(1) Let $R = \mathbb{Z}[\sqrt{-2}] = \left\{ m + ni\sqrt{2} \mid m, n \in \mathbb{Z} \right\}$ (subring of $\mathbb{C}$).

(a) Prove that R is a Euclidean ring.

Set $\delta(m+ni\sqrt{2}) = |m+ni\sqrt{2}|^2 = m^2+2n^2$. Then $\delta(ab) = \delta(a)\delta(b) \geq \delta(a)$. If $a, b \in R$, $b \neq 0$, then $\frac{a}{b} = u + vi\sqrt{2}$, where $u, v \in \mathbb{Q}$. Choose $u_0, v_0 \in \mathbb{Z}$ such that $|u - u_0| \leq \frac{1}{2}$ and $|v - v_0| \leq \frac{1}{2}$ and set $q = u_0 + v_0 i\sqrt{2}$, $c = \frac{a}{b} - q$, $r = bc$. Then $|c| \leq \frac{3}{4}$ and $a = bq + r$. Therefore, $r = a - bq \in R$ and $\delta(r) = \delta(bc) < \delta(b)$. Thus R is Euclidean.

(b) Prove that a prime $p \in \mathbb{Z}$ is irreducible in R if and only if there is no integer a such that $a^2 \equiv -2 \pmod{p}$.

Suppose that $p = ab$, where neither a nor b is a unit. Then $\delta(a) \neq 1$ and $\delta(b) \neq 1$. Since $p^2 = \delta(p) = \delta(a)\delta(b)$, it follows that $\delta(a) = p$, that is, if $a = m + ni\sqrt{2}$, then $p = m^2 + 2n^2$. Obviously, $n \neq 0$. Then there is $k \in \mathbb{Z}$ such that $kn \equiv 1 \pmod{p}$. Hence $(mk)^2 \equiv -2 \pmod{p}$. Therefore, if there is no integer x such that $x^2 \equiv -2 \pmod{p}$, p is irreducible in R.
On the contrary, if $x^2 \equiv -2 \pmod{p}$, then $p \mid x^2 + 2 = (x + i\sqrt{2})(x - i\sqrt{2})$. Since $p \nmid x \pm i\sqrt{2}$, p is not irreducible in the Euclidean domain R.

(c) Prove that the equation $x^2 + 2y^2 = p$, where $p \in \mathbb{Z}$ is a prime, has an integral solution if and only if there is an integer a such that $a^2 \equiv -2 \pmod{p}$.

We have already seen that if there is an integer a such that $a^2 \equiv -2 \pmod{p}$, then p is not irreducible, and if p is not irreducible, there are $m, n \in \mathbb{Z}$ such that $m^2 + 2n^2 = p$, and vice versa.

(2) (a) Calculate $[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}]$.

We have $[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}(\sqrt{3})][\mathbb{Q}(\sqrt{3}) : \mathbb{Q}]$. Since the minimal polynomial for $\sqrt{3}$ over $\mathbb{Q}$ is $x^2 - 3$, the second factor equals 2. The minimal polynomial for i over $\mathbb{Q}(\sqrt{3})$ is x^2+1, since this polynomial is of degree 2 and has no roots in $\mathbb{Q}(\sqrt{3})$ (it has no real roots at all). Therefore, the first factor is also 2 and $[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}] = 4$.

(b) Find the minimal polynomial of $\sqrt{3} - i$ over $\mathbb{Q}$.

$\theta = \sqrt{3} - i$ is a root of $(x - \sqrt{3} + i)(x - \sqrt{3} + i) = x^2 - 2\sqrt{3}x + 4$. To eliminate $\sqrt{3}$, consider

$$p(x) = (x^2 - 2\sqrt{3}x + 4)(x^2 + 2\sqrt{3}x + 4) = x^4 - 4x^2 + 16.$$

Certainly, $p(\theta) = 0$. Prove that $p(x)$ is irreducible over $\mathbb{Q}$. It has no roots, so it can only decompose as $(x^2 + ax + b)(x^2 + cx + d)$ with integral a, b, c, d. It gives the relations:

$$\begin{aligned} a + c &= 0, \\ b + d + ac &= -4, \\ ad + bc &= 0, \\ bd &= 16. \end{aligned}$$

Hence, $a = -c$, so $a^2 = b + d + 4$ and $a(b - d) = 0$. If $a = 0$, then $b + d = -4$ and $bd = 16$, which is impossible. So $a \neq 0$ and $b = d$. Since $bd = 16$, then $b = d = \pm 4$, hence either $a^2 = 12$ or $a^2 = -4$, both impossible. Therefore, $p(x)$ is irreducible over $\mathbb{Q}$, so it is the minimal polynomial of θ over $\mathbb{Q}$.

(c) Prove that $\mathbb{Q}(\sqrt{3}, i) = \mathbb{Q}(\sqrt{3} - i)$.

Obviously, $\mathbb{Q}(\sqrt{3} + i) \subseteq \mathbb{Q}(\sqrt{3}, i)$. Then

$$4 = [\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}(\sqrt{3} + i)][\mathbb{Q}(\sqrt{3} + i) : \mathbb{Q}].$$

Since the minimal polynomial of $\sqrt{3} + i$ over $\mathbb{Q}$ is of degree 4, we have $[\mathbb{Q}(\sqrt{3} + i) : \mathbb{Q}] = 4$, wherefrom $[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}(\sqrt{3} + i)] = 1$. It means that $\mathbb{Q}(\sqrt{3}, i) = \mathbb{Q}(\sqrt{3} + i)$.

(3) Find $[\mathbb{Q}(\theta) : \mathbb{Q}]$, where θ is a root of the polynomial $x^5 - 10x^4 + 20x^2 + 10x - 20$. Does it depend on the choice of a root θ? Why?

This polynomial is irreducible over $\mathbb{Q}$ (apply the Eisenstein criterion with $p = 5$). Hence, $[\mathbb{Q}(\theta) : \mathbb{Q}] = 5$ and does not depend on the choice of a root.

(4) Does $[\mathbb{Q}(\theta) : \mathbb{Q}]$, where θ is a root of the polynomial $x^4 + 4x^3 + 6x^2 + 12x + 9$, depend on the choice of the root θ? Why?

This polynomial is not irreducible: it has rational roots -1 and -3. Hence, $x^4 + 4x^3 + 6x^2 + 12x + 9 = (x+1)(x+3)(x^2+3)$, the last factor being irreducible over $\mathbb{Q}$. Therefore $[\mathbb{Q}(\theta) : \mathbb{Q}] = 1$ if $\theta = -1$ or $\theta = -3$, while $[\mathbb{Q}(\theta) : \mathbb{Q}] = 2$ if $\theta = \pm i\sqrt{3}$ (the roots of $x^2 + 3$).