MATH 4576 Rings and Fields — QUIZ 1

Prof. I. Drozd — Spring 2011

(1) [6] Let $R = Z_{10}$ (the ring of residues modulo 10).

(a) List all units of R and their multiplicative inverses.

We have in Z_{10}: $1 \cdot 1 = 1$, $3 \cdot 7 = 1$, $9 \cdot 9 = 1$, hence, $1, 3, 7, 9$ are units. Lower we'll see that other elements are zero divisors, so cannot be units.

(b) List all zero divisors of R.

We have in Z_{10}: $2 \cdot 5 = 0$, $4 \cdot 5 = 0$, $6 \cdot 5 = 0$, $8 \cdot 5 = 0$, hence, $2, 4, 5, 6, 8$ are zero divisors. We have seen that all other nonzero elements are units, so cannot be zero divisors.

(c) Is the subset $\{ [0], [1], [2], [3], [5], [6], [9] \}$ a subring of R?

No, since [3] and [5] belong to this set, while $[3] + [5] = [8]$ does not.

(d) Is the subset $S = \{ [0], [2], [4], [6], [8] \}$ a subring of R?

Yes, since one can verify (do it) that if $[a], [b] \in S$, then also $[a] + [b] = [a + b]$ and $[a][b] = [ab]$ belong to S. The same is true for $-[a]$ and $[0]$.

Another way: one easily sees that S consists of all classes modulo 10 which contain even numbers (if a is even and b is odd, $a \not\equiv b \pmod{10}$). Since a sum and a product of even numbers is even, a sum or a product of classes from S belongs to S.

(e) Which of the maps $f, g : Z_5 \to S$ is a ring isomorphism?

$$f([a]_5) = [2a]_{10}\,, \qquad g([a]_5) = [6a]_{10}\,.$$

One easily verifies that both f and g are bijections and respect addition. So we only have to check which of them respect the multiplication.

We have

$$f([a]_5)f([b]_5) = [2a]_{10} \cdot [2b]_{10} = [4ab]_{10},$$

while

$$f([a]_5 \cdot [b]_5) = f([ab]_5) = [2ab]_{10} \neq [4ab]_{10},$$

for instance, if $a = b = 1$. Hence, f is not an isomotphism.

On the other hand,

$$g([a]_5)g([b]_5) = [6a]_{10} \cdot [6b]_{10} = [36ab]_{10} = [6ab]_{10},$$

and

$$g([a]_5 \cdot [b]_5) = g([ab]_5) = [6ab]_{10},$$

so g is an isomotphism.

(2) [1] Give an example of a one-to-one map $f : \mathbb{Z} \to \mathbb{Z}$ which is *not* a ring isomorphism.

There is a large choive. For instance, $f(x) = x + 1$ is one-to-one, but $f(x+y) = x+y+1$, while $f(x)+f(y) = x+1+y+1 = x + y + 2 \neq f(x + y)$, hence f is not an isomotphism.

(3) [3] Let R be the ring of 3×3 matrices with rational entries. Which of the following subsets are subrings of R? Which of them are commutative?

$$S_1 = \left\{ \begin{pmatrix} a & b & 0 \\ 2b & a & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \; a, b, c \in \mathbb{Q} \right\},$$

$$S_2 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{Z} \right\}.$$

Take two matrices from S_1:

$$A = \begin{pmatrix} a & b & 0 \\ 2b & a & 0 \\ 0 & 0 & c \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} a' & b' & 0 \\ 2b' & a' & 0 \\ 0 & 0 & c' \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} a + a' & b + b' & 0 \\ 2b + 2b' & a + a' & 0 \\ 0 & 0 & c + c' \end{pmatrix} \text{ belongs to } S_1,$$

$$AB = \begin{pmatrix} aa' + 2bb' & ab' + ba' & 0 \\ 2ba' + 2ab' & aa' + 2bb' & 0 \\ 0 & 0 & cc' \end{pmatrix} \text{ belongs to } S_1.$$

Obviously, also $-A \in S_1$ and the zero matrix belogs to S_1. Therefore, S_1 is a subring. Moreover,

$$A'A = \begin{pmatrix} a'a + 2b'b & a'b + b'a & 0 \\ 2b'a + 2a'b & a'a + 2b'b & 0 \\ 0 & 0 & c'c \end{pmatrix} = AA',$$

so S_1 is commutative.

Take two matrices from S_2:

$$A = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} a' & b' & c' \\ 0 & a' & d' \\ 0 & 0 & a' \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} a + a' & b + b' & c + c' \\ 0 & a + a' & d + d' \\ 0 & 0 & a + a' \end{pmatrix} \text{ belongs to } S_1,$$

$$AB = \begin{pmatrix} aa' & ab' + ba' & ac' + bd' + ca' \\ 0 & aa' & ad' + da' \\ 0 & 0 & aa' \end{pmatrix} \text{ belongs to } S_2.$$

Obviously, also $-A \in S_2$ and the zero matrix belogs to S_2. Therefore, S_2 is a subring. On the other hand, if

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

then

$$AA' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A'A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so $AA' \neq A'A$. Since $A \in S_2$, $A' \in S_2$, S_2 is n ot commutative.