REDUCTION ALGORITHM AND REPRESENTATIONS OF BOXES AND ALGEBRAS

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Abstract. This paper is a survey of applications of the reduction algorithm for boxes to the representation theory of finite dimensional algebras. The topic seems important in two respects. First of all, the main advantage of the notion of box is just the possibility to study representations inductively, reducing the corresponding matrices step by step. Second, there are several principal facts in the representation theory that cannot be proved (at least have never been proved till now) without using representations of boxes and the reduction algorithm. I have chosen for the presentation here three main results. They are:

- tame–wild dichotomy [12, 6];
- relation between tameness and generic modules [7];
- coverings of tame boxes and algebras [14].

Since there is a certain prejudice to the notion of box and especially to the reduction algorithm, I have decided to give some technical details of the main constructions and to sketch proofs. I hope that they are not so complicated and understandable well enough, and the astonishing resemblance of these proofs is itself a good publicity for the techniques of boxes.

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1. Categories and functors

In this article we consider *linear categories* (in particular, algebras, which we identify with the categories with one object) over a fixed field $\mathbf{k}$.[1] It means that the sets of morphisms $\mathsf{A}(A,B)$ between two objects of such a category A are vector spaces over $\mathbf{k}$ and the multiplication of morphisms is $\mathbf{k}$-bilinear. All functors (bifunctors) between such categories are also supposed $\mathbf{k}$-linear (bilinear). We denote by Vec (vec) the category of vector spaces (respectively finite dimensional vector spaces) over $\mathbf{k}$. A *module* over a category A is, by definition, a (linear) functor $M : \mathsf{A} \to \mathsf{Vec}$; an A-B-*bimodule* is, by definition, a (bilinear) functor $\mathsf{A}^\circ \times \mathsf{B} \to \mathsf{Vec}$, where A° denotes the *opposite* (or *dual*) category to A. We often say A-*bimodule* instead of A-A-bimodule. In particular, any A-module can be considered as an A-$\mathbf{k}$-bimodule. We write dim, Hom, $\otimes$, etc. instead of $\dim_{\mathbf{k}}$, $\mathrm{Hom}_{\mathbf{k}}$, $\otimes_{\mathbf{k}}$, etc., and denote by DV the *dual vector space* $\mathrm{Hom}(V,\mathbf{k})$. If V is an A-B-bimodule, we write bva instead of $V(a,b)v$ for $v \in V(A,B)$, $a \in \mathsf{A}(A',A)$, $b \in \mathsf{B}(B,B')$ (it is an element from $V(A',B')$). Every category A (rather its set of morphisms) can be considered as an A-bimodule, which we call the *regular* A-*bimodule.*

An additive category A is said to be *fully additive* if every idempotent in it splits, i.e. corresponds to a decomposition of the object into a direct sum (equivalently, every idempotent has a kernel). For any category A, there is a unique (up to equivalence) fully additive category $\mathsf{add}\,\mathsf{A}$ containing A. It can be defined either as the category of matrix idempotents over A or as the category of *finitely generated projective* A°-modules. Every functor $F : \mathsf{A} \to \mathsf{B}$ prolongs uniquely (up to isomorphism) to a functor $\mathsf{add}\,\mathsf{A} \to \mathsf{add}\,\mathsf{B}$, which we denote by the same letter F. In particular, the categories of A-modules and $\mathsf{add}\,\mathsf{A}$-modules are equivalent.

Just as for usual bimodules over rings, one can define operations such as Hom or $\otimes$. Formally, if M is an A-B-bimodule and N is an C-A-bimodule, we define their tensor product $M \otimes_{\mathsf{A}} N$ as the C-B-bimodule such that $(M \otimes_{\mathsf{A}} N)(C,B)$ is the factor space of the direct sum $\bigoplus_{A \in \mathrm{Ob}\,\mathsf{A}} M(A,B) \otimes N(C,A)$ modulo the subspace generated by all differences $ua \otimes v - u \otimes av$, where $u \in M(A,B)$, $v \in N(C,A')$ and $a \in \mathsf{A}(A',A)$ for some objects $A, A' \in \mathrm{Ob}\,\mathsf{A}$. On the other hand, for an A-B-bimodule M and an A-C-bimodule N, the B-C-bimodule $\mathrm{Hom}_{\mathsf{A}}(M,N)$ has the values $(\mathrm{Hom}_{\mathsf{A}}(M,N))(B,C) = \mathrm{Hom}_{\mathsf{A}}(M(\_\,,B), N(\_\,,C))$, the right side being the space of morphisms

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[1] Further we shall mostly suppose that $\mathbf{k}$ is algebraically closed, but it is not essential for the first definitions.

of functors $\mathsf{A}\to\mathsf{Vec}$. One can easily check that the usual identities (cf. [5, Chapter IX, § 2]) for $\otimes$ and Hom hold, especially:

$$\begin{gathered}L\otimes_{\mathsf B}(M\otimes_{\mathsf A}N)\simeq(L\otimes_{\mathsf B}M)\otimes_{\mathsf A}N,\ \text{ where } {}_{\mathsf D}L_{\mathsf B},\ {}_{\mathsf B}M_{\mathsf A},\ {}_{\mathsf A}N_{\mathsf C}\,;\\ \mathrm{Hom}_{\mathsf B}(M\otimes_{\mathsf A}N,L)\simeq(\mathrm{Hom}_{\mathsf A}(M,\mathrm{Hom}_{\mathsf B}(N,L)),\ \text{ where } {}_{\mathsf D}L_{\mathsf B},{}_{\mathsf C}M_{\mathsf A},{}_{\mathsf A}N_{\mathsf B}\end{gathered}$$

(both are isomorphisms of C-D-bimodules). We shall freely use these isomorphisms as well as the analogous ones established for bimodules over rings in [5, 19].

If $F:\mathsf{A}\to\mathsf{B}$ is a functor and V is a B-C-bimodule (or a C-B-bimodule), one can define the A-C-bimodule V^F such that $V^F(A,C)=V(FA,C)$ (respectively the C-A-bimodule ${}^FV$ such that ${}^FV(C,A)=V(C,FA)$). We often omit the superscript ^{F} if the sense of the notation is quite clear. Especially one can consider the A-B-bimodule B^F, or the B-A-bimodule ${}^F\mathsf{B}$, or the A-bimodule ${}^F\mathsf{B}^F$. Certainly, if $M:\mathsf{B}\to\mathsf{Vec}$ is a B-module, the A-module ${}^FM$ is just the composition MF. It is easy to see that $V^F\simeq\mathrm{Hom}_{\mathsf B}({}^F\mathsf{B},V)\simeq V\otimes_{\mathsf B}\mathsf{B}^F$ for every B-C-bimodule V (respectively ${}^FV\simeq\mathrm{Hom}_{\mathsf B}(\mathsf{B}^F,V)\simeq{}^F\mathsf{B}\otimes_{\mathsf B}V$ for every C-B-bimodule V). Therefore, in particular,

$$\begin{aligned}\mathrm{Hom}_{\mathsf{A}\text{-}\mathsf{C}}(W_1,V^F)&\simeq\mathrm{Hom}_{\mathsf{B}\text{-}\mathsf{C}}(W_1\otimes_{\mathsf A}{}^F\mathsf{B},V),\\ \mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{A}}(W_2,{}^FV)&\simeq\mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{B}}(\mathsf{B}^F\otimes_{\mathsf A}W_2,V),\\ \mathrm{Hom}_{\mathsf{A}\text{-}\mathsf{A}}(W,{}^FV^F)&\simeq\mathrm{Hom}_{\mathsf{B}\text{-}\mathsf{B}}(\mathsf{B}^F\otimes_{\mathsf A}W\otimes_{\mathsf A}{}^F\mathsf{B},V),\end{aligned}$$

where W_1 (respectively W_2 and W) is a A-C-bimodule (respectively C-A-bimodule and A-bimodule).

Let Γ be an oriented graph (or a *quiver*), perhaps with multiple arrows and loops. Remind that the *(linear) category* $\mathbf{k}\Gamma$ *freely generated* by Γ is defined as follows:

- The objects of $\mathbf{k}\Gamma$ are the vertices of the graph Γ.
- The vector space of morphisms from a vertex A to another vertex B has a basis consisting of all *paths* starting from A and ending at B, that is words $p=a_n\dots a_2a_1$, where a_i are arrows of the graph Γ, the source of a_{i+1} coincide with the target of a_i for $i=1,\dots,n-1$, the source of a_1 is A and the target of a_n is B. We write $p:A\to B$. If $A=B$, we allow an "empty" path ι_A (i.e. with $n=0$) starting and ending at A.
- The product of two paths $p=a_n\dots a_2a_1:A\to B$ and $q=b_m\dots b_2b_1:C\to A$ is defined as their concatenation $pq=a_n\dots a_1b_m\dots b_1$. Certainly, if $q=\iota_A$ (or $p=\iota_A$), one has

$pq = p$ (respectively $pq = q$). The products of any morphisms are defined by linearity.

A category A is called *free* if it is isomorphic (not simply equivalent!) to a category of the form $\mathbf{k}\Gamma$ for some graph Γ. The images of the arrows of Γ under an isomorphism $\mathbf{k}\Gamma \to \mathsf{A}$ are called *a set of free generators* of the category A. Just as for free algebras, one can check that $\mathbf{k}\Gamma \simeq \mathbf{k}\Gamma'$ if and only if $\Gamma \simeq \Gamma'$, hence, there is a one-to-one correspondence between isomorphism classes of graphs and of free categories. On the other hand, there can be plenty of sets of free generators in the same free category (it is always the case if there are oriented cycles in Γ or there is an arrow $a : A \to B$ and a path $p : A \to B$ such that $p \neq a$). We denote by $\mathsf{add}\,\Gamma$ the fully additive category $\mathsf{add}\,\mathbf{k}\Gamma$.

Especially, if the graph Γ is *trivial*, i.e. has no arrows, the category $\mathbf{k}\Gamma$ is the *trivial category* whose set of objects equals the set of vertices of the graph Γ. It means that there are no morphisms between different objects and the endomorphism ring of every object coincides with $\mathbf{k}$.

We shall also use *semi-free* categories defined as follows. Let Γ be an oriented graph, $\mathfrak{S}$ be the set of loops from Γ and $g : a \mapsto g_a$ be a mapping $\mathfrak{S} \to \mathbf{k}[t]$ such that neither of g_a is zero. The category $\mathsf{A} = \mathbf{k}\Gamma[g_a(a)^{-1} \,|\, a \in \mathfrak{S}]$ is called a *semi-free category*, the arrows of Γ are called a set of *semi-free generators* of A. Evidently, we can (and shall always) suppose that all polynomial g_a are *unital* (with the leading coefficient 1). The polynomial g_a is called the *marking polynomial* of the loop a. The set of arrows of Γ is called a set of *semi-free generators* of A. If $g_a \neq 1$ the loop a is called a *marked loop* of A. Especially, if Γ only contain loops and there is at most one loop $a : A \to A$ for every object A, the corresponding semifree category is called a *minimal category.*

A *free module* over a category A is, by definition, a module isomorphic to a direct sum of *representable* (or *principal*) modules, i.e. those of the form $\mathsf{A}(A, \_)$. If $M \simeq \bigoplus_i \mathsf{A}(A_i, \_)$, the images in M of the identity morphisms 1_{A_i} are called a *set of free generators* of M. Just in the same way, a *free* A-B-*bimodule* is a bimodule V isomorphic to a direct sum $\bigoplus_i \mathsf{B}(B_i, \_) \otimes \mathsf{A}(\_, A_i)$ and the images in V of the elements $1_{B_i} \otimes 1_{A_i}$ are called a *set of free generators* of V.

A category A is said to be *skeletal* if:

- there are no nontrivial idempotents in $\mathsf{A}(A, A)$ for each $A \in \operatorname{Ob} \mathsf{A}$;
- each object from $\mathsf{add}\,\mathsf{A}$ decomposes into a direct sum of objects from A;

- if $\bigoplus_{i=1}^{n} A_i \simeq \bigoplus_{j=1}^{m} B_j$ in $\mathsf{add}\,\mathsf{A}$, where $A_i, B_j \in \operatorname{Ob}\mathsf{A}$, then $n = m$ and there is a permutation σ such that $A_i \simeq B_{\sigma i}$ for all $i = 1, \dots, n$.

For instance, if the category A is *local*, i.e. all algebras $\mathsf{A}(A, A)$ are local, and has no isomorphic objects, it is skeletal (cf. [1, Theorem 3.6]). Each semi-free category is skeletal too (in a bit different setting it is proved in [20]).

2. Boxes and their representations

A *coalgebra over a category* A is defined as an A-bimodule V together with homomorphisms of A-bimodules $\Delta : \mathsf{V} \to \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V}$ (*comultiplication*) and $\varepsilon : \mathsf{V} \to \mathsf{A}$ (*counit*) such that the following diagrams are commutative:

$$\begin{tikzcd}
\mathsf{V} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} \arrow[d, "\Delta\otimes 1"] \\
\mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} \arrow[r, "1\otimes\Delta"'] & \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{V} \arrow[r, "\sim"] \arrow[d, "\Delta"'] & \mathsf{A} \otimes_{\mathsf{A}} \mathsf{V} \arrow[d, "1"] & \mathsf{V} \arrow[r, "\sim"] \arrow[d, "\Delta"'] & \mathsf{V} \otimes_{\mathsf{A}} \mathsf{A} \arrow[d, "1"] \\
\mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} \arrow[r, "\varepsilon\otimes 1"'] & \mathsf{A} \otimes_{\mathsf{A}} \mathsf{V} & \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} \arrow[r, "1\otimes\varepsilon"'] & \mathsf{V} \otimes_{\mathsf{A}} \mathsf{A}
\end{tikzcd}$$

(the first rows of the last two diagrams are the natural isomorphisms).

A *box* is defined as a pair $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$, where A is a category and V is an A-coalgebra. The kernel $\overline{\mathsf{V}} = \operatorname{Ker}\varepsilon$ of the counit is called the *kernel* of the box $\mathfrak{A}$. If $v \in \mathsf{V}(A, B)$, we often write $v : A \dashrightarrow B$. If C is a category, we define the *category of representations* $\mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ of the box $\mathfrak{A}$ in the category C in the following way:

- The *objects* of this category are functors $M : \mathsf{A} \to \mathsf{C}$.
- A *morphism* from M to N in $\mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ is a homomorphism of A-bimodules $f : \mathsf{V} \to \operatorname{Hom}_{\mathsf{C}}({}^{M}\mathsf{C}, {}^{N}\mathsf{C})$. We denote the set of all morphisms from M to N by $\operatorname{Hom}_{\mathsf{C}\text{-}\mathfrak{A}}(M, N)$.
- The *product* of two morphisms, $f \in \operatorname{Hom}_{\mathsf{C}\text{-}\mathfrak{A}}(M, N)$ and $g \in \operatorname{Hom}_{\mathsf{C}\text{-}\mathfrak{A}}(L, M)$, is defined as the composition

$$\mathsf{V} \xrightarrow{\Delta} \mathsf{V} \otimes_{\mathsf{A}} \mathsf{V} \xrightarrow{f\otimes g} \operatorname{Hom}_{\mathsf{C}}({}^{M}\mathsf{C}, {}^{N}\mathsf{C}) \otimes_{\mathsf{A}} \operatorname{Hom}_{\mathsf{C}}({}^{L}\mathsf{C}, {}^{M}\mathsf{C}) \xrightarrow{mult} \operatorname{Hom}_{\mathsf{C}}({}^{L}\mathsf{C}, {}^{N}\mathsf{C}),$$

where *mult* denotes the multiplication of morphisms of functors.

- The *identity morphism* of a representation M is defined as the composition $\mathsf{V} \xrightarrow{\varepsilon} \mathsf{A} \to \mathrm{Hom}_{\mathsf{C}}({}^M\mathsf{C}, {}^M\mathsf{C})$, where the second homomorphism maps $a : A \to B$ to $\mathsf{C}(\_, M(a)) : \mathsf{C}(\_, MA) \to \mathsf{C}(\_, NB)$.

One can easily check that in this way we obtain indeed a category. If $\mathsf{C} = \mathsf{Proj}\text{-}\mathbf{R}$, the category of right projective modules over an algebra $\mathbf{R}$, we write $\mathsf{Rep}(\mathfrak{A}, \mathbf{R})$ instead of $\mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ and $\mathrm{Hom}_{\mathbf{R}\text{-}\mathfrak{A}}(M, N)$ instead of $\mathrm{Hom}_{\mathsf{C}\text{-}\mathfrak{A}}(M, N)$. If $\mathbf{R} = \mathbf{k}$, we omit it at all and write $\mathsf{Rep}(\mathfrak{A})$ and $\mathrm{Hom}_{\mathfrak{A}}(M, N)$.

Sometimes it is convenient to identify $\mathrm{Hom}_{\mathsf{A}\text{-}\mathsf{A}}(\mathsf{V}, \mathrm{Hom}_{\mathsf{C}}({}^M\mathsf{C}, {}^N\mathsf{C}))$ with $\mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{A}}(\mathsf{V} \otimes_{\mathsf{A}} {}^M\mathsf{C}, {}^N\mathsf{C})$ and we shall do it freely.

A *principal box* is one of the form $\mathfrak{A} = (\mathsf{A}, \mathsf{A})$, where the coalgebra is the regular bimodule with identity comultiplication and counit. It is easy to see that the category of representations of this box coincide with that of the category A; in particular, $\mathsf{Rep}(\mathfrak{A}) = \mathsf{A}\text{-}\mathsf{Mod}$. We always identify a principal box with the corresponding category; it allows to consider (formally) the representation theory of algebras as a partial case of that of boxes.

A *morphism of boxes* $\Phi : \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ and $\mathfrak{B} = (\mathsf{B}, \mathsf{W})$, is a pair (Φ_0, Φ_1), where $\Phi_0 : \mathsf{A} \to \mathsf{B}$ is a functor and $\Phi_1 : \mathsf{V} \to {}^{\Phi_0}\mathsf{W}^{\Phi_0}$ is a morphism of A-bimodules compatible in the evident sense with comultiplication and counit. We usually omit indices and write $\Phi(a)$ both for $a \in \mathsf{A}$ and for $a \in \mathsf{V}$. Such a morphism induces the *inverse image* functor $\Phi^* : \mathsf{Rep}(\mathfrak{B}, \mathsf{C}) \to \mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ for each category C: it maps a representation M to the composition $M\Phi_0$ and a morphism $f \in \mathrm{Hom}_{\mathsf{C}\text{-}\mathfrak{B}}(M, N)$, i.e. a homomorphism of bimodules $\mathsf{W} \to \mathrm{Hom}_{\mathsf{C}}({}^M\mathsf{C}, {}^N\mathsf{C})$, to the morphism $M\Phi_0 \to N\Phi_0$, i.e. the homomorphism $\Phi^* f : \mathsf{V} \to \mathrm{Hom}_{\mathsf{C}}({}^{M\Phi_0}\mathsf{C}, {}^{N\Phi_0}\mathsf{C})$, such that $\Phi^* f(v) = f(\Phi_1 v)$.

Suppose given a box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ and a functor $F : \mathsf{A} \to \mathsf{B}$. We define the new box $\mathfrak{A}^F = (\mathsf{B}, \mathsf{W})$ in the following way:

- $\mathsf{W} = \mathsf{B}^F \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} {}^F\mathsf{B}$.
- The comultiplication $\mathsf{W} \to \mathsf{W} \otimes_{\mathsf{B}} \mathsf{W} \simeq \mathsf{B}^F \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} {}^F\mathsf{B}^F \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} {}^F\mathsf{B}$ maps $a \otimes v \otimes b$ to $\sum_i a \otimes v_i^{(1)} \otimes 1 \otimes v_i^{(2)} \otimes b$, where $\Delta(v) = \sum_i v_i^{(1)} \otimes v_i^{(2)}$.
- The counit $\mathsf{W} \to \mathsf{B}$ maps $a \otimes v \otimes b$ to $aF(\varepsilon(v))b$.

The functor F can be prolonged to the morphism $\mathfrak{A} \to \mathfrak{A}^F$, which we denote by F too, setting, for $v \in \mathsf{V}(A, B)$, $F(v) = F(1_B) \otimes v \otimes F(1_A) \in \mathsf{W}(FA, FB)$.

Theorem 2.1. *Let* $\mathfrak{A}=(\mathsf{A},\mathsf{V})$ *be a box,* $F:\mathsf{A}\to\mathsf{B}$ *be a functor and* $\mathfrak{A}^F$ *be the above defined box. The inverse image functor* F^* *corresponding to the morphism of boxes* $F:\mathfrak{A}\to\mathfrak{A}^F$ *induces an equivalence of the category* $\mathsf{Rep}(\mathfrak{A}^F,\mathsf{C})$ *onto the full subcategory* $\mathsf{Rep}(\mathfrak{A},\mathsf{C}\,|\,F)\subseteq\mathsf{Rep}(\mathfrak{A},\mathsf{C})$ *consisting of all representations that are isomorphic to the composition* MF *for some functor* $M:\mathsf{B}\to\mathsf{C}$. *In particular, if every representation is isomorphic to such a composition, the functor* F^* *establishes an equivalence between* $\mathsf{Rep}(\mathfrak{A},\mathsf{C})$ *and* $\mathsf{Rep}(\mathfrak{A}^F,\mathsf{C})$.

Proof. It follows immediately from the isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathsf{A}\text{-}\mathsf{A}}(\mathsf{V},\mathrm{Hom}_{\mathsf{C}}({}^{MF}\mathsf{C},{}^{NF}\mathsf{C}))&\simeq\mathrm{Hom}_{\mathsf{A}\text{-}\mathsf{A}}(\mathsf{V},{}^{F}\,\mathrm{Hom}_{\mathsf{C}}({}^{M}\mathsf{C},{}^{N}\mathsf{C})^{F})\\&\simeq\mathrm{Hom}_{\mathsf{B}\text{-}\mathsf{B}}(\mathsf{B}^{F}\otimes_{\mathsf{A}}\mathsf{V}\otimes_{\mathsf{A}}{}^{F}\mathsf{B},\mathrm{Hom}_{\mathsf{C}}({}^{M}\mathsf{C},{}^{N}\mathsf{C})).\end{aligned}$$

□

We shall often use the following corollary of this theorem.

Corollary 2.2. *Let* $\mathfrak{A}=(\mathsf{A},\mathsf{V})$ *be a box,* A' *be a subcategory of* A *and* $F':\mathsf{A}'\to\mathsf{B}'$ *be a functor. Denote by* $\mathsf{B}=\mathsf{A}\coprod^{\mathsf{A}'}\mathsf{B}'$ *the amalgamation (or pullback) of* A *and* B' *under* A' *and by* $F:\mathsf{A}\to\mathsf{B}$ *the natural functor. Then* F^* *induces an equivalence between* $\mathsf{Rep}(\mathfrak{A}^F,\mathsf{C})$ *and the full subcategory* $\mathsf{Rep}(\mathfrak{A},\mathsf{C}\,|\,\mathsf{A}',F')\subseteq\mathsf{Rep}(\mathfrak{A},\mathsf{C})$ *consisting of all representations* M *such that the restriction of* M *onto* A' *can be factored through* F'.

For every box $\mathfrak{A}=(\mathsf{A},\mathsf{V})$ we denote by $\mathsf{add}\,\mathfrak{A}$ the box $(\mathsf{add}\,\mathsf{A},\mathsf{V})$ (we denote by the same letter V the prolongation of V onto $\mathsf{add}\,\mathsf{A}$). For every fully additive category C (e.g. for Vec) there is an equivalence $\mathsf{Rep}(\mathfrak{A},\mathsf{C})\simeq\mathsf{Rep}(\mathsf{add}\,\mathfrak{A},\mathsf{C})$ and we shall identify these categories.

A box $\mathfrak{A}=(\mathsf{A},\mathsf{V})$ is called *skeletal* if so is the category A. Then a representation $M\in\mathsf{Rep}(\mathfrak{A},\mathbf{R})$, where $\mathbf{R}$ is an algebra, is said to be *finite* (or *of finite rank*) if:

- For any object $A\in\mathrm{Ob}\,\mathsf{A}$, $MA\in\mathsf{proj}\text{-}\mathbf{R}$, the category of finitely generated projective (right) $\mathbf{R}$-modules.
- The *support* of M, i.e. the set $\mathsf{supp}\,M=\{\,A\in\mathrm{Ob}\,\mathsf{A}\,|\,MA\neq 0\,\}$, is finite.

If $\mathbf{R}=\mathbf{k}$ (hence, $\mathsf{proj}\text{-}\mathbf{R}=\mathsf{vec}$), they also call finite representations *finite dimensional.* The category of all finite representations of $\mathfrak{A}$ over $\mathbf{R}$ is denoted by $\mathsf{rep}(\mathfrak{A},\mathbf{R})$ ($\mathsf{rep}(\mathfrak{A})$ if $\mathbf{R}=\mathbf{k}$). Let $|\mathsf{proj}\text{-}\mathbf{R}|$ be the set of isomorphism classes of finitely generated projective $\mathbf{R}$-modules. The function $\mathbf{dim}\,M:\mathrm{Ob}\,\mathsf{A}\to|\mathsf{proj}\text{-}\mathbf{R}|$ mapping A to the isomorphism class of MA is called the *vector dimension* of M. If all projective $\mathbf{R}$-modules are free of unique rank (e.g. if $\mathbf{R}=\mathbf{k}$), we identify $|\mathsf{proj}\text{-}\mathbf{R}|$ with $\mathbb{N}$, the set of nonnegative integers. If, moreover, the set $\operatorname{Ob}\mathsf{A}$ is finite, we consider $\mathbf{dim}\,M$ just as a vector with entries from $\mathbb{N}$. We denote by $\mathsf{ind}(\mathfrak{A},\mathbf{R})$ the set of isomorphism classes of indecomposable finite representations of $\mathfrak{A}$ over $\mathbf{R}$ and by $\mathsf{ind}_{\mathbf{d}}(\mathfrak{A},\mathbf{R})$ the subset of $\mathsf{ind}(\mathfrak{A},\mathbf{R})$ consisting of the classes of representations of vector dimension $\mathbf{d}$. Note that there are boxes such that $\mathsf{ind}_{\mathbf{d}}(\mathfrak{A})\cap\mathsf{ind}_{\mathbf{d}'}(\mathfrak{A})\neq\emptyset$ for some vector dimensions $\mathbf{d}\neq\mathbf{d}'$.

If $\mathbf{d}$, $\mathbf{c}$ are two vector dimensions, we write $\mathbf{d}\le\mathbf{c}$ if $\mathbf{d}(A)\le\mathbf{c}(A)$ for all objects A.

3. TYPES OF BOXES

In the representation theory (especially over algebraically closed fields), as well as in most other applications, they mainly use the so-called *normal free boxes* in the following sense.

A *section* of a box $\mathfrak{A}=(\mathsf{A},\mathsf{V})$ is, by definition, a set of elements $\omega=\{\,\omega_A\in\mathsf{V}(A,A)\mid A\in\operatorname{Ob}\mathsf{A}\,\}$ such that $\varepsilon(\omega_A)=1_A$. This section is said to be *normal* if $\Delta\omega_A=\omega_A\otimes\omega_A$ for all objects A. A box is called *normal* if it has a normal section. Evidently, the element $\partial a=\omega_B a-a\omega_A$ belongs to the kernel $\overline{\mathsf{V}}$ of the box $\mathfrak{A}$. Moreover, if $v\in\overline{\mathsf{V}}(A,B)$, the element $\partial v=\mu(v)-v\otimes\omega_A-\omega_B\otimes v$ belongs to $\overline{\mathsf{V}}\otimes_{\mathsf{A}}\overline{\mathsf{V}}$. We call ∂ the *differential* of the (normal) box $\mathfrak{A}$. Note that it depends on the section. We prolong ∂ to the tensor square $\overline{\mathsf{V}}^{\otimes 2}=\overline{\mathsf{V}}\otimes_{\mathsf{A}}\overline{\mathsf{V}}$ setting $\partial(u\otimes v)=\partial u\otimes v-u\otimes\partial v\in\overline{\mathsf{V}}^{\otimes 3}$. We often omit the sign $\otimes$ and set $\overline{a}=0$ for $a\in\operatorname{Mor}\mathsf{A}$, $\overline{v}=1$ for $v\in\mathsf{V}$. Then the mapping ∂ has the following properties:

- $\partial(xy)=(\partial x)y+(-1)^{\overline{x}}x(\partial y)$ (Leibniz rule);
- $\partial^2=0$.

Note that if φ is an isomorphism of representations of a normal box, then $\varphi(\omega_A)$ is an isomorphism for each object A. Especially, the *vector dimensions* of isomorphic finite representations coincide.

A normal box $\mathfrak{A}=(\mathsf{A},\mathsf{V})$ is called *free* (*semi-free*) if A is a free (semi-free) category, $\partial a=0$ for each marked loop of A and the kernel $\overline{\mathsf{V}}$ is a free A-bimodule. If Σ_0 is a set of free (semi-free) generators of the category A and Σ_1 is a set of free generators of the A-bimodule $\overline{\mathsf{V}}$, their union $\Sigma=\Sigma_0\cup\Sigma_1$ is called a set of free (semi-free) generators of the box $\mathfrak{A}$. The elements of Σ_0 are usually called *solid arrows* and those of Σ_1 *dotted arrows* of the box $\mathfrak{A}$. Thus, to a semi-free box we associate a *bigraph*, i.e. a graph whose arrows are of two types: solid and dotted. The *marked loops* and the *marking polynomials* of a semi-free box $\mathfrak{A}$ are just those of the semi-free category A. The morphisms from $\mathsf{A}(A,B)$, or the elements from $\overline{\mathsf{V}}(A,B)$, or from $\overline{\mathsf{V}}^{\otimes 2}$ can be considered as linear combinations of paths of the arrows of the corresponding bigraph and the inverse morphisms $a^* = g_a(a)^{-1}$ for the marked loops a such that all arrows of the paths are solid, respectively, each path contains exactly one, or exactly two dotted arrows.

A semi-free box $\mathfrak{A} = (\mathsf{A},\mathsf{V})$ is called *so-trivial* if the category A is trivial. If A is a minimal category, we call the box $\mathfrak{A}$ *so-minimal.*

We fix a section $\omega : \mathsf{V} \to \mathsf{A}$ and consider the differential ∂ with respect to this section. A semi-free box $\mathfrak{A}$ is called *triangular* if there is a set of semi-free generators Σ and a function $\nu : \Sigma \to \mathbb{N}$ such that, for every arrow $a \in \Sigma$, its differential ∂a is a linear combination of paths only containing the arrows b with $\nu(b) < \nu(a)$. Such a set of generators is also called *triangular.* Triangular semi-free boxes have a lot of good features that are not valid in general. Especially, the following important results hold.

Proposition 3.1 (cf. [18]).[2] *Let $\mathfrak{A} = (\mathsf{A},\mathsf{V})$ be a triangular semi-free box with normal section ω and a triangular set of semi-free generators Σ.*

1. *A morphism $f : M \to N$ of representations from $\mathsf{Rep}(\mathfrak{A},\mathsf{C})$ is an isomorphism if and only if $f(\omega_A)$ is an isomorphism for every object A.*
2. *If the category C is fully additive, so is $\mathsf{Rep}(\mathfrak{A},\mathsf{C})$.*
3. *Suppose that $M \in \mathsf{Rep}(\mathfrak{A},\mathsf{C})$, $\{\, N_A \,|\, A \in \operatorname{Ob}\mathsf{A} \,\}$ is a set of objects from C, for each in $\operatorname{Ob}\mathsf{A}$ an isomorphism $\gamma_A : MA \to N_A$ and for each dotted arrow $v : A \dashrightarrow B$ from Σ_1 a morphism $\gamma_v : MA \to N_B$ are given. There is a representation $N \in \mathsf{Rep}(\mathsf{A},\mathsf{C})$ and an isomorphism $\gamma : M \to N$ such that $NA = N_A$, $\gamma(\omega_A) = \gamma_A$ for each object A and $\gamma(v) = \gamma_v$ for each dotted arrow v.*

Usually we impose additional conditions on the considered boxes. Namely, we say that a box $\mathfrak{A} = (\mathsf{A},\mathsf{V})$ is *locally finitely generated* if for every object $A \in \operatorname{Ob}\mathsf{A}$ the A-modules $\mathsf{A}(A,\_), \mathsf{V}(A,\_)$ as well as A°-modules $\mathsf{A}(\_,A), \mathsf{V}(\_,A)$ are finitely generated. If the box $\mathfrak{A}$ is semi-free, it means that in the corresponding bigraph there are finitely many arrows ending or starting at each vertex. Denote by s^0_{AB} and s^1_{AB} respectively the number of solid and dotted arrows starting at A

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[2] In [18] these properties are proved for *free differential graded categories*, but it is well known that this setting is quite equivalent to that of free boxes. Moreover, the proofs for semi-free boxes are the same as for free ones

and ending at B and set, for every function $\mathbf{d} : \operatorname{Ob} \mathsf{A} \to \mathbb{R}$ with finite support $\operatorname{\mathsf{supp}} \mathbf{d} = \{\, A \,|\, \mathbf{d}(A) \neq 0 \,\}$,

$$\begin{aligned} Q^{+}_{\mathfrak{A}}(\mathbf{d}) &= \sum_{A \in \operatorname{Ob} \mathsf{A}} \mathbf{d}(A)^2 + \sum_{A,B \in \operatorname{Ob} \mathsf{A}} s^{1}_{AB}\mathbf{d}(A)\mathbf{d}(B), \\ Q^{-}_{\mathfrak{A}}(\mathbf{d}) &= \sum_{A,B \in \operatorname{Ob} \mathsf{A}} s^{0}_{AB}\mathbf{d}(A)\mathbf{d}(B), \\ Q_{\mathfrak{A}}(\mathbf{d}) &= Q^{+}_{\mathfrak{A}}(\mathbf{d}) - Q^{-}_{\mathfrak{A}}(\mathbf{d}). \end{aligned}$$

They call $Q_{\mathfrak{A}}$ the *Tits form* of the box $\mathfrak{A}$.

Corollary 3.2. *Let $\mathfrak{A}$ be a locally finitely generated semi-free box. The set of representation $M \in \mathsf{rep}(\mathfrak{A})$ of vector dimension $\mathbf{d}$ can be identified with the points of an affine variety $\mathsf{rep}_{\mathbf{d}}(\mathfrak{A})$ of dimension $Q^{-}_{\mathfrak{A}}(\mathbf{d})$ over the field $\mathbf{k}$ (actually, with a principal open subset in the affine space $\mathbb{A}^{Q^{-}_{\mathfrak{A}}(\mathbf{d})}_{\mathbf{k}}$). The isomorphism classes of representations are connected locally closed subsets in $\mathsf{rep}_{\mathbf{d}}(\mathfrak{A})$ of dimensions $d \leq Q^{+}_{\mathfrak{A}}(\mathbf{d})$.*

Actually, in most applications they only deal with free boxes. Nevertheless, semi-free ones seem unavoidable in the reduction algorithm described in Section 6, especially when we study *tame boxes.*

4. Boxes, bimodules and algebras

Let A be a category, U be an A-bimodule. Define the new category $\mathsf{El}(\mathsf{U})$ of *elements of the bimodule* U (or *matrices* over U) as follows:

- $\operatorname{Ob} \mathsf{El}(\mathsf{U}) = \bigcup_{A \in \operatorname{Ob} \mathsf{add}\,\mathsf{A}} \mathsf{U}(A, A)$.
- A morphism from $u \in \mathsf{U}(A, A)$ to $u' \in \mathsf{U}(A', A')$ is a morphism $a \in \mathsf{A}(A, A')$ such that $au = u'a$ (both elements are from $\mathsf{U}(A, A')$).

This category is fully additive. The *zero elements* $0 \in \mathsf{U}(A, A)$ form a fully additive subcategory $\mathsf{El}_0(\mathsf{U})$. If the category A is skeletal, each zero element decomposes uniquely into a direct sum of indecomposable zero elements, which belong to $\mathsf{U}(A, A)$, where $A \in \operatorname{Ob} \mathsf{A}$.

We call a category A *locally finite dimensional* if all morphism spaces $\mathsf{A}(A, B)$ are finite dimensional and for every object A the set $\{\, B \,|\, \mathsf{A}(A, B) \neq 0 \text{ or } \mathsf{A}(B, A) \neq 0 \,\}$ is finite. A skeletal locally finite dimensional category is called *basic.* For every locally finite dimensional category A there is a unique basic category A_0 such that $\mathsf{add}\,\mathsf{A} \simeq \mathsf{add}\,\mathsf{A}_0$. Therefore, in the representation theory of locally finite dimensional categories we may restrict ourselves by basic ones. A bimodule U over a category A is called *locally finite dimensional* if all spaces $\mathsf{U}(A,B)$ are finite dimensional and for every object A the set $\{\, B \mid \mathsf{U}(A,B) \neq 0 \text{ or } \mathsf{U}(B,A) \neq 0 \,\}$ is finite.

Theorem 4.1. *Suppose that the field* $\mathbf{k}$ *is algebraically closed. Let* U *be a locally finite dimensional bimodule over a locally finite dimensional category* C. *There is a free triangular locally finitely generated box* $\mathfrak{A}$ *such that* $\mathsf{El}(\mathsf{U}) \simeq \mathsf{rep}(\mathfrak{A})$.

Proof. Without loss of generality we suppose C basic. Then the set $\mathsf{R} = \operatorname{rad} \mathsf{C}$ of all noninvertible morphisms is an ideal of C called its *radical*. Moreover, it is easy to check that $\bigcap_{n=1}^{\infty} \mathsf{R}^n = 0$. For every nonzero morphism $c \in \mathsf{C}$ set $\nu(c) = \max\{\, n \mid c \in \mathsf{R}^n \,\}$. Define subbimodules U_n setting $\mathsf{U}_0 = \mathsf{U}$, $\mathsf{U}_{n+1} = \mathsf{R}\mathsf{U}_n + \mathsf{U}_n\mathsf{R}$, and set, for every nonzero $u \in \mathsf{U}$, $\nu(u) = \max\{\, n \mid u \in \mathsf{U}_n \,\}$ (again $\bigcap_{n=1}^{\infty} \mathsf{U}_n = 0$). For each two objects A, B and each $n \geq 0$ choose a basis $\mathfrak{E}^0_n(A,B)$ of $\mathsf{R}^n(A,B)$ modulo $\mathsf{R}^{n+1}(A,B)$ and a basis $\mathfrak{E}^1_n(A,B)$ of $\mathsf{U}_n(A,B)$ modulo $\mathsf{U}_{n+1}(A,B)$. Then $\mathfrak{E}^0(A,B) = \bigcup_{n=1}^{\infty} \mathfrak{E}^0_n(A,B)$ and $\mathfrak{E}^1(A,B) = \bigcup_{n=0}^{\infty} \mathfrak{E}^1_n(A,B)$ are bases respectively of $\mathsf{R}(A,B)$ and $\mathsf{U}(A,B)$. Consider the dual spaces $\mathsf{D}\mathsf{R}(A,B)$ and $\mathsf{D}\mathsf{U}(A,B)$ with bases $\mathfrak{F}^1(A,B)$ and $\mathfrak{F}^0(A,B)$ dual respectively to $\mathfrak{E}^0(A,B)$ and $\mathfrak{E}^1(A,B)$. For $f \in \mathfrak{F}^i(A,B)$ $(i = 0,1)$ set $\nu(f) = \nu(e)$, where e is the element from $\mathfrak{E}^{1-i}$ dual to f. Let $a \in \mathfrak{E}^i(A,B)$, $b \in \mathfrak{E}^j(C,A)$, where (i,j) is $(0,0)$, or $(0,1)$, or $(1,0)$, and $k = i + j$. Then the elements $\lambda(a,b,c)$ for $\in \mathfrak{E}^k(C,B)$ are uniquely determined such that

$$ab = \sum_{c \in \mathfrak{E}^k(C,B)} \lambda(a,b,c)c. \tag{1}$$

For elements $a' \in \mathfrak{F}^{1-i}(A,B)$, $b' \in \mathfrak{F}^{1-j}(C,A)$, $c' \in \mathfrak{F}^{1-k}(C,B)$ dual respectively to a, b, c set $\lambda(a',b',c') = \lambda(a,b,c)$.

Consider the free normal box $\mathfrak{A}$ with the set of vertices $\operatorname{Ob} \mathsf{C}$, the solid (dotted) arrows from A to B being $\mathfrak{F}^0(A,B)$ (respectively $\mathfrak{F}^1(A,B)$) and the differential:

$$\partial a = \sum_{C} \left(\sum_{b \in \mathfrak{F}^0(C,B), v \in \mathfrak{F}^1(A,C)} \lambda(b,v,a)bv - \sum_{b \in \mathfrak{F}^0(A,C), v \in \mathfrak{F}^1(C,B)} \lambda(v,b,a)vb \right)$$

for $a \in \mathfrak{F}^0(A,B)$,

$$\partial v = \sum_{C} \sum_{u \in \mathfrak{F}^1(C,B), w \in \mathfrak{F}^1(A,C)} \lambda(u,w,v)u \otimes w \quad \text{for } v \in \mathfrak{F}^1(A,B).$$

On can easily check that $\mathfrak{A}$ is triangular with respect to the function ν defined above, representations of $\mathfrak{A}$ are in a natural one-to-one correspondence with the objects from $\mathsf{El}(\mathsf{U})$ and their morphisms are in one-to-one correspondence with the morphisms from $\mathsf{El}(\mathsf{U})$. ☐

It is often useful to consider an A-B-bimodule U as an $(\mathsf{A}\times\mathsf{B})$-bimodule setting $\mathsf{U}\left((A,B),(A',B')\right)=\mathsf{U}(A,B')$. Then we call U a *bipartite* bimodule. In particular, every A-bimodule (e.g. every ideal of the category A) can be considered as a bipartite $(\mathsf{A}\times\mathsf{A})$-bimodule. Such bimodules are especially used in relation with the following result.

Theorem 4.2. *Let* A *be a locally finite dimensional category,* $\mathsf{R}=\operatorname{rad}\mathsf{A}$ *considered as bipartite* $\mathsf{A}\times\mathsf{A}$*-bimodule. There is a functor* $\operatorname{Cok}:\mathsf{El}(\mathsf{R})\to\mathsf{rep}(\mathsf{A})$ *with the following properties:*

- Cok *is full and dense.*
- *The set* $\operatorname{Ker}\operatorname{Cok}=\{\,u\in\mathsf{ind}(\mathsf{R})\mid\operatorname{Cok}u=0\,\}$ *only consists of some zero elements.*
- *The restriction of* Cok *onto the full subcategory* $\mathsf{El}^*(\mathsf{R})$ *consisting of the objects that have no direct summands from* $\operatorname{Ker}\operatorname{Cok}$ *maps nonisomorphic objects to nonisomorphic ones.*

They often say that the restriction of Cok onto $\mathsf{El}^*(\mathsf{R})$ is a *representation equivalence.*

Proof. We suppose the category A basic and identify $\mathsf{add}\,\mathsf{A}$ with the category dual to that of finitely generated left projective A-modules. Then $\mathsf{R}(P^\circ,Q^\circ)$ can be identified with $\operatorname{Hom}_{\mathsf{A}}(Q,P\mathsf{R})$. For every finite A-module M there is a *minimal projective presentation*, i.e. a short exact sequence $Q\xrightarrow{\varphi}P\to M\to 0$ with $\operatorname{Im}\varphi\subseteq\mathsf{R}P$, $\operatorname{Ker}\varphi\subseteq\mathsf{R}Q$. We can consider φ as an element from $\mathsf{R}(P^\circ,Q^\circ)$. Moreover, any two minimal projective presentations give isomorphic elements from $\mathsf{El}(\mathsf{R})$. Conversely, if $\varphi\in\mathsf{R}(P^\circ,Q^\circ)$, we can consider it as a homomorphism $Q\to\mathsf{R}P$; thus, setting $\operatorname{Cok}\varphi=\operatorname{Coker}\varphi$, we get a full and dense functor $\mathsf{El}(\mathsf{R})\to\mathsf{rep}(\mathsf{A})$. Note that if the condition $\operatorname{Ker}\varphi\subseteq\mathsf{R}Q$ does not hold, one can decompose $Q=Q_0\oplus Q_1$ so that $Q_0\subseteq\operatorname{Ker}\varphi$ and $\operatorname{Ker}\varphi\cap Q_1\subseteq\mathsf{R}Q_1$. Therefore, as an element from $\mathsf{El}(\mathsf{R})$, φ decomposes as $\varphi_0\oplus\varphi_1$, where φ_1 arises from a minimal projective presentation of $\operatorname{Cok}\varphi$ while φ_0 is a zero morphism $Q_0\to 0$. ☐

We denote by $\mathfrak{R}_{\mathsf{A}}$ the box corresponding to the bipartite $\mathsf{A}\times\mathsf{A}$-bimodule R via Theorem 4.1 and by the same symbol Cok the functor $\mathsf{rep}(\mathfrak{R}_{\mathsf{A}})\to\mathsf{rep}(\mathsf{A})$ that is the composition of the equivalence $\mathsf{rep}(\mathfrak{R}_{\mathsf{A}})\simeq\mathsf{El}(\mathsf{R})$ and the functor Cok from Theorem 4.2. We also denote by $\mathsf{rep}^*(\mathfrak{R}_{\mathsf{A}})$ the image in $\mathsf{rep}(\mathfrak{R}_{\mathsf{A}})$ of $\mathsf{El}^*(\mathsf{R})$; thus, the restriction of Cok onto $\mathsf{rep}^*(\mathfrak{R}_{\mathsf{A}})$ is a representation equivalence. Note that $\mathsf{El}^*(\mathsf{R})$ consists of all representations that have no zero direct summands from $\mathsf{R}(A,0)$ $(A \in \operatorname{Ob}\mathsf{A})$.

5. Representation types

From now on *we suppose the field* $\mathbf{k}$ *algebraically closed,* though in the definition of representation finite type it is not necessary and in the definition of representation discrete type we only need that $\mathbf{k}$ be infinite. Moreover, we suppose that *all boxes are locally finitely generated.*

Let $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ be a skeletal box. We say that it is *representation (locally) finite* if there is a set $\mathfrak{M} \subseteq \mathsf{rep}(\mathfrak{A})$ of its indecomposable representations such that $\mathsf{add}\,\mathfrak{M} = \mathsf{rep}(\mathfrak{A})$ and for every object $A \in \operatorname{Ob}\mathsf{A}$ the set $\mathfrak{M}_A = \{\, M \in \mathfrak{M} \,|\, MA \neq 0 \,\}$ is finite. If $\mathfrak{A}$ is finitely generated, it just means that the set $\mathfrak{M}$ is finite. (We usually omit the word "locally" and say that $\mathfrak{A}$ is representation finite.)

We say that $\mathfrak{A}$ is *representation discrete* if there is a set $\mathfrak{M} \subseteq \mathsf{rep}(\mathfrak{A})$ such that $\mathsf{add}\,\mathfrak{M} = \mathsf{rep}(\mathfrak{A})$ and for each vector dimension $\mathbf{d}$ the set $\{\, M \in \mathfrak{M} \,|\, \mathbf{dim}\, M = \mathbf{d} \,\}$ is finite.

Note that if the category $\mathsf{rep}(\mathfrak{A})$ is fully additive (hence, Krull–Schmidt), one can always take for $\mathfrak{M}$ the set $\mathsf{ind}(\mathfrak{A})$ of all nonisomorphic indecomposable representations.

A deep and difficult theorem proved in [2, 3] claims that for finite dimensional algebras over an algebraically closed (hence, over an infinite perfect) field *representation discrete* implies *representation finite* (it had been known before as the Second Brauer–Thrall conjecture). On the other hand, the free category defined by the graph

$$\cdots \longrightarrow A_{-1} \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

is representation discrete but not finite. A problem remains whether a *finite* free box (i.e. with finite bigraph) is representation discrete if and only if it is representation finite.

The following result follows evidently from Corollary 3.2.

Corollary 5.1. *If a free box* $\mathfrak{A}$ *is representation discrete, its Tits form* $Q_{\mathfrak{A}}$ *is* weakly positive*, i.e.* $Q_{\mathfrak{A}}(\mathbf{d}) > 0$ *for each nonzero vector* $\mathbf{d}$ *with nonnegative entries.*

A representation $M \in \mathsf{rep}(\mathfrak{A}, \mathbf{R})$ is said to be *strict* if, for any finite dimensional representations N, N' of the algebra $\mathbf{R}$,

- if $M \otimes_{\mathbf{R}} N \simeq M \otimes_{\mathbf{R}} N'$, then $N \simeq N'$;
- if N is indecomposable, so is also $M \otimes_{\mathbf{R}} N$.

Figuratively, it means that the classification of representations of the box $\mathfrak{A}$ "contains" that of algebra $\mathbf{R}$. They often say that the functor $M\otimes_{\mathbf{R}}\_ : \mathsf{rep}(\mathbf{R})\to\mathsf{rep}(\mathfrak{A})$ is a *representation embedding.*

Example 5.2. Let $a : A\to A$ be a minimal loop of a semi-free box $\mathfrak{A}$ such that there are no marked loops $b : A\to A,\ b\neq a$. The following representation $J^a\in\mathsf{rep}(\mathfrak{A},\mathbf{R})$, where $\mathbf{R}=\mathbf{k}[t,g_a(t)^{-1}]$, is strict:

$$\begin{aligned} J^a(A)&=\mathbf{R}, \quad J^a(B)=0 \quad \text{if } B\neq A,\\ J^a(a)&=t, \quad J^a(b)=0 \quad \text{if } b\neq a \end{aligned}$$

(here t is identified with the multiplication by t in $\mathbf{R}$). We denote by $J^a_n(\lambda)$ the representation $J^a\otimes_{\mathbf{R}}\mathbf{R}/(t-\lambda)^n$, where $g_a(\lambda)\neq 0$. All of them are indecomposable and pairwise nonisomorphic. In particular, if there is a minimal loop in a semi-free box, it is *representation strongly infinite*, i.e. the set of vector dimensions $\mathbf{d} : \operatorname{Ob}\mathsf{A}\to\mathbb{N}$ such that $\mathsf{ind}_{\mathbf{d}}(\mathfrak{A})$ is infinite is infinite itself.

A box $\mathfrak{A}$ is called *(representation) wild* if, for any finitely generated algebra $\mathbf{R}$, there is a strict representation $M\in\mathsf{rep}(\mathfrak{A},\mathbf{R})$. The following easy (and well known, cf. [17, 10, 12]) results show that to prove the wildness it is enough to construct a strict representation over *one* test algebra.

Proposition 5.3. *Suppose that an algebra* $\mathbf{R}_0$ *is wild and there is a strict representation of a box* $\mathfrak{A}$ *over* $\mathbf{R}_0$. *Then* $\mathfrak{A}$ *is also wild.*

Corollary 5.4. *A box* $\mathfrak{A}$ *is wild if and only if there is a strict representation* $M\in\mathsf{rep}(\mathfrak{A},\mathbf{R}_0)$, *where* $\mathbf{R}_0$ *is one of the following algebras:*

$\mathbf{k}\langle x,y\rangle$, the free algebra in 2 generators;

$\mathbf{k}[x,y]$, the polynomial algebra in 2 variables;

$\mathbf{k}[[x,y]]$, the power series algebra in 2 variables;

$\mathbf{k}[x,y]/(x^2,y^3,xy^2)$;

$\mathbf{k}\Gamma_2$ or $\mathbf{k}\Gamma_2^0$, where Γ_2 is the quiver:

$$\begin{tikzcd} A \arrow[loop left, "a"] \arrow[r, "b"] & B \end{tikzcd} ;$$

$\mathbf{k}\Gamma_5$ or $\mathbf{k}\Gamma_5^\circ$, where Γ_5 is the quiver:

$$\begin{tikzcd} & & A_0 \arrow[dll, "a_1"'] \arrow[dl, "a_2"'] \arrow[d, "a_3"'] \arrow[dr, "a_4"] \arrow[drr, "a_5"] & & \\ A_1 & A_2 & A_3 & A_4 & A_5\,. \end{tikzcd}$$

A *rational algebra* is, by definition, an algebra of the form $\mathbf{R} = \mathbf{k}[t, g(t)^{-1}]$, where $g(t)$ is a nonzero polynomial. A strict representation M of a box $\mathfrak{A}$ over such an algebra is called a *rational family* of its representations. They say that the representations $M \otimes_{\mathbf{R}} L$, where $L \in \mathsf{ind}(\mathbf{R})$, *belong to the rational family* M. (Note that any indecomposable representation of a rational algebra $\mathbf{R} = \mathbf{k}[t, f(t)^{-1}]$ is of the form $J_m(\lambda) = \mathbf{R}/(t-\lambda)^m$, where $f(\lambda) \neq 0$.)

Suppose that a box $\mathfrak{A}$ is skeletal. We call it *(representation) tame* if there is a set $\mathfrak{M}$ of its representations such that:

- each $M \in \mathfrak{M}$ is a strict representation of $\mathfrak{A}$ of finite rank over a rational algebra $\mathbf{R}_M$ (it may depend on M);
- for each vector dimension $\mathbf{d} : \operatorname{Ob} \mathsf{A} \to \mathbb{N}$ there is only finitely many $M \in \mathfrak{M}$ with $\mathbf{dim}\, M = \mathbf{d}$;
- for each vector dimension $\mathbf{d}$ almost all representations from $\mathsf{ind}_{\mathbf{d}}\, \mathfrak{A}$ (i.e. all but a finite number of them) are isomorphic to $M \otimes_{\mathbf{R}_M} N$ for some $M \in \mathfrak{M}$ and some finite dimensional representation N of $\mathbf{R}_M$.

Such a set $\mathfrak{M}$ is called a *parametrizing set* of representations of the box $\mathfrak{A}$. We denote by $|\mathfrak{M}|$ the set $\{\, M \otimes_{\mathbf{R}_M} N \mid M \in \mathfrak{M},\ N \in \mathsf{ind}(\mathbf{R}_M) \,\}$. Note that the set $\mathfrak{M}$ may be empty; thus all representation discrete boxes are by definition also tame.

Let $\mathfrak{M}$ runs through all possible parametrizing sets and $\mu_{\mathfrak{A}}(\mathbf{d})$ be the minimum number of elements in the set $\{\, M \in \mathfrak{M} \mid \mathbf{dim}\, M = \mathbf{d} \,\}$. Call a tame box $\mathfrak{A}$ *bounded* if there is a constant such that $\mu_{\mathfrak{A}}(\mathbf{d}) \leq C$ for all $\mathbf{d}$ and *unbounded* otherwise.

If a box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is tame, then for every dimension $\mathbf{d} : \operatorname{Ob} \mathsf{A} \to \mathbb{N}$ of its finite dimensional representations there is a constructible subset $\mathfrak{R}_{\mathbf{d}} \subseteq \mathsf{rep}_{\mathbf{d}}(\mathfrak{A})$ of dimension at most $|\mathbf{d}| = \sum_{A \in \operatorname{Ob} \mathsf{A}} \mathbf{d}(A)$ such that $\mathfrak{R}_{\mathbf{d}}$ intersects all isomorphism classes from $\mathsf{rep}_{\mathbf{d}}(\mathfrak{A})$. Some easy geometrical considerations imply the following result [11].

Proposition 5.5. *Neither skeletal box can be both tame and wild.*

Again, Corollary 3.2 together with some elementary geometrical observations (cf. [11]) implies the following result.

Corollary 5.6. *If a semi-free box $\mathfrak{A}$ is tame, its Tits form $Q_{\mathfrak{A}}$ is* weakly nonnegative*, i.e. $Q_{\mathfrak{A}}(\mathbf{d}) \geq 0$ for every vector $\mathbf{d}$ with nonnegative entries.*

The relation between the representations of a locally finite dimensional category C and the box $\mathfrak{R}_{\mathsf{C}}$ corresponding to the bipartite $\mathsf{C}\times\mathsf{C}$-bimodule $\mathsf{R} = \operatorname{rad} \mathsf{C}$ (cf. Section 4) implies the following

Corollary 5.7. *The representation type of a locally finite dimensional category* C *coincides with that of the box* $\mathfrak{R}_{\mathsf{C}}$ *.*

Let $\mathbf{R}=\mathbf{k}[t,g(t)^{-1}]$ be a rational algebra and $\mathfrak{A}=(\mathsf{A},\mathsf{V})$ be a skeletal box with a generating set $\mathfrak{G}$ of morphisms from A (for instance, $\mathfrak{A}$ is semi-free and $\mathfrak{G}$ is a set of its solid arrows). A representation $M\in\mathsf{rep}(\mathfrak{A},\mathbf{R})$ is said to be *linear* if a basis can be chosen in each MA (it is a free $\mathbf{R}$-module) such that all entries of matrices corresponding to the homomorphisms Ma $(a\in\mathfrak{G})$ with respect to these bases are linear polynomials in t. We shall see later that each tame semi-free box or tame locally finite dimensional category has a parametrizing family consisting of linear representations.

6. Reduction algorithm

Theorem 2 and especially Corollary 2.2 are used for the so-called "*reduction algorithm.*" The existence of this algorithm is the main advantage of boxes in the representation theory.

Suppose that $\mathfrak{A}=(\mathsf{A},\mathsf{V})$ is a semi-free triangular box with a semi-free triangular set of generators $\Sigma=\Sigma_0\cup\Sigma_1$ with respect to a function $\nu:\Sigma\to\mathbb{N}$. Let $a:A\to B$ be an element from Σ_0 with the smallest value of $\nu(a)$. There are three possibilities:

1. $\partial a\neq 0$; then ∂a is just a linear combination of elements from Σ_1: $\partial a=\sum_i\lambda_i v_i$, where $\lambda_i\in\mathbf{k}$, $v_i\in\Sigma_1$ and $\nu(v_i)<\nu(a)$. If $\lambda_j\neq 0$, we can replace v_j by ∂a getting a new triangular semi-free set of generators that contains ∂a. In this case we call a a *superfluous arrow* and always suppose that $\partial a\in\Sigma_1$.[3]
2. $\partial a=0$ and $A\neq B$. Then we call a a *minimal edge.*
3. $\partial a=0$ and $A=B$. Then we call a a *minimal loop.*

Certainly, these notions depend on the chosen set of generators and the function ν. We often call a a superfluous arrow, or a minimal edge, or a minimal loop if there is a set of generators containing a and a function ν such that a is so with respect to this set and this function.

The following result explains the term "*superfluous.*"

Theorem 6.1 (cf. [18, 12, 6])**.** *Let* a *be a superfluous arrow,* $\mathsf{B}=\mathsf{A}/(a)$ *and* $F:\mathsf{A}\to\mathsf{B}$ *be the natural projection. Then:*

1. $F^*:\mathsf{Rep}(\mathfrak{A},\mathsf{C})\to\mathsf{Rep}(\mathfrak{A}^F,\mathsf{C})$ *is an equivalence for any category* C.
2. *The box* $\mathfrak{A}^F$ *is again semi-free (free if so is* $\mathfrak{A}$*) triangular.*

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[3] In [18, 12] they call such an arrow *nonregular*, but the word "superfluous" seems more appropriate to the situation.

(3) *The bigraph of the box* $\mathfrak{A}^F$ *can be obtained from that of the box* $\mathfrak{A}$ *by deleting the solid arrow* a *and the dotted arrow* ∂a.

(4) *The differential of the box* $\mathfrak{A}^F$ *can be obtained from that of the box* $\mathfrak{A}$ *by omitting all terms containing* a *or* ∂a.

(5) *If* $M \simeq F^*N$ *and* $MA \neq 0$, $MB \neq 0$, *then* $Q^-_{\mathfrak{B}}(\mathbf{dim}\, N) < Q^-_{\mathfrak{A}}(\mathbf{dim}\, M)$.

Suppose now that a is a *minimal edge* and there are no marked loops in $\mathsf{A}(A,A) \cup \mathsf{A}(B,B)$. Consider the subcategory $\mathsf{A}' \subseteq \mathsf{A}$ consisting of two objects A, B and one arrow a. Denote by B' the trivial category with three objects A_0, B_0, AB and consider the functor $F' : \mathsf{A}' \to \operatorname{\mathsf{add}} \mathsf{B}'$ that maps $A \mapsto A_0 \oplus AB$, $B \mapsto B_0 \oplus AB$ and $a \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} : A_0 \oplus AB \to B_0 \oplus AB$. We often write A_1 or B_1 for AB.

Proposition 6.2. *The category* $\widetilde{\mathsf{B}} = \operatorname{\mathsf{add}}(\mathsf{A} \coprod^{\mathsf{A}'} \mathsf{B}')$ *is equivalent to* $\operatorname{\mathsf{add}} \mathsf{B}$, *where* B *is again a semi-free (free if so is* A *) category.*

Proof. The category $\widetilde{\mathsf{B}}$ can be defined up to equivalence as a fully additive category with the following universal property:

- *There is a commutative diagram*

$$\begin{tikzcd} \mathsf{A}' \arrow[r, "F'"] \arrow[d, "\mathrm{em}"'] & \operatorname{\mathsf{add}} \mathsf{B}' \arrow[d, "\widetilde{E}"'] \\ \mathsf{A} \arrow[r, "\widetilde{F}"] & \widetilde{\mathsf{B}}, \end{tikzcd} \tag{2}$$

  where em *is the embedding, such that for any pair of functors* $G : \mathsf{A} \to \mathsf{C}$, $H' : \mathsf{B}' \to \mathsf{C}$, *where* C *is fully additive and* $G \cdot \mathrm{em} = H'E$, *there is a unique functor* $H : \widetilde{\mathsf{B}} \to \mathsf{C}$ *such that* $G = HF$ *and* $H' = HE$.

Consider the semi-free category B and the functors $F : \mathsf{A} \to \operatorname{\mathsf{add}} \mathsf{B}$, $E : \operatorname{\mathsf{add}} \mathsf{B}' \to \operatorname{\mathsf{add}} \mathsf{B}$ defined as follows:

- $\operatorname{Ob} \mathsf{B} = (\operatorname{Ob} \mathsf{A} \setminus \{ A, B \}) \cup \{ A_0, B_0, AB \}$.
- The set of arrows of B consists of:
  - the arrows $b : C \to D$ from A such that $\{ C, D \} \cap \{ A, B \} = \emptyset$;
  - for each arrow $b : C \to D$ (or $D \to C$), where $C \in \{ A, B \}$, $D \notin \{ A, B \}$, two arrows $b_0 : C_0 \to D$ and $b_1 : C_1 \to D$ (respectively $b_0 : D \to C_0$ and $b_1 : D \to C_1$);
  - for any arrow $b : C \to D$, where $C, D \in \{ A, B \}$ and $b \neq a$, four arrows $b_{ij} : C_j \to D_i$ ($i, j = 0, 1$).

- The marking polynomials for loops in B are the same as in A. (Here we use the assumption that there are no marked loops at A and B).
- E is induced by the natural embedding $\mathsf{B}' \to \mathsf{B}$.
- $F(A) = A_0 \oplus AB$, $F(B) = B_0 \oplus AB$, $F(C) = C$ if $C \notin \{A, B\}$.
- $F(a) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, while $F(b)$ for an arrow $b \neq a$, $b : C \to D$, is defined as follows:
  - if $\{C, D\} \cap \{A, B\} = \emptyset$, then $F(b) = b$;
  - if $C \in \{A, B\}$, $D \notin \{A, B\}$ (or $D \in \{A, B\}$, $C \notin \{A, B\}$), then $F(b) = \begin{pmatrix} b_0 & b_1 \end{pmatrix}$ (respectively $F(b) = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$);
  - if $C, D \in \{A, B\}$, then $F(b) = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$.

Obviously, the diagram

$$\begin{tikzcd}
\mathsf{A}' \arrow[r, "F'"] \arrow[d, "\mathrm{em}"'] & \operatorname{\mathsf{add}} \mathsf{B}' \arrow[d, "E"'] \\
\mathsf{A} \arrow[r, "F"] & \operatorname{\mathsf{add}} \mathsf{B}
\end{tikzcd}$$

commutes and has the same universal property as the diagram (2). Hence, $\widetilde{\mathsf{B}} \simeq \operatorname{\mathsf{add}} \mathsf{B}$. ☐

Evidently, every functor $M : \mathsf{A}' \to \mathsf{Vec}$ can be factored through F'. Namely, if $M_0 = \operatorname{Ker} Ma$, M_1 is a complement of M_0 in MA and M_2 is a complement of $\operatorname{Im} Ma$ in MB, then $M \simeq NF'$, where $NA_0 = M_0, NB_0 = M_2, NAB = M_1$. Therefore, Corollary 2.2 implies the first claim of the following theorem.

Theorem 6.3. *In the above situation*

1. $F^* : \mathsf{Rep}(\mathfrak{A}^F) \to \mathsf{Rep}(\mathfrak{A})$ *is an equivalence.*
2. *The box* $\mathfrak{A}^F$ *is equivalent to* $\operatorname{\mathsf{add}} \mathfrak{B}$, *where* $\mathfrak{B} = (\mathsf{B}, \mathsf{W})$ *is again a semi-free (free is so is* $\mathfrak{A}$*) triangular box.*
   We denote by $\hat{F}$ the induced equivalence $\mathsf{Rep}(\mathfrak{B}) \to \mathsf{Rep}(\mathfrak{A})$.
3. If $M \simeq \hat{F}N$ and $MA \neq 0, MB \neq 0$, then $Q^-_{\mathfrak{B}}(\mathbf{dim}\, N) < Q^-_{\mathfrak{B}}(\mathbf{dim}\, M)$.

Proof. Certainly, we can take for W the restriction onto B of the coalgebra $\mathsf{V}^F = (\operatorname{\mathsf{add}} \mathsf{B})^F \otimes_{\mathsf{A}} \mathsf{V} \otimes_{\mathsf{A}} {}^F(\operatorname{\mathsf{add}} \mathsf{B})$. We only have to show that the box $\mathfrak{B}$ is indeed semi-free triangular. For every element $v \in \mathsf{V}(C, D)$, consider the matrix presentation of Fv with respect to the decomposition of FC and FD into a direct sum of objects from B. If v runs through a set of generators of V, the matrix elements of such presentations form a set of generators of W. We take the natural set of generators of V consisting of the dotted arrows from Σ and of the elements ω_C $(C \in \operatorname{Ob}\mathsf{A})$. There are the following possibilities for $v : C \dashrightarrow D$:

(1) $\{C, D\} \cap \{A, B\} = \emptyset$. Then $Fv : C \to D$ coincides with its own matrix presentation and we denote it by the same letter v.

(2) $C \in \{A, B\}$, $D \notin \{A, B\}$ (or $D \in \{A, B\}$, $C \notin \{A, B\}$). Then the matrix presentation of Fv is $\begin{pmatrix} v_0 & v_1 \end{pmatrix}$ with $v_0 : C_0 \dashrightarrow D$, $v_1 : C_1 \dashrightarrow D$ (respectively $\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$ with $v_0 : C \dashrightarrow D_0$, $v_1 : C \dashrightarrow D_1$).

(3) $C, D \in \{A, B\}$ but $v \neq \omega_C$. Then the matrix presentation of Fv is $\begin{pmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{pmatrix}$, where $v_{ij} : C_j \dashrightarrow D_i$.

(4) $v = \omega_A$ or $v = \omega_B$. Then we denote its matrix presentation by $\begin{pmatrix} \xi_{00} & \xi_{01} \\ \xi_{10} & \xi_{11} \end{pmatrix}$, respectively by $\begin{pmatrix} \eta_{00} & \eta_{01} \\ \eta_{10} & \eta_{11} \end{pmatrix}$.

The relations for these generators of W are just the corollaries of those from V, which are only

$$\omega_D b - b\omega_C = \partial b, \tag{3}$$

where b runs through the solid arrows from Σ, $b : C \to D$. Therefore, there are no relations at all for the matrix elements originated from the dotted arrows from Σ. For $b = a$ the relation (3) becomes

$$\begin{pmatrix} \eta_{00} & \eta_{01} \\ \eta_{10} & \eta_{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_{00} & \xi_{01} \\ \xi_{10} & \xi_{11} \end{pmatrix},$$

that is $\eta_{01} = \xi_{10} = 0$, $\eta_{11} = \xi_{11}$. Note that $\varepsilon\omega_A = 1_A$ implies $\varepsilon\xi_{ii} = 1_{Ai}$, $\varepsilon\xi_{ij} = 0$ if $i \neq j$ and the same is valid for η. Denote $\eta_{10} = \eta$, $\xi_{01} = \xi$, $\xi_{00} = \omega_{A_0}$, $\eta_{00} = \omega_{B_0}$, $\xi_{11} = \eta_{11} = \omega_{AB}$. Moreover, the matrix equality $\mu(\xi_{ij}) = (\xi_{ij}) \otimes (\xi_{ij})$ means that $\mu\omega_{A_0} = \omega_{A_0} \otimes \omega_{A_0}$, $\mu\omega_{AB} = \omega_{AB} \otimes \omega_{AB}$, $\mu\xi = \omega_{A_0}\xi + \xi\omega_{AB}$, and $\mu(\eta_{ij}) = (\eta_{ij}) \otimes (\eta_{ij})$ means that $\mu\omega_{B_0} = \omega_{B_0} \otimes \omega_{B_0}$, $\mu\eta = \omega_{AB}\eta + \eta\omega_{B_0}$. Hence, ω is a normal section, so $\underline{\mathfrak{B}}$ is a normal box. As there are no relations for ξ and η, the kernel $\overline{\mathsf{W}}$ of this box is free as B-bimodule with a set of free generators consisting of the matrix elements originated from the dotted arrows from Σ and the elements ξ, η. Moreover, $\partial\xi = \partial\eta = 0$.

If $b : C \to D$ and $\{C, D\} \cap \{A, B\} = \emptyset$, the relation (3) remains unaltered for the corresponding elements from B. If, $b : A \to D$, $D \notin$

$\{A, B\}$, it becomes

$$\omega_D \begin{pmatrix} b_0 & b_1 \end{pmatrix} = \begin{pmatrix} b_0 & b_1 \end{pmatrix} \begin{pmatrix} \omega_{A_0} & \xi \\ 0 & \omega_{AB} \end{pmatrix}$$

or $\omega_D b_0 = b_0 \omega_{A_0}$, $\omega_D b_1 = b_1 \omega_{AB} + b_0 \xi$, i.e. $\partial b_0 = 0$, $\partial b_1 = b_0 \xi$. Just in the same way one can calculate the differentials of the other solid arrows from B. For instance, if $b : B \to A$, then $\partial b_{11} = 0$, $\partial b_{01} = -\xi b_{11}$, $\partial b_{10} = b_{11}\eta$, $\partial b_{00} = b_{01}\eta - \xi b_{10}$.

On the other hand, the equations $\mu(Fv) = Fv \otimes \omega_C + \omega_D Fv + \partial(Fv)$ give the values of ∂w for the matrix components w of Fv. For instance, if $v : C \dashrightarrow B$, we get $\partial v_0 = 0$, $\partial v_1 = \eta v_0$, etc. These calculations imply immediately that the constructed set of generators of the box $\mathfrak{B}$ is triangular. $\square$

Suppose now that $a : A \to A$ is a minimal loop and there is no marked loop $c \neq a$ in $\mathsf{A}(A, A)$. Let $\mathfrak{X}$ be a finite subset of $\mathbf{k}$ and n be a positive integer. Denote by A' the subcategory of A with the unique object A and the algebra of morphisms $\mathbf{k}[a, g_a(a)^{-1}]$. Consider the minimal category B' with the set of objects $\{ A_0, A_{m\lambda} \mid 1 \leq m \leq n,\ \lambda \in \mathfrak{X} \}$ and the unique loop $a_0 : A_0 \to A_0$ with the marking polynomial $g_{a_0}(t) = g_a(t) \prod_{\lambda \in \mathfrak{X}} (t - \lambda)$. Define the functor $F' : \mathsf{A}' \to \mathsf{B}'$ setting $F'(A) = A_0 \oplus \left(\bigoplus_{m,\lambda} m A_{m\lambda} \right)$, $F'(a) = \begin{pmatrix} a_0 & 0 \\ 0 & J \end{pmatrix}$, where the matrix J is a direct sum of Jordan cells

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix} \qquad (m \times m \text{ matrix}).$$

For any linear mapping $\varphi : V \to V$ of a finite dimensional vector space V, one can consider the Fitting decomposition $V = V_0 \oplus V_1$ such that both V_0 and V_1 are invariant under φ, the restriction $\varphi|_{V_0}$ has no eigenvalues from $\mathfrak{X}$ and the minimal polynomial of the restriction $\varphi|_{V_1}$ is of the form $\prod_{\lambda \in \mathfrak{X}} (t - \lambda)^{k_\lambda(\varphi)}$. Let $M : \mathsf{A} \to \mathsf{vec}$ be a functor. Then the restriction of M onto A' can be factored through F' if and only if $k_\lambda(Ma) \leq n$ for all $\lambda \in \mathfrak{X}$. In particular, it is the case if $\dim MA \leq n$.

Now the calculations quite analogous (though more cumbersome) to those used in the proofs of Proposition 6.2 and Theorem 6.3 give the following result.

Theorem 6.4. *In the above situation, there is a functor* $F : \mathsf{A} \to \mathsf{add}\,\mathsf{B}$, *where* B *is a semi-free category such that:*

1. *The functor* $F^* : \mathsf{rep}(\mathfrak{A}^F) \to \mathsf{rep}(\mathfrak{A})$ *induces an equivalence between* $\mathsf{rep}(\mathfrak{A}^F)$ *and the full subcategory of* $\mathsf{rep}(\mathfrak{A})$ *consisting of all representation* M *such that* $k_\lambda(Ma) \le n$ *for all* $\lambda \in \mathfrak{X}$. *Especially, the image of* F^* *contains all representation* M *with* $\dim MA \le n$.
2. *The box* $\mathfrak{A}^F$ *is equivalent to* $\mathsf{add}\,\mathfrak{B}$, *where* $\mathfrak{B} = (\mathsf{B}, \mathsf{W})$ *is again a semi-free triangular box.*
   We denote by $\hat{F}$ the induced functor $\mathsf{rep}(\mathfrak{B}) \to \mathsf{rep}(\mathfrak{A})$.
3. If $M \simeq \hat{F}N$ and Ma has an eigenvalue from $\mathfrak{X}$, then $Q^-_{\mathfrak{B}}(\mathbf{dim}\,N) < Q^-_{\mathfrak{A}}(\mathbf{dim}\,M)$.

Note that in this case the box $\mathfrak{B}$ is no more free even if so was the box $\mathfrak{A}$. Actually, it was the reason why semi-free boxes were introduced in [12].[4]

One more variant of reduction occurs in studying coverings (cf. Section 10) and deals with *minimal lines.* By definition, a *minimal line* L in a semi-free box $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is a set of pairwise different vertices $\{\, A_n \mid n \in \mathbb{Z} \,\}$ and of arrows $\{\, a_n : A_n \to A_{n+1} \,\}$ with $\partial a_n = 0$ for all n. Denote by M_{mn} $(m, n \in \mathbb{Z},\ m \le n)$ the following functor $\mathsf{L} \to \mathsf{vec}$:

$$M_{mn}A = \begin{cases} \mathbf{k} & \text{if } A = A_k,\ m \le k \le n, \\ 0 & \text{otherwise;} \end{cases}$$

$$M_{mn}a = \begin{cases} 1 & \text{if } a = a_k,\ m \le k < n, \\ 0 & \text{otherwise.} \end{cases}$$

Fix an integer r. Let B' be the trivial category with the set of objects $\{\, B_{mn} \mid |m-n| \le r \,\}$ and $F' : \mathsf{L} \to \mathsf{add}\,\mathsf{B}'$ be the functor such that:

- $F'A_k = \bigoplus_{m \le k \le n,\, |m-n| \le r} B_{mn}$,
- with respect to this decomposition, $F'a_k : \bigoplus_{m \le k \le n} B_{m'n'} \to \bigoplus_{m' \le k+1 \le n'} B_{m'n'}$ is the matrix with the entries

$$\alpha_{mn,m'n'} = \begin{cases} 1 & \text{if } m = m',\ n = n' \\ 0 & \text{otherwise.} \end{cases}$$

The same observations as before give the following result [14].

Theorem 6.5. *Let* L *be a minimal line in a semi-free box* $\mathfrak{A}$ *such that there are no marked loops at the objects* A_n *belonging to this line.*

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[4] Their definition in [12] was a bit different and more complicated. The present one is a combination of [12] and [6].

There is a functor $F : \mathsf{A} \to \mathsf{add}\,\mathsf{B}$ *, where* B *is a semi-free category, such that:*

(1) *The functor* $F^* : \mathsf{Rep}(\mathfrak{A}^F) \to \mathsf{Rep}(\mathfrak{A})$ *induces an equivalence between* $\mathsf{Rep}(\mathfrak{A}^F)$ *and the full subcategory of* $\mathsf{Rep}(\mathfrak{A})$ *consisting of all representation* M *such that the restriction of* M *onto* L *decomposes into a direct sum of representations* M_{mn} *with* $|m-n| \le r$ *.*

(2) *The box* $\mathfrak{A}^F$ *is equivalent to* $\mathsf{add}\,\mathfrak{B}$ *, where* $\mathfrak{B} = (\mathsf{B}, \mathsf{W})$ *is again a semi-free triangular box.*
We denote by $\hat{F}$ the induced functor $\mathsf{Rep}(\mathfrak{B}) \to \mathsf{Rep}(\mathfrak{A})$.

(3) If $M \simeq \hat{F}N$ and $MA_n \ne 0,\ MA_{n+1} \ne 0$ for some n , then $Q^-_{\mathfrak{B}}(\mathbf{dim}\,N) < Q^-_{\mathfrak{A}}(\mathbf{dim}\,M)$.

The following immediate observation is sometimes useful.

Proposition 6.6. *Let* $\hat{F}$ *be one of the functors from Theorems 6.1, 6.3, 6.4 or 6.5,* M *be a linear representation of* $\mathfrak{B}$ *over a rational algebra* $\mathbf{R}$ *. Then* $\hat{F}M$ *is a linear representation of* $\mathfrak{A}$ *over* $\mathbf{R}$ *.*

7. Finite type

The first application of the reduction algorithm is that to the representation discrete boxes. The following result was proved in [18] (it had been known before as the First Brauer–Thrall conjecture).

Theorem 7.1. *Suppose that a semi-free triangular box* $\mathfrak{A}$ *is not representation discrete.*[5]

(1) $\mathfrak{A}$ *is representation strongly infinite.*

(2) $\mathfrak{A}$ *has an indecomposable infinite dimensional representation with finite support.*

Proof. Let the set $\mathsf{ind}_{\mathbf{d}}(\mathfrak{A})$ be infinite for some vector dimension $\mathbf{d}$. We prove the theorem using the induction on $q = Q^-_{\mathfrak{A}}(\mathbf{d})$. Obviously, one can suppose that this vector dimension is *sincere*, i.e. $\mathbf{d}(A) \ne 0$ for each object A (especially, $\mathfrak{A}$ only has finitely many objects). If $q = 0$, there are no solid arrows at all, i.e. the box is so-trivial and has finitely many representations of any vector dimension. Thus, the claim is true for $q = 0$. Suppose that it is true for each semi-free box $\mathfrak{B}$ and each vector dimension $\mathbf{c}$ such that $Q^-_{\mathfrak{B}}(\mathbf{c}) < q$. If $q > 0$, there are solid arrows, hence, there is either a superfluous arrow, or a minimal edge, or a minimal loop $a : A \to A$. In the latter case the box is representation strongly infinite (cf. Example 5.2). Moreover, we

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[5] As we have already seen, a representation discrete semi-free box is actually free.

can define an indecomposable infinite dimensional representation J_∞ setting

$$\begin{gathered} J_\infty A = \mathbf{k}(t), \quad J_\infty B = 0 \text{ if } B \neq A, \\ J_\infty a \text{ is the multiplication by } t, \quad J_\infty b = 0 \text{ if } b \neq a. \end{gathered}$$

If $a : A \to B$ is a minimal edge, consider the functor $\hat{F}$ from Theorem 6.3. For any representation $M \in \mathsf{ind}_{\mathbf{d}}(\mathfrak{A})$ there is a representation $N \in \mathsf{ind}_{\mathbf{d}}(\mathfrak{B})$ such that $M \simeq \hat{F}N$. Set $\mathbf{c} = \mathbf{dim}\, N$. There is finitely many possibilities for $\mathbf{c}$, therefore, there is at least one such dimension with infinite set $\mathsf{ind}_{\mathbf{c}}(\mathfrak{B})$. Since $Q^-_{\mathfrak{B}}(\mathbf{c}) < q$, the box $\mathfrak{B}$, hence also $\mathfrak{A}$, is representation strongly infinite and has an indecomposable infinite dimensional representation. Just the same observation works for a superfluous arrow (use Theorem 6.1). $\square$

Corollary 7.2. *If the Tits form of a semi-free triangular box $\mathfrak{A}$ is not weakly positive, the box $\mathfrak{A}$ is representation strongly infinite.*

Theorem 7.1 and Corollary 5.7 immediately imply the following result.

Corollary 7.3. *If a locally finite dimensional category is not representation discrete, it is representation strongly infinite and has an infinite dimensional representation with finite support.*[6]

Theorem 7.4. *Suppose that a semi-free triangular box $\mathfrak{A}$ is representation (locally) finite. Then every representation $M \in \mathsf{Rep}(\mathfrak{A})$ with a finite support is a direct sum of finite dimensional representations.*

Proof. Obviously, we may suppose that $\mathfrak{A}$ only has finitely many objects, hence, finitely many indecomposable finite dimensional representations. Then we can just follow the proof of Theorem 7.1 using the induction on $q = \max \left\{ Q^-_{\mathfrak{A}}(\mathbf{dim}\, N) \mid N \in \mathsf{ind}(\mathfrak{A}) \right\}$. $\square$

Corollary 7.5. *Suppose that a locally finite dimensional category C is representation (locally) finite. Then every representation $M \in \mathsf{Rep}(\mathsf{C})$ with a finite support is a direct sum of finite dimensional representations.*

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[6] It follows from [2, 3] that one can replace here "representation discrete" by "representation finite."

8. Tame and wild type

Theorem 8.1 (Tame–wild dichotomy, cf. [12])**.** *If a semi-free triangular box is not wild, it is tame. Moreover, it has a parametrizing set consisting of linear representations.*[7]

Proof. We shall prove that if $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ is not wild, then for every dimension $\mathbf{d} : \mathrm{Ob}\,\mathsf{A} \to \mathbb{N}$ there is a finite set $\mathfrak{M}_{\mathbf{d}}$ such that

- each $M \in \mathfrak{M}_{\mathbf{d}}$ is a strict linear representation of $\mathfrak{A}$ of vector dimension $\mathbf{d}_M \le \mathbf{d}$ over a rational algebra $\mathbf{R}_M$ (it may depend on M);
- if $\mathbf{d}' \le \mathbf{d}$, almost all representations from $\mathsf{ind}_{\mathbf{d}'}\mathfrak{A}$ (i.e. all but a finite number of them) are isomorphic to $M \otimes_{\mathbf{R}_M} N$ for some $M \in \mathfrak{M}_{\mathbf{d}}$ and some finite dimensional representation N of $\mathbf{R}_M$;
- if $\mathbf{d} \le \mathbf{c}$ then $\mathfrak{M}_{\mathbf{d}} \subseteq \mathfrak{M}_{\mathbf{c}}$.

Certainly, then one can put $\mathfrak{M} = \bigcup_{\mathbf{d}} \mathfrak{M}_{\mathbf{d}}$.

We suppose $\mathbf{d}$ sincere and use induction on $q = Q^-_{\mathfrak{A}}(\mathbf{d})$. Again the case $q = 0$ is trivial, so we may suppose that $q > 0$ and the claim is true for all boxes $\mathfrak{B}$ and all dimensions of their representations $\mathbf{c}$ with $Q^-_{\mathfrak{B}}(\mathbf{c}) < q$. If there is a minimal edge or a superfluous arrow in $\mathfrak{A}$, the proof just repeats that of Theorem 7.1 (using Proposition 6.6 for linearity). Hence, we may suppose that there are neither minimal edges nor superfluous arrows in $\mathfrak{A}$, only minimal loops. Note that if there is a minimal loop $a : A \to A$ and a solid arrow $b : A \to B$ or $b : B \to A$ with $\partial b = 0$, the box $\mathfrak{A}$ is wild due to Corollary 5.4. If every solid arrow from $\mathfrak{A}$ is a minimal loop, set $\mathfrak{M} = \{\, J^a \mid a \text{ is a minimal loop} \,\}$, where J^a has been defined in Example 5.2. Evidently, $|\mathfrak{M}| = \mathsf{ind}(\mathfrak{A})$ and all representations J^a are linear.

If there are solid arrows that are not minimal loops, the triangularity implies that there is one of them, say $b : A \to B$, such that ∂b only contains minimal loops (and dotted arrows). First suppose that there is a (unique) minimal loop $a : A \to A$ and no minimal loops $c : B \to B$ (or vice versa). Then $\partial b = \sum_{i=1}^k v_i f_i(a)$ for some dotted arrows v_i and some nonzero polynomials $f_i(t)$, and, choosing a new set of generators, we may suppose that $k = 1$, i.e. $\partial b = v f(a)$. Use Theorem 6.4 for the set $\mathfrak{X} = \{\, \lambda \in \mathbf{k} \mid f(\lambda) = 0 \,\}$ and $n = \mathbf{d}(A)$. Note that $\mathsf{ind}_{\mathbf{d}}(\mathfrak{A}) = \mathfrak{N}_1 \cup \mathfrak{N}_2$, where $\mathfrak{N}_1$ consists of representations N such that NA has eigenvalues from $\mathfrak{X}$ and $\mathfrak{N}_2$ of all other representations.

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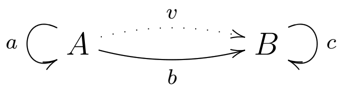
[7] The latter property was first noticed in [15], though it easily follows from the reduction algorithm.

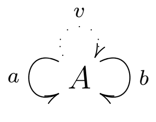
Theorem 6.4 and the induction conjecture implies that almost all representations from $\mathfrak{R}_1$ can be obtained from a finite set of strict linear representation over rational algebras. All representations from $\mathfrak{R}_2$ are isomorphic to G^*N, where $G : \mathsf{A} \to \mathsf{A}[f(a)^{-1}]$. The box $\mathfrak{A}^G$ is also semi-free with the same sets of objects and arrows, but the arrow b is superfluous in $\mathfrak{A}^G$. Hence, we can use again the induction hypothesis and claim that almost all representations from $\mathfrak{R}_2$ (thus almost all representations from $\mathsf{ind}_{\mathsf{d}}(\mathfrak{A})$) can be obtained from a finite set of strict linear representations over rational algebras.

Suppose now that there is both a minimal loop $a \in \mathsf{A}(A, A)$ and a minimal loop $c \in \mathsf{A}(B, B)$, both unique since $\mathfrak{A}$ is not wild (perhaps $A = B$, then $a = c$). We consider $\mathsf{V}(A, B)$ as $\mathbf{k}[x, y]$-bimodule: for $v \in \mathsf{V}(A, B)$ and $f(x, y) = \sum_{ij} \lambda_{ij} x^i y^j$ set $f(x, y)v = \sum_{ij} \lambda_{ij} c^j v a^i$. Then $\partial b = \sum_k f_k(x, y)v_k$ for some dotted arrows $v_k : A \cdot\cdot> B$). Let $d(x, y)$ be the greatest common divisor of all $f_k(x, y)$. There are polynomials $g_k(x, y)$ and $h(x)$ such that $h(x)d(x, y) = \sum_k g_k(x, y)f_k(x, y)$. Using Theorem 6.4 for the loop a and the set $\mathfrak{X} = \{\, \lambda \in \mathbf{k} \mid h(\lambda) = 0 \,\}$, we are able to reduce the situation, just as in the preceding paragraph, to the case when $h(a)$ is invertible. Then, changing the set of dotted arrows, we can suppose that $\partial b = d(x, y)v$ for some dotted arrow v. If $d(x, y) = 1$, b is superfluous, so we can use the inductive procedure. Otherwise the following lemma accomplishes the proof.

Lemma 8.2. *Let* $\mathfrak{A}_0$ *be a triangular semi-free box with the bigraph*



or



such that $\partial a = 0, \partial c = 0$ *and* b *is not superfluous. Then* $\mathfrak{A}_0$ *is wild.*

The proof of this lemma is just an explicit construction of strict representations of the given boxes over the wild algebra $\mathbf{k}\Gamma_5$ from Corollary 5.4. For instance, if $\mathfrak{A}_0$ is free with the first of the given bigraphs and $\partial b = va - cv$ (it is a typical case), a strict representation M from $\mathsf{rep}(\mathfrak{A}_0, \mathbf{k}\Gamma_5)$ can be defined as follows. We denote by P_i the indecomposable projective module corresponding to the vertex A_i of the graph Γ_5, identify the arrows a_i with homomorphisms $P_0 \to P_i$)

and set:

$$\begin{aligned} MA &= 9P_0, \\ MB &= \bigoplus_{i=1}^{5} (2i-1)P_i, \\ Ma &= J_9, \\ Mc &= \bigoplus_{i=1}^{5} J_{2i-1}, \end{aligned}$$

$$Mb = \left(\begin{array}{cccc} a_1 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ a_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_3 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_4 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_5 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{array} \right),$$

where J_m denotes the $m \times m$ nilpotent Jordan cell. □

Corollary 8.3. *If a locally finite dimensional category is not wild, it is tame and has a parametrizing family that consists of linear representations.*

Corollary 8.4. *If the Tits form of a semi-free triangular box is not weakly nonnegative, this box is wild.*

Remark 8.5. One can easily see from the proof of Theorem 8.1 that if a semi-free triangular box $\mathfrak{A}$ is wild, it has a strict representation over any free algebra $\mathbf{k}\langle x_1, x_2, \dots, x_m\rangle$ that is also *linear*, i.e. all entries of matrices corresponding to the homomorphisms Ma (a runs through solid arrows) with respect to some bases chosen in all modules MA (which are free [1, Chapter IV, § 5]) are linear in $x_1, x_2, \dots, x_n$. The same is true for locally finite dimensional categories.[8]

Indeed, the proof of Theorem 8.1 also gives the following results that are sometimes useful.

Proposition 8.6. *Suppose that a semi-free triangular box $\mathfrak{A}$ is not wild, a is a minimal loop from $\mathfrak{A}$ and $\mathbf{d}$ is a vector dimension of representations of $\mathfrak{A}$.*

1. *There is a finite subset $\mathfrak{X} \subseteq \mathbf{k}$ such that, for each $M \in \mathsf{ind}_{\mathbf{d}}(\mathfrak{A})$, $M \not\simeq J_n^a(\lambda)$, the set of eigenvalues of Ma is contained in $\mathfrak{X}$.*
2. *There is a morphism $\Phi : \mathfrak{A} \to \mathsf{add}\,\mathfrak{T}$, where $\mathfrak{T}$ is a so-minimal box, such that the functor $\Phi^* : \mathsf{rep}(\mathfrak{T}) \to \mathsf{rep}(\mathfrak{A})$ is full and faithful and its image contains all representations of dimensions $\mathbf{d}' \leq \mathbf{d}$.*

9. Generic modules

Let $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ be a normal box, $M \in \mathsf{Rep}(\mathfrak{A})$ and $\mathcal{E} = \mathrm{Hom}_{\mathfrak{A}}(M, M)$. For each object $A \in \mathrm{Ob}\,\mathsf{A}$ and each element $\alpha \in \mathcal{E}$, $\alpha(\omega_A)$ is an endomorphism of the vector space MA and $\alpha\beta(\omega_A) = \alpha(\omega_A)\beta(\omega_A)$. Hence, we can consider MA as $\mathcal{E}$-module setting $\alpha u = \alpha(\omega_A)u$. Suppose that $\mathfrak{A}$ is skeletal. They say that M is of *finite endolength* if $\mathsf{supp}\, M$ is finite and $\mathrm{length}_{\mathcal{E}}\, MA < \infty$ for each object A. Let $\mathsf{fel}(\mathfrak{A})$ be the category of all representations from $\mathsf{Rep}(\mathfrak{A})$ of finite endolength. A representation $M \in \mathsf{fel}(\mathfrak{A})$ is said to be *generic* if it is indecomposable and infinite dimensional (i.e. $\dim MA = \infty$ for at least one object A). We denote by $\mathbf{e\text{-}len}\, M$ and call the *vector endolength* of M the function $\mathrm{Ob}\,\mathsf{A} \to \mathbb{N}$ mapping A to $\mathrm{length}_{\mathcal{E}}(MA)$ and set $\text{e-len}\, M = \sum_{A \in \mathrm{Ob}\,\mathsf{A}} \mathrm{length}_{\mathcal{E}}(MA)$. Let $\mathsf{gen}(\mathfrak{A})$ denote the set of isomorphism classes of all generic representations of $\mathfrak{A}$ and $\mathsf{gen}_{\mathbf{d}}(\mathfrak{A})$ those of generic representations with vector endolength $\mathbf{d}$. In particular, these definitions are valid if we consider a locally finite dimensional category instead of a box. Thus it contains, in particular, representations of finite dimensional algebras.

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[8] It was also first observed in [15].

Example 9.1. Let M be a strict representation of $\mathfrak{A}$ over a rational algebra $\mathbf{R}$. Denote by $M^m(t)$ the representation $M\otimes_{\mathbf{R}} J_m(t)$, where $J_m(t)$ is the $\mathbf{k}(t)$-$\mathbf{R}$-bimodule such that its underlying right $\mathbf{k}(t)$-module is $m\mathbf{k}(t)$ and the left multiplication by t is given by the matrix

$$\begin{pmatrix} t & 1 & 0 & \dots & 0 & 0\\ 0 & t & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & t & 1\\ 0 & 0 & 0 & \dots & 0 & t \end{pmatrix}.$$

Then $M^m(t)$ is a generic representation of the box $\mathfrak{A}$ with $\mathbf{e\text{-}len}\,M^m = m\,\mathbf{dim}\,M$. Moreover, its endomorphism algebra $\mathcal{E}$ is a finite dimensional $\mathbf{k}(t)$-algebra.[9]

If a box (or an algebra) is representation finite, it has no generic modules (cf. Corollary 7.5). Moreover, the following result can be obtained just following the proof of Theorem 7.1.

Proposition 9.2. *If a semi-free triangular box (or a locally finite dimensional category) has a generic representation, it is representation strongly infinite.*

The following refined version of tame–wild dichotomy was actually proved in [7] (though the original formulation was a bit different there).

Theorem 9.3. *For a semi-free triangular box* $\mathfrak{A}=(\mathsf{A},\mathsf{V})$ *the following conditions are equivalent:*

1. $\mathfrak{A}$ *is not wild.*
2. $\mathfrak{A}$ *is tame.*
3. *For each vector dimension* $\mathbf{d}:\operatorname{Ob}\mathsf{A}\to\mathbb{N}$ *the set* $\mathsf{gen}_{\mathbf{d}}(\mathfrak{A})$ *is finite.*
4. *There is a parametrizing set* $\mathfrak{M}$ *of representations of* $\mathfrak{A}$ *such that every generic representation* $N\in\mathsf{gen}(\mathfrak{A})$ *is isomorphic to* $M^m(t)$ *for some* $M\in\mathfrak{M}$.

Moreover, the representations from $\mathfrak{M}$ *can be chosen linear.*

Proof. $(1)\Leftrightarrow(2)$ is already known. $(4)\Rightarrow(3)$ is trivial.

$(3)\Rightarrow(1)$: Consider a strict representation M of $\mathfrak{A}$ over $\mathbf{k}[x,y]$. Denote by $N(\lambda)$ $(\lambda\in\mathbf{k})$ the $\mathbf{k}(t)$-$\mathbf{k}[x,y]$-bimodule such that the underlying $\mathbf{k}(t)$-module is just $\mathbf{k}(t)$, x acts as multiplication by λ and y as multiplication by t. Then $M\otimes_{\mathbf{k}[x,y]}N(\lambda)$ are generic pairwise nonisomorphic representations of $\mathfrak{A}$.

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[9] Certainly, it is a mistake: only $M^1(t)$ is indecomposable, so generic!

$(1) \Rightarrow (4)$. Using induction on $Q^-_{\mathfrak{A}}(\mathbf{d})$, we find finite sets of rational families $\mathfrak{M}_{\mathbf{d}}$ such that

- $\mathbf{dim}\, M \le \mathbf{d}$ for each $M \in \mathfrak{M}_{\mathbf{d}}$;
- every generic representation of vector endolength $\mathbf{d}' \le \mathbf{d}$ is isomorphic to $M^m(t)$ for some $M \in \mathfrak{M}_{\mathbf{d}}$;
- if $\mathbf{d} < \mathbf{c}$ then $\mathfrak{M}_{\mathbf{d}} \subseteq \mathfrak{M}_{\mathbf{c}}$.

Certainly, then one can put $\mathfrak{M} = \bigcup_{\mathbf{d}} \mathfrak{M}_{\mathbf{d}}$. The inductive procedure uses again the reduction algorithm and is quite analogous to that of the proof of Theorem 8.1. The main new ingredient is to check that vector endolength behave during the reduction in the same way as vector dimension. It follows easily from the fact that if $F : \mathfrak{A} \to \mathsf{add}\, \mathfrak{B}$ is one of the morphisms of boxes described in Section 6 and $FA = \bigoplus_i B_i$, one can arrange indices so that the image $F\omega_A$ be a triangular matrix with the diagonal entries ω_{B_i}. Hence, if $M = F^*N$ and $\mathcal{E} = \mathrm{Hom}_{\mathfrak{A}}(M, M) \simeq \mathrm{Hom}_{\mathfrak{B}}(N, N)$, then $\mathrm{length}_{\mathcal{E}}(MA) = \sum_i \mathrm{length}_{\mathcal{E}}(NB_i)$ We refer to [7] for details. □

Theorem 9.3 together with the results from Section 4 implies immediately its analogue for algebras [7].

Corollary 9.4. *Let* C *be a locally finite dimensional category (for instance, a finite dimensional algebra). The following conditions are equivalent:*

1. C *is not wild.*
2. C *is tame.*
3. *For every* n *there is only finitely many generic* C*-modules of endolength* n *(up to equivalence).*
4. *There is a parametrizing set* $\mathfrak{M}$ *of representations of* C *such that every generic representation* $N \in \mathsf{gen}(\mathfrak{A})$ *is isomorphic to* $M^m(t)$ *for some* $M \in \mathfrak{M}$.

Moreover, the representations from $\mathfrak{M}$ *can be chosen linear.*

10. COVERINGS

Let $\mathfrak{A} = (\mathsf{A}, \mathsf{V})$ be a box, $\mathbf{G}$ be a group. We say that $\mathbf{G}$ *acts* on $\mathfrak{A}$ if it acts on the sets of objects and morphisms of A as well as on the set of elements of V so that

- if $A, B \in \mathrm{Ob}\, \mathsf{A}$, $a \in \mathsf{A}(A, B)$, $v \in \mathsf{V}(A, B)$ and $g \in \mathbf{G}$, then $ga \in \mathsf{A}(gA, gB)$, $gv \in \mathsf{V}(gA, gB)$;
- $g(ab) = (ga)(gb)$, $g(a + b) = ga + gb$, $g(\lambda a) = \lambda(ga)$, where $g \in \mathbf{G}$, $\lambda \in \mathbf{k}$ and a, b are elements from $\mathrm{Mor}\, \mathsf{A}$ or V such that the left parts of the corresponding equations are defined;
- $\mu(gv) = g\mu(v)$, where $g(u_1 \otimes u_2) = (gu_1 \otimes gu_2)$;

- $\varepsilon(gv) = g\varepsilon(v)$.

If, moreover, A is skeletal and $gA \neq A$ for each $A \in \operatorname{Ob}\mathsf{A}$, $g \in \mathbf{G}$, $g \neq 1$, we say that $\mathbf{G}$ *acts freely* on $\mathfrak{A}$.

If $\mathbf{G}$ acts freely on $\mathfrak{A}$, the *orbit box* $\mathbf{G}\backslash\mathfrak{A} = (\mathbf{G}\backslash\mathsf{A}, \mathbf{G}\backslash\mathsf{V})$ is defined in the following way:

- $\operatorname{Ob}(\mathbf{G}\backslash\mathsf{A})$ is the set of orbits of $\mathbf{G}$ on $\operatorname{Ob}\mathsf{A}$;
- $(\mathbf{G}\backslash\mathsf{A})(\mathbf{G}A, \mathbf{G}B) = \bigoplus_{g,h\in\mathbf{G}} \mathsf{A}(gA, hB)/U_{\mathsf{A}}$, where U_{A} is the subspace generated by all differences $a - ga$ $(g \in \mathbf{G})$;
- $(\mathbf{G}\backslash\mathsf{V}) = (\mathbf{G}A, \mathbf{G}B) \bigoplus_{g,h\in\mathbf{G}} \mathsf{V}(gA, hB)/U_{\mathsf{V}}$, where U_{V} is the subspace generated by all differences $v - gv$ $(g \in \mathbf{G})$;
- $(\mathbf{G}a)(\mathbf{G}b) = (\mathbf{G}ab')$, where b' is the unique element from $\mathbf{G}b$ such that its target coincide with the source of a;
- $\varepsilon(\mathbf{G}v) = \mathbf{G}\varepsilon(v)$;
- $\mu(\mathbf{G}v) = \mathbf{G}\mu(v)$.

Let $\Pi : \mathfrak{A} \to \mathbf{G}\backslash\mathfrak{A}$ be the natural projection. It defines the inverse image functor $\mathsf{Rep}(\mathbf{G}\backslash\mathfrak{A}, \mathsf{C}) \to \mathsf{Rep}(\mathfrak{A}, \mathsf{C})$ for each category C. On the other hand, if C is additive, the *direct image* functor $\Pi_* : \mathsf{rep}(\mathfrak{A}, \mathsf{C}) \to \mathsf{rep}(\mathbf{G}\backslash\mathfrak{A}, \mathsf{C})$ is induced by the tensor product $(\mathbf{G}\backslash\mathsf{A})^{\Pi} \otimes_{\mathsf{A}} \_$. Moreover, if $\mathbf{G}$ is finite or C has infinite direct sums, the functor Π_* is defined for *all* representations, not only finite. The following description of Π and Π_* is straightforward.

Proposition 10.1. *For every objects $\hat{A}, \hat{B}$ from $\mathbf{G}\backslash\mathsf{A}$ and for each representatives $A_0 \in \hat{A}$, $B_0 \in \hat{B}$,*

$$\begin{aligned}
(\mathbf{G}\backslash\mathsf{A})(\hat{A}, \hat{B}) &\simeq \bigoplus_{A\in\hat{A}} \mathsf{A}(A, B_0) \simeq \bigoplus_{B\in\hat{B}} \mathsf{A}(A_0, B);\\
(\mathbf{G}\backslash\mathsf{V})(\hat{A}, \hat{B}) &\simeq \bigoplus_{A\in\hat{A}} \mathsf{V}(A, B_0) \simeq \bigoplus_{B\in\hat{B}} \mathsf{V}(A_0, B);\\
(\Pi_* M)\hat{A} &\simeq \bigoplus_{A\in\hat{A}} MA.
\end{aligned}$$

If a group $\mathbf{G}$ acts freely on a box $\mathfrak{A}$ and $\overline{\mathfrak{A}} \simeq \mathbf{G}\backslash\mathfrak{A}$, they say that $\mathfrak{A}$ is a *Galois covering* of $\overline{\mathfrak{A}}$ with *Galois group* $\mathbf{G}$.

The construction from Section 4 immediately implies the following result.

Proposition 10.2. *If a group $\mathbf{G}$ acts freely on a basic category C and $\mathfrak{R}_{\mathsf{C}}$ is the box corresponding to the radical of C via Theorem 4.1, then $\mathbf{G}$ acts freely on $\mathfrak{R}_{\mathsf{C}}$ and there is a natural equivalence $\mathbf{G}\backslash(\mathfrak{R}_{\mathsf{C}}) \simeq \mathfrak{R}_{\mathbf{G}\backslash\mathsf{C}}$, which commutes with the functors* Cok.

Any action of a group $\mathbf{G}$ on a box $\mathfrak{A}$ induces an action of $\mathbf{G}$ on its representation categories: for any $M \in \mathsf{Rep}(\mathfrak{A},\mathsf{C})$ and $g \in \mathbf{G}$, gM is the representation such that $(gM)A = M(g^{-1}A)$. In general, this action is not free even if so is the action of $\mathbf{G}$ on $\mathfrak{A}$; nevertheless, it is free on the categories of finite representations if $\mathbf{G}$ is torsion free.

For representation finite and sometimes for tame boxes there are good relations between the representations of a box and those of its Galois coverings.

Theorem 10.3. *Let a group* $\mathbf{G}$ *acts freely on a semi-free triangular box* $\mathfrak{A}$.

1. $\mathfrak{A}$ *is representation locally finite if and only if so is* $\overline{\mathfrak{A}} = \mathbf{G}\backslash\mathfrak{A}$.
2. *If these boxes are representation locally finite,* $\mathbf{G}$ *acts freely on* $\mathsf{rep}(\mathfrak{A})$ *and the functor* Π_* *induces an equivalence* $\mathbf{G}\backslash \mathsf{rep}(\mathfrak{A}) \simeq \mathsf{rep}(\overline{\mathfrak{A}})$.

Corollary 10.4 (cf. [4])**.** *Let a group* $\mathbf{G}$ *acts freely on a locally finite dimensional category* C.

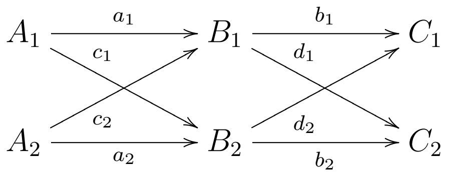
1. C *is representation locally finite if and only if so is* $\overline{\mathsf{C}} = \mathbf{G}\backslash\mathsf{C}$.
2. *If these categories are representation locally finite,* $\mathbf{G}$ *acts freely on* $\mathsf{rep}(\mathsf{C})$ *and the functor* Π_* *induces an equivalence* $\mathbf{G}\backslash \mathsf{rep}(\mathsf{C}) \simeq \mathsf{rep}(\overline{\mathsf{C}})$.

Remark 10.5. If $\mathfrak{A}$ is *representation discrete*, it may not be the case for $\overline{\mathfrak{A}}$. The easiest example is the free category with the graph

$$\cdots \longrightarrow A_{-1} \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

and the obvious action of the group $\mathbf{Z}$. The orbit category consists of one loop, hence, is representation strongly infinite.

If $\mathfrak{A}$ is tame, $\mathbf{G}\backslash\mathfrak{A}$ may not be so. The easiest example is perhaps that of finite dimensional category $\mathsf{C} = \mathbf{k}\Gamma/I$, where Γ is the graph



and I is the ideal generated by the set

$$\{\, d_1a_1 - b_2c_1,\ d_2a_2 - b_1c_2,\ b_1a_1 - d_2c_1,\ b_2a_2 - d_1c_2 \,\},$$

with the evident free action of the group $\mathbf{G}$ of order 2. It is not difficult to check that C is tame, but if $\operatorname{char}\mathbf{k} = 2$ the orbit category $\mathbf{G}\backslash\mathsf{C}$ is wild [16].

Nevertheless, the situation becomes much better if the group $\mathbf{G}$ is torsion free.

Theorem 10.6 (cf. [14]). *Suppose that a torsion free group* $\mathbf{G}$ *acts freely on a semi-free triangular box* $\mathfrak{A}$.

1. $\mathfrak{A}$ *is tame if and only if so is* $\overline{\mathfrak{A}} = \mathbf{G}\backslash\mathfrak{A}$.
2. *If these boxes are tame, then:*
   - (a) $\mathsf{ind}(\overline{\mathfrak{A}}) = \mathsf{ind}_0 \sqcup \mathsf{ind}_1$, *where* $\mathsf{ind}_0 = \Pi_*(\mathsf{ind}(\mathfrak{A})) \simeq \mathbf{G}\backslash \mathsf{rep}(\mathfrak{A})$ *and* $\mathsf{ind}_1 = |\mathfrak{N}|$, *where* $\mathfrak{N}$ *is a set of strict linear representations of* $\mathfrak{A}$ *over the algebra* $\mathbf{T} = \mathbf{k}[t, t^{-1}]$.
   - (b) $\mathrm{Hom}_{\mathfrak{A}}(M, M') \subseteq \mathrm{rad}^\infty(\overline{\mathfrak{A}})$ *if* $M \in \mathsf{ind}_0$, $M' \in \mathsf{ind}_1$, *or vice versa, or* M, M' *belong to different rational families from* $\mathfrak{N}$.
   - (c) *If* $M \simeq N \otimes_{\mathbf{T}} L$, $M' \simeq N \otimes_{\mathbf{T}} L'$ *for some* $N \in \mathfrak{N}$ *and* $L, L' \in \mathsf{ind}(\mathbf{T})$, *then* $\mathrm{Hom}_{\mathfrak{A}}(M, M') = 1 \otimes \mathrm{Hom}_{\mathbf{T}}(L, L') \oplus H$, *where* $H = \mathrm{Hom}_{\mathfrak{A}}(M, M') \cap \mathrm{rad}^\infty(\overline{\mathfrak{A}})$.

   Here rad^∞ denotes the intersection of all powers of the radical of the category of representations.

Corollary 10.7. [10] *Suppose that a torsion free group* $\mathbf{G}$ *acts freely on a locally finite dimensional category* C.

1. C *is tame if and only if so is* $\overline{\mathsf{C}} = \mathbf{G}\backslash\mathsf{C}$.
2. *If these boxes are tame, then*
   - (a) $\mathsf{ind}(\overline{\mathsf{C}}) = \mathsf{ind}_0 \sqcup \mathsf{ind}_1$, *where* $\mathsf{ind}_0 = \Pi_*(\mathsf{ind}(\mathsf{C})) \simeq \mathbf{G}\backslash \mathsf{rep}(\mathsf{C})$ *and* $\mathsf{ind}_1 = |\mathfrak{N}|$, *where* $\mathfrak{N}$ *is a set of strict linear representations of* C *over the algebra* $\mathbf{T} = \mathbf{k}[t, t^{-1}]$.
   - (b) $\mathrm{Hom}_{\mathsf{C}}(M, M') \subseteq \mathrm{rad}^\infty(\overline{\mathsf{C}})$ *if* $M \in \mathsf{ind}_0$, $M' \in \mathsf{ind}_1$, *or vice versa, or* M, M' *belong to different rational families from* $\mathfrak{N}$.
   - (c) *If* $M \simeq N \otimes_{\mathbf{T}} L$, $M' \simeq N \otimes_{\mathbf{T}} L'$ *for some* $N \in \mathfrak{N}$ *and* $L, L' \in \mathsf{ind}(\mathbf{T})$, *then* $\mathrm{Hom}_{\mathsf{C}}(M, M') = 1 \otimes \mathrm{Hom}_{\mathbf{T}}(L, L') \oplus H$, *where* $H = \mathrm{Hom}_{\mathsf{C}}(M, M') \cap \mathrm{rad}^\infty(\overline{\mathsf{C}})$.

Proof. The proofs of Theorems 10.3 and 10.6 are based on the procedure of “equivariant reduction.” Namely, we find a solid arrow a of the box $\overline{\mathfrak{A}}$ that is either superfluous, or a minimal edge, or a minimal loop. In the first two cases one can lift a to a set $\widetilde{a}$ of superfluous arrows, respectively minimal edges of the box $\mathfrak{A}$. Then we apply the corresponding step of the reduction algorithm (cf. Section 6) both to the arrow a and to all arrows of the set $\widetilde{a}$. As the result, we obtain a new box $\mathfrak{B}$ with a free action of the same group $\mathbf{G}$

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[10] Partial cases of this theorem were proved in [8, 9]

and functors $F : \mathfrak{A} \to \mathsf{add}\,\mathfrak{B}$, $\overline{F} : \mathfrak{B} \to \mathsf{add}\,\overline{\mathfrak{B}}$, where $\overline{\mathfrak{B}} = \mathbf{G}\backslash\mathfrak{B}$, such that $F(gA) = g(FA)$ and both F and $\overline{F}$ induce equivalences $\mathsf{rep}(\mathfrak{B}) \simeq \mathsf{rep}(\mathfrak{A})$, $\mathsf{rep}(\overline{\mathfrak{B}}) \simeq \mathsf{rep}(\overline{\mathfrak{A}})$ so that the diagram

$$\begin{tikzcd}
\mathsf{rep}(\mathfrak{B}) \arrow[r, "\sim"] \arrow[d, "\Pi_*"'] & \mathsf{rep}(\mathfrak{A}) \arrow[d, "\Pi_*"] \\
\mathsf{rep}(\overline{\mathfrak{B}}) \arrow[r, "\sim"] & \mathsf{rep}(\overline{\mathfrak{A}})
\end{tikzcd}$$

is commutative. Therefore, we can proceed inductively as in the proofs of Sections 7 and 8.

If a is a minimal loop, there are two possibilities: either a is lifted to $\mathfrak{A}$ as a set of minimal loops or as a set of *minimal lines* (cf. Section 6). In the former case we again use equivariant reduction and induction, while in the latter case the following lemma works.

Lemma 10.8. *If the box $\mathfrak{A}$ is not wild and a minimal loop a of $\overline{\mathfrak{A}}$ is lifted to minimal lines, then, for every indecomposable representation $M \in \mathsf{ind}(\overline{\mathfrak{A}})$, either Ma is nilpotent or $M \simeq M'$ such that $M'b = 0$ for each arrow $b \neq a$.*

The representations of the second kind belong to the rational family $J^a \in \mathsf{rep}(\mathfrak{A}, \mathbf{T})$. For those of the first kind we again use an equivariant reduction, namely, apply Theorem 6.4 to the loop a (setting $\mathfrak{X} = \{\,0\,\}$) and Theorem 6.5 to the minimal lines that form the preimage of a.

The proof of Lemma 10.8 is the most intricate. Here we use a new class of boxes called *quasi-triangular*. Roughly speaking, a box is quasi-triangular if it becomes semi-free triangular after making invertible the arrows of several lines that become minimal after this procedure. Actually, in [14, Lemma 8.4] we prove a generalization of Lemma 10.8 for a quasi-triangular box $\mathfrak{A}$ using a generalized version of reduction algorithm to arrange an equivariant reduction (we refer to [14] for technical details). □

As we have already noticed, if $\mathbf{G}$ has elements of finite order, there is not a simple relation between representations of $\mathfrak{A}$ and $\mathbf{G}\backslash\mathfrak{A}$. Nevertheless, there is evidence that the following result might hold.

Conjecture 10.9. *Suppose that a group $\mathbf{G}$, which has no elements of order equal to $\operatorname{char}\mathbf{k}$, acts freely on a semi-free triangular box $\mathfrak{A}$.*

1. *$\mathfrak{A}$ is tame if and only if so is $\overline{\mathfrak{A}} = \mathbf{G}\backslash\mathfrak{A}$.*
2. *If these boxes are tame, then*
   - (a) *$\mathsf{ind}(\overline{\mathfrak{A}}) = \mathsf{ind}_0 \sqcup \mathsf{ind}_1$, where ind_0 consists of direct summands of images $\{\,\Pi_*M \mid M \in \mathsf{ind}(\mathfrak{A})\,\}$ and $\mathsf{ind}_1 = |\mathfrak{N}|$,*

where $\mathfrak{N}$ *is a set of strict representations of* $\mathfrak{A}$ *over the algebra* $\mathbf{T}=\mathbf{k}[t,t^{-1}]$.

(b) $\mathrm{Hom}_{\mathfrak{A}}(M,M')\subseteq\mathrm{rad}^{\infty}(\overline{\mathfrak{A}})$ *if* $M\in\mathsf{ind}_0$, $M'\in\mathsf{ind}_1$, *or vice versa, or* M,M' *belong to different rational families from* $\mathfrak{N}$.

(c) *If* $M\simeq N\otimes_{\mathbf{T}}L$, $M'\simeq N\otimes_{\mathbf{T}}L'$ *for some* $N\in\mathfrak{N}$ *and* $L,L'\in\mathsf{ind}(\mathbf{T})$ *, then* $\mathrm{Hom}_{\mathfrak{A}}(M,M')=1\otimes\mathrm{Hom}_{\mathbf{T}}(L,L')\oplus H$ *, where* $H=\mathrm{Hom}_{\mathfrak{A}}(M,M')\cap\mathrm{rad}^{\infty}(\overline{\mathfrak{A}})$.

The same is true for representations of locally finite dimensional categories.

(For instance, these properties always hold if $\mathrm{char}\,\mathbf{k}=0$ *.)*

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