lambda(P_1, P_2) = U(P_1, P_2)$ and are stable under direct sums and summands. As Λ_1 is artinian, any Λ_1-module X possesses a projective cover whence we obtain the following lemma:

Lemma 2. *If $\varphi \in \mathbf{X}$, then $\underline{\operatorname{Im}}\,\varphi \in C_0$ and the functor $\underline{\operatorname{Im}}\colon \mathbf{X} \to C_0$ is full, dense and reflects isomorphisms and indecomposability.*

Identify according to Theorem 1, $U(K)$ with Rep(**a**) for a free triangular bocs **a**. Then **X** becomes an open subcategory in Rep(**a**), thus Theorem 2 is applicable, i.e. **X** is either tame or wild.

Let $u \in \mathbf{X}(R)$ for some algebra R. Then $u\colon P_1 \to P_2$ where P_i is a projective $\Lambda_i \otimes R^{op}$-module. Call u *good* provided $P_i \simeq \tilde{P}_i/I\tilde{P}_i$ where $\tilde{P}_1$ (resp. $\tilde{P}_2$) is a projective $\Lambda \otimes R^{op}$-module (resp. $\Lambda' \otimes R^{op}$-module) and Coker u is flat over R. In this case denote $\tilde{u}\colon \tilde{P}_1 \to \tilde{P}_2$ some homomorphism for which $u = \tilde{u} (\operatorname{mod} I)$.

Lemma 3. (a) *If $u \in \mathbf{X}(R)$ is good and $M = \operatorname{Im}\tilde{u}$, then $M \in \mathrm{CM}(\Lambda, R)$.*

(b) *If $\{u_i | i \in I,\ u_i \in \mathbf{X}(R_i)\}$ is a strict set, all u_i are good and $M_i = \operatorname{Im}\tilde{u}_i$, then $\{M_i | i \in I\}$ is also a strict set.*

Proof. (a) Remark that $\operatorname{Coker} u \simeq \operatorname{Coker}\tilde{u}$, so we have an exact sequence

$$0 \to M \to \tilde{P}_2 \to N \to 0$$

with R-flat N and hence an exact sequence

$$0 \to M \otimes_R L \to \tilde{P}_2 \otimes_R L \to N \otimes_R L \to 0$$

for any $L \in \operatorname{Rep}(R)$ where $\tilde{P}_2 \otimes_R L$ is Λ'-projective. This does imply all properties (M1)-(M4) for M.

(b) follows directly from Lemmas 1 and 2.

Lemma 4. *Let $u \in \mathbf{X}(R)$ for a finitely generated commutative domain R. Then there exists a non-zero $f \in R$ such that $u_f \in \mathbf{X}(R_f)$ is good.*

Proof. Denote by F the quotient field of R. Then $(\Lambda/\operatorname{rad}\Lambda) \otimes F$ is semi-simple [B1], hence $\operatorname{rad}(\Lambda \otimes F) = (\operatorname{rad}\Lambda) \otimes F$ and $(\Lambda \otimes F)/\operatorname{rad}(\Lambda \otimes F) \simeq (\Lambda/\operatorname{rad}\Lambda) \otimes F$. Hence in $\Lambda \otimes F$ idempotents can be lifted modulo radical and any projective $(\Lambda \otimes F)$-module is of the form $P \otimes F$ for some projective Λ-module P. The same is true for the algebras Λ' and $\Lambda_i (i = 1, 2)$. As $\Lambda_1 = \Lambda/I$ and $I \subset \operatorname{rad}\Lambda$, any projective $(\Lambda_1 \otimes F)$-module is of the form $(P \otimes F)/I(P \otimes F)$. Therefore, if P is a projective $\Lambda_1 \otimes R$-module, there exists a non-zero $f \in R$ such that $P_f \simeq \tilde{P}/I\tilde{P}$ for a projective $\Lambda_1 \otimes R_f$-module $\tilde{P}$. So if $u \in \mathbf{X}(R)$, $u\colon P_1 \to P_2$, we can find $f \in R$ for which $(P_i)_f \simeq \tilde{P}_i/I\tilde{P}_i$. But as Λ_i are finite-dimensional, $N = \operatorname{Coker} u_f$ is finitely generated over R_f and there exists a non-zero $g \in R$ such that N_g is flat [B2], thus u_{fg} is good.

Corollary 1. *If **X** is wild, then Λ is wild.*

Proof. Let $u \in \mathbf{X}(R)$, $R = K[x, y]$, be strict. Find $f \in R$ such that u_f is good and a maximal ideal $m \subset R$ such that $f \notin m$. As the m-adique completion of R is isomorphic to $\hat{R} = K[|x, y|]$ u_f provides a good and strict element $\hat{u} \in \mathbf{X}(\hat{R})$. Then Lemma 3 implies that Λ is CM-wild.

Corollary 2. *If Λ' is hereditary and **X** is tame, then Λ is* CM-*tame.*

Proof. Let $\{u_i | i \in I, u_i \in \mathbf{X}(R_i)\}$ be a strict set satisfying conditions (4) of Theorem 2. Remark that if R is a rational algebra, then $\operatorname{Rep}_d(R) - \operatorname{Rep}_d(R_f)$ is finite for any non-zero $f \in R$ and any dimension d. Therefore, Lemma 4 allows us to suppose all u_i good. But as Λ' is hereditary, $\mathrm{CM}(\Lambda | \Lambda') = \mathrm{CM}(\Lambda)$. Hence, Lemmas 1–3 imply that the set $\{M_i | i \in I\}$ with $M_i = \operatorname{Im}\tilde{u}_i$ satisfies condition (4) of Theorem 3.

Now (1) $\Rightarrow$ (4) follows from Corollaries 1 and 2.

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