MATH 4576 Rings and Fields — QUIZ 4

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(1) Let $I = \{ f(x) \in \mathbb{Z}[x] \mid f(1) \text{ is divisible by } 3 \}$. Proof that I is an ideal in $\mathbb{Z}[x]$ and $\mathbb{Z}[x]/I \simeq Z_3$.

Consider the map $\phi : \mathbb{Z}[x] \to Z_3$ such that $\phi(f(x)) = [f(1)]_3$. It is a homomorphism. It is surjective, since we can take for $f(x)$ any integer constant, so $\operatorname{Im}\phi = Z_3$. On the other hand, $\operatorname{Ker}\phi = \{ f(x) \mid [f(1)]_3 = [0] \} = I$. Therefore, by the 1st isomorphism theorem, I is an ideal and $\mathbb{Z}[x]/I \simeq Z_3$.

(2) Which of the following quotient rings are fields? If not, find a zero divisor.

(a) $\mathbb{Q}[x]/(x^3 - 3x^2 + 3x + 3)$.

This polynomial is irreducible by the Eisenstein criterion (take $p = 3$). Hence, this quotient ring is a field.

(b) $\mathbb{Q}[x]/(x^3 - 3x^2 + x + 2)$

This polynomial has a rational root 2, hence decomposes as $(x-2)(x^2-x-1)$. Therefore, this quotient ring is not a field. Moreover, $([x]-2)([x]^2-[x]-1) = [x^3-3x^2+x+2] = [0]$, so $[x] - 2$ is a zero divisor.

(c) $Z_3[x]/(x^3 + 2x + 2)$

This polynomial is of degree 3 and has no roots in Z_3, hence it is irreducible and the quotient ring is a filed.

(3) Let θ be a complex root of the polynomial $x^3 + 4x + 2$. Express $(1 + \theta)^{-1}$ as a polynomial in θ with rational coefficients.

First, we present 1 as $u(x)(x^3 + 4x + 2) + v(x)(x + 1)$:

$$x^3 + 4x + 2 = (x^2 - x + 5)(x + 1) - 3,$$

$$1 = -\frac{1}{3}(x^3 + 4x + 2) + \frac{1}{3}(x^2 - x + 5)(x + 1),$$

wherefrom

$$1 = -\frac{1}{3}(\theta^3 + 4\theta + 2) + \frac{1}{3}(x^2 - x + 5)(\theta + 1) = \frac{1}{3}(x^2 - x + 5)(\theta + 1),$$

and

$$(1 + \theta)^{-1} = \frac{1}{3}x^2 - \frac{1}{3}x + \frac{5}{3}.$$

(4) Prove that $\mathbb{R}[x]/(x^2 + x + 1)$ is a field. List its elements and write the multiplication rule for them.

This polynomial has no roots in $\mathbb{R}$, Since it is of degree 2, it is irreducible and the quotient ring is a field. Every element of this field can be uniquely presented as $a + b[x]$, where $a, b \in \mathbb{R}$. Then

$$(a + b[x])(c + d[x]) = ac + (ad + bc)[x] + ad[x]^2,$$

$$adx^2 + (ad + bc)x + ac = ad(x^2 + x + 1) + (ad + bc - ad)x + (ac - bd),$$

hence

$$(a + b[x])(c + d[x]) = (ac - bd) + (ad + bc - ad)[x].$$

(5) List all units and all zero divisors in $Z_3[x]/(x^2 + 2)$.

$x^2 + 2$ has a root 1 in Z_3, so $x^2 + 2 = (x + 1)(x + 2)$. Every coset from $Z_3/(x^2 + 2)$ is of the form $a + b[x]$. It is a zero divisor if and only if $a + bx$ has nonconstant common divisors with $x^2 + 2$. Therefore the zero divisors are $1 + [x], 2 + [x], 2(1 + [x]) = 2 + 2[x], 2(2 + [x]) = 1 + 2[x]$. All other nonzero elements are units. They are: $1, 2, x, 2x$.