(1) [4] Find $d(x) = \gcd(f(x), g(x))$ in $F[x]$ and a Bézout presentation $d(x) = u(x)f(x) + v(x)g(x)$, where

(a) $F = \mathbb{R}$, $f(x) = 2x^4 + 3x^3 - 4x + 1$, $g(x) = x^2 - 1$.

We use the Euclidean Algorithm and calculate:

$$f(x) = (2x^2 + 3x + 2)g(x) + (-x + 3), \text{ set } r(x) = -x + 3,$$
$$g(x) = (-x - 3)r(x) + 8, \text{ set } r_1(x) = 8,$$
$$r(x) = \Big(-\frac{1}{8}x + \frac{3}{8}\Big)r_1(x).$$

Therefore, $\gcd(f(x), g(x)) = 1$ (the monic polynomial associated to $r_1(x)$). Now,

$$\begin{aligned} 1 &= \frac{1}{8}g(x) + \Big(\frac{1}{8}x + \frac{3}{8}\Big)r(x) = \\ &= \frac{1}{8}g(x) + \Big(\frac{1}{8}x + \frac{3}{8}\Big)(f(x) - \big(2x^2 + 3x + 2\big)g(x)) = \\ &= \Big(\frac{1}{8}x + \frac{3}{8}\Big)f(x) - \Big(\frac{1}{4}x^3 + \frac{9}{8}x^2 + \frac{11}{8}x + \frac{5}{8}\Big)g(x). \end{aligned}$$

(b) $F = Z_3$, $f(x) = 2x^4 + x^2 + x + 2$, $g(x) = x^3 + x^2 + 1$.

As above:

$$f(x) = (2x + 1)g(x) + (2x + 1), \text{ set } 2x + 1 = r(x),$$
$$g(x) = (2x^2 + x + 1)r(x),$$

hence, $\gcd(f(x), g(x)) = x + 2$ (divide $r(x)$ by 2 to get 1 for the leading coefficient). Now,

$$x + 2 = 2(2x + 1) = 2f(x) - (x + 2)g(x).$$

(2) [2] Decompose the polynomial $x^4 + x^3 - x - 1$ into a product of irreducible polynomial in the ring $\mathbb{Q}[x]$.

$f(1) = 0$, so $x-1 \mid f(x)$. Get $f(x) = (x-1)(x^3+2x^2+2x+1)$. The seconf factor has a root -1, so is divisible by $x + 1$. Get $x^3 + 2x^2 + 2x + 1 = (x + 1)(x^2 + x + 1)$. Here the second factor has no roots and os of degree 2, hence is irreducible. Therefore, the decomposition is:

$$f(x) = (x - 1)(x + 1)(x^2 + x + 1).$$

(3) [2] Show that $x^3 + 3x^2 + 4$ is irreducible in $Z_7[x]$.

Since $\deg f(x) = 3$, we obly have to check that $f(x)$ has no roots in Z_7. We have:

$f(0) = 4,\ f(1) = 1,\ f(2) = 3,\ f(3) = 6,\ f(4) = f(-3) = 4,$

$f(5) = f(-2) = 1,\ f(6) = f(-1) = 6,$

so $f(x)$ has no roots indeed.

(4) [2] Give an example of a polynomial $f(x)$ which is a unit in $Z_4[x]$ though $\deg f(x) > 0$.

For instance, $2x+1$, since $(2x+1)(2x+1) = 4x^2+4x+1 = 1$ in Z_4.