On Schappert's characterization of strictly unimodal plane curve singularities

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Introduction

The representation theory of curve singularities (more precisely, their local rings) turns out to be closely related to their "deformation" properties. Namely, as was shown in [6],[8],[7], such ring R is of *finite type*, that is has only finitely many torsion-free indecomposable modules (up to isomorphism), if and only if it dominates one of the so called *simple plane curve singularities* in the sense of [1]. In [4] the authors have shown that R is of *tame type*, that is has only 1-parametre families of indecomposable torsion-free modules, if and only if it dominates one of the *unimodal plane curve singularities* of type T_{pq} (again in the classification of [1]).

These singularities form the "serial" part of the list of all unimodal plane curve singularities. There are also 14 "exceptional" ones, which happen to be wild, that is possesses n-parametre families of (non-isomorphic) indecomposable modules for arbitrary large n. Of course, the *bimodal* plane curve singularities in the sense of [1] are also wild. Nevertheless, in [10] was shown that all uni- and bimodal plane curve singularities possess only 1-parametre families of *ideals*. Remark that in [12] these singularities are called *strictly unimodal* and we prefer to use this terminology.

The aim of our paper is to show that the strictly unimodal plane curve singularities are in some sense "universal" among those having not more than 1-parametre families of ideals. Namely, we prove that a curve singularity has this property if and only if it dominates one of the strictly unimodal plane curve singularities.

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Moreover, we prove this result for the curve singularities of arbitrary characteristic using, instead of the definition of such singularities by corresponding relations, their definition via *parametrization* given in [10]. Remark that it follows from [6],[8],[7] that a curve singularity has only finitely many non-isomorphic ideals if and only if it is of finite type (in contrast with 1-parametre case).

The proof of this theorem follows the same scheme as that of the main result on tameness from [4]. Namely, we first introduce some "overring conditions" for the ring R and show that whenever they do not hold, R possess 2-parametre families of non-isomorphic ideals. Then we show that these conditions imply that R dominates a strictly unimodal plane curve singularity. To accomplish the proof, we need also to show that any strictly unimodal plane curve singularity has not more than 1-parametre families of ideals. But indeed, one can calculate all ideals of these rings. It has been done in [10]. Though Schappert used the "definition via relations", one can verify (and we do it here for three most complicated examples) that his calculations depend only on the parametrization of these rings. This calculation of ideals shows that all strictly unimodal plane curve singularities really have only 1-parametre families of ideals.

1. Preliminaries

Notation 1.1. Throughout this article we use the following notations:

1. R denotes a complete local noetherian ring without nilpotent elements.
2. Q its full ring of fractions.
3. $\mathbf{m} = \operatorname{rad} \mathsf{R}$ its unique maximal ideal.
4. $\mathbf{k} = \mathsf{R}/\mathbf{m}$, the residue field of R.
5. R_0 its *normalization*, i.e. its integral closure in Q.
6. $\mathsf{R}_i = \mathbf{m}^i \mathsf{R}_0 + \mathsf{R}$.
7. $\mathbf{m}_i = \mathbf{m}^i \mathsf{R}_0 + \mathbf{m}$ (obviously, it is the maximal ideal of R_i for $i > 0$).
8. $\mathsf{d}\,(M)$ the minimal number of generators of an R-module M or, the same, $\dim\,(M/\mathbf{m}M)$ (over the residue field $\mathbf{k}$).
9. $\mathsf{d}_i = \mathsf{d}\,(\mathsf{R}_i)$.

Later on we suppose $\mathbf{k}$ to be algebraically closed.

Definition 1.2. R is said to be a *curve singularity* provided it satisfies the following conditions:

1. R is $\mathbf{k}$-algebra and $\mathsf{R}/\mathbf{m} = \mathbf{k}$.
2. R is of Krull dimension 1.

Really, such rings are just the completions of the local rings of points of arbitrary (reduced) algebraic curves over the field $\mathbf{k}$.

It is known (cf. [3]) that in this case d_0 is finite and, moreover, $\mathsf{d}(I) \le \mathsf{d}_0$ for each R-ideal I.

Remind the definitions related to the families of R-modules (cf. [5],[9]). Really, we will consider here only *full* R-*ideals*, i.e. such ideals I, that $\mathsf{Q}I = \mathsf{Q}$ (later on we omit the epithet "full').

Definition 1.3. Let X be an algebraic variety over $\mathbf{k}$ and $\mathcal{I}$ an $\mathsf{R} \otimes \mathcal{O}_X$-ideal, such that $\mathsf{Q}\mathcal{I} = \mathsf{Q} \otimes \mathcal{O}_X$. Call $\mathcal{I}$ a *family of ideals with the base* X if it is flat over $\mathcal{O}_X$ and, moreover, $\mathcal{I}/r\mathcal{I}$ is $\mathcal{O}_X$-flat for each non-zero-divisor $r \in \mathsf{R}$.

A series of such families, which are in some sense *universal*, can be constructed as follows. Consider the subvariety $\mathsf{B}(d)$ of the Grassmannian $\mathsf{Gr}(d, \mathsf{R}_0/\mathsf{R})$, consisting of those subspaces, which are R-submodules in R_0/R. The pre-image $\mathcal{I}(d)$ in $\mathsf{R}_0 \otimes \mathcal{O}_{\mathsf{B}(d)}$ of the canonical vector bundle on $\mathsf{B}(d)$ is then a family of R-ideals and any other family can be "glued" from the inverse images of the families $\mathcal{I}(d)$ (cf. [5], Proposition 3.4 and Corollary 3.5). Hence, we are able to define, following [9], the *number of parametres for* R-*ideals* $\mathsf{par}(1, \mathsf{R})$.

Definition 1.4. Denote $\mathsf{B}(d,i)$ the subset of $\mathsf{B}(d)$ consisting of such points x that the set (which is locally closed) $\{\, y \in \mathsf{B}(d) \,|\, \mathcal{I}(d)(y) \simeq \mathcal{I}(d)(x) \,\}$ has the dimension i and define:

$$\mathsf{par}(1, \mathsf{R}) = \max_{d,i}\{\, \dim \mathsf{B}(d,i) - i \,\}.$$

(Remark that $\mathsf{B}(d,i)$ is also locally closed in $\mathsf{B}(d)$).

They say that a ring R_1 *dominates* R if $\mathsf{R} \subseteq \mathsf{R}_1 \subseteq \mathsf{R}_0$,. In this case, evidently, $\mathsf{par}\,(1, \mathsf{R}_1) \leq \mathsf{par}\,(1, \mathsf{R})$. It follows from [7], [3] that $\mathsf{par}\,(1, \mathsf{R}) = 0$ if and only if R dominates one of the so-called *simple (or 0-modal) plane curve singularities* in the sense of [1] (cf. also [12]). We are going to prove an analogous fact concerning the *strictly unimodal* plane curve singularities (cf. [1], [12]) [1], whose list is given in Table 1. Remind some notions related to it.
As $\mathbf{k}$ is algebraically closed, we may suppose, that $\mathsf{R}_0 = \prod_{i=1}^{s} \mathsf{D}_i$, where $\mathsf{D}_i = \mathbf{k}[[t_i]]$ (formal power series rings). The number s is called *the number of branches* of R. Denote $t = (t_1, t_2, \dots, t_s)$ and v_i the standard valuation in D_i. For any element $r = (r_1, r_2, \dots, r_s) \in \mathsf{R}_0$ define its *(multi-)valuation* as the vector $\mathbf{v}(r) = (\,v_1(r_1), v_2(r_2), \dots, v_s(r_s)\,)$. In Table 1 we prefer to present the plane curve singularities in a "parametrization" form, i.e. given by their generators x, y as of *complete* subalgebras of R_0. Such presentation has an advantage that we may use it as a *definition* if $\mathsf{char}\,\mathbf{k} \neq 0$. Really, in the table the *valuations* $\mathbf{v}(x)$ and $\mathbf{v}(y)$ are given.

Later on we often write SUS instead of the words "strictly unimodal plane curve singularity". We also use the following definition and notation.

Definition 1.5. Let $\{\,a_1, a_2, \dots, a_d\,\}$ be a basis of $\mathbf{m}/(\mathbf{m} \cap t\mathbf{m}\mathsf{R}_0)$, $\mathbf{v}_j = \mathbf{v}(a_j)$. The set $\{\,\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d\,\}$ will be called *a valuation type* of R and denoted by $\mathrm{val}\,(\mathsf{R})$.

2. Main theorem

We pass now to the main theorem. In addition to the Notations 1.1, denote $\tilde{I} = t^2\mathbf{m}\mathsf{R}_0 + \mathbf{m}$, $\tilde{\mathsf{R}} = \mathsf{End}\,\tilde{I}$ and A_0 the 4-dimensional $\mathbf{k}$-algebra having a basis $\{\,1, a, b, ab\,\}$ with $a^2 = b^2 = 0$ (these notations will be used only in the case $\mathsf{char}\,\mathbf{k} = 2$).

Theorem 2.1. Let R be a curve singularity. The following conditions are equivalent:

1. $\mathsf{par}\,(1, \mathsf{R}) \leq 1$.

2. R dominates a simple or strictly unimodal plane curve singularity.

[1] In [1] these singularities are called "uni-" and "bimodal", while in [12] they are called "strictly unimodal". We use the latter terminology.

Table 1.

Type	s	$\mathbf{v}(x)$	$\mathbf{v}(y)$	
E	1 2 3	(3) $(1,2)$ $(1,1,1)$	(l) (∞,l) (∞,k,l)	$l\in\{7,8,10,11\}$ $4\le l\le 7$ $l=2$ or $l=3$
T	2 3 4	$(2,k)$ $(1,1,k)$ $(1,\infty,1,k)$	$(l,2)$ $(\infty,l,2)$ $(\infty,1,l,1)$	k,l odd k odd
W	1 2 2 3	(4) $(1,3)$ $(2,2)$ $(1,1,2)$	(l) (∞,l) $(3,l)$ $(\infty,l,3)$	$5\le l\le 7$ * $l=4$ or $l=5$ *
Z	2 3 4	$(1,l)$ $(1,\infty,2)$ $(1,\infty,1,1)$	$(\infty,3)$ $(\infty,1,l)$ $(\infty,1,2,l)$	$l\in\{4,5,7,8\}$ $2\le l\le 5$

* If $\mathsf{char}\,\mathbf{k}=2$, then, in case W, for $s=1$, necessarily $l=5$ and, for $s=2$, if $\mathbf{v}(x)=(2,2)$ and $l=3$, necessarily $x^2-y^3\notin t^7\mathsf{R}_0$.

3. (a) $\mathsf{d}(\mathsf{R}_0) \leq 4$;
   (b) $\mathsf{d}(\mathsf{R}_1) \leq 3$;
   (c) $\mathsf{d}(\mathsf{R}_2 + e\mathsf{R}) \leq 3$ for any such idempotent $e \in \mathsf{R}_0$, that $\mathsf{d}(e\mathsf{R}_0) = 1$ (provided it exists);
   (d) if $\mathsf{d}(\mathsf{R}_0) = 3$, then $\mathsf{d}(\mathsf{R}_3) \leq 2$.
   (e) if $\mathsf{char}\,\mathbf{k} = 2$, then $\tilde{R}/\tilde{I} \not\simeq \mathsf{A}_0$.

Remark 2.2. One can see, that the condition 3(c) of the theorem means that either $\mathsf{d}(\mathsf{R}_2) \leq 2$ or $e\mathbf{m} \not\subseteq \mathbf{m} + \mathbf{m}^3\mathsf{R}_0$.

Proof. $1 \Rightarrow 3$. Suppose first, that $d = \mathsf{d}_0 \geq 5$. Consider the factoralgebra $A = \mathsf{R}_0/\mathbf{m}\mathsf{R}_0$. If V is any subspace in A, then its pre-image $M(V)$ in R_0 is an R-submodule. Moreover, if V and U are two subspaces in A such that $AV = AU = A$, then, evidently, $M(V) \simeq M(U)$ if and only if $U = aV$ for some invertible element $a \in A$. Consider now the subset $\mathsf{Gr}_0(m, A)$ of the Grassmannian $\mathsf{Gr}(m, A)$, consisting of all such subspaces V that $AV = A$. Obviously, it is an open subset, hence, algebraic variety over $\mathbf{k}$ of the dimension $m(d-m)$. The group $\mathsf{G} = A^*/k^*$ is acting on this variety, and different orbits of this action correspond to non-isomorphic R-ideals. But, as $d > 4$, $\dim \mathsf{G} \leq \dim \mathsf{Gr}_0(2, A) - 2$. In view of [5] (Corollary 3.8), this implies $\mathsf{par}(1, \mathsf{R}) \geq 2$.

Let now $\mathsf{d}(\mathsf{R}_1) \geq 4$. Remark that $\mathsf{rad}\,\mathsf{R}_1 = \mathbf{m}\mathsf{R}_0$ and $\mathsf{R}_1/\mathsf{rad}\,\mathsf{R}_1 = \mathbf{k}$. Thus, the algebra $A' = \mathsf{R}_1/\mathbf{m}\mathsf{R}_1$ is local with the radical $\mathsf{J} = \mathbf{m}\mathsf{R}_0/\mathbf{m}\mathsf{R}_1$. Moreover, $\mathbf{m}\mathsf{R}_1 \supseteq (\mathbf{m}\mathsf{R}_0)^2$, whence $\mathsf{J}^2 = 0$. Then, for any subspace $W \subseteq \mathsf{J}$, the subspace $V = \mathbf{k} + W$ is a subalgebra in A and its pre-image $M(V)$ is a subring of R_0. Hence, taking different subspaces $W \subseteq \mathsf{J}$, we get non-isomorphic R-modules. As $\dim \mathsf{J} \geq 3$, $\dim \mathsf{Gr}(2, \mathsf{J}) \geq 2$ and $\mathsf{par}(1, \mathsf{R}) \geq 2$, too.

Remark 2.3. The same observations show that $\mathsf{par}(1, \mathsf{R}) \geq [\frac{\mathsf{d}_0 - 2}{2}]^2$ and if $\mathsf{d}_0 = \mathsf{d}_1$, then $\mathsf{par}(1, \mathsf{R}) \geq [\frac{\mathsf{d}_0 - 1}{2}]^2$.

To complete the proof, we need two simple lemmas.

Lemma 2.4. Let $J \subset \mathsf{R}$ be any ideal and $d = \dim(\mathsf{R}_0/J\mathsf{R}_0)$. Then $\dim(I/JI) \leq d$ for each R-ideal I.

Proof. Find an R_0-ideal $J_0 \subseteq J$. Then $\dim(I/JI) \leq \dim(\mathsf{R}_0 I/J_0 I) \leq (\mathsf{R}_0/J_0)$ as all R_0-ideals are principal. Hence, we can find an ideal I with the maximal value $m_0 = \mathsf{dim}\,(I/JI)$. Now, if M is any torsion-free R-module of rank n, we have $\mathsf{dim}\,(M/JM) \le nm_0$. But $\mathsf{R}_0 I \simeq \mathsf{R}_0$, hence, there exists an exact sequence of the form

$$0 \longrightarrow M \longrightarrow nI \longrightarrow \mathsf{R}_0 \longrightarrow 0$$

for some n and some module M of rank $n-1$. Taking factors modulo J, we get:

$$nm_0 = \mathsf{dim}\,(nI/nJI) \le (n-1)m_0 + d\,,$$

whence $m_0 \le d$ **q.e.d.**

Lemma 2.5. Suppose that $\mathsf{d}_0 = \mathsf{d}_1 = \ldots = \mathsf{d}_k$. Then there exists an element $r \in \mathbf{m}$, such that:

1. Elements $r, r^2, \ldots, r^k$ form a basis of $\mathbf{m}_{k+1}/\mathbf{m}^{k+1}\mathsf{R}_0$.

2. $r\mathbf{m}_k = \mathbf{m}\mathbf{m}_k$.

Proof. To prove the first assertion, take $r \in \mathbf{m} \setminus \mathbf{m}^2\mathsf{R}_0$, consider the dimensions $c_j = \mathsf{dim}\,(\mathbf{m}_j/\mathbf{m}^j\mathsf{R}_0)$ and remark, that

$$\mathsf{d}_j = 1 + \mathsf{dim}\,(\mathbf{m}_j/\mathbf{m}_{j+1}) = 1 + c_j + \mathsf{dim}\,(\mathbf{m}^j\mathsf{R}_0/\mathbf{m}^{j+1}\mathsf{R}_0) - c_{j+1}\,.$$

Evidently, $\mathsf{dim}\,(\mathbf{m}^j\mathsf{R}_0/\mathbf{m}^{j+1}\mathsf{R}_0) = \mathsf{d}_0$ for all j. So, we have $c_{j+1} = c_j + 1$ for $j \le k$, whence, obviously, $c_j = j - 1$. In particular, $\mathsf{dim}\,(\mathbf{m}_{k+1}/\mathbf{m}^{k+1}\mathsf{R}_0) = k$. As, of course, the elements $r, r^2, \ldots, r^k$ are linear independent modulo $\mathbf{m}^{k+1}\mathsf{R}_0$, they really form a basis of this vector space.

Now remark, that $r\mathbf{m}_k \subseteq \mathbf{m}\mathbf{m}_k$ and $\mathsf{dim}\,(\mathbf{m}_k/r\mathbf{m}_k) \le \mathsf{dim}\,(\mathsf{R}_0/\mathbf{m}\mathsf{R}_0) = \mathsf{d}_0$ in view of Lemma 2.4. But the just obtained result implies, that

$$\begin{aligned}
\mathsf{dim}\,(\mathbf{m}_k/\mathbf{m}\mathbf{m}_k) &= \mathsf{dim}\,(\mathbf{m}_k/\mathbf{m}_{k+1}) + \mathsf{dim}\,(\mathbf{m}_{k+1}/\mathbf{m}\mathbf{m}_k) = \\
&= \mathsf{d}_0 - 1 + \mathsf{dim}\,(\mathbf{m}_{k+1}/(\mathbf{m}^2 + \mathbf{m}^{k+1}\mathsf{R}_0) = \mathsf{d}_0\,.
\end{aligned}$$

Therefore, $r\mathbf{m}_k = \mathbf{m}\mathbf{m}_k$ **q.e.d.**

Remark, that the proof of Lemma 2.5 implies also the following corollary.

Corollary 2.6. $\mathsf{d}_{j+1} \le \mathsf{d}_j$ for every $j \ge 0$.

Proof. We need to prove that $c_{j+1}-c_j \leq c_{j+2}-c_{j+1}$. But $c_j = \dim(\mathbf{m}_j/\mathbf{m}^j\mathsf{R}_0) = \dim(\mathbf{m}/\mathbf{m}\cap\mathbf{m}^j\mathsf{R}_0)$, hence

$$c_{j+1}-c_j = \dim(\mathbf{m}\cap\mathbf{m}^j\mathsf{R}_0)/\dim(\mathbf{m}\cap\mathbf{m}^{j+1}\mathsf{R}_0) = \dim(\mathbf{m}\cap\mathbf{m}^j\mathsf{R}_0 + m^{j+1}\mathsf{R}_0/\mathbf{m}^{j+1}\mathsf{R}_0).$$

Therefore,

$$(c_{j+2}-c_{j+1})-(c_{j+1}-c_j) = \dim(\mathbf{m}^{j+1}\mathsf{R}_0/\mathbf{m}^{j+2}\mathsf{R}_0) - \dim(L_j/L_{j+1}),$$

where $L_j = \mathbf{m}\cap\mathbf{m}^j\mathsf{R}_0 + m^{j+1}\mathsf{R}_0$. But $\dim(\mathbf{m}^{j+1}\mathsf{R}_0/\mathbf{m}^{j+2}\mathsf{R}_0) = \mathsf{d}_0$ and $\mathbf{m}L_j \subseteq L_{j+1}$, whence $\dim(L_j/L_{j+1}) \leq \dim(L_j/\mathbf{m}L_j) \leq \mathsf{d}_0$ **q.e.d.**

Now suppose, that $\mathsf{d}_0 = \mathsf{d}_3 = 3$. Consider the factoralgebra $F = \mathsf{R}_2/\mathbf{m}^2\mathbf{m}_2$. Choosing r as in Lemma 2.5, we see, that $\mathbf{m}^i\mathbf{m}_2 = r^i\mathbf{m}_2$ and $\dim(\mathbf{m}_2/r\mathbf{m}_2) = 3$. Of course, $r \notin r\mathbf{m}_2$, so we can choose $r,u,v \in \mathbf{m}_2$ linear independent modulo $r\mathbf{m}_2$. Then $\{1,r,u,v,ru,rv\}$ is a basis of F. Now F contains a 2-parametres family of subalgebras, containing the image of R (i.e. 1 and r), namely, the subalgebras $A(\lambda,\mu)$ with the bases $\{1,r,u+\lambda v+\mu rv, ru+\lambda rv\}$. Then their pre-images in R_2 form a 2-parametres family of overrings of R, hence, of pairwise non-isomorphic R-ideals.

Remark 2.7. Really, as it follows from [3], in this case we could get only *families of overrings*, as there are at most two non-isomorphic ideals with a fixed endomorphism ring.

At last, suppose that $\mathsf{d}(\mathsf{R}_2+e\mathsf{R}) = 4$ for some idempotent $e \in \mathsf{R}_0$, such that $\mathsf{d}(e\mathsf{R}_0) = 1$. As we have remarked, that means: $\mathsf{d}(\mathsf{R}_2) = 3$ and $e\mathbf{m} \subseteq \mathbf{m}+\mathbf{m}^3\mathsf{R}_0$. Of course, the idempotent e is primitive and $e\mathsf{R} = e\mathsf{R}_0$. Denote $\mathsf{R}' = (1-e)\mathsf{R}$; $\mathsf{R}'_k = (1-e)\mathsf{R}_k$; $\mathsf{d}'_k = \mathsf{d}(\mathsf{R}'_k)$. Then $\mathsf{d}'_0 = \mathsf{d}'_1 = \mathsf{d}'_2 = 3$. Hence, we can apply Lemma 2.5 and choose an element $r \in \mathbf{m}$, such that $\{r, r^2\}$ form a basis of $\mathbf{m}\mathsf{R}'_2/\mathbf{m}^3\mathsf{R}'_0$. Consider the factoralgebra $F = (e\mathsf{R}+\mathsf{R}_1)/(\mathbf{m}^2+\mathbf{m}^3\mathsf{R}_0)$. If $\{r,u,v\}$ is a basis of $\mathbf{m}\mathsf{R}'_0/\mathbf{m}^2\mathsf{R}'_0$, then a basis of F can be chosen in the form: $\{1,e,r,u,v,ru,rv\}$. The subspaces $V(\lambda) = V(\lambda_0,\lambda_1,\lambda_2)$ with the bases $\{1, e+\lambda_0 u+\lambda_1 v+\lambda_2 rv, r, \lambda_0 ru+\lambda_1 rv\}$, where $\lambda_0 \neq 0$,, form a 3-dimensional family of R-submodules in F. Thus, they define a 3-dimensional family $M(\lambda)$ of R-ideals. But it follows from [5] (v. the proof of Proposition 3.6), that the ideals, isomorphic to some fixed $M(\lambda)$, form the subvariety of dimension

$\mathsf{dim}\,(V(\lambda)/S(\lambda))$, where $S(\lambda) = \{\, a \in F \,|\, aV(\lambda) \subseteq V(\lambda)\,\}$. As $\mathsf{dim}\,V(\lambda) = 4$ and $S(\lambda) \supset \{1, r, \lambda_0 ru + \lambda_1 rv\}$, this dimension is not more than (really, equals) 1. Hence, again $\mathsf{par}\,(1, \mathsf{R}) \geq 2$.

If $\mathsf{char}\,\mathbf{k} = 2$ and $\tilde{\mathsf{R}}/\tilde{I} \simeq \mathsf{A}_0$, consider the subspaces $A(\lambda, \mu) \subset \mathsf{A}_0$ with bases $\{\,1, a + \lambda b + \mu ab\,\}$. They are subalgebras in A_0 (as $\mathsf{char}\,\mathbf{k} = 2$), hence, their pre-images in $\tilde{\mathsf{R}}$ form a 2-parametres family of overrings of R, hence, of non-isomorphic R-ideals **q.e.d.**

$3 \Rightarrow 2$. Take any ring R satisfying the conditions 3(a–e). It is known (cf. [3],[7]), that if R has only finitely many non-isomorphic ideals then it dominates one of the simple pane curve singularities. So, we may suppose, that R has infinitely many non-isomorphic ideals, i.e. $\mathsf{d}_0 \geq 3$ and if $\mathsf{d}_0 = 3$, then also $\mathsf{d}_1 = 3$ (cf. ibid.). Suppose first that $s = 1$, where s is the number of branches. Then the condition (a) implies that $\mathrm{val}\,(\mathsf{R}) = \{\,3\,\}$ or $\{\,4\,\}$. In the first case the condition (d) easily implies that R contains also an element of the valuation $l \in \{\,7, 8, 10, 11\,\}$. But then it dominates a SUS of type E (cf. the list). If $\mathrm{val}\,(\mathsf{R}) = \{\,4\,\}$, then the condition (b) implies, that R contains an element of the valuation $5 \leq l \leq 7$, hence, dominates a SUS of type W. If $\mathsf{char}\,\mathbf{k} = 2$, the condition (e) implies also that R contains an element of the valuation 5: otherwise $\tilde{I} = t^6\mathsf{R}_0 + \mathbf{k} + \mathbf{k}t^4$, hence $\tilde{\mathsf{R}} = t^2\mathsf{R}_0 + \mathbf{k}$ and $\tilde{\mathsf{R}}/\tilde{I} \simeq \mathsf{A}_0$.

Let now $s = 2$. If $\mathsf{d}_0 = 3$ then $\mathrm{val}\,(\mathsf{R}) = \{\,(1,2)\,\}$ (up to the numbering of the branches; later on we omit this notice). Again the condition (d) implies that R contains an element of the valuation (∞, l) with $4 \leq l \leq 7$, hence, dominates a SUS of type E. Suppose that $\mathsf{d}_0 = 4$. Then the following cases can occur:

- $\mathrm{val}\,(\mathsf{R}) = \{\,(1,3)\,\}$. Then the condition (b) implies that R contains an element with the valuation either $(\infty, 4)$ or $(\infty, 5)$, hence, dominate a SUS of type W.

- $\mathrm{val}\,(\mathsf{R}) = \{\,(2,2)\,\}$. Again (b) implies that R contains an element with the valuation $(3, l)$, i.e. dominates a SUS of type W. Again, if $\mathsf{char}\,\mathbf{k} = 2$ and $x^3 - y^2 \in t^7\mathsf{R}_0$, we get that $\tilde{\mathsf{R}}/\tilde{I} \simeq \mathsf{A}_0$.

- $\mathrm{val}\,(\mathsf{R}) = \{\,(2,k), (l,2)\,\}$. Then R dominates a SUS of type T.

- $\mathrm{val}\,(\mathsf{R}) = \{\,(1,l), (\infty,3)\,\}$. Now the condition (c) obviously implies that $l \leq 8$ (and not 6), hence, R dominates a SUS of type Z.

If $s = \mathsf{d}_0 = 3$, then $\operatorname{val}(\mathsf{R}) = \{(1,1,1)\}$ and the condition (d) implies that R contains am element of the valuation $(\infty, 3, l)$, i.e. dominates a SUS of type E. Let $s = 3$, $\mathsf{d}_0 = 4$. If $\operatorname{val}(\mathsf{R})$ consists of only one vector, then it is $(1,1,2)$ and the condition (b) implies immediately that R contains also an element of the valuation $(\infty, 3, l)$, hence, dominate a SUS of type W. If $\operatorname{val}(\mathsf{R})$ consists of 2 vectors, then there are the following possibilities:

- $\operatorname{val}(\mathsf{R}) = \{(1,1,k), (\infty, l, 2)\}$. Then R dominates a SUS of type T.
- $\operatorname{val}(\mathsf{R}) = \{(1,\infty,l), (\infty,1,2)\}$. Then the condition (c) implies that $l \le 5$, hence, R dominates a SUS of type Z.

If $\operatorname{val}(\mathsf{R})$ consists of 3 vectors, they may be chosen as

$$\{(1,\infty,k), (\infty,1,l),\ (\infty,\infty,2)\}$$

and R dominates a SUS of type T.

At last, let $s = 4$. The condition (b) implies that $\operatorname{val}(\mathsf{R})$ has at least 2 vectors. If there are really only 2 of them, then either $\operatorname{val}(\mathsf{R}) = \{(1,\infty,k,1), (\infty,1,1,l)\}$ or $\operatorname{val}(\mathsf{R}) = \{(1,\infty,1,1), (\infty,1,k,l)\}$. In the first case R dominates one of the SUS of type T, while in the latter case the condition (c) implies that $k \le 2$, thus R dominates a SUS of type Z. Finally, if $\operatorname{val}(\mathsf{R})$ contains 3 vectors, one can easily see that R dominates a SUS of type T **q.e.d.**

3. Ideals of strictly unimodal plane curve singularities

Now we have to prove the implication $2 \Rightarrow 1$, that is to show that any SUS from Table 1 has only 1-parametre families of ideas. Indeed, it has been done in the Schappert's work [10]. Though Schappert supposed that $\operatorname{char} \mathbf{k} = 0$ and used another definition of SUS, one can check that his calculations are valid for our list too. independently of the characteristics. To precise it, we show below examples of such calculations (in somewhat different form). Moreover, we have chosen the most complicated cases. Our calculations are based on the following simple observation (cf. [3]). Let R be a curve singularity, $\mathbf{m}$ its maximal ideal and $\mathsf{S} = \operatorname{End}_{\mathsf{R}}(\mathbf{m}) = \{ a \in \mathsf{Q} \mid a\mathbf{m} \subseteq \mathbf{m} \}$. We keep these notations through the whole section and also put $\mathbf{n} = \operatorname{rad} \mathsf{S}$, $\mathsf{S}' = \operatorname{End}_{\mathsf{S}}(\mathbf{n})$. For any R-ideal I, $\mathsf{S}I$ is an S-ideal and $\mathbf{m}I = \mathbf{m}\mathsf{S}I \subset I \subseteq \mathsf{S}I$. Consider the factor $V = V(I) = I/\mathbf{m}I$. It is a *generating subspace* in $W = \mathsf{S}I/\mathbf{m}\mathsf{S}I$, i.e. such that $FV = W$, where

$F = \mathsf{S}/\mathbf{m}$, and one can easily check that $I \simeq I'$ if and only if $\bar{\gamma}V(I) = V(I')$ for some map $\bar{\gamma} \to FF$ induced by an automorphisms γ of $\mathsf{S}I$. Moreover, $\mathsf{S} \neq \mathsf{R}$, whenever R is not discrete valuation ring. Therefore, we can calculate the ideals "inductively", ascending by overrings. Remark also that any plane curve singularity R is *Gorenstein* [2], i.e. $\mathrm{inj.dim}_{\mathsf{R}}\mathsf{R} = 1$. Hence, R has the only minimal overring R' and any R-ideal is either principal or R'-ideal. In particular, $\mathsf{par}\,(1, \mathsf{R}) = \mathsf{par}\,(1, \mathsf{R}')$.

Remark that all SUS of type T are known to be *tame*, i.e. have at most 1-parametre families of indecomposable torsion-free modules of any rank. So we have to consider only the SUS of types W,Z and E.

3.1. Ideals of singularities of type W

Here we consider the case, when $s = 1$ and R contains elements x, y with $v(x) = 4$, $v(y) = 7$ ("type W_{18}" in Arnold's classification). It is convenient to suppose here that $t = x^2/y$. Of course, we suppose also that $\mathsf{char}\,\mathbf{k} \neq 2$. It is easy to verify that then $\mathsf{R} \supset t^{18}\mathsf{R}_0$ and, if R is a plane curve singularity, its minimal overring contains even $t^{14}\mathsf{R}_0$. Hence, we may restrict ourselves by the case, when R is the smallest subalgebra of R_0 containing $t^{14}\mathsf{R}_0$ and generated modulo $t^{14}\mathsf{R}_0$ by x and y. Thus,

$$\mathsf{R} = \langle\, 1, x, y, x^2, xy, x^3 \,\rangle + t^{14}\mathsf{R}_0\,,$$

where, as usual, we denote by $\langle\, a_1, a_2, \dots, a_m \,\rangle$ the k-subspace generated by $a_1, a_2, \dots, a_m$. An obvious calculation shows that in this case

$$\mathsf{S} = \langle\, 1, x, y, x^2 \,\rangle + t^{10}\mathsf{R}_0$$

and

$$\mathsf{S}' = \langle\, 1, z, x \,\rangle + t^{6}\mathsf{R}_0\,,$$

where $z = y/x$. As $v(z) = 3$, it follows from [6] or [8] that S' has only finitely many non-isomorphic ideals (it is the simple plane curve singularity of type E_6). Moreover, in these articles the precise list of such ideals is given. Namely, they are, except R_0 and S' itself:

$$\begin{aligned} \mathsf{A} &= \mathsf{S}'\langle\, 1, t^5 \,\rangle\,, \\ \mathsf{A}^* &= \mathsf{S}'\langle\, 1, t \,\rangle\,, \\ \mathsf{A}' &= \mathsf{S}'\langle\, 1, t^2 \,\rangle\,. \end{aligned}$$

Here A and A' are overrings of S' and A^* the module dual to A^* with respect to the duality described e.g. in [3].

Now, for each of these ideals, say M, we have to calculate $M/\mathbf{n}M$. Here is their list (we write "r" for the image of an element $r \in M$ in $M/\mathbf{n}M$ too):

$$\begin{aligned}
\mathsf{S}' &= \langle 1, z, z^2, z^3 \rangle, \\
\mathsf{A} &= \langle 1, z, t^5, t^6 \rangle, \\
\mathsf{A}^* &= \langle 1, t, z, t^6 \rangle, \\
\mathsf{A}' &= \langle 1, t^2, z, t^5 \rangle, \\
\mathsf{R}_0 &= \langle 1, t, t^2, t^3 \rangle.
\end{aligned}$$

Now one can easily write down all generating subspaces V from each $W = M/\mathbf{n}M$ (up to automorphisms of M). Consider the case $M = \mathsf{A}'$ (the most complicated). First, as V is generating, it has to contain at least two elements of the form: $a_1 = 1+\lambda_1 z+\lambda_2 t^5$ and $a_2 = t^2+\mu_1 z+\mu_2 t^5$. Dividing by a_1 (which is the image of an invertible element from A'), we may suppose that $a_1 = 1$. Suppose that $V = \langle a_1, a_2 \rangle$. If $\mu_1 \neq 0$, one can replace V by $(1 - (\mu_1 + \mu_2)(2\mu_1)^{-1})V$, thus obtaining a subspace of the same form but with $\mu_2 = 0$. Hence, we get two 1-parametre families of S-ideals:

$$\begin{aligned}
F_1(\mu) &= \mathsf{S}\langle 1, t^2 + \mu z \rangle, \\
F_2(\mu) &= \mathsf{S}\langle 1, t^2 + \mu t^5 \rangle.
\end{aligned}$$

Suppose now that V contains another element, which can be chosen in the form $b = \eta_1 z+\eta^2 t^5$. If $\eta_1 \neq 0$, we may suppose that $\mu_1 = 0$ and then, multiplying by $1 - \eta_2/\eta_1 z$, also $\eta_2 = 0$, i.e. $b = z$. But then, multiplying by $1 - \mu_2 t^3$, we get $\mu_2 = 0$, which gives only one more ideal:

$$I_1 = \mathsf{S}\langle 1, t^2, z \rangle.$$

If $\eta_1 = 0$, we may suppose $\mu_2 = 0$, obtaining 1-parametre family:

$$F_3(\mu) = \mathsf{S}\langle 1, t^2 + \mu z, t^5 \rangle.$$

Of course, we have to add to those families also the ideal A' itself (corresponding to $V = W$). But we can remark that $\mathsf{d}(\mathsf{A}') \leq 4$, hence $\dim \mathsf{A}'/\mathbf{m}\mathsf{A}' \leq 4$ and $\mathbf{m}\mathsf{A}' = \mathbf{n}\mathsf{A}'$. Thus, any generating subspace in $\mathsf{A}'/\mathbf{m}\mathsf{A}'$ (with respect to S) must coincide with $\mathsf{A}'/\mathbf{m}\mathsf{A}'$ itself. Therefore, we need not consider the case $\mathsf{S}I = \mathsf{A}'$ when calculating the R-ideals. The same argument is valid, of course, in each case, when we have such S-ideal L that $\mathbf{n}L = \mathbf{m}L$ (it is always the case, when $\dim(L/\mathbf{n}\mathsf{S}) = \mathsf{d}_0$): we may exclude them while calculating the R-ideals. In particular, here we may exclude all S'-ideals.

Quite analogous observations give us the following list of S-ideals (which are not S'-ideals):

$\mathsf{S}'L = \mathsf{S}'$:

$$\begin{aligned}
F_4(\mu) &= \mathsf{S}\langle 1, z+\mu z^3 \rangle\,,\\
F_5(\mu) &= \mathsf{S}\langle 1, z^2+\mu z^3 \rangle\,,\\
\mathsf{S} &= \mathsf{S}\langle 1 \rangle\,,\\
I_2 &= \mathsf{S}\langle 1, z^3 \rangle\,,\\
I_3 &= \mathsf{S}\langle 1, z, z^2 \rangle\,,\\
I_4 &= \mathsf{S}\langle 1, z, z^3 \rangle\,,\\
I_5 &= \mathsf{S}\langle 1, z^2, z^3 \rangle\,,
\end{aligned}$$

$\mathsf{S}'L = \mathsf{A}$:

$$\begin{aligned}
F_6(\mu) &= \mathsf{S}\langle 1, z+\mu t^5 \rangle\,,\ \mu \neq 0\,,\\
F_7(\mu) &= \mathsf{S}\langle 1, z+\mu t^5, t^6 \rangle\,, \mu \neq 0\,,\\
F_8(\mu) &= \mathsf{S}\langle 1, t^5+\mu t^6 \rangle\,,\\
F_9(\mu) &= \mathsf{S}\langle 1, z, t^5+\mu t^6 \rangle\,,\\
I_6 &= \mathsf{S}\langle 1, t^5, t^6 \rangle\,.
\end{aligned}$$

$\mathsf{S}'L = \mathsf{A}^*$:

$$\begin{aligned}
F_{10}(\mu) &= \mathsf{S}\langle 1, t+\mu z \rangle\,,\\
F_{11}(\mu) &= \mathsf{S}\langle 1, t+\mu z, t^6 \rangle\,,\\
I_7 &= \mathsf{S}\langle 1, t, z \rangle\,.
\end{aligned}$$

$\mathsf{S}'L = \mathsf{R}_0$:

$$I_8 = \mathsf{S}\langle 1, t, t^2 \rangle\,.$$

Now we can pass to R-ideals I. Put $M = \mathsf{S}I$. It would be one of the ideals $F_{1-11}(\mu)$, I_{1-8} or S. One can easily check, that in the following cases $\mathbf{m}M = \mathbf{n}M$, so we need not to consider them:

$$\begin{array}{ll} F_i(\mu) & \text{for } i \neq 4, 8, 10 \ , \\ F_i(\mu) & \text{for } i = 4, 10 \text{ and } \mu \neq 0 \ , \\ I_i & \text{for } i \neq 2 \ . \end{array}$$

In the case $M = F_4(0)$ we have: $M/\mathbf{m}M = \langle 1, z, t^{13} \rangle$. Hence, the only proper generating subspace is $\langle 1, z \rangle$, which gives one new ideal:

$$I_9 = \mathsf{R}\langle 1, z \rangle \ .$$

Analogously, in the case $M = F_{10}(0)$ we obtain also one new ideal:

$$I_{10} = \mathsf{R}\langle 1, t \rangle \ .$$

In the case $M = F_8$ we have $M/\mathbf{m}M = \langle 1, t^5 + \mu t^6, t^{10} \rangle$. Hence, there is again only one proper generating subspace, namely, $\langle 1, t^5 + \mu t^6 \rangle$ and we obtain a new 1-parametre family:

$$F_{12}(\mu) = \mathsf{R}\langle 1, u(\mu) \rangle \ , \quad \text{where } u(\mu) = t^5 + \mu t^6 \ .$$

In the case $M = \mathsf{S}$ we have: $M/\mathbf{m}M = \langle 1, t^{10}, t^{13} \rangle$. As there is no element with valuation 3 in S, we get here a new family parametrized by the projective line:

$$F_{13}(\lambda_0 : \lambda_1) = \mathsf{R}\langle 1, \lambda_0 t^{10} + \lambda_1 t^{13} \rangle \ .$$

At last, the case $M = I_2$ also gives a new 1-parametre family:

$$F_{14}(\mu) = \mathsf{R}\langle 1, z^3 + \mu t^{10}, t^{13} \rangle \ .$$

Thus, we have described all R-ideals and proved that $\mathsf{par}\,(1, \mathsf{R}) = 1$. Quite similar (mainly, easier) calculations show that $\mathsf{par}\,(1, \mathsf{R}) = 1$ all other singularities of type W.

Remark: In the list of Schappert [10] the ideals $F_2(\mu)$ (which are indeed overrings of R) and I_3 are missed.

3.2. Ideals of singularities of type Z

Now consider the singularities of type Z. Here we suppose that $s = 2$ and R contains elements x, y with $\mathbf{v}(x) = (1, 8)$, $\mathbf{v}(y) = (\infty, 3)$ (SUS of type Z_{19}). One can check that in this case $\mathsf{R} \supset (t_1^3, t_2^{17})\mathsf{R}_0$ and, moreover, its minimal overring contains (t_1^2, t_2^{14}). Hence, we may suppose that

$$\mathsf{R} = \langle\, 1, x, y, y^2, y^3, xy, y^4 \,\rangle + (t_1^2, t_2^{14})\mathsf{R}_0 \,.$$

It is convenient to take for t_1 the first component of x and choose t_2 in such way that $y^3 = (0, t_2)x$. Now

$$\mathsf{S} = \langle\, 1, x, y, y^2, y^3 \,\rangle + (t_1, t_2^{11})\mathsf{R}_0$$

and

$$\mathsf{S}' = \mathsf{D}_1 \oplus \mathsf{S}_2 \,,$$

where $\mathsf{D}_1 = \mathbf{k}[[t_1]]$ and $\mathsf{S}_2 = \mathbf{k}[[t_2^3, t_2^5]]$ (the simple plane curve singularity of type E_8). Here is the list of all S_2-ideals (cf. [6],[8]) given by their generators over S_2 and over S:

Ideal	S_2-generators	S-generators
S_2	1	$1,\ t_2^5,\ t_2^{10}$
A	$1,\ t_2^7$	$1,\ t_2^5,\ t_2^7$
A^*	$1,\ t_2^2$	$1,\ t_2^2,\ t_2^7$
B	$1,\ t_2^4$	$1,\ t_2^4, t_2^5$
B^*	$1,\ t_2$	$1,\ t_2,\ t_2^5$
B'	$1,\ t_2^2,\ t_2^4$	$1,\ t_2^2,\ t_2^4$
D_2	$1,\ t_2,\ t_2^2$	$1,\ t_2,\ t_2^2$

Any (full) S'-ideal is of the form $\mathsf{D}_1 \oplus N$, where N is an S_2-ideal. Consider first the S-ideals I such that $\mathsf{S}'I = \mathsf{D}_1 \oplus \mathsf{A}$. Then $\mathbf{n}I = \mathbf{n} + \langle\, (0, t_2^{10}) \,\rangle$,

$$\mathsf{S}'I/\mathbf{n}I = W = \langle\, (1,0), (0,1), (0,t_2^5), (0,t_2^7) \,\rangle$$

and any generating subspace $V \subseteq W$ contains an element of the form $(1,a)$ and also elements of the form $(\alpha_1, 1{+}\mu_1 t_2^5), (\alpha_2, t_2^7{+}\mu_2 t_2^5)$. Multiplying by $(1, 1{-}\mu_1 t_2^5)$, we may suppose that V contains $(\alpha_1, 1)$. Then, if $\dim V = 2$, there are only two possibilities:

$$V = \langle\, (1,1), (0, \mu t_2^5 + t_2^7) \,\rangle \quad \text{and} \quad V = \langle\, (0,1), (1, \mu t_2^5 + t_2^7) \,\rangle \,.$$

If $\mathsf{dim}\, V = 3$, we can add also an element of the form (β, t_2^5) or $(1,0)$ It gives three more possibilities:

$$\begin{aligned} V &= \langle\, (1,1), (0,t_2^5), (0,t_2^7)\,\rangle\,, \\ V &= \langle\, (0,1), (\lambda_0,t_2^5), (\lambda_1,t_2^7)\,\rangle\,, \\ V &= \langle\, (1,0), (0,1), (\mu t_2^5 + t_2^7)\,\rangle\,, \end{aligned}$$

where $(\lambda_0 : \lambda_1)$ is a point of the projective line. Hence, here is the list of the corresponding S-ideals (which are not S'-ideals):

$$\begin{aligned} F_1(\mu) &= \mathsf{S}\langle\, (1,1), (0,\mu t_2^5 + t_2^7)\,\rangle\,, \\ F_2(\mu) &= \mathsf{S}\langle\, (0,1), (1,\mu t_2^5 + t_2^7)\,\rangle\,, \\ F_3(\lambda_0 : \lambda_1) &= \mathsf{S}\langle\, (0,1), (\lambda_0,t_2^5), (\lambda_1,t_2^7)\,\rangle\,, \\ F_4(\mu) &= \mathsf{S}\langle\, (1,0), (0,1), (\mu t_2^5 + t_2^7)\,\rangle\,, \\ I_1 &= \mathsf{S}\langle\, (1,1), (0,t_2^5), (0,t_2^7)\,\rangle\,. \end{aligned}$$

Remark that the ideals $F_4(\mu)$ are decomposable (as modules), while all other ideals of this list are indecomposable.

Analogous calculations give the following list of all S-ideals I, which are not S'-ideals:

$\mathsf{S}'I = \mathsf{S}'$:

$$\begin{aligned} F_5(\lambda_0 : \lambda_1) &= \mathsf{S}\langle\, (\lambda_0,1), (\lambda_1,t_2^5)\,\rangle\,, \\ \mathsf{S} &= \mathsf{S}\langle\, (1,1)\,\rangle\,, \\ I_2 &= \mathsf{S}\langle\, (1,1), (0,t_2^{10})\,\rangle\,, \\ I_3 &= \mathsf{S}\langle\, (1,1), (0,t_2^5), (0,t_2^{10})\,\rangle\,, \\ I_4 &= \mathsf{S}\langle\, (0,1), (1,t_2^5), (0,t_2^{10})\,\rangle\,, \\ I_5 &= \mathsf{S}\langle\, (0,1), (1,t_2^{10})\,\rangle\,, \\ I_6 &= \mathsf{S}\langle\, (0,1), (0,t_2^5), (1,t_2^{10})\,\rangle\,, \\ I_7 &= \mathsf{S}\langle\, (1,0), (0,1)\,\rangle\,, \\ I_8 &= \mathsf{S}\langle\, (1,0), (0,1), (0,t_2^5)\,\rangle\,, \\ I_9 &= \mathsf{S}\langle\, (1,0), (0,1), (0,t_2^{10})\,\rangle\,, \end{aligned}$$

$\mathsf{S}'I = \mathsf{D}_1 \oplus \mathsf{A}^*$, $\mathbf{n}I = \mathbf{n} + \langle\, (0,t_2^5), (0,t_2^{10})\,\rangle$:

$$
\begin{aligned}
F_6(\lambda_0:\lambda_1) &= \mathsf{S}\langle\,(\lambda_0,1),(\lambda_1,t_2^2)\,\rangle\,,\\
F_7(\lambda_0:\lambda_1) &= \mathsf{S}\langle\,(\lambda_0,1),(\lambda_1,t_2^2),(0,t_2^7)\,\rangle\,,\\
I_{10} &= \mathsf{S}\langle\,(0,1),(0,t_2^2),(1,t_2^7)\,\rangle\,,\\
I_{11} &= \mathsf{S}\langle\,(1,0),(0,1),(0,t_2^2)\,\rangle\,.
\end{aligned}
$$

$\mathsf{S}'I=\mathsf{D}_1\oplus\mathsf{B}\,,\ \mathbf{n}I=\mathbf{n}+\langle\,(0,t_2^7),(0,t_2^{10})\,\rangle:$

$$
\begin{aligned}
F_8(\mu) &= \mathsf{S}\langle\,(1,1),(0,t_2^4+\mu t_2^5)\,\rangle\,,\\
F_9(\mu) &= \mathsf{S}\langle\,(0,1),(1,t_2^4+\mu t_2^5)\,\rangle\,,\\
F_{10}(\lambda_0:\lambda_1) &= \mathsf{S}\langle\,(0,1),(\lambda_0,t_2^4),(\lambda_1,t_2^5)\,\rangle\,,\\
F_{11}(\mu) &= \mathsf{S}\langle\,(1,0),(0,1),(0,t_2^4+\mu t_2^5)\,\rangle\,,\\
I_{12} &= \mathsf{S}\langle\,(1,1),(0,t_2^4),(0,t_2^5)\,\rangle\,.
\end{aligned}
$$

$\mathsf{S}'I=\mathsf{D}_1\oplus\mathsf{B}^*\,,\ \mathbf{n}I=n+\langle\,(0,t_2^4),(0,t_2^7),(0,t_2^{10})\,\rangle:$

$$
\begin{aligned}
F_{12}(\lambda_0:\lambda_1) &= \mathsf{S}\langle\,(\lambda_0,1),(\lambda_1,t_2)\,\rangle\,,\\
F_{13}(\lambda_0:\lambda_1) &= \mathsf{S}\langle\,(\lambda_0,1),(\lambda_1,t_2),(0,t_2^5)\,\rangle\,,\\
I_{13} &= \mathsf{S}\langle\,(0,1),(0,t_2),(1,t_2^5)\,\rangle\,,\\
I_{14} &= \mathsf{S}\langle\,(1,0),(0,1),(0,t_2)\,\rangle\,.
\end{aligned}
$$

$\mathsf{S}'I=\mathsf{D}_1\oplus\mathsf{B}'\,,\ \mathbf{n}I=n+\langle\,(0,t_2^5),(0,t_2^7),(0,t_2^{10})\,\rangle:$

$$
\begin{aligned}
I_{15} &= \mathsf{S}\langle\,(1,1),(0,t_2^2),(0,t_2^4)\,\rangle\\
I_{16} &= \mathsf{S}\langle\,(0,1),(1,t_2^2),(0,t_2^4)\,\rangle\\
I_{17} &= \mathsf{S}\langle\,(0,1),(0,t_2^2),(1,t_2^4)\,\rangle\,.
\end{aligned}
$$

$\mathsf{S}'I=\mathsf{R}_0:$

$$
I_{18} = \mathsf{S}\langle\,(1,1),(0,t_2),(0,t_2^2)\,\rangle
$$

$$\begin{aligned} I_{19} &= \mathsf{S}\langle\, (0,1),(1,t_2),(0,t_2^2)\,\rangle \\ I_{20} &= \mathsf{S}\langle\, (0,1),(0,t_2),(1,t_2^2)\,\rangle\,. \end{aligned}$$

Here μ denotes an element of $\mathbf{k}$ and $(\lambda_0 : \lambda_1)$ a point of the projective line over $\mathbf{k}$.

Now pass to the calculation of R-ideals, which are not S-ideals. Again one can verify that for the following S-ideals M we have $\mathbf{m}M = \mathbf{n}M$, so we do not need to consider the case, when $\mathsf{S}I = M$:

$$\begin{array}{rl} F_i(\mu) & \text{for } i \neq 1,8\ , \\ F_i(\lambda_0 : \lambda_1) & \text{for all } i\ , \\ F_8(\mu) & \text{for } \mu \neq 0\ , \\ I_i & \text{for } i \neq 2,7\ . \end{array}$$

For $M = F_1(\mu)$, $M/\mathbf{m}M = \langle\, (1,1),(0,\mu t_2^5 + t_2^7),(0,t_2^8)\,\rangle$. If $\mu \neq 0$, the multiplication by itself maps the second of these elements to the third one, which is congruent modulo $\mathbf{m}M$ to $\mu^{-1}t_2^{10}$. Hence, here we obtain only the following new 1-parametre family of ideals:

$$F_{14}(\mu) = \mathsf{R}\langle\, (1,1),(0,\mu t_2^5 + t_2^7)\,\rangle\,,\ \mu \neq 0\,.$$

If $\mu = 0$, we obtain the following new family:

$$F_{15}(\mu) = \mathsf{R}\langle\, (1,1),(t_2^7 + \mu t_2^8)\,\rangle\,.$$

In the case $M = F_8(0)$, $M/\mathbf{m}M = \langle\, (1,1),(0,t_2^4 + \mu t_2^5),(0,t_2^8)\,\rangle$. Hence, we also obtain only one new family of ideals:

$$F_{16}(\mu) = \mathsf{R}\langle\, (1,1),(t_2^4 + \mu t_2^5)\,\rangle\,.$$

The remaining cases: S and I_i ($i = 2,7$), give the ring R itself and the following ideals:

$$\begin{aligned} F_{17}(\lambda_0 : \lambda_1) &= \mathsf{R}\langle\, (1,1),(\lambda_0 t_1, \lambda_1 t_2^{13})\,\rangle\,, \\ F_{18}(\mu) &= \mathsf{R}\langle\, (1,1),(\mu t_1, t_2^{10})\,\rangle\,, \\ I_{21} &= \mathsf{R}\langle\, (1,0),(0,1)\,\rangle\,, \end{aligned}$$

$$I_{22} \;=\; \mathsf{R}\langle\,(1,t_2^{13}),(0,1)\,\rangle\,.$$

Therefore, we have proved that in this case also $\mathsf{par}\,(1,\mathsf{R}) = 1$. Quite similar calculations prove the same for all other singularities of type Z.

Remark: Here also the family F_{13} fails in the Schappert's list of ideals.

3.3. Ideals of singularities of type E

Now consider the case of singularities E. In this case $\mathsf{d}_0 = 3$, so it follows from [3] that each R-ideal is isomorphic either to an overring of R or to its dual module. Therefore, we only need to find all overrings. But if I is an overring of R then $\mathsf{S}I$ is also an overring of S. Hence, at each stage of our inductive process we may restrict ourselves by considering only overrings. To be complete, we always mark, which of these overrings are Gorenstein (i.e. self-dual).

Here we suppose that $s = 3$ and R contains elements x, y with $\mathbf{v}(x) = (1,1,1)$ and $\mathbf{v}(y) = (\infty,k,3)$ with $k > 3$ (SUS of type $\mathsf{E}_{3,p}$). Of course, we suppose here that $x = t = (t_1,t_2,t_3)$. Again, passing to the minimal overring, we may suppose that R contains $(t_1^{k+2},t_2^{k+2},t_3^5)\mathsf{R}_0$ and is generated by x and y modulo this ideal. Then S contains $(t_1^{k+1},t_2^{k+1},t_3^4)\mathsf{R}_0$ and is generated modulo this ideal by x and z, where $y = xz$ and $\mathbf{v}(z) = (\infty,k-1,2)$ (it is a SUS of type $\mathsf{E}_{1,p}$). The ring S' contains $(t_1^k,t_2^k,t_3^3)\mathsf{R}_0$ and is generated modulo this ideal by x and z. Put now $\mathbf{n}' = \mathsf{rad}\,\mathsf{S}'$, $\mathsf{S}'' = \mathsf{End}\,(\mathbf{n}')$. Then S'' is generated by x and z', where $\mathbf{v}(z') = (\infty,k-2,1)$ and $z = xz'$. Hence, S'' has only finitely many ideals up to isomorphism (it is a simple plane curve singularity of type D), cf. [6],[8]. Namely, here is the list of the overrings of S'' (except S'' itself):

$$\begin{aligned}
\mathsf{A}_i &= \mathsf{S}''\langle\,1,(0,t_2^i,0)\,\rangle \quad (1 \le i \le k-2)\,,\\
\mathsf{B}_i &= \mathsf{S}''\langle\,1,(0,0,1),(0,t_2^i,0)\,\rangle \quad (0 \le i \le k-2)\,,\\
\mathsf{B}_{01} &= \mathsf{S}''\langle\,1,(1,0,0)\,\rangle\,,\\
\mathsf{B}_{02} &= \mathsf{S}''\langle\,1,(0,1,0)\,\rangle\,,\\
\mathsf{R}_0 &= \mathsf{S}''\langle\,(1,0,0),(0,1,0),(0,0,1)\,\rangle\,.
\end{aligned}$$

Among them, only A_i are non-Gorenstein. Moreover, $M\mathbf{n}'' = M\mathbf{n}'$ for $M = \mathsf{R}_0$ or $M = \mathsf{B}_i,\ i < k-2$, where $\mathbf{n}'' = \mathsf{rad}\,\mathsf{S}''$, so we do not need further to consider these overrings. Remark also that in the case of B_{k-2} the generator $(0,t_2^i,0)$ above is superfluous.

As S is Gorenstein, its overrings, except S itself, are those of S'. Here is the list of factoralgebas $M/\mathbf{n}'M$ for the overrings M of S'':

$$\begin{aligned}
S''/\mathbf{n}' &= \langle\, 1, z', (0,0,t_3^2)\,\rangle\,, \\
\mathsf{A}_i/\mathbf{n}'\mathsf{A}_i &= \langle\, 1, (0,0,t_3), (0,t_2^i,0)\,\rangle\,, \\
\mathsf{B}_{k-2}\mathbf{n}'\mathsf{B}_{k-2} &= \langle\, 1, (0,0,1), (0,t_2^{k-2},0)\,\rangle\,, \\
\mathsf{B}_{01}/\mathbf{n}'\mathsf{B}_{01} &= \langle\, 1, (1,0,0), (0,t_2,0)\,\rangle\,, \\
\mathsf{B}_{02}/\mathbf{n}'\mathsf{B}_{02} &= \langle\, 1, (0,1,0), (0,t_2,0)\,\rangle\,.
\end{aligned}$$

It is easy now to find all proper subalgebras of these algebras and, hence, the overrings of S (except S itself), which are not overrings of S'':

$$\begin{aligned}
\mathsf{S}' &= \mathsf{S}\langle\, 1, (0,0,t_3^3)\,\rangle\,, \\
\mathsf{A}_{k-1} &= \mathsf{S}\langle\, 1, (0,0,t_3^2)\,\rangle\,, \\
F_i(\mu) &= \mathsf{S}\langle\, 1, (0,t_2^i,\mu t_3)\,\rangle \quad (1 \le i \le k-2)\,, \\
\mathsf{B}_{k-1} &= \mathsf{S}\langle\, 1, (0,0,1)\,\rangle\,, \\
\mathsf{B}'_{01} &= \mathsf{S}\langle\, 1, (1,0,0)\,\rangle\,, \\
\mathsf{B}'_{02} &= \mathsf{S}\langle\, 1, (0,1,0)\,\rangle\,.
\end{aligned}$$

Among them only S', A_{k-1} and $F_i(0)$ are non-Gorenstein. Remark also that $M\mathbf{m} = M\mathbf{n}'$ for all overrings of S'', so we do not need to consider them further. To find all overrings of R, which are not overrings of S, calculate the factoralgebras $M/\mathbf{m}M$ for the rings of the preceding list and S:

$$\begin{aligned}
\mathsf{S}/\mathbf{m} &= \langle\, 1, z, (0,0,t_3^4)\,\rangle\,, \\
\mathsf{S}'/\mathbf{m}\mathsf{S}' &= \langle\, 1, z, (0,0,t_3^3)\,\rangle\,, \\
\mathsf{A}_{k-1}/\mathbf{m}\mathsf{A}_{k-1} &= \langle\, 1, (0,0,t_3^2), (0,0,t_2^{k-1})\,\rangle\,, \\
F_i(\mu)/\mathbf{m}F_i(\mu) &= \langle\, 1, (0,t_2^i,\mu t_3), (0,0,t_3^2)\,\rangle\,, \\
\mathsf{B}_{k-1}/\mathbf{m}\mathsf{B}_{k-1} &= \langle\, 1, (0,0,1), (0,t_2^{k-1})\,\rangle\,, \\
\mathsf{B}'_{01}/\mathbf{m}\mathsf{B}'_{01} &= \langle\, 1, (1,0,0), (0,t_2^2,0)\,\rangle\,, \\
\mathsf{B}'_{02}/\mathbf{m}\mathsf{B}'_{02} &= \langle\, 1, (0,1,0), (t_1^2,0,0)\,\rangle\,.
\end{aligned}$$

It gives us the following list of the overrings of R, which are not overrings of S (except R itself):

$$
\begin{aligned}
\mathsf{R}' &= \mathsf{R}\langle\, 1, (0,0,t_3^4)\,\rangle\,, \\
F'_{k-1}(\mu) &= \mathsf{R}\langle\, 1, \mu z + (0,0,t_3^3)\,\rangle\,, \\
F_{k-1}(\mu) &= \mathsf{R}\langle\, 1, (0,\mu t_2^{k-1}, t_3^2)\,\rangle \quad (\mu \neq 1)\,, \\
F'_i(\mu) &= \mathsf{R}\langle\, 1, (0, t_2^i, t_3^2)\,\rangle \quad (1 \le i \le k-2)\,, \\
\mathsf{B}_k &= \mathsf{R}\langle\, 1, (0,0,1)\,\rangle\,, \\
\mathsf{B}''_{01} &= \mathsf{R}\langle\, 1, (1,0,0)\,\rangle\,, \\
\mathsf{B}''_{02} &= \mathsf{R}\langle\, 1, (0,1,0)\,\rangle\,.
\end{aligned}
$$

Here $F_{k-1}(\mu)$, B_k, B''_{01} and B''_{02} are Gorenstein.

Thus, we have proved that $\mathsf{par}\,(1,\mathsf{R}) = 1$. Analogous calculations show the same for all other SUS of type E, which accomplishes the proof of Theorem 2.1.

References

[1] Arnold V. I., Varchenko A. N., Gusein-Zade S. M., Singularities of Differentiable Maps, Vol. 1, Birkhäus er, Boston-Basel-Stuttgart, 1985.

[2] Bass H., On the ubiquity of Gorenstein rings,

[3] Drozd Yu. A., Ideals of commutative rings, Mat. Sbornik, 101 (1976) 334–348.

[4] Drozd Yu. A., Greuel G.-M., Cohen-Macauley module type, Compositio Math. 89 (1993) 315–338.

[5] Drozd Yu. A., Greuel G.-M., Semicontinuity for representations of Cohen-Macaulay rings, Preprint Nr. 247, Fachbereich Math. Univ. Kaiserslautern, 1993.

[6] Drozd Yu. A., Roiter A. V., Commutative rings with a finite number of indecomposable integral representations Izv. Aad. Nauk SSSR. Ser. Mat. 31 (1967) 783–798.

[7] Greuel G.-M., Knörrer H., Einfache Kurvensingularitäten und torsionfreie Moduln, Math. Ann. 270 (1985) 417–425.

[8] Jacobinski H., Anneaux commutatifs avec un nombre fini de reseaux indecomposable, Acta Math. (1967)

[9] Knörrer H., Torsionfreie Moduln bei Deformation von Kurvensingularitäten, In: Greuel G.-M., Trautmann G. (ed.) Singularties, Representations of Algebras and Vector Bundles, Lambrecht 1985. Lecture Notes in Math., Vol. 1273, Springer, Berlin-Heidelberg-New York (1987) 150–155.

[10] Schappert A., A characterization of strict unmodal plane cure singularities, In: Greuel G.-M., Trautmann G. (ed.) Singularties, Representations of Algebras and Vector Bundles, Lambrecht 1985. Lecture Notes in Math., Vol. 1273, Springer, Berlin-Heidelberg-New York (1987) 168–177.

[11] Schappert A., Kurvensingularitäten und Isomorphieklassen von Moduln, Dissertation, Universität Kaiserslautern, 1990.

[12] Wall C. T. C., Classification of unimodal isolated singularities of complete intersections, In: Orlik P. (ed.) Singuarities, Arcata 1981. Proc. Sympos. Pure Math. 40(2) (1983) 625–640.