REPRESENTATIONS OF BOXES AND THEIR APPLICATIONS

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1. Categories

If not specified another, all categories are supposed *preadditive* and all functors *additive*. Recall that an additive category $\mathscr{A}$ is said to be *fully additive* (or *Karoubian*) if all idempotents in $\mathscr{A}$ *split*, i.e. if $e \in \mathscr{A}(A, A)$ is an idempotent, there are morphisms $\pi : A \to B$ and $\iota : B \to A$ such that $e = \iota\pi$ and $\pi\iota = 1_B$. Since $1 - e$ is also an idempotent, there are also $\pi' : A \to B'$ and $\iota' : B' \to A$ such that $1 - e = \iota'\pi'$ and $\pi'\iota' = 1_B$. Therefore, $A \simeq B \oplus B'$. One can easy embed any preadditive category $\mathscr{A}$ into a fully additive category $\mathscr{A}^{\oplus}$ called the *fully additive hull* of $\mathscr{A}$, such that every object in $\mathscr{A}^{\omega}$ is isomorphic to a direct summand of a direct sum of objects from $\mathscr{A}$. This category can be constructed as the category of *matrix idempotetnts*. Below we will give another description of $\mathscr{A}^{\oplus}$.

A category $\mathscr{A}$ is said to be *local*, if every object $A \in \mathscr{A}$ decomposes as $A \simeq A_1 \oplus A_2 \oplus \ldots \oplus A_n$, where all rings $\mathscr{A}(A_i, A_i)$ are local. It is well-known [1, Theorem I.3.6] that a local category is fully additive and *Krull–Schmidt*. It means that if $A \simeq A_1 \oplus A_2 \oplus \ldots \oplus A_n \simeq A'_1 \oplus A'_2 \oplus \ldots \oplus A'_m$, where all A_i and A'_j are indecomposable, then $n = m$ and $A_i \simeq A'_i$ up to a renumeration of A'_j's.

Let $\Bbbk$ be a commutative ring. We say that a category $\mathscr{A}$ is a $\Bbbk$-*category* if all groups $\mathscr{A}(A, B)$ are actually $\Bbbk$-modules and the multiplication of morphisms is $\Bbbk$-bilinear. A $\Bbbk$-category $\mathscr{A}$ is said to be *locally finite over* $\Bbbk$, or $\Bbbk$-*lof* if all $\Bbbk$-modules $\mathscr{A}(A, B)$ are finitely generated. If the ring $\Bbbk$ is noetherian, local and complete while $\mathscr{A}$ is a fully additive $\Bbbk$-lof, then $\mathscr{A}$ is local. We will mainly consider the case when $\Bbbk$ is a field and $\mathscr{A}$ is a $\Bbbk$-lof.

A (left) *module over a category* $\mathscr{A}$, or an $\mathscr{A}$*-module*, is an *module* (additive) functor $M : \mathscr{A} \to \mathsf{Ab}$, the category of abellian groups. If $x \in M(A)$ and $\alpha : A \to B$, we write αx instead of $M(\alpha)x \in M(B)$. Analogously, we call a functor $N : \mathscr{A}^{\mathrm{op}}$ a *right* $\mathscr{A}$*-module* and write $y\beta$ instead of $N(\beta)y \in N(B)$ for $y \in N(A)$ and $\beta \in \mathscr{A}(B, A)$. If M is an $\mathscr{A}$-module and $\mathscr{A}^\omega$ is the fully additive hull of $\mathscr{A}$, one can extend M to an $\mathscr{A}^\omega$-module (uniquely up to isomorphism), which we denote by the same letter M. We denote the category of $\mathscr{A}$-modules by $\mathscr{A}$-Mod. For any subset $S \subseteq \bigcup_A M(A)$ we denote by $S(A)$ the intersection $S \cap M(A)$. A *set of generators* of an $\mathscr{A}$-module M is, by definition, a subset $G \subseteq \bigcup_A M(A)$ such that every element $x \in M(B)$ can be presented as a sum $\sum_{g \in G} \in G\alpha_g g$, where $\alpha_g : A \to B$ if $g \in G(A)$ and almost all $\alpha_g = 0$. If one can choose a finite set of generators, the module M is said to be *finitely generated*. We denote by $\mathscr{A}$-mod the category of finiktely generated $\mathscr{A}$-modules. Both categories $\mathscr{A}$-Mod and $\mathscr{A}$-mod are abelian, where kernels and cokernels of a morphism $f : M \to M'$ are defined “componentwise,” i.e. $(\ker f)(A) = \ker f(A)$ and $(\operatorname{Coker} f)(A) = \operatorname{Coker} f(A)$. In particular, a sequence

$$\cdots \to M_{k-1} \to M_k \to M_{k+1} \to \dots$$

is exact if and only if so are all sequences

$$\cdots \to M_{k-1}(A) \to M_k(A) \to M_{k+1}(A) \to \dots ,$$

where A runs through objects of $\mathscr{A}$.

For every object $A \in \mathscr{A}$ we denote by $\mathscr{A}^A$ the *representable (left)*$\mathscr{A}$*-module* $\mathscr{A}(A, \_)$ and by $\mathscr{A}_A$ the *representable right* $\mathscr{A}$*-module* $\mathscr{A}(\_\,, A)$. Obviously, these modules are finitely generated: the set of generators consists of a unique element 1_A. The well-known Yoneda Lemma claims that mapping A to $\mathscr{A}_A$ we get a full embedding $\mathscr{A} \to \mathscr{A}^{\mathrm{op}}$-mod. Moreover, $\mathscr{A}_A$ are projective in the category $\mathscr{A}$-Mod (or $\mathscr{A}$-mod) and every projective module from $\mathscr{A}$-mod is isimorphic to a direct summand of a direct sum of representable modules. Therefore, one can identify the fully additive hull $\mathscr{A}^\omega$ with the category $\mathscr{A}^{\mathrm{op}}$-proj of finitely generated projective right $\mathscr{A}$-modules.

Given a left $\mathscr{A}$-module M and a right $\mathscr{A}$-module N, we define their *tensor product* $N \otimes_{\mathscr{A}} M$ as the factorgroup of $\bigoplus_A N(A) \otimes M(A)$ by the subgroup generated by all differences $x\alpha \otimes y - x \otimes \alpha y$, where $x \in N(B)$, $y \in M(A)$, $\alpha : A \to B$. This operation has usual properties of tensor product of modules over a ring (and coincide with the latter if $\mathscr{A}$ only contains one object, so is actually a ring).

An *$\mathscr{A}$-$\mathscr{B}$-bimodule* is, by definition a biadditive functor $V : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathsf{Ab}$. If $v \in V(A,B)$, we often write $v : A \dashrightarrow B$, and we write $\alpha v \beta$ instead of $V(\alpha,\beta)v \in V(A',B')$, where $\alpha : A' \to B,\ \beta : B \to B'$. It matches the usual rule for "multiplication of arrows," since we have the sequence of arrows

$$A' \xrightarrow{\alpha} A \overset{v}{\dashrightarrow} B \xrightarrow{\beta} B' .$$

We only have to remember that there is at most one dashed arrow in any product and if there is one, the whole product is also dashed. If $\mathscr{A} = \mathscr{B}$, we speak about *$\mathscr{A}$-bimodules*. Again, any $\mathscr{A}$-$\mathscr{B}$-bimodule V can be extended to an $\mathscr{A}^{\omega}$-$\mathscr{B}^{\omega}$-bimodule, uniquely up to isomorphism, and we denote this extended bimodule by V too. Obviously, if $\mathscr{A}$ is a $\Bbbk$-category, every left (right) $\mathscr{A}$-module can be considered as $\Bbbk$-$\mathscr{A}$-bimodule (respectively, as $\mathscr{A}$-$\Bbbk$-bimodule).

Given an $\mathscr{A}$-$\mathscr{B}$-bimodule V and a $\mathscr{B}$-$\mathscr{C}$-bimodule U, one can define their *tensor product* $U \otimes_{\mathscr{B}} V$, which is an $\mathscr{A}$-$\mathscr{C}$-bimodule, setting

$$(U \otimes_{\mathscr{B}} V)(A,C) = U(\_\,,C) \otimes_{\mathscr{B}} V(A,\_\,).$$

Again, this operation has usual properties of tensor product of bimodules over rings, including the *adjointness* formula:

$$\mathrm{Hom}_{\mathscr{A}\text{-}\mathscr{C}}(U \otimes_{\mathscr{B}} V, W) \simeq \mathrm{Hom}_{\mathscr{B}\text{-}\mathscr{C}}(U, \mathrm{Hom}_{\mathscr{A}}(V,W)),$$

where U,V are as above, W is an $\mathscr{A}$-$\mathscr{C}$-bimodule and $\mathrm{Hom}_{\mathscr{A}}(V,W)$ is the $\mathscr{B}$-$\mathscr{C}$-bimodule such that

$$\mathrm{Hom}_{\mathscr{A}}(V,W)(B,C) = \mathrm{Hom}_{\mathscr{A}}(V(\_\,,B), W(\_\,,C)).$$

Let $\mathscr{A}$ be a $\Bbbk$-categfory. We define the *principle $\mathscr{A}$-bimodule $\mathscr{A}_A^B$* as $\mathscr{A}^B \otimes_{\Bbbk} \mathscr{A}_A$, i.e.

$$\mathscr{A}_B^A(X,Y) = \mathscr{A}(B,Y) \otimes_{\Bbbk} \mathscr{A}(X,A).$$

The element $1_B \otimes 1_A$ is a generator of this bimodule. Direct sums of principle bimodules are called *free bimodules*. They are projective in the category of $\mathscr{A}$-$\mathscr{B}$-bimodules and every finitely generated projective is a direct summand of a free bimodule.

Let $F : \mathscr{A} \to \mathscr{B}$ be a functor and V be a $\mathscr{B}$-$\mathscr{C}$-bimodule (or a $\mathscr{C}$-$\mathscr{B}$-bimodule). One can consider the $\mathscr{A}$-$\mathscr{C}$-bimodule V^F such that $V^F(A,C) = V(FA,C)$ (respectively, the $\mathscr{C}$-$\mathscr{A}$-bimodule ${}^FV$ such that ${}^FV(C,A) = V(C,FA)$). If V is a $\mathscr{B}$-bimodule, we also can define the $\mathscr{A}$-bimodule ${}^FV^F$. Especially, we can define the $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathscr{B}^F$ as well as the $\mathscr{B}$-$\mathscr{A}$-bimodule ${}^F\mathscr{B}$ and the $\mathscr{A}$-bimodule ${}^F\mathscr{B}^F$. Moreover, one easily sees that $V^F \simeq V \otimes_{\mathscr{B}} \mathscr{B}^F \simeq \mathrm{Hom}_{\mathscr{B}}({}^F\mathscr{B}, V)$ and ${}^FV \simeq$

$V \otimes_{\mathscr{B}} {}^F\mathscr{B} \simeq \mathrm{Hom}_{\mathscr{B}}(\mathscr{B}^F, V)$. Sometimes we omit the superscript ^{F}, when it implies no ambiguity.

Let Γ be a *quiver* (an oriented graph) and $\Bbbk$ be a commutative ring. We define the *path category* $\Bbbk\Gamma$ as the $\Bbbk$-category with the set of objects $\mathrm{Ver}\,\Gamma$ (the set of vertices of Γ) and such that $\Bbbk\Gamma(x, y)$ is the free $\Bbbk$-module with the basis consisting of all paths from x to y in the quiver Γ. If $x = y$, wew also count the *empty path* from x to x (containing no arrows), which we denote by 1_x. The product ab of paths $a : x \to y$ and $b : z \to x$ is just their concatenation; especially $a1_x = a$ and $1_x b = b$. This definition gives the multiplication of morphisms from $\Bbbk\Gamma$ by $\Bbbk$-linearity. We often call $\Bbbk\Gamma$ the *free* $\Bbbk$*-category generated by the quiver* Γ.

In what follows, we often consider *biquivers.* A biquiver Γ consists of the set of vertices $\mathrm{Ver}\,\Gamma$ and for each pair (x, y) of vertices two sets $\Gamma_0(x, y)$ and $\Gamma_1(x, y)$. We call elements of $\Gamma_0(x, y)$ the *solid arrows* from x to y and the elements of $\Gamma_1(x, y)$ the *dashed arrows* from x to y. We also denote by Γ_0 the usual quiver with the set of vertices $\mathrm{Ver}\,\Gamma$ and with $\Gamma_0(x, y)$ as the set of arrows from x to y (the *solid part* of Γ). For every path p in Γ we define its *degree* $\deg p$ as the number of dashed arrows in p. Now we consider the free $\Bbbk$-category $\Bbbk\Gamma_0$ and define the $\Bbbk\Gamma_0$-bimodule $\Bbbk\Gamma_1$ taking for $(\Bbbk\Gamma_1)(x, y)$ the set of all paths of degree 1 from x to y. This bimodule is generated by the dashed arrows from Γ_1, and one easily sees that

$$\Bbbk\Gamma_1 \simeq \bigoplus_{\substack{\alpha\in\Gamma_1\\ \alpha: x \dashrightarrow y}} (\Bbbk\Gamma_0)^y_x$$

(just map the arrow $\alpha : x \dashrightarrow y$ to $1_y \otimes 1_x$). So every biquiver defines a free $\Bbbk$-category and a free bimodule over this category.

2. Boxes, representations and change of rings

s2

21 **Definition 2.1.** (1) A *box* is a quadruple $\mathfrak{A} = (\mathscr{A}, \mathscr{V}, \mu, \varepsilon)$, where

- $\mathscr{A}$ is a category;
- $\mathscr{V}$ is an $\mathscr{A}$-bimodule;
- $\mu : \mathscr{V} \to \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V}$ and $\varepsilon : \mathscr{V} \to \mathscr{A}$

are homomorphisms of $\mathscr{A}$-bimodules such that the diagrams

$$\begin{tikzcd}
\mathscr{V} \arrow[r, "\mu"] \arrow[d, "\mu"'] & \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \arrow[d, "1_V \otimes \mu"] \\
\mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \arrow[r, "\mu \otimes 1_V"'] & \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V},
\end{tikzcd}$$

as well as

$$\begin{tikzcd} \mathscr{V} \arrow[r, "\mu"] \arrow[d, equal] & \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \arrow[d, "1_V \otimes \varepsilon"] \\ \mathscr{V} \arrow[r, "id_r"'] & \mathscr{V} \otimes_{\mathscr{A}} \mathscr{A} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{V} \arrow[r, "\mu"] \arrow[d, equal] & \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \arrow[d, "\varepsilon \otimes 1_V"] \\ \mathscr{V} \arrow[r, "id_l"'] & \mathscr{A} \otimes_{\mathscr{A}} \mathscr{V} \end{tikzcd}$$

are commutative, where id_r and id_l are natural identifications, mapping v, respectively, to $v \otimes 1$ and to $1 \otimes v$.

In other words, μ and ε establish an $\mathscr{A}$*-coalgebra* structure on the bimodule $\mathscr{V}$.

We often write $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ not mentioning μ and ε.

(2) A *morphism of boxes* $\mathfrak{A} = (\mathscr{A}, \mathscr{V}, \mu, \varepsilon) \to \mathfrak{A}' = (\mathscr{A}', \mathscr{V}', \mu', \varepsilon')$ is a pair $F = (F_0, F_1)$, where $F_0 : \mathscr{A} \to \mathscr{A}'$ is a functor and $F_1 : \mathscr{V} \to {}^F\mathscr{V}'^F$ is a homomorphism of $\mathscr{A}$-bimodules compatable with comultiplication and counit, i.e. such that the diagrams

$$\begin{tikzcd} \mathscr{V} \arrow[r, "\mu"] \arrow[d, "F_1"'] & \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \arrow[d, "F_1 \otimes F_1"] \\ \mathscr{V}' \arrow[r, "\mu'"'] & \mathscr{V}' \otimes_{\mathscr{A}'} \mathscr{V}' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{V} \arrow[r, "\varepsilon"] \arrow[d, "F_1"'] & \mathscr{A} \arrow[d, "F_0"] \\ \mathscr{V}' \arrow[r, "\varepsilon'"'] & \mathscr{A}' \end{tikzcd}$$

are commutative.

We usually omit indices and write F instead of F_0 and F_1 when it is not ambiguous. The kernel of the homomorphism ε is called the *kernel of the box*.

If $\mathscr{V} = \mathscr{A}$, $\varepsilon = 1_{\mathscr{A}}$ and $\mu : \mathscr{A} \to \mathscr{A} \otimes_{\mathscr{A}} \mathscr{A}$ is the natural isomorphim, the box $\mathfrak{A} = (\mathscr{A}, \mathscr{A})$ is called *principal.*

22 **Definition 2.2.** Given a box $\mathfrak{A}$, we define the *category of* $\mathfrak{A}$*-modules* $\mathfrak{A}$-Mod as follows:

- *Objects* of $\mathfrak{A}$-Mod are $\mathscr{A}$-modules.
- The set of *morphisms* $\mathrm{Hom}_{\mathfrak{A}}(M, N)$ is defined as $\mathrm{Hom}_{\mathscr{A}}(\mathscr{V} \otimes_{\mathscr{A}} M, N)$.
- The *product* gf of morphisms $f \in \mathrm{Hom}_{\mathfrak{A}}(M, N)$ and $g \in \mathrm{Hom}_{\mathfrak{A}}(N, L)$, i.e. $\mathscr{A}$-homomorphisms $f : \mathscr{V} \otimes_{\mathscr{A}} M \to N$ and $g : \mathscr{V} \otimes_{\mathscr{A}} N \to L$, is defined as the composition $g(1 \otimes f)(\mu \otimes 1)$:
$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\mu \otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{1 \otimes f} \mathscr{V} \otimes_{\mathscr{A}} N \xrightarrow{g} L.$$
- The *identity morphism* $1_M \in \mathrm{Hom}_{\mathfrak{A}}(M, M)$ is defined as the composition $id_l^{-1}(\varepsilon \otimes 1)$:
$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\varepsilon \otimes 1} \mathscr{A} \otimes_{\mathscr{A}} M \xrightarrow{id_l^{-1}} M.$$

We must check that this product is associative and 1_M is indeed an identity morphism, i.e. $f1_M = f$ and $1_M f' = f'$ whenever these products are defined. Let $h \in \mathrm{Hom}_{\mathfrak{A}}(L, K)$, i.e. h is an $\mathscr{A}$-homomorphism $\mathscr{V} \otimes_{\mathscr{A}} L \to K$. Then $h(gf)$ is the composition

$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\mu\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{1\otimes gf} \mathscr{V} \otimes_{\mathscr{A}} L \xrightarrow{h} K,$$

that is, the composition

$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\mu\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} M \xrightarrow{1\otimes\mu\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \to \\ \xrightarrow{1\otimes 1\otimes f} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} N \xrightarrow{1\otimes g} \mathscr{V} \otimes_{\mathscr{A}} L \xrightarrow{h} K.$$

while $(hg)f$ is the composition

$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\mu\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{1\otimes f} \mathscr{V} \otimes_{\mathscr{A}} N \xrightarrow{hg} K,$$

that is, the composition

$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\mu\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{1\otimes f} \mathscr{V} \otimes_{\mathscr{A}} N \to \\ \xrightarrow{\mu\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} N \xrightarrow{1\otimes g} \mathscr{V} \otimes_{\mathscr{A}} L \xrightarrow{h} K.$$

Note that in the composition

$$\mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{1\otimes f} \mathscr{V} \otimes_{\mathscr{A}} N \xrightarrow{\mu\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} N$$

both μ in $\mu \otimes 1$ and 1 in $1 \otimes f$ act on the first multiplier $\mathscr{V}$. Therefore, it is the same as the composition

$$\mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\mu\otimes 1\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} N \xrightarrow{1\otimes 1\otimes f} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} N.$$

After this identification, the product $(hg)f$ becomes the compositon

$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\mu\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\mu\otimes 1\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \to \\ \xrightarrow{1\otimes 1\otimes f} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} N \xrightarrow{1\otimes g} \mathscr{V} \otimes_{\mathscr{A}} L \xrightarrow{h} K.$$

Since $\mu(\mu \otimes 1) = \mu(1 \otimes \mu)$, hence $(\mu \otimes 1)(\mu \otimes 1 \otimes 1) = (\mu \otimes 1)(1 \otimes \mu \otimes 1)$, this composition equals that for $h(gf)$ above. Just in the same way (even easier) one verifies that $f1_M = f$ and $1_M f' = f'$ whenever these products are defined. We leave it to the reader. Thus $\mathfrak{A}$-Mod is indeed a category.

There is a natural functor $\mathscr{A}$-Mod $\to$ $\mathfrak{A}$-Mod which is identity on objects and maps an $\mathscr{A}$-homomorphism $\alpha : M \to N$ to the compositon

$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\varepsilon\otimes 1} \mathscr{A} \otimes_{\mathscr{A}} M \xrightarrow{id_l^{-1}} M \xrightarrow{\alpha} N.$$

In particular, every diagram of direct sum in $\mathscr{A}$-Mod gives rise to a diagram of direct sum in $\mathfrak{A}$-Mod, so the latter category is always additive. Further we shall show some conditions for it being fully additive (it is not always the case).

Note that if $\mathfrak{A} = (\mathscr{A}, \mathscr{A})$ is a principal box, the category $\mathfrak{A}$-Mod coincide with $\mathscr{A}$-Mod. So we can (and will) identify such a principal box with the category $\mathscr{A}$.

If $F : \mathfrak{A} \to \mathfrak{B}$ is a morphism of boxes, where $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ and $\mathfrak{B} = (\mathscr{B}, \mathscr{W})$, it induces a functor $F^* : \mathfrak{B}\text{-Mod} \to \mathfrak{A}\text{-Mod}$ which maps a $\mathscr{B}$-module M to $\mathscr{A}$-module $M^F = \mathscr{B} \otimes_{\mathscr{B}} M \simeq \mathrm{Hom}_{\mathscr{B}}(\mathscr{B}, M)$ and a morphism $f \in \mathrm{Hom}_{\mathfrak{B}}(M, N)$, i.e. a homomorphism $f : \mathscr{W} \otimes_{\mathscr{B}} M \to N$ to the morphism $F^*f \in \mathrm{Hom}_{\mathfrak{A}}(M^F, N^F)$ given by the composition $V \otimes_{\mathscr{A}} M \xrightarrow{F_1 \otimes 1} W \otimes_{\mathscr{B}} M \xrightarrow{f} N$. In other words, F^*f maps an element $v \otimes x \in \mathscr{V} \otimes_{\mathscr{A}} M$ to $f(Fv \otimes x) \in N$.

We consider a special case of morphisms of boxes arising in "change of rings." Let $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ be a box and $F : \mathscr{A} \to \mathscr{B}$ is a functor. We define a new box $\mathfrak{A}^F = (\mathscr{B}, \mathscr{B} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B})$ with comultiplication given as the composition

$$\begin{aligned}\mathscr{B} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B} \xrightarrow{1 \otimes \mu \otimes 1} & \mathscr{B} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B} \to \\ & \xrightarrow{1 \otimes ins \otimes 1} \mathscr{B} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B} \simeq \\ & \simeq (\mathscr{B} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B}) \otimes_{\mathscr{B}} (\mathscr{B} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B}),\end{aligned}$$

where $ins : \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \to \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B} \otimes_{\mathscr{A}} \mathscr{V}$ maps $u \otimes v$ to $u \otimes 1 \otimes v$. Then the pair (F, F_1), where $F_1(v) = 1 \otimes v \otimes 1$, becomes a morphism $\mathfrak{A} \to \mathfrak{A}^F$. We denote it by the same label F. Now we get the following "change-of-ring theorem."

23 **Theorem 2.3.** *For any functor $F : \mathfrak{A} \to \mathfrak{B}$ the morphism of boxes $F : \mathfrak{A} \to \mathfrak{A}^F$ induces a fully faithful functor $F^* : \mathfrak{A}^F\text{-Mod} \to \mathfrak{A}\text{-Mod}$. Its image consists of all modules $M : \mathscr{A} \to \mathsf{Ab}$ that factor through F.*

The proof is quite evident, since, for any two $\mathscr{B}$-modules M, N

$$\begin{aligned}\mathrm{Hom}_{\mathscr{B}}(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B} \otimes_{\mathscr{B}} M, N) &\simeq \mathrm{Hom}_{\mathscr{A}}(\mathscr{V} \otimes_{\mathscr{A}} M^F, \mathrm{Hom}_{\mathscr{A}}(\mathscr{B}, N)) \\ &\simeq \mathrm{Hom}_{\mathscr{A}}(\mathscr{V} \otimes_{\mathscr{A}} M^F, N^F).\end{aligned}$$

□

Note that even if $\mathfrak{A} = (\mathscr{A}, \mathscr{A})$ is a principal box, the induced box $\mathfrak{A}^F = (\mathscr{B}, \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B})$ is, as a rule, non-principal.

We usually use Theorem 2.3 in connection to the *pushout* construction. Let $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$, $\mathscr{A}'$ be a subcategory of $\mathscr{A}$ and $F' : \mathscr{A}' \to \mathscr{B}'$

be a functor. We consider the pushout diagram of categories

$$\begin{tikzcd} \mathscr{A}' \arrow[r, "emb"] \arrow[d, "F'"'] & \mathscr{A} \arrow[d, "F"] \\ \mathscr{B}' \arrow[r] & \mathscr{B}, \end{tikzcd}$$

where *emb* is the embedding of $\mathscr{A}'$. It gives the induced box $\mathfrak{A}^F$ and the fully faithful functor $F^* : \mathfrak{A}^F\text{-Mod} \to \mathfrak{A}\text{-Mod}$. Obviously, the image of F^* consists of all $\mathscr{A}$-modules $M : \mathscr{A} \to \mathsf{Ab}$ such that the restriction $M|\mathscr{A}' : \mathscr{A}' \to \mathsf{Ab}$ factors through F'.

3. Free boxes and differential biquivers

s3

The most used class of boxes are the so called *free normal boxes*. We fix a commutative ring $\Bbbk$ and consider $\Bbbk$-categories. All functors are then supposed $\Bbbk$-linear (bifunctors are $\Bbbk$-bilinear).

31 **Definition 3.1.** Let $\mathfrak{A} = (\mathscr{A}, \mathscr{V}, \mu, \varepsilon)$ be a box.

(1) The box $\mathfrak{A}$ is said to be *free* (over $\Bbbk$) if $\mathscr{A}$ is a free $\Bbbk$-category and the kernel $\bar{\mathscr{V}} = \ker \varepsilon$ is a free $\mathscr{A}$-bimodule.
(2) A *section* ω of the box $\mathfrak{A}$ is a set of elements $\{\, \omega_A \in \mathscr{V}(A, A) \,\}$, where A runs through the objects of $\mathscr{A}$, such that $\varepsilon(\omega_A) = 1_A$ for every object A.
(3) A section ω is said to be *normal* (or *group-like*) if $\mu(\omega_A) = \omega_A \otimes \omega_A$ for every A.
(4) A box is said to be *normal* if it has a normal section.

As we have seen in Section 1, the pair $(\mathscr{A}, \bar{\mathscr{V}})$, where $\mathscr{A}$ is a free category and $\bar{\mathscr{V}}$ is a free $\mathscr{A}$-bimodule can be given by a biquiver Γ. Then $\mathscr{A} = \Bbbk\Gamma_0$ and $\mathscr{V} = \Bbbk\Gamma_1$. If $\mathfrak{A}$ is a free box with a section ω, a set of generators of the bimodule $\mathscr{V}$ consists of the elements ω_A and free generators of $\bar{\mathscr{V}}$, i.e. the arrows from Γ_1. Moreover, since $\mathscr{V}/\bar{\mathscr{V}} \simeq \mathscr{A}$, to know the whole bimodule structure on $\mathscr{V}$ we only have to know the differences $\partial a = a\omega_A - \omega_B a$ for every arrow $a \in \Gamma_0(A, B)$. This difference belongs to $\bar{\mathscr{V}}(A, B)$, since $\varepsilon(\partial a) = a1_A - 1_B a = 0$. So we get a map $\partial : \mathscr{A} \to \bar{\mathscr{V}}$. One easily check that it it is a *derivation*, i.e. satisfies the Leibniz rule $\partial(ab) = (\partial a)b + a(\partial b)$.

Note that every element from $\mathscr{V}(A, B)$ can be presented as a sum $\alpha\omega_A + v_1$ as well as a sum $\omega_B\alpha + v_2$, where $\alpha = \varepsilon(v)$ and $v_1, v_2 \in \bar{\mathscr{V}}(A, B)$. Therefore, every element $w \in \mathscr{V}^{\otimes 2}(A, B)$ can be presented as

$$w = \omega_B \otimes \alpha\omega_A + v_2 \otimes \omega_A + \omega_B \otimes v_1 + \widetilde{w},$$

where $\alpha : A \to B$, $v_1, v_2 : A \dashrightarrow B$ and $w \in \bar{\mathscr{V}}^{\otimes 2}$. Suppose that $w = \mu(v)$, where $v \in \bar{\mathscr{V}}$, and apply $\varepsilon \otimes 1$. Since $(\varepsilon \otimes 1)\mu = 1_{\mathscr{V}}$, we get $\alpha\omega_A + v_1 = v$, so $\alpha = 0$ and $v_1 = v$. Applying $1 \otimes \varepsilon$, we get $v_2 = v$, therefore

$$\mu(v) = v \otimes \omega_A + \omega_B v + \partial v, \quad \text{where} \ \ \partial v \in \bar{\mathscr{V}}^{\otimes 2},$$

where $\partial v \in \bar{\mathscr{V}}^{\otimes 2}$. If $b : B \to C$, then

$$\begin{aligned} \partial(bv) &= \mu(bv) - bv \otimes \omega_A - \omega_C \otimes bv = \\ &= b\mu(v) - bv \otimes \omega_A - b\omega_B \otimes v + \partial b \otimes v = \\ &= b(\partial v) + \partial b \otimes v, \end{aligned}$$

taking into account that $\omega_C b = b\omega_B b + \partial b$. Analogously, if $a : C \to A$, we also get

$$\partial(va) = (\partial v)a - v \otimes \partial a.$$

All these rules can be formulated as the *graded Leibniz rule*

e31 (1)
$$\partial(\alpha\beta) = (\partial\alpha)\beta + (-1)^{\deg \alpha}\alpha(\partial\beta),$$

where α and β can be either both from $\mathscr{A}$ or one from $\mathscr{A}$ and the other from $\bar{\mathscr{V}}$, and we omit the sign $\otimes$ between the elements from $\bar{\mathscr{V}}$. Now we *define* a map $\partial : \bar{\mathscr{V}}^{\otimes 2} \to \bar{\mathscr{V}}^{\otimes 3}$ using the graded Leibniz rule (1) as the definition. Thus we set, for $v : A \dashrightarrow C$ and $u : C \dashrightarrow B$,

$$\begin{aligned} \partial(u \otimes v) &= \partial u \otimes v - u \otimes \partial v = \\ &= (\mu(u) - u \otimes \omega_C - \omega_B \otimes u) \otimes v - \\ &\qquad - u \otimes (\mu(v) - v \otimes \omega_A - \omega_C \otimes v) = \\ &= (\mu \otimes 1)(u \otimes v) - (1 \otimes \mu)(u \otimes v) + u \otimes v \otimes \omega_A - \omega_B \otimes u \otimes v. \end{aligned}$$

Therefore, for every $w \in \bar{\mathscr{V}}^{\otimes 2}(A, B)$, we have

$$\partial(w) = (\mu \otimes 1)(w) - (1 \otimes \mu)(w) + w \otimes \omega_A - \omega_B \otimes w.$$

32 **Proposition 3.2.** *If the section ω is normal, then $\partial^2\alpha = 0$ for every element $\alpha \in \mathscr{A}$ or $\alpha \in \mathscr{V}$.*

Proof. Let $\alpha : A \to B$, so $\partial\alpha = \alpha \otimes \omega_A - \omega_B \otimes \alpha$. Then

$$\begin{aligned} \partial^2\alpha &= \mu(\partial\alpha) - \partial\alpha \otimes \omega_A - \omega_B \otimes \partial\alpha = \\ &= \alpha\omega_A \otimes \omega_A - \omega_B \otimes \omega_B\alpha - \alpha\omega_A \otimes \omega_A + \\ &\quad + \omega_B\alpha \otimes \omega_A - \omega_B \otimes \alpha\omega_A + \omega_B \otimes \omega_B\alpha = 0. \end{aligned}$$

If $\alpha : A \dashrightarrow B$, so $\partial\alpha = \mu(\alpha) - \alpha \otimes \omega_A - \omega_B \otimes \alpha$, then

$$\begin{aligned}\partial^2\alpha = (\mu \otimes 1)(\partial\alpha) - (1 \otimes \mu)(\partial\alpha) + \partial\alpha \otimes \omega_A - \omega_B \otimes \partial\alpha = \\ = (\mu \otimes 1)\mu(\alpha) - \mu(\alpha) \otimes \omega_A - \omega_B \otimes \omega_B \otimes \alpha - \\ - (1 \otimes \mu)\mu(\alpha) + \alpha \otimes \omega_A \otimes \omega_A - \omega_B \otimes \mu(\alpha) - \\ + \mu(\alpha) \otimes \omega_A - \alpha \otimes \omega_A \otimes \omega_A - \omega_B \otimes \alpha \otimes \omega_A - \\ - \omega_B \otimes \mu(\alpha) + \omega_B \otimes \alpha \otimes \omega_A + \omega_B \otimes \omega_b \otimes \alpha = 0,\end{aligned}$$

since $(\mu \otimes 1)\mu = (1 \otimes \mu)\mu$. □

Thus, to define the bimodule structure and the coalgebra structure on a free box $\Bbbk\Gamma$, we have to define ∂a for every arrow of Γ, both solid and dashed. Then the value of ∂ on every path can be obtained usinig Leibniz rule. Moreover, to verify that $\partial^2 = 0$, one only has to check it for every arrow. Indeed, since ∂ increases $\deg\alpha$ by 1, we have

$$\begin{aligned}\partial^2(\alpha\beta) = \partial((\partial\alpha)\beta + (-1)^{\deg\alpha}\alpha(\partial\beta)) = \\ = (\partial^2\alpha)\beta + (-1)^{\deg\alpha+1}(\partial\alpha)(\partial\beta) + \\ + (-1)^{\deg\alpha}(\partial\alpha)(\partial\beta) + (-1)^{\deg\alpha}\alpha(\partial^2\beta) = 0\end{aligned}$$

as soon as $\partial^2\alpha = \partial^2\beta = 0$.

33 **Definition 3.3.** A pair (Γ, ∂), where Γ is a bigraph and ∂ is map sending every arrow $a \in \Gamma(i,j)$ to a $\Bbbk$-linear combination of paths from i to j of degree $\deg a + 1$ such that, calculated by the graded Leibniz rule, $\partial^2 a = 0$ for every arrow a, is called a *differential bigraph* (over the ring $\Bbbk$).

Thus, we have one-to-one correspondence between free normal boxes and differential bigraphs over $\Bbbk$.

Given a differential biquiver (Γ, ∂), we calculate the category of modules $\mathfrak{A}$-Mod of the corresponding box $\mathfrak{A}$ as follows. Its objects are the representation of the solid part Γ_0 of the biquiver. In other words, such an object M consists of $\Bbbk$-modules $M(i)$, where i runs through $\operatorname{Ver}\Gamma$ and of $\Bbbk$-linear maps $M(a) : M(i) \to M(j)$ given for every solid arrow $a : i \to j$. To define a morphism $M \to N$, i.e. an $\mathscr{A}$-homomorphism $\mathscr{V} \otimes_{\mathscr{A}} M \to N$, we need some observations. Since $\bar{\mathscr{V}} = \bigoplus_{\substack{\alpha\in\Gamma_1 \\ \alpha:x\dashrightarrow y}} \mathscr{A}_x^y$, there is an exact sequence

$$0 \to \bigoplus_{\substack{\alpha\in\Gamma_1 \\ \alpha:i\dashrightarrow j}} \mathscr{A}_i^j \to \mathscr{V} \xrightarrow{\varepsilon} \mathscr{A} \to 0,$$

and there is a (right) section $\omega_r : \mathscr{A} \to \mathscr{V}$ mapping $a : i \to j$ to $\omega_j a$. Note that ω_l is not a bimodule homomorphism: it only respects the right multiplication by morphisms from $\mathscr{A}$. Since $\mathscr{A}_i^j = \mathscr{A}^j \otimes_{\Bbbk} \mathscr{A}_i$ and $\mathscr{A}_i \otimes_{\mathscr{A}} M \simeq M(i)$, there is an exact sequence of left $\mathscr{A}$-modules

$$0 \to \bigoplus_{\substack{\alpha\in\Gamma_1 \\ \alpha:i \dashrightarrow j}} \mathscr{A}^j \otimes_{\Bbbk} M(i) \to \mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\varepsilon\otimes 1} M \to 0.$$

It also has a section $\omega_r \otimes 1 : M \to \mathscr{V} \otimes_{\mathscr{A}} M$ mapping $x \in M(i)$ to $\omega_i \otimes x$. This section is also not an $\mathscr{A}$-homomophism; it only respects multiplication by elements from $\Bbbk$. Therefore, to define an $\mathscr{A}$-homomorphism $f : \mathscr{V} \otimes_{\mathscr{A}} M \to N$, we havce to prescribe the values $f(\alpha \otimes x)$ and $f(\omega_i \otimes x)$, which we denote, respectively, by $f(\alpha)x$ and $f(\omega_i)x$. So, we get $\Bbbk$-homomorphisms $f(om_i) : M(i) \to N(i)$ for every $i \in \operatorname{Ver}\Gamma$ and $f(\alpha) : M(i) \to N(j)$ for every $\alpha : i \dashrightarrow j$. On the other hand, suppose given such homomorphisms $f(\omega_i)$ and $f(\alpha)$. In order that they define an $\mathscr{A}$-homomorphism, they must be compatable with the multiplication by arrows from Γ_0. Since α is a free generator of $\mathscr{A}_j^i$, it just gives a definiton of $f(p\alpha q)$ for any solid paths $p : j \to k$ and $q : l \to i$. Namely, $f(p\alpha q) = N(p)f(\alpha)M(q)$. For $f(\omega_i)$ it gives, for each solid arrow $a : i \to j$,

$$N(a)f(\omega_i)x = f(a\omega_i)x = f(\omega_i a + \partial a)x = f(\omega_i)M(a) + f(\partial a)x,$$

i.e.

e23 (2) $$N(a)f(\omega_i) = f(\omega_j)M(a) + f(\partial a).$$

Note that, since $\partial a \in \bar{\mathscr{V}}(i,j)$, we have already calculated it above.

The equation (2) shows the difference between morphisms in $\mathscr{A}$-Mod and $\mathfrak{A}$-Mdd. It consists in the extra term $f(\partial a)$.

Now we calculate the rule of composition. Let $f : \mathscr{V} \otimes_M \to N$ and $g : \mathscr{V} \otimes_{\mathscr{A}} L \to M$ are given by the sets $\{ f(\omega_i), f(\alpha) \}$ and $\{ g(\omega_i), g(\alpha) \}$. Then

$$\begin{aligned}(fg)(\omega_i \otimes x) &= f(1 \otimes g)(\mu \otimes 1)(\omega_i \otimes x) = \\ &= f(1 \otimes g)(\omega_i \otimes \omega_i \otimes x) = f(\omega_i \otimes g(\omega_i)x) = f(\omega_i)g(\omega_i)x,\end{aligned}$$

so

$$(fg)(\omega_i) = f(\omega_i)g(\omega_i).$$

Let $\alpha : i \dashrightarrow j$ with $\partial\alpha = \sum_r p_r \otimes q_r$, where p_r, q_r are paths of degree 1. Then

$$\begin{aligned}(fg)(\alpha \otimes x) &= f(1 \otimes g)(\mu \otimes 1)(\alpha \otimes x) = \\ &= f(g \otimes 1)(\omega_j \otimes al \otimes x + \alpha \otimes \omega_i \otimes x + (\partial\alpha) \otimes x) = \\ &= f(\omega_j \otimes g(\alpha)x + \alpha \otimes g(\omega_i)x + \textstyle\sum_r p_r \otimes g(q_r)x) = \\ &= f(\omega_j)g(\alpha)x + f(\alpha)g(\omega_i)x + \textstyle\sum_r f(p_r)g(q_r)x,\end{aligned}$$

so

$$(fg)(\alpha) = f(\omega_j)g(\alpha) + f(\alpha)g(\omega_i) + (f * g)(\partial\alpha),$$

where $(f * g)(u \otimes v) = f(u)g(v)$. Note that if f is an isomorphism, all $f(\omega_i)$ are also isomorphism. The comverse is not true in general case, as we shall see below.

34 **Example 3.4.** Consider the differential bigraph

$$\begin{tikzcd} & 1 \arrow[dl, "a"'] \arrow[dr, "b"] & \\ 2 \arrow[rr, dashed, "\xi"'] & & 3 \end{tikzcd} \qquad \partial a = 0,\ \partial b = \xi a,\ \partial\xi = 0.$$

Let $\mathfrak{A}$ be the corresponding free normal box. An $\mathscr{A}$-module M is given by a diagram of $\Bbbk$-modules

$$M(2) \xleftarrow{M(a)} M(1) \xrightarrow{M(b)} M(3).$$

If N is another module, an $\mathfrak{A}$-morphism $f : M \to N$ is given by a diagram

$$\begin{tikzcd} M(2) \arrow[d, "f_2"'] \arrow[drr, dashed, "X"] & M(1) \arrow[l, "M(a)"'] \arrow[r, "M(b)"] \arrow[d, "f_1"'] & M(3) \arrow[d, "f_3"] \\ N(1) & N(2) \arrow[l, "N(a)"] \arrow[r, "N(b)"'] & N(3), \end{tikzcd}$$

where we set $f_i = f(\omega_i)$, $X = f(\xi)$. Since $\partial a = 0$, the left square of this diagram should be commutative: $N(a)f_1 = f_2M(a)$, but since $\partial b = \xi a$, the right square is not. It is "commutative up to ∂b," i.e. $N(b)f_1 = f_3M(b) + XM(a)$ (note that $f(\partial b) = f(\xi a) = XM(a)$). The product fg of morphisms is given by the rules:

$$(fg)_i = f_ig_i,$$
$$(fg)(\xi) = f_3g(\xi) + f(\xi)g_2.$$

35 **Example 3.5.** Let the differential bigraph Γ be

$$\begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[loop right, dashed, "\xi"] \end{tikzcd}, \quad \partial a = \xi a,\ \partial\xi = \xi^2.$$

Then a representation of $\mathfrak{A}$ is a $\Bbbk$-module M with a fixed endomorphism A. A morphism $f : (M, A) \to (N, B)$ is a pair (f, X) of $\Bbbk$-homomorphisms $M \to N$ such that $Bf = fA + XA$. Consider the case $M = \Bbbk$, $A = 0$. Then the pair $e = (0, 1_M)$ is an endomorphism of this module. Moreover, the product $(f, X)(g, Y)$ is the pair $(fg, fY + Xg + XY)$, so $e^2 = e$ and e is a nontrivial idempotent. It cannot split. Indeed, if $e = (f, X)(g, Y)$ and $(g, X)(f, Y) = 1_{(N,B)}$ for some N, then $gf = 1_N$ and $fg = 0$, which is imposiible. Therefore, the category $\mathfrak{A}$-Mod is not fully additive.

Consider now the representation N given by the pair $(\Bbbk, 1)$. Then the pair $(1, -1)$ defines a morphism $f : N \to M$. But the product $(1, -1)(g, Y)$ is given by the pair $(g, -g)$, which never equals the pair $(1, 0)$, which defines the identity morphism. Therefore, f is not an isomorphism, though $f(\omega_1)$ is invertible.

36 **Example 3.6** (Repsesentations of posets)**.** Let S be a *poset* (partially ordered set), o be a new symbol, not belonging to S. We consider the differential bigraph $\hat{S}$ with the set of vertices $S \cup \{o\}$, solid arrows $a_i : i \to o$ for every element $i \in S$, dashed arrows $\gamma_{ij} : j \dashrightarrow i$ for each pair of elements $i, j \in S,\ i < j$ and the derivation ∂ defined by the rules:

$$\partial a_j = \sum_{i<j} a_i \gamma_{ij},$$

$$\partial \gamma_{ij} = - \sum_{i<k<j} \gamma_{ik}\gamma_{kj},$$

We denote by $\mathfrak{A}(S)$ the corresponding free normal box. Then a representation M of $\mathfrak{A}$ is a diagram of $\Bbbk$-modules

$$\begin{tikzcd}
 & & M(o) & & \\
M(1) \arrow[urr, "M_1"] & M(2) \arrow[ur, "M_2"'] & \dots & M(k) \arrow[ul, "M_k"'] & \dots
\end{tikzcd},$$

where the indices in the lower row are the elements of S and $M_i = M(a_i)$. A morphism f from M to another representation N is a set of homomorphisms (f_o, f_i, g_{ij}), where $i, j \in S$, $i < j$, $f_o : M(o) \to N(o)$, $f_i : M(i) \to N(i)$, $g_{ij} : M(j) \to N(i)$ such that

$$N_j f_j = f_o M_j + \sum_{i<j} N_i g_{ij} \quad \text{for every } j \in S.$$

If a set (f'_o, f'_i, g'_{ij}) defines another morphism f', the product ff' is given by the set $(f_o f'_o, f_i f'_i, h_{ij})$, where $h_{ij} = f_i g'_{ij} + g_{ij} f'_j - \sum_{i<k<j} g_{ik} g'_{ij}$.

If $\Bbbk$ is a field and all vector spaces $M(i)$ are finite dimensional, we can rewrite it using matrices. Then a representation is given by a set of matrices $\{\, M(i) \mid i \in S \,\}$ having the same number of rows. Two such representations are equivalent if they can be transformed to each other by the following operations:

- Elementary transformations of rows common to all matrices $M(i)$.
- Elementary transformations of columns inside each matrix $M(i)$.
- Adding multiples of columns of $M(i)$ to those of $M(j)$ for each pair $i < j$.

It is just the original definition of Nazarova–Roiter [5].

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