MATH 4576 RINGS AND FIELDS QUIZ 6
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(1) Find $[\mathbb{Q}(\sqrt{\sqrt{6}-1}\,) : \mathbb{Q}]$.

Let $\theta = \sqrt{\sqrt{6}-1}$. Then $\theta^2 = \sqrt{6}-1$, so $(\theta^2+1)^2 = 6$ and θ is a root of $p(x) = x^4 + 2x^2 - 5$. It is easy to verify that $p(x)$ is irreducible in $\mathbb{Q}[x]$, so it is the minimal polynomial of θ over $\mathbb{Q}$. Therefore $[\mathbb{Q}(\theta) : \mathbb{Q}] = \deg p(x) = 4$.

(2) Find a basis of $\mathbb{Q}(\sqrt{3}, \sqrt[3]{3})$ over $\mathbb{Q}$.

Consider the extensions $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{\ }) \subseteq \mathbb{Q}(\sqrt{3}, \sqrt[3]{3})$. Since $x^2 - 3$ is the minimal polynomial of $\sqrt{3}$ over $\mathbb{Q}$, the basis of $\mathbb{Q}(\sqrt{3})$ over $\mathbb{Q}$ has is $\{\,1, \sqrt{3}\,\}$. The polynomial $x^3 - 3$ has no roots in $\mathbb{Q}(\sqrt{3})$ (it is easy to check). Since it is of degree 3, it is irreducible over $\mathbb{Q}(\sqrt{3})$, so it is the minimal polynomial of $\sqrt[3]{3}$ over $\mathbb{Q}(\sqrt{3})$. Therefore, a basis of $\mathbb{Q}(\sqrt{3}, \sqrt[3]{3})$ over $\mathbb{Q}(\sqrt{3})$ is $\{\,1, \sqrt[3]{3}, \sqrt[3]{9}\,\}$. Then a basis of $\mathbb{Q}(\sqrt{3}, \sqrt[3]{3})$ over $\mathbb{Q}$ consists of the products $\{\,1, \sqrt{3}, \sqrt[3]{3}, \sqrt{3}\sqrt[3]{3}, \sqrt[3]{9}, \sqrt{3}\sqrt[3]{9}\,\}$.

(3) Let θ be a root of the polynomial $f(x) = x^3 + x + 1 \in Z_5[x]$. Find $[Z_5(\theta) : Z_5]$ and prove that $Z_5(\theta)$ is a splitting field for $f(x)$ over Z_5.

$f(x)$ has no roots in Z_5 (it is easy to check) and is of degree 3. Therefore it is irreducible and $[Z_5(\theta) : Z_5] = \deg f(x) = 3$. Now

$$f(x) = (x-\theta)g(x), \text{ where } g(x) = x^2 + \theta x + (\theta^2+1).$$

One can check that $g(-\theta^2+\theta+1) = 0$, so $g(x)$ splits into linear factors over $Z_5(\theta)$. Therefore, $f(x)$ also splits into linear factors over $Z_5(\theta)$, which means that $Z_5(\theta)$ is a splitting field of $f(x)$.