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DERIVED TAME LOCAL AND TWO-POINT ALGEBRAS

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Abstract. We determine derived representation type of complete finitely generated local and two-point algebras over an algebraically closed field.

Introduction

We consider finitely generated unital (associative) algebras over an algebraically closed field $\Bbbk$.

One of the main problems in the representation theory of algebras is a classification of indecomposable finitely generated modules. The dichotomy theorem [14] divides all finite dimensional algebras according to their *representation type* into *tame* and *wild*. In the case of tame algebras a classification of indecomposable modules is relatively easy, for each dimension d they admit a parametrization of d-dimensional indecomposable modules by a finite number of 1-parameter families. The situation is much more complicated in the case of wild algebras. This singles out the problem of establishing the representation type of a given algebra. The answer is fully known for complete local algebras (those algebras whose quiver contains a single vertex) [6, 7, 13, 21, 23, 28, 31, 32] and for finite dimensional two-point algebras [4, 8, 18, 19, 20, 25, 27]. In the case of infinite dimensional two-point algebras the problem is still open, except the pure noetherian algebras [15].

During the last years there has been an active study of derived categories. In particular, a notion of *derived representation type* was introduced for finite dimensional algebras [22]. The tame-wild dichotomy for derived categories over finite dimensional algebras was established in [3]. The structure of the derived category is known for a few classes of finite dimensional algebras (e.g. [9, 10, 24, 26]).

On the other hand, certain infinite dimensional algebras and their derived categories play an important role in applications, in particular in the study of singularities of projective curves (cf. [11]). First results on derived representation type in the infinite dimensional case were obtained in [10].

In the present paper we determine representation type of the bounded derived category of finitely generated modules over finitely generated complete local and two-point algebras.

Our main results are the following classification theorems

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Theorem A. *Let* $\mathbf{A}$ *be a complete local algebra over algebraically closed field* $\mathbb{k}$*. Then* $\mathbf{A}$ *is derived tame if and only if* $\mathbf{A}$ *is isomorphic to one of the following algebras:*

- $\mathbf{L}_1 = \mathbb{k}$.
- *The* algebra of dual numbers $\mathbf{L}_2 = \mathbb{k}[x]/(x^2)$.
- *The power series algebra* $\mathbf{L}_3 = \mathbb{k}[[x]]$.
- $\mathbf{L}_4 = \mathbb{k}[[x, y]]/(xy)$ *- the local ring of a* simple node *of an algebraic curve over a field* $\mathbb{k}$*.*
- *The* dihedral algebra $\mathbf{L}_5 = \mathbb{k}\langle\langle x, y\rangle\rangle/(x^2, y^2)$.

Moreover, the first algebra is derived finite and the second and third are derived discrete.

Theorem B. *Let* $\mathbf{A}$ *be the completion of a two-point algebra* $\mathbb{k}\mathcal{Q}/\mathcal{I}$ *over algebraically closed field* $\mathbb{k}$*. Then*

(1) The following conditions are equivalent:

(i) $\mathbf{A}$ *is derived tame.*

(ii) $\mathbf{A}$ *is either a gentle algebra or a nodal non-gentle algebra (see Section 2 for definitions) or one of the algebras* $\mathbf{D}_1$*,* $\mathbf{D}_2$ *(see Section 2.4).*

(iii) $\mathbf{A}$ *is isomorphic to one of the algebras from Table 2 or to the algebra* (9) *from Table 1 or to one of the algebras* $\mathbf{D}_1$*,* $\mathbf{D}_2$ [1].

(2) $\mathbf{A}$ *is derived discrete if and only if* $\mathbf{A}$ *is isomorphic to one of the algebras* $(1), (3) - (5), (10) - (13), (16) - (17)$ *from Table 2.*

(3) $\mathbf{A}$ *is derived finite if and only if* $\mathbf{A}$ *is isomorphic to the algebra* (1) *from Table 2.*

Remark.

- Our definition (Definition 2.5) of a gentle algebra is slightly different from the standard one: we do not require the finite dimensionality of such algebra but instead require its completeness.
- Note that the classes of nodal and gentle algebras are not disjoint. In the case of two-point algebras there exists exactly one (up to isomorphism) nodal algebra which is not gentle. Obviously, there are many gentle algebras which are not nodal.

The structure of the paper is as follows. In Section 1 preliminary results about derived categories and derived representation type are given.

In Section 2 we recall the definitions of nodal and gentle algebras, and classify all such algebras in local and two-point cases.

In Section 3 we prove Theorems A and B.

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[1]Note that all algebras from Table 1, except (9), are also in Table 2

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1. Derived representation type

Let $\mathbf{A}$ be a *semi-perfect* [2] associative finitely generated $\Bbbk$-algebra. We denote by $\mathbf{A}$-mod the category of left finitely generated $\mathbf{A}$-modules and by $\mathcal{D}(\mathbf{A})$ the derived category $\mathcal{D}^b(\mathbf{A}\text{-mod})$ of bounded complexes over $\mathbf{A}$-mod. As usually, it can be identified with the homotopy category $\mathcal{K}^{-,b}(\mathbf{A}\text{-pro})$ of (right bounded) complexes of (finitely generated) projective $\mathbf{A}$-modules with bounded cohomologies. Since $\mathbf{A}$ is semi-perfect, each complex from $\mathcal{K}^{-,b}(\mathbf{A}\text{-pro})$ is homotopic to a *minimal* one, i.e. to a complex $C_\bullet = (C_n, d_n)$ such that $\operatorname{Im} d_n \subseteq \operatorname{rad} C_{n-1}$ for all n. If $C_\bullet$ and $C'_\bullet$ are two minimal complexes, they are isomorphic in $\mathcal{D}(\mathbf{A})$ if and only if they are isomorphic as complexes. Moreover, any morphism $f : C_\bullet \to C'_\bullet$ in $\mathcal{D}(\mathbf{A})$ can be presented by a morphism of complexes, and f is an isomorphism if and only if the latter one is. We denote by $\mathcal{P}_{\min}(\mathbf{A})$ the category of minimal right bounded complexes of (finitely generated) projective $\mathbf{A}$-modules with bounded cohomologies.

Let $A_1, A_2, \dots, A_t$ be all pairwise non-isomorphic indecomposable projective $\mathbf{A}$-modules (all of them are direct summands of $\mathbf{A}$). If P is a finitely generated projective $\mathbf{A}$-module, it uniquely decomposes as

$$P = \bigoplus_{i=1}^{t} p_i A_i.$$

Denote by $\mathbf{r}(P)$ the vector $(p_1, p_2, \dots, p_t)$. The sequence

$$(\dots, \mathbf{r}(P_n), \mathbf{r}(P_{n-1}), \dots)$$

(it has only finitely many nonzero entries) is called the *vector rank* $\mathbf{r}_\bullet(P_\bullet)$ of a bounded complex $P_\bullet$ of projective $\mathbf{A}$-modules.

The following definition is analogous to the definitions of *derived tame* and *derived wild* type for finite dimensional algebras [3].

Definition 1.1.
1. We call a *rational family* of bounded minimal complexes over $\mathbf{A}$ a bounded complex $(P_\bullet, d_\bullet)$ of finitely generated projective $\mathbf{A}\otimes\mathsf{R}$-modules, where R is a *rational algebra*, i.e. $\mathsf{R} = \Bbbk[t, f(t)^{-1}]$ for a nonzero polynomial $f(t)$, and $\operatorname{Im} d_n \subseteq \mathsf{J}P_{n-1}$, where $\mathsf{J} = \operatorname{rad}\mathbf{A}$. For a rational family $(P_\bullet, d_\bullet)$ we define the complex $P_\bullet(m,\lambda) = (P_\bullet \otimes_{\mathsf{R}} \mathsf{R}/(t-\lambda)^m, d_\bullet \otimes 1)$ of projective $\mathbf{A}$-modules, where $m \in \mathbb{N}$, $\lambda \in \Bbbk$, $f(\lambda) \neq 0$. Set $\mathbf{r}_\bullet(P_\bullet) = \mathbf{r}_\bullet(P_\bullet(1,\lambda))$ ($\mathbf{r}_\bullet$ does not depend on λ).
2. We call an algebra $\mathbf{A}$ *derived tame* if there is a set $\mathfrak{P}$ of rational families of bounded complexes over $\mathbf{A}$ such that:
   (a) For each vector rank $\mathbf{r}_\bullet$ the set $\mathfrak{P}(\mathbf{r}_\bullet) = \{\, P_\bullet \in \mathfrak{P} \mid \mathbf{r}_\bullet(P_\bullet) = \mathbf{r}_\bullet \,\}$ is finite.
   (b) For each vector rank $\mathbf{r}_\bullet$ all indecomposable complexes $(P_\bullet, d_\bullet)$ of projective $\mathbf{A}$-modules of this vector rank, except finitely many isomorphism classes, are isomorphic to $P_\bullet(m,\lambda)$ for some $P_\bullet \in \mathfrak{P}$ and some m, λ.

The set $\mathfrak{P}$ is called a *parameterizing set* of $\mathbf{A}$-complexes.

3. We call an algebra $\mathbf{A}$ *derived wild* if there is a bounded complex $(P_\bullet, d_\bullet)$ of projective modules over $\mathbf{A} \otimes \Sigma$, where Σ is the free $\Bbbk$-algebra in 2 variables, such that $\operatorname{Im} d_n \subseteq \mathsf{J}P_{n-1}$ and, for any finite dimensional Σ-modules L, L',
   (a) $P_\bullet \otimes_\Sigma L \simeq P_\bullet \otimes_\Sigma L'$ if and only if $L \simeq L'$.
   (b) $P_\bullet \otimes_\Sigma L$ is indecomposable if and only if so is L.

Note that, according to these definitions, every *derived discrete* (in particular, *derived finite* [9]) algebra [35] is derived tame (with the empty set $\mathfrak{P}$).

It is proved in [3] that every finite dimensional algebra over an algebraically closed field is either derived tame or derived wild.

2. Some classes of algebras

2.1. **Quivers with relations.** A *quiver* $\mathcal{Q}$ is a tuple $(\mathcal{Q}_0, \mathcal{Q}_1, \mathfrak{s}, \mathfrak{t})$ consisting of a set $\mathcal{Q}_0$ of *vertices*, a set $\mathcal{Q}_1$ of *arrows*, and maps $\mathfrak{s}, \mathfrak{t} : \mathcal{Q}_1 \to \mathcal{Q}_0$ which specify the *starting* and *ending* vertices. A *path* p in $\mathcal{Q}$ of length $\ell(p) = n \geq 1$ is a sequence of arrows $a_n, \dots, a_1$ such that $\mathfrak{s}(a_{i+1}) = \mathfrak{t}(a_i)$ for $1 \leq i < n$. Note that we write paths from right to left for convenience. For a path p set $\mathfrak{s}(p) = \mathfrak{s}(a_1)$ and $\mathfrak{t}(p) = \mathfrak{t}(a_n)$. Then the concatenation $p'p$ of two paths p, p' is defined in the natural way whenever $\mathfrak{s}(p') = \mathfrak{t}(p)$. Every vertex $i \in \mathcal{Q}_0$ determines a path e_i (of length 0) with $\mathfrak{s}(e_i) = i$ and $\mathfrak{t}(e_i) = i$. A quiver $\mathcal{Q}$ determines the path algebra $\Bbbk\mathcal{Q}$, which has an $\Bbbk$-basis consisting of the paths of $\mathcal{Q}$ with multiplication given by the concatenation of paths. The algebra $\Bbbk\mathcal{Q}$ is finite-dimensional precisely when $\mathcal{Q}$ does not contain an oriented cycle. An ideal $\mathcal{I} \subseteq \Bbbk\mathcal{Q}$ is called *admissible* if $\mathcal{I} \subseteq \operatorname{rad}^2(\Bbbk\mathcal{Q})$ where $\operatorname{rad}(\Bbbk\mathcal{Q})$ is the radical of the algebra $\Bbbk\mathcal{Q}$. It is well-known that if $\Bbbk$ is algebraically closed, any finite-dimensional $\Bbbk$-algebra is Morita equivalent to a quotient $\Bbbk\mathcal{Q}/\mathcal{I}$ where $\mathcal{I}$ is an admissible ideal. By a slight abuse of notation we identify paths in the quiver $\mathcal{Q}$ with their cosets in $\Bbbk\mathcal{Q}/\mathcal{I}$.

2.2. **Nodal algebras.**

Definition 2.1. A semi-perfect noetherian algebra $\mathbf{A}$ is called *nodal* if it is *pure noetherian* (i.e. has no minimal ideals), and there is a hereditary algebra $\mathbf{H} \supseteq \mathbf{A}$, which is semi-perfect and pure noetherian such that

- $\operatorname{rad} \mathbf{A} = \operatorname{rad} \mathbf{H}$.
- $\operatorname{lenght}_{\mathbf{A}}(\mathbf{H} \otimes_{\mathbf{A}} U) \leq 2$ for every simple left $\mathbf{A}$-module U.
- $\operatorname{lenght}_{\mathbf{A}}(V \otimes_{\mathbf{A}} \mathbf{H}) \leq 2$ for every simple right $\mathbf{A}$-module V.

It was shown in [15] that nodal algebras are the only pure noetherian algebras such that the classification of their modules of finite length is tame (all others being wild).

Proposition 2.2. *(1) Let $\mathbf{A}$ be a local $\Bbbk$-algebra. Then $\mathbf{A}$ is nodal if and only if it is isomorphic to one of the following algebras:*

- *The algebra $\Bbbk[[x]]$ of power series.*

- *The local ring* $\Bbbk[[x,y]]/(xy)$ *of a* simple node *of an algebraic curve over* $\Bbbk$.
- *The* dihedral algebra $\Bbbk\langle\langle x,y\rangle\rangle/(x^2,y^2)$.

(2) Let $\mathbf{A}=\Bbbk\mathcal{Q}/\mathcal{I}$ *be a two-point algebra. Then the following conditions are equivalent:*

- $\mathbf{A}$ *is nodal.*
- $\mathbf{A}$ *is isomorphic to the completion of one of the algebras from Table 1 below.*

TABLE 1. Nodal two-point algebras

<table>
<tr>
<td>$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd}$
(1) $\mathcal{I}=0$</td>
<td>$\begin{tikzcd} 1 \arrow[r, bend left=60, "a"] \arrow[r, bend left=20, "b"'] & 2 \arrow[l, bend left=20, "c"'] \arrow[l, bend left=60, "d"] \end{tikzcd}$
(2) $\mathcal{I}=\langle ca,db,ac,bd\rangle$
(3) $\mathcal{I}=\langle ca,db,bc,ad\rangle$</td>
</tr>
<tr>
<td>$\begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[r, bend left, "b"] & 2 \arrow[l, bend left, "c"] \end{tikzcd}$
(4) $\mathcal{I}=\langle a^2,bc\rangle$
(5) $\mathcal{I}=\langle ba,ac\rangle$</td>
<td>$\begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[r, bend left, "c"] & 2 \arrow[l, bend left, "d"] \arrow[loop right, "b"] \end{tikzcd}$
(6) $\mathcal{I}=\langle a^2,b^2,dc,cd\rangle$
(7) $\mathcal{I}=\langle a^2,db,bc,cd\rangle$
(8) $\mathcal{I}=\langle ca,db,bc,ad\rangle$
(9) $\mathcal{I}=\langle a^2-dc,b^2-cd,ca-bc,db-ad\rangle$</td>
</tr>
</table>

Proof. All algebras under consideration are of the form $\mathbf{A}=\widehat{\Bbbk\mathcal{Q}}/\mathcal{I}$ for some finite connected quiver $\mathcal{Q}$ and some admissible ideal $\mathcal{I}\subseteq\widehat{\Bbbk\mathcal{Q}}$. In particular, $\dim_{\Bbbk}U=1$ for every simple $\mathbf{A}$-module U. Recall first that every hereditary pure noetherian algebra of this form is isomorphic to a direct product of algebras of type $\widehat{\Bbbk\mathcal{Q}_n}$, where $\mathcal{Q}_n$ is a cycle

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[lll, bend left] \end{tikzcd},$$

or, equivalently, subalgebras $\mathbf{H}_n$ in $\mathrm{Mat}(n,\mathbf{S})$, where $\mathbf{S}=\Bbbk[[t]]$, consisting of all matrices (a_{ij}) such that $a_{ij}(0)=0$ for $i<j$. If $\mathbf{A}$ satisfies the conditions of Definition 2.1, then the algebra $\mathbf{H}$ is Morita-equivalent to an algebra of this form. Let $\mathbf{J}=\operatorname{rad}\mathbf{A}=\operatorname{rad}\mathbf{H}$. Note that $\mathbf{H}/\mathbf{J}\simeq\mathbf{H}\otimes_{\mathbf{A}}(\mathbf{A}/\mathbf{J})$ as left $\mathbf{H}$-module.

If $\mathbf{A}$ is local, then $d=\dim_{\Bbbk}\mathbf{H}/\mathbf{J}\le 2$. If $d=1$, $\mathbf{A}=\mathbf{H}\simeq\mathbf{S}$. If $d=2$, then either $\mathbf{H}\simeq\mathbf{S}\times\mathbf{S}$ or $\mathbf{H}\simeq\mathbf{H}_2$. In both cases $\mathbf{H}/\mathbf{J}\simeq\Bbbk\times\Bbbk$ and $\mathbf{A}/\mathbf{J}\simeq\Bbbk$ can be embedded into $\mathbf{H}/\mathbf{J}$ only diagonally. Therefore, in the former case

$\mathbf{A}$ is identified with the subalgebra in $\mathbf{S}\times\mathbf{S}$ consisting of all pairs (a,b) such that $a(0)=b(0)$, i.e. $\mathbf{A}\simeq \Bbbk[[x,y]]/(xy)$ (take $(t,0)$ for x and $(0,t)$ for y). In the latter case $\mathbf{A}$ is identified with the subalgebra in $\mathbf{H}_2$ consisting of matrices (a_{ij}) such that $a_{11}(0)=a_{22}(0)$, i.e. $\mathbf{A}\simeq\langle x,y\rangle/(x^2,y^2)$ (take e_{21} for x and te_{12} for y).

If $\mathbf{A}$ is two-point, i.e. $\mathbf{A}/\mathbf{J}\simeq\Bbbk^2$, then $d=\dim_{\Bbbk}\mathbf{H}/\mathbf{J}\le 4$. Note that if $d=2$, then $\mathbf{A}=\mathbf{H}\simeq\widehat{\Bbbk\mathcal{Q}}_2$. So we can assume that $d=3$ or 4. There are the following possibilities (taking into account that $\mathbf{A}$ is connected):

Case 1. $\mathbf{H}=\mathrm{Mat}(2,\mathbf{S})$. Then $\mathbf{H}/\mathbf{J}\simeq\mathrm{Mat}(2,\Bbbk)$. Any subalgebra of $\mathrm{Mat}(2,\Bbbk)$ isomorphic to $\Bbbk^2$ is conjugate to the subalgebra of diagonal matrices. Therefore, $\mathbf{A}$ is isomorphic to the subalgebra of $\mathrm{Mat}(2,\mathbf{S})$ consisting of matrices (a_{ij}) such that $a_{12}(0)=a_{21}(0)=0$, i.e. to the algebra (9) from Table 1 (take te_{11} for a, te_{22} for b, te_{21} for c and te_{12} for d).

Case 2. $\mathbf{H}=\mathbf{H}_3$. Then $\mathbf{H}/\mathbf{J}\simeq\Bbbk^3$ and the embedding $\Bbbk^2\to\Bbbk^3$ (up to a permutation of components) maps (α,β) to (α,α,β). Therefore, $\mathbf{A}$ is isomorphic to the subalgebra of $\mathbf{H}_3$ consisting of matrices (a_{ij}) such that $a_{ii}(0)=a_{jj}(0)$ for some choice of two different indices $i,j\in\{1,2,3\}$. One can check that all choices lead to isomorphic algebras, namely, to the algebra (4) from Table 1 (if $i=1$, $j=2$, take e_{21} for a, e_{32} for b and te_{13} for c).

Case 3. $\mathbf{H}=\mathbf{S}\times\mathbf{H}_2$. Again $\mathbf{H}/\mathbf{J}\simeq\Bbbk^3$ and the embedding $\Bbbk^2\to\Bbbk^3$ (up to a permutation of components) maps (α,β) to (α,α,β). Therefore, $\mathbf{A}$ is isomorphic to the subalgebra of $\mathbf{H}$ consisting of all pairs $(a,(b_{ij}))$ such that $a(0)=b_{ii}(0)$ for some $i\in\{1,2\}$. Again both choices lead to isomorphic algebras, namely, to the algebra (5) from Table 1 (if $i=1$, take the pair $(t,0)$ for a, $(0,e_{21})$ for b and $(0,te_{12})$ for c).

Case 4. $\mathbf{H}=\mathbf{H}_4$. Then $\mathbf{H}\simeq\Bbbk^4$ and the embedding $\Bbbk^2\to\Bbbk^4$ (up to a permutation of components) maps (α,β) to $(\alpha,\alpha,\beta,\beta)$ or to $(\alpha,\alpha,\alpha,\beta)$. The latter case is impossible, since the length of $\mathbf{H}\otimes_{\mathbf{A}}U$ equals 3, where U is the simple $\mathbf{A}$-module on which the first component of $\Bbbk^2$ acts nontrivially. Hence, to define $\mathbf{A}$ up to an isomorphism we need to choose an index $k\in\{2,3,4\}$; then $\mathbf{A}$ is isomorphic to the subalgebra of $\mathbf{H}$ consisting of all matrices (a_{ij}) such that $a_{11}(0)=a_{kk}(0)$ and $a_{ii}(0)=a_{jj}(0)$, where $\{1,2,3,4\}=\{1,k,i,j\}$. One easily sees that the choices $k=2$ and $k=4$ lead to isomorphic algebras, and they are isomorphic to the algebra (6) from Table 1 (for $k=2$ take e_{21} for a, e_{43} for b, e_{32} for c and te_{14} for d). The case $k=3$ gives the algebra (3) from Table 1 (take te_{14} for a, e_{32} for b, e_{43} for c and e_{21} for d).

Case 5. $\mathbf{H}=\mathbf{S}\times\mathbf{H}_3$. The same considerations as in Case 4 show that $\mathbf{A}$ is isomorphic to the subalgebra of $\mathbf{H}$ consisting of all pairs $(a,(b_{ij}))$ such that $a(0)=b_{11}(0)$ and $a_{33}(0)=a_{22}(0)$, i.e. to the algebra (7) from Table 1 (take the pair $(0,e_{32})$ for a, $(t,0)$ for b, $(0,te_{13})$ for c and $(0,e_{21})$ for d).

Case 6. $\mathbf{H}=\mathbf{H}_2\times\mathbf{H}_2$. It follows, as above, that $\mathbf{A}$ is isomorphic to the subalgebra of $\mathbf{H}$ consisting of all pairs $((a_{ij}),(b_{ij}))$ such that $a_{ii}(0)=b_{ii}(0)$ for $i=1,2$, i.e. to the algebra (2) from Table 1 (take the pair $(0,te_{12})$ for a, $(te_{12},0)$ for b, $(e_{21},0)$ for c and $(0,e_{21})$ for d).

CASE 7. $\mathbf{H} = \mathbf{S} \times \mathbf{S} \times \mathbf{H}_2$. Then $\mathbf{A}$ is isomorphic to the subalgebra of $\mathbf{H}$ consisting of triples $(a, b, (c_{ij}))$ such that $a(0) = c_{11}(0)$ and $b(0) = c_{22}(0)$, i.e. to the algebra (8) from Table 1 (take the triple $(t, 0, 0)$ for a, $(0, t, 0)$ for b, $(0, 0, te_{12})$ for c and $(0, 0, e_{21})$ for d).

□

2.3. **Gentle algebras.** Let $\mathcal{Q}$ be a quiver and $\mathcal{I}$ an admissible ideal in the path algebra $\Bbbk\mathcal{Q}$.

Definition 2.3. The pair $(\mathcal{Q}, \mathcal{I})$ is said to be *special biserial* if the following holds:

(G1) At every vertex of $\mathcal{Q}$ at most two arrows end and at most two arrows start.
(G2) For each arrow b there is at most one arrow a with $\mathfrak{t}(a) = \mathfrak{s}(b)$ and $ba \notin \mathcal{I}$ and at most one arrow c with $\mathfrak{t}(b) = \mathfrak{s}(c)$ and $cb \notin \mathcal{I}$.

Definition 2.4. The pair $(\mathcal{Q}, \mathcal{I})$ is said to be *gentle* if it is special biserial, and moreover the following holds:

(G3) $\mathcal{I}$ is generated by zero relations of length 2.
(G4) For each arrow b there is at most one arrow a with $\mathfrak{t}(a) = \mathfrak{s}(b)$ and $ba \in \mathcal{I}$ and at most one arrow c with $\mathfrak{t}(b) = \mathfrak{s}(c)$ and $cb \in \mathcal{I}$.

Definition 2.5. A $\Bbbk$-algebra $\mathbf{A}$ is called *special biserial* (respectively, *gentle*), if it is Morita equivalent to the completion of an algebra $\Bbbk\mathcal{Q}/\mathcal{I}$, where the pair $(\mathcal{Q}, \mathcal{I})$ is special biserial (respectively, gentle).

Remark. Note that Definitions 2.3, 2.4, 2.5 do not require the finite dimensionality of the algebra $\mathbf{A}$. In the finite dimensional case special biserial algebras were defined in [34], while gentle algebras were defined in [1]. Also note that gentle algebras without completion appeared in [12] under the name *locally gentle* algebras.

The proof of the following statement is straightforward.

Proposition 2.6. *(1) Let $\mathbf{A}$ be a complete local algebra over $\Bbbk$. Then $\mathbf{A}$ is gentle if and only if $\mathbf{A}$ is isomorphic to one of the following algebras:*

- $\mathbf{L}_1 = \Bbbk$.
- $\mathbf{L}_2 = \Bbbk[x]/(x^2)$.
- $\mathbf{L}_3 = \Bbbk[[x]]$.
- $\mathbf{L}_4 = \Bbbk[[x, y]]/(xy)$.
- $\mathbf{L}_5 = \Bbbk\langle\langle x, y\rangle\rangle/(x^2, y^2)$.

(2) Let $\Bbbk\mathcal{Q}/\mathcal{I}$ be a two-point algebra over $\Bbbk$ and $\mathbf{A}$ its completion. Then the following conditions are equivalent:

- $\mathbf{A}$ *is gentle.*
- $\mathbf{A}$ *is isomorphic to one of the algebras from Table 2 below.*

Remark. Note that algebras (3), (8), (9), (14), (15) and (22) − (24) from Table 2 are nodal. Note also that algebras (7), (11), (13), (17) and (19)−(21) are infinite dimensional but not nodal.

TABLE 2. Gentle two-point algebras

$\mathcal{Q}1: \begin{tikzcd} 1 \arrow[r, "a"] & 2 \end{tikzcd}$ (1) $\mathcal{I} = 0$	$\mathcal{Q}2: \begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 2 \end{tikzcd}$ (2) $\mathcal{I} = 0$
$\mathcal{Q}3: \begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd}$ (3) $\mathcal{I} = 0$ (4) $\mathcal{I} = \langle ba \rangle$ (5) $\mathcal{I} = \langle ba, ab \rangle$	$\mathcal{Q}4: \begin{tikzcd} 1 \arrow[r, bend left=60, "a"] \arrow[r, bend left, "b"] & 2 \arrow[l, bend left, "c"] \end{tikzcd}$ (6) $\mathcal{I} = \langle ca, bc \rangle$ (7) $\mathcal{I} = \langle ca, ac \rangle$
$\mathcal{Q}5: \begin{tikzcd} 1 \arrow[r, bend left=60, "a"] \arrow[r, bend left, "b"] & 2 \arrow[l, bend left, "c"] \arrow[l, bend left=60, "d"] \end{tikzcd}$ (8) $\mathcal{I} = \langle ca, db, ac, bd \rangle$ (9) $\mathcal{I} = \langle ca, db, bc, ad \rangle$	$\mathcal{Q}6: \begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[r, "b"] & 2 \end{tikzcd}$ (10) $\mathcal{I} = \langle a^2 \rangle$ (11) $\mathcal{I} = \langle ba \rangle$
$\mathcal{Q}7: \begin{tikzcd} 1 \arrow[r, "b"] & 2 \arrow[loop above, "a"] \end{tikzcd}$ (12) $\mathcal{I} = \langle a^2 \rangle$ (13) $\mathcal{I} = \langle ab \rangle$	$\mathcal{Q}8: \begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[r, bend left, "b"] & 2 \arrow[l, bend left, "c"] \end{tikzcd}$ (14) $\mathcal{I} = \langle a^2, bc \rangle$ (15) $\mathcal{I} = \langle ba, ac \rangle$ (16) $\mathcal{I} = \langle a^2, bc, cb \rangle$ (17) $\mathcal{I} = \langle ba, ac, cb \rangle$
$\mathcal{Q}9: \begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[r, "c"] & 2 \arrow[loop above, "b"] \end{tikzcd}$ (18) $\mathcal{I} = \langle a^2, b^2 \rangle$ (19) $\mathcal{I} = \langle a^2, bc \rangle$ (20) $\mathcal{I} = \langle ca, b^2 \rangle$ (21) $\mathcal{I} = \langle ca, bc \rangle$	$\mathcal{Q}10: \begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[r, bend left, "c"] & 2 \arrow[l, bend left, "d"] \arrow[loop right, "b"] \end{tikzcd}$ (22) $\mathcal{I} = \langle a^2, b^2, dc, cd \rangle$ (23) $\mathcal{I} = \langle a^2, db, bc, cd \rangle$ (24) $\mathcal{I} = \langle ca, db, bc, ad \rangle$

It was shown in [30] that any finite dimensional gentle algebra is derived tame. The proof of this result from [30] can not be adapted for the case of infinite dimensional gentle algebras. On the other hand, in [9] a different approach was used to obtain a classification of indecomposable objects in derived categories over finite dimensional gentle algebras. This approach is based on the reduction of the classification problem to a matrix problem considered by Bondarenko in [5]. We note that with minor modifications the same reduction works in the case of infinite dimensional gentle algebras. Hence we immediately obtain the following result.

Theorem 2.7. *Any gentle algebra is derived tame.*

2.4. **Two deformations of gentle algebras.** Consider the following quiver $\mathcal{Q}$:

$$\begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[r, "c"] & 2 \arrow[loop above, "b"] \end{tikzcd}$$

Let $\mathbf{D}_i = \Bbbk\mathcal{Q}/\mathcal{I}_i$, $i = 1, 2$, where $\mathcal{I}_1 = \langle a^2, ca - bc\rangle$ and $\mathcal{I}_2 = \langle b^2, ca - bc\rangle$. These two algebras are anti-isomorphic.

Consider $\mathbf{A}^1_\lambda = \Bbbk\mathcal{Q}/\mathcal{I}_1$, $\mathbf{A}^2_\lambda = \Bbbk\mathcal{Q}/\mathcal{I}_2$, where $\mathcal{I}_1 =< a^2, bc - \lambda ca >$, $\mathcal{I}_2 =< b^2, ca - \lambda bc >$. Note that $\mathbf{A}^1_\lambda$ is a deformation of (20), while $\mathbf{A}^2_\lambda$ is a deformation of (21) from Table 2. Clearly, $\mathbf{A}^1_\lambda \simeq \mathbf{D}_1$ and $\mathbf{A}^2_\lambda \simeq \mathbf{D}_2$ for any $\lambda \neq 0$.

Lemma 2.8. *Algebras $\mathbf{D}_1$ and $\mathbf{D}_2$ are derived tame.*

Proof. Let $\mathbf{A} = \begin{bmatrix} \mathbf{S} & t\mathbf{S} \\ t\mathbf{S} & \mathbf{S} \end{bmatrix}$, where $\mathbf{S} = \Bbbk[[t]]$.

Then we have the following short exact sequence:

$$0 \longrightarrow P_1 \xrightarrow{t} P_2 \longrightarrow L \longrightarrow 0$$

where $P_1 = \begin{bmatrix} \mathbf{S} \\ t\mathbf{S} \end{bmatrix}$, $P_2 = \begin{bmatrix} t\mathbf{S} \\ \mathbf{S} \end{bmatrix}$ are indecomposable projective left $\mathbf{A}$-modules and $L = \begin{bmatrix} 0 \\ \frac{\mathbf{S}}{t^2\mathbf{S}} \end{bmatrix}$.

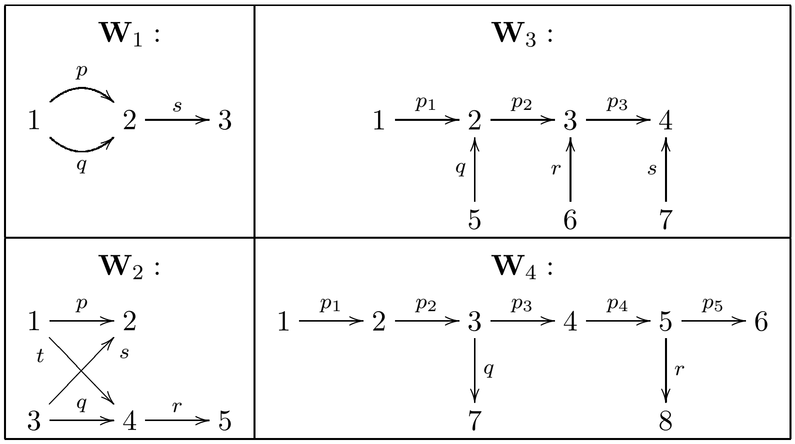
Define a complex $T_\bullet = T_0 \oplus T_1$ of $\mathbf{A}$-modules as follows. Let $T_0 : 0 \to L \to 0$ (in degree -1) and $T_1 : 0 \to P_1 \to 0$ (in degree 0). It is easy to check that the complex $T_\bullet$ is tilting (see [33] for definition) and the endomorphism algebra $\mathrm{End}_{\mathcal{D}^b(\mathbf{A}\text{-mod})}(T_\bullet)$ is isomorphic to $\mathbf{D}_1$. Since $\mathbf{A}$ is isomorphic to the algebra (9) from the Table 1, it is derived tame by [10]. Therefore algebra $\mathbf{D}_1$ is also derived tame. The case of the algebra $\mathbf{D}_2$ is similar. $\square$

Remark. Both algebras (20) and (21) are derived tame by [9]. It is known that in the finite dimensional case the tameness of an algebra implies the tameness of its deformations [17]. But it is an open question in the infinite dimensional case.

3. Classification

3.1. **Derived wildness.** We will need the following hereditary algebras which are used in the next sections.

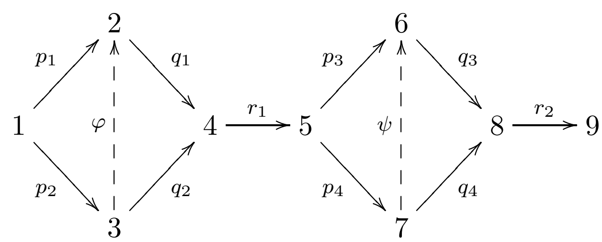
Table 3. Some wild hereditary algebras



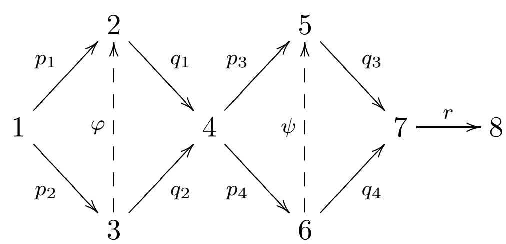
It is well known that the algebras $\mathbf{W}_1 - \mathbf{W}_4$ are wild [29].

We also need the following boxes (see [14] for definition), which will be used in the proof of Theorem A and Theorem B:

$\mathbf{W}_5$:



$\mathbf{W}_6$:



Let f be the quadratic form corresponding to the box $\mathbf{W}_5$ (resp., $\mathbf{W}_6$) (see [14] for definition). Consider the following dimension vector $\mathbf{d} = (d_i)_{i=1}^9 = (2,2,2,4,4,2,2,2,1)$ (resp., $\mathbf{d} = (d_i)_{i=1}^8 = (2,2,2,4,2,2,2,1)$). Since $f(\mathbf{d}) = -1$, it follows from [14] that $\mathbf{W}_5$ and $\mathbf{W}_6$ are wild.

We will use the following notations. Let $\mathbf{B}$ be one of the algebras $\mathbf{W}_1 - \mathbf{W}_4$ or one of the boxes $\mathbf{W}_5 - \mathbf{W}_6$. Since $\mathbf{B}$ is wild, there exists $\mathbf{B}$-$\Bbbk\langle x,y\rangle$-bimodule $M = M(\mathbf{B})$, finitely generated and free over $\Bbbk\langle x,y\rangle$ such that the functor $M \otimes_{\Bbbk\langle x,y\rangle} \_$, from the category of finite dimensional $\Bbbk\langle x,y\rangle$-modules to the category of $\mathbf{B}$-modules, preserves indecomposability and isomorphism classes. We denote by d_i^M the rank of $M(i)$ over $\Bbbk\langle x,y\rangle$.

From now on let $\mathbf{A}$ be the completion of an algebra $\Bbbk\mathcal{Q}/\mathcal{I}$ for some finite quiver $\mathcal{Q}$ and some admissible ideal $\mathcal{I}$. We denote by A_i the indecomposbale projective $\mathbf{A}$-module corresponding to the vertex i of $\mathcal{Q}$ and set $\widetilde{A}_i = A_i \otimes_{\Bbbk} \Bbbk\langle x,y\rangle$.

The following technical lemmas are needed for the proof.

Lemma 3.1. *Let $\mathbf{B}$ be a full subalgebra of $\mathbf{A}$ (i.e., a subalgebra of the form $e\mathbf{A}e$ for some idempotent e). If $\mathbf{B}$ is derived wild then $\mathbf{A}$ is derived wild.*

Proof. Obvious. □

Lemma 3.2. *Suppose that there exist $a,b\in\mathcal{Q}_1$ and $w=\sum_i\lambda_i w_i\neq 0$, where w_i are some paths of length ≥ 1, such that $\mathfrak{s}(w_i)=\mathfrak{s}(w_j)$ and $\mathfrak{t}(w_i)=\mathfrak{t}(w_j)$ for all i,j, $\lambda_i\in\Bbbk$, $\mathfrak{s}(a)=\mathfrak{s}(b)$, $\mathfrak{t}(a)=\mathfrak{t}(b)$, $\mathfrak{t}(a)=\mathfrak{s}(w)$ (resp. $\mathfrak{s}(a)=\mathfrak{t}(w)$) and $wa,wb\in\mathcal{I}$ (resp., $aw,bw\in\mathcal{I}$). Then $\mathbf{A}$ is derived wild.*

Proof. We assume that $\mathfrak{s}(a)=\mathfrak{t}(w)$ (the other case is similar). Let $M=M(\mathbf{W}_1)$. Denote by $N_\bullet$ the following complex of $\mathbf{A}-\Bbbk\langle x,y\rangle$-bimodules:

$$\begin{tikzcd} d_1\widetilde{A}_{\mathfrak{t}(a)} \arrow[r, bend left, "aM(p)"] \arrow[r, bend right, "bM(q)"'] & d_2\widetilde{A}_{\mathfrak{s}(a)} \arrow[r, "wM(s)"] & d_3\widetilde{A}_{\mathfrak{s}(w)} \end{tikzcd}$$

or, equivalently,

$$\cdots \longrightarrow 0 \longrightarrow d_1\widetilde{A}_{\mathfrak{t}(a)} \xrightarrow{aM(p)+bM(q)} d_2\widetilde{A}_{\mathfrak{s}(a)} \xrightarrow{wM(s)} d_3\widetilde{A}_{\mathfrak{s}(w)} \longrightarrow 0 \longrightarrow \cdots$$

It is not difficult to verify that the functor $N_\bullet\otimes_{\Bbbk\langle x,y\rangle}-$, which acts from the category of finite dimensional $\Bbbk\langle x,y\rangle$-modules to the category $\mathcal{P}_{\min}(\mathbf{A})$, preserves indecomposability and the isomorphism classes. Hence, $\mathbf{A}$ is derived wild. □

Lemma 3.3. *Suppose that there exist $a,b\in\mathcal{Q}_1$ such that $\mathfrak{s}(a)=\mathfrak{t}(a)=\mathfrak{t}(b)$ (resp., $\mathfrak{s}(a)=\mathfrak{t}(a)=\mathfrak{s}(b)$) and $a^2,ab\in\mathcal{I}$ (resp., $a^2,ba\in\mathcal{I}$). Then $\mathbf{A}$ is derived wild.*

Proof. We assume that $\mathfrak{s}(a)=\mathfrak{t}(a)=\mathfrak{s}(b)$ (the other case is similar). Let $M=M(\mathbf{W}_3)$ be as above. Let us denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x,y\rangle$-bimodules.

$$\begin{tikzcd} d_1\widetilde{A}_{\mathfrak{s}(a)} \arrow[r, "aM(p_1)"] & d_2\widetilde{A}_{\mathfrak{s}(a)} \arrow[r, "aM(p_2)"] & d_3\widetilde{A}_{\mathfrak{s}(a)} \arrow[r, "aM(p_3)"] & d_4\widetilde{A}_{\mathfrak{s}(a)} \\ d_5\widetilde{A}_{\mathfrak{t}(b)} \arrow[ur, "bM(q)"] & d_6\widetilde{A}_{\mathfrak{t}(b)} \arrow[ur, "bM(r)"] & d_7\widetilde{A}_{\mathfrak{t}(b)} \arrow[ur, "bM(s)"] & \end{tikzcd}$$

Here each column presents direct summands of a non-zero component N_n (in our case $n = 3, 2, 1, 0$) and the arrows show the non-zero components of the differential. Again applying the functor $N_\bullet \otimes_{\Bbbk\langle x,y\rangle} -$ we immediately obtain that $\mathbf{A}$ is derived wild. $\square$

3.2. **Proof of Theorem A.**

Proof. " $\Rightarrow$."

Suppose first that $\mathbf{A}$ is pure noetherian. Since $\mathbf{A}$ is derived tame, it is tame and hence nodal by [15]. Then it follows from Proposition 2.2 that $\mathbf{A}$ is isomorphic to one of the algebras $\mathbf{L}_3 - \mathbf{L}_5$.

Suppose now that $\mathbf{A}$ has some minimal ideal $\mathcal{J}$. If $\mathcal{Q}_1 = \emptyset$ then $\mathbf{A}$ is isomorphic to the algebra $\mathbf{L}_1$. Suppose that there exist $a, b \in \mathcal{Q}_1$, $a \neq b$. Consider any $0 \neq z \in \mathcal{J}$. Then $\mathbf{A}$ satisfies the conditions of Lemma 3.2, where $a = a, b = b$ and $w = z$, hence $\mathbf{A}$ is derived wild.

Therefore we can assume that $\mathcal{Q}_1$ has only one arrow, say a. Then $a^n \in \mathcal{J}$ for some $n \in \mathbb{N}, n > 1$. If $n = 2$ then $\mathbf{A}$ is isomorphic to $\mathbf{L}_2$. Assume that $n > 2$. Let $M = M(\mathbf{W}_6)$. Denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x, y\rangle$-bimodules:

$$\begin{tikzcd}
 & d_2\widetilde{A} \arrow[dr, "a^{n-1}M(q_1)"] & & d_5\widetilde{A} \arrow[dr, "a^{n-1}M(q_3)"] & & \\
d_1\widetilde{A} \arrow[ur, "a^nM(p_1)"] \arrow[dr, "a^{n-1}M(p_2)"'] & & d_4\widetilde{A} \arrow[ur, "a^nM(p_3)"] \arrow[dr, "a^{n-1}M(p_4)"'] & & d_7\widetilde{A} \arrow[r, "a^nM(r)"] & d_8\widetilde{A} \\
 & d_3\widetilde{A} \arrow[ur, "a^nM(q_2)"'] & & d_6\widetilde{A} \arrow[ur, "a^nM(q_4)"'] & &
\end{tikzcd}$$

Again it is easy to check that the functor $N_\bullet \otimes_{\Bbbk\langle x,y\rangle} -$ preserves indecomposability and the isomorphism classes. We conclude that $\mathbf{A}$ is derived wild.

" $\Leftarrow$." Since $\mathbf{L}_1$ and $\mathbf{L}_3$ are hereditary, it follows from [24] that $\mathbf{L}_1$ is derived finite and $\mathbf{L}_3$ is derived discrete but not derived finite. Since $\mathbf{L}_2$ is gentle, it follows from [9] that $\mathbf{L}_2$ is derived discrete but not derived finite. Since $\mathbf{L}_4$ and $\mathbf{L}_5$ are nodal algebras, it follows from [10] that $\mathbf{L}_4$ and $\mathbf{L}_5$ are derived tame but not derived discrete. $\square$

3.3. **Proof of Theorem B.**

Proof. (1) $(i) \Rightarrow (iii)$.

Since $\mathbf{A}$ is derived tame, then $\mathbf{A}$ is tame and hence $\Bbbk\mathcal{Q}/\operatorname{rad}^2(\Bbbk\mathcal{Q})$ is tame. Then we conclude that $\mathcal{Q}$ is one of the quivers from Table 2.

Let us consider all cases.

Case 1. $\mathcal{Q} = \mathcal{Q}1$. Then $\mathbf{A}$ is isomorphic to the algebra (1) from Table 2.

Case 2. $\mathcal{Q} = \mathcal{Q}2$. Then $\mathbf{A}$ is isomorphic to the algebra (2) from Table 2.

Case 3. $\mathcal{Q} = \mathcal{Q}3$. It follows from Theorem A and Lemma 3.1 that for $i \in \{1, 2\}$ we have $e_i\mathbf{A}e_i \cong \mathbf{L}_j$ for some $j \in \{1, 2, 3\}$. If $e_i\mathbf{A}e_i \cong \mathbf{L}_3$ for some i, then $\mathbf{A}$ is isomorphic to the algebra (3) from Table 2. If $e_i\mathbf{A}e_i \cong \mathbf{L}_1$ for $i = 1, 2$, then $\mathbf{A}$ is isomorphic to the algebra (5) from Table 2. If $e_i\mathbf{A}e_i \cong \mathbf{L}_1$ and $e_j\mathbf{A}e_j \cong \mathbf{L}_2$ for $i, j \in \{1, 2\}$, then $\mathbf{A}$ is isomorphic to the algebra (4) from Table 2. Suppose finally that $e_i\mathbf{A}e_i \cong \mathbf{L}_2$ for $i = 1, 2$. Then $baba, abab \in \mathcal{I}$, $ab \notin \mathcal{I}$ and $ba \notin \mathcal{I}$. Therefore, $aba \in \mathcal{I}$ or $bab \in \mathcal{I}$ or $\mathcal{I} = \langle abab, baba\rangle$. Let us consider all cases.

(a) $aba \in \mathcal{I}$, $bab \notin \mathcal{I}$.

Let $M = M(\mathbf{W}_5)$. Let us denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x, y\rangle$-bimodules:

$$\begin{tikzcd}
& d_2\widetilde{A}_2 \arrow[dr, "aM(q_1)"] & & & d_6\widetilde{A}_2 \arrow[dr, "aM(q_3)"] & & \\
d_1\widetilde{A}_2 \arrow[ur, "abM(p_1)"] \arrow[dr, "aM(p_2)"'] & & d_4\widetilde{A}_1 \arrow[r, "babM(r_1)"] & d_5\widetilde{A}_2 \arrow[ur, "abM(p_3)"] \arrow[dr, "aM(p_4)"'] & & d_8\widetilde{A}_1 \arrow[r, "baM(r_2)"] & d_9\widetilde{A}_1 \\
& d_3\widetilde{A}_1 \arrow[ur, "baM(q_2)"'] & & & d_7\widetilde{A}_1 \arrow[ur, "baM(q_4)"'] & &
\end{tikzcd}$$

Since the functor $N_\bullet \otimes_{\Bbbk\langle x,y\rangle} \_$ preserves indecomposability and the isomorphism classes, we conclude that $\mathbf{A}$ is derived wild.

(b) $bab \in \mathcal{I}$, $aba \notin \mathcal{I}$. This case is similar to the case (a).

(c) $\mathcal{I} = \langle aba, bab\rangle$.

Let $M = M(\mathbf{W}_5)$. Let us denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x, y\rangle$-bimodules:

$$\begin{tikzcd}
& d_2\widetilde{A}_1 \arrow[dr, "bM(q_1)"] & & & d_6\widetilde{A}_2 \arrow[dr, "aM(q_3)"] & & \\
d_1\widetilde{A}_1 \arrow[ur, "baM(p_1)"] \arrow[dr, "bM(p_2)"'] & & d_4\widetilde{A}_2 \arrow[r, "abM(r_1)"] & d_5\widetilde{A}_2 \arrow[ur, "abM(p_3)"] \arrow[dr, "aM(p_4)"'] & & d_8\widetilde{A}_1 \arrow[r, "baM(r_2)"] & d_9\widetilde{A}_1 \\
& d_3\widetilde{A}_2 \arrow[ur, "abM(q_2)"'] & & & d_7\widetilde{A}_1 \arrow[ur, "baM(q_4)"'] & &
\end{tikzcd}$$

Applying the functor $N_\bullet \otimes_{\Bbbk\langle x,y\rangle} \_$ we conclude that $\mathbf{A}$ is derived wild.

(d) $\mathcal{I} = \langle abab, baba\rangle$.

Let $M = M(\mathbf{W}_5)$. Let us denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x, y\rangle$-bimodules:

$$
\begin{tikzcd}
 & d_2\widetilde{A}_1 \arrow[dr, "baM(q_1)"] & & & d_6\widetilde{A}_1 \arrow[dr, "baM(q_3)"] & & \\
d_1\widetilde{A}_2 \arrow[ur, "abaM(p_1)"] \arrow[dr, "abM(p_2)"'] & & d_4\widetilde{A}_1 \arrow[r, "babM(r_1)"] & d_5\widetilde{A}_2 \arrow[ur, "abaM(p_3)"] \arrow[dr, "abM(p_4)"'] & & d_8\widetilde{A}_1 \arrow[r, "baM(r_2)"] & d_9\widetilde{A}_1 \\
 & d_3\widetilde{A}_2 \arrow[ur, "abaM(q_2)"'] & & & d_7\widetilde{A}_2 \arrow[ur, "abaM(q_4)"'] & &
\end{tikzcd}
$$

Again applying the functor $N_\bullet \otimes_{\Bbbk\langle x,y\rangle}$ - we conclude that $\mathbf{A}$ is derived wild.

Case 4. $\mathcal{Q} = \mathcal{Q}4$. Up to isomorphism we may assume that $ca + cf_1 \in \mathcal{I}$ and $ac + f_2c \in \mathcal{I}$ or $bc + f_3 \in \mathcal{I}$ for some $f_i \in \operatorname{rad}^2 \mathbf{A}$; otherwise $\mathbf{A}/\operatorname{rad}^3 \mathbf{A}$ is wild by [12] and hence $\mathbf{A}$ is wild. Replacing $a + f_1$ with a we can assume in both cases that $ca \in \mathcal{I}$. Let us consider all cases.

(a) $ca, ac + f_2c \in \mathcal{I}$. Then it follows from Lemma 3.2 that $cb \notin \mathcal{I}$. Since $f_2 = ag_1(cb) + bg_2(cb)$ for some polynomials g_i, we have $ac = h(bc)$ for some polynomial h. Then it follows from Lemma 3.1 that $e_1\mathbf{A}e_1 \cong \mathbf{L}_2$, hence $cbcb \in \mathcal{I}$. If $bcbc \notin \mathcal{I}$ or $acbc \notin \mathcal{I}$, $\mathbf{A}$ is derived wild by Lemma 3.2. Therefore $f_2 \in \mathcal{I}$ and hence $ac \in \mathcal{I}$. Hence $\mathbf{A}$ is isomorphic to the algebra (7) from Table 2.

(b) $ca, bc + f_3c \in \mathcal{I}$. Replacing $b + f_3$ with b we can assume that $bc \in \mathcal{I}$. Then it follows from Lemma 3.2 that $ac \notin \mathcal{I}$, $cb \notin \mathcal{I}$ and $acb \notin \mathcal{I}$, hence $\mathbf{A}$ is isomorphic to the algebra (6) from Table 2.

Case 5. $\mathcal{Q} = \mathcal{Q}5$. Suppose first that $\mathbf{A}$ is pure noetherian. Since $\mathbf{A}$ is derived tame, it is tame and hence nodal by [15]. Then it follows from Proposition 2.2 that $\mathbf{A}$ is isomorphic to one of the algebras (8) or (9) from Table G.

Suppose finally that $\mathbf{A}$ has some minimal ideal $\mathcal{J}$. Given $0 \neq z \in \mathcal{J}$. We suppose that $s(z) = e(a)$ (the case $s(z) = s(a)$ is similar). Then $\mathbf{A}$ satisfies the conditions of Lemma 3.2, where $u = a, v = b$ and $w = z$, hence $\mathbf{A}$ is derived wild.

Case 6. $\mathcal{Q} = \mathcal{Q}6$. Suppose first that $e_1\mathbf{A}e_1$ is finite-dimensional. Then it follows from Theorem A and Lemma 3.1 that $a^2 \in I$. From Lemma 3.3 we conclude that $ba \notin \mathcal{I}$. Hence $\mathbf{A}$ is isomorphic to the algebra (10) from Table 2.

Suppose finally that $e_1\mathbf{A}e_1$ is infinite-dimensional. Then $e_1\mathbf{A}e_1 \cong \mathbf{L}_2$. If $ba \notin \mathcal{I}$ then $\mathbf{A}$ is wild, since the finite-dimensional algebra $\mathbf{A}/\left\langle a^7, ba^2 \right\rangle$ is wild by [27] and hence $\mathbf{A}$ is derived wild. Therefore $ba \in \mathcal{I}$ and $\mathbf{A}$ is isomorphic to the algebra (11) from Table 2.

Case 7. $\mathcal{Q} = \mathcal{Q}7$. This case is dual to the previous case. Then we obtain in this case that $\mathbf{A}$ is isomorphic to one of the algebras (12) or (13) from Table 2.

Case 8. $\mathcal{Q} = \mathcal{Q}8$. Then it follows from Theorem A and Lemma 3.1 that $e_1\mathbf{A}e_1$ is isomorphic to one of the algebras $\mathbf{L}_2 - \mathbf{L}_5$. Then one of the following situations occur:

(a) $e_1\mathbf{A}e_1$ is isomorphic to one of the algebras $\mathbf{L}_4 - \mathbf{L}_5$. Then $cb \notin \Bbbk[[a]]$ (in particular, $cb \notin \mathcal{I}$).

Suppose first that $\mathbf{A}$ is pure noetherian. Since $\mathbf{A}$ is derived tame, it is tame and hence nodal by [15]. Then it follows from Proposition 2.2 that $\mathbf{A}$ is isomorphic to one of the algebras (14) – (15) from Table 2.

Suppose finally that $\mathbf{A}$ has some minimal ideal $\mathcal{J}$. Fix $0 \neq z \in \mathcal{J}$. Then one of the following situations occur:

(aa) $\mathfrak{t}(z) = 1$. Let $M = M(\mathbf{W}_1)$ (see Section 3.1). Let us denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x, y\rangle$-bimodules.

$$\begin{tikzcd} d_1\widetilde{A}_1 \arrow[r, bend left, "aM(p)"] \arrow[r, bend right, "cbM(q)"'] & d_2\widetilde{A}_1 \arrow[r, "zM(s)"] & d_3\widetilde{A}_{\mathfrak{s}(z)} \end{tikzcd}$$

Applying the functor $N_\bullet \otimes_{\Bbbk\langle x,y\rangle}$ _ we conclude that $\mathbf{A}$ is derived wild.

(ab) $\mathfrak{s}(z) = 1$. This case is dual to the case (aa).

(ac) $\mathfrak{s}(z) = \mathfrak{t}(z) = 2$. Since algebras $\mathbf{L}_3 - \mathbf{L}_5$ are pure noetherian, we obtain from Theorem A and Lemma 3.1 that algebra $e_2\mathbf{A}e_2$ is isomorphic to the algebra $\mathbf{L}_2$.

Then we can assume that $z = bc$ or $bc \in \mathcal{I}$ and $z = bfc$ for some $f \in \operatorname{rad}\mathbf{A}$.

Suppose first that $z = bc$.

Let $M = M(\mathbf{W}_5)$. Let us denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x, y\rangle$-bimodules:

$$\begin{tikzcd} & d_2\widetilde{A}_1 \arrow[dr, "cM(q_1)"] & & & d_6\widetilde{A}_2 \arrow[dr, "bM(q_3)"] & & \\ d_1\widetilde{A}_1 \arrow[ur, "cbM(p_1)"] \arrow[dr, "cM(p_2)"'] & & d_4\widetilde{A}_2 \arrow[r, "bcM(r_1)"] & d_5\widetilde{A}_2 \arrow[ur, "bcM(p_3)"] \arrow[dr, "bM(p_4)"'] & & d_8\widetilde{A}_1 \arrow[r, "cbM(r_2)"] & d_9\widetilde{A}_1 \\ & d_3\widetilde{A}_2 \arrow[ur, "bcM(q_2)"'] & & & d_7\widetilde{A}_1 \arrow[ur, "cbM(q_4)"'] & & \end{tikzcd}$$

It shows that $\mathbf{A}$ is derived wild.

Suppose finally that $bc \in \mathcal{I}$ and $z = bfc$ for some $f \in \operatorname{rad}\mathbf{A}$.

Let $M = M(\mathbf{W}_5)$. Let us denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x, y\rangle$-bimodules:

$$\begin{tikzcd}
 & d_2\widetilde{A}_1 \arrow[dr, "fcM(q_1)"] & & & d_6\widetilde{A}_2 \arrow[dr, "bM(q_3)"] & & \\
d_1\widetilde{A}_1 \arrow[ur, "cbM(p_1)"] \arrow[dr, "cM(p_2)"'] & & d_4\widetilde{A}_2 \arrow[r, "bfcM(r_1)"] & d_5\widetilde{A}_2 \arrow[ur, "bfcM(p_3)"] \arrow[dr, "bfM(p_4)"'] & & d_8\widetilde{A}_1 \arrow[r, "cM(r_2)"] & d_9\widetilde{A}_2 \\
 & d_3\widetilde{A}_2 \arrow[ur, "bfcM(q_2)"'] & & & d_7\widetilde{A}_1 \arrow[ur, "cbM(q_4)"'] & &
\end{tikzcd}$$

We immediately see that $\mathbf{A}$ is derived wild.

(b) $e_1\mathbf{A}e_1$ is isomorphic to $\mathbf{L}_2$. Then $a^2, cb \in \mathcal{I}$ and hence $ba \notin \mathcal{I}$ and $ac \notin \mathcal{I}$ by Lemma 3.3. Suppose first that $bc \in \mathcal{I}$ and $bac \notin \mathcal{I}$. Then $\mathbf{A}$ is isomorphic to the algebra (16) from Table 2.

Suppose next that $bac \in \mathcal{I}$. Let $M = M(\mathbf{W}_6)$. Denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x, y\rangle$-bimodules:

$$\begin{tikzcd}
 & d_2\widetilde{A}_1 \arrow[dr, "cM(q_1)"] & & d_5\widetilde{A}_1 \arrow[dr, "cM(q_3)"] & & \\
d_1\widetilde{A}_2 \arrow[ur, "baM(p_1)"] \arrow[dr, "bM(p_2)"'] & & d_4\widetilde{A}_2 \arrow[ur, "baM(p_3)"] \arrow[dr, "bM(p_4)"'] & & d_7\widetilde{A}_2 \arrow[r, "bM(r)"] & d_8\widetilde{A}_1 \\
 & d_3\widetilde{A}_1 \arrow[ur, "acM(q_2)"'] & & d_6\widetilde{A}_1 \arrow[ur, "acM(q_4)"'] & &
\end{tikzcd}$$

which shows that $\mathbf{A}$ is derived wild.

Suppose, finally that $bc \notin \mathcal{I}$ and $bac \notin \mathcal{I}$. If $bc = \lambda bac$ (resp., $bac = \lambda bc$) for some $\lambda \in \Bbbk, \lambda \neq 0$, we can reduce this case to previous one replacing $b(1-a)$ (resp., $b(a-1)$) with b. Therefore it remains to consider the case when bc and bac are linearly independent. But in this case $e_2\mathbf{A}e_2 \cong \mathbf{L}_5/(xy - yx)$, hence $\mathbf{A}$ is wild by Theorem A and Lemma 3.1.

(c) $e_1\mathbf{A}e_1$ is isomorphic to $\mathbf{L}_3$.

(ca) Suppose first that $cb \in \mathcal{I}$. If $ba \notin \mathcal{I}$ then $\mathbf{A}$ is wild, since the finite-dimensional algebra $\mathbf{A}/\langle a^7, ba^2, c\rangle$ is wild by [27] and hence $\mathbf{A}$ is derived wild. The case $ac \notin \mathcal{I}$ is similar. Therefore we obtain that $ba, ac \in \mathcal{I}$. If $bc \notin \mathcal{I}$, then $\mathbf{A}$ is isomorphic to the algebra (17) from Table 2. Therefore it remains to consider the case when $bc \in \mathcal{I}$.

Let $M = M(\mathbf{W}_4)$. Let us denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x, y\rangle$-bimodules:

$$\begin{tikzcd} d_1\widetilde{A}_1 \arrow[r, "cM(p_1)"] & d_2\widetilde{A}_2 \arrow[r, "bM(p_2)"] & d_3\widetilde{A}_1 \arrow[r, "cM(p_3)"] \arrow[dr, "aM(q)"] & d_4\widetilde{A}_2 \arrow[r, "bM(p_4)"] & d_5\widetilde{A}_1 \arrow[r, "cM(p_5)"] \arrow[dr, "aM(r)"] & d_6\widetilde{A}_2 \\ & & & d_7\widetilde{A}_1 & & d_8\widetilde{A}_1 \end{tikzcd}$$

We conclude that $\mathbf{A}$ is derived wild.

(cb) Suppose finally that $cb \notin \mathcal{I}$. Then $cb = f(a)$ for some polynomial f such that $f(0) = 0$. Then for any $w \in \operatorname{rad}\mathbf{A}$ there exist $u, v \in \mathbf{A}$ such that $vwu = g(a)$ for some polynomial g and hence $\mathbf{A}$ is pure noetherian. Since $\mathbf{A}$ is derived tame, it is tame and hence nodal by [15]. But it follows from Proposition 2.2 that if $\mathbf{A}$ is nodal algebra with quiver $\mathcal{Q} = \mathcal{Q}8$ than $e_1\mathbf{A}e_1$ is isomorphic to one of the algebras $\mathbf{L}_4 - \mathbf{L}_5$, hence this case is impossible.

Case 9. $\mathcal{Q} = \mathcal{Q}9$. Then one of the following situations occurs:

(a) $e_i\mathbf{A}e_i$ is finite-dimensional for $i = 1, 2$. Then it follows from Theorem A and Lemma 3.1 that $a^2, b^2 \in \mathcal{I}$. Then we conclude from Lemma 3.3 that $ca \notin \mathcal{I}$ and $bc \notin \mathcal{I}$ and hence $\mathbf{A}$ is isomorphic to the algebra (18) from Table 2.

(b) $e_i\mathbf{A}e_i$ is infinite-dimensional for $i = 1, 2$. Then it follows from Theorem A and Lemma 3.1 that $e_i\mathbf{A}e_i \cong \mathbf{L}_2$ for $i = 1, 2$. If $ca \notin \mathcal{I}$ or $bc \notin \mathcal{I}$ then it follows from [27] that $\mathbf{A}/\left\langle a^5, b^5\right\rangle$ is wild, therefore $\mathbf{A}$ is wild and hence $\mathbf{A}$ is derived wild. Hence we obtain that $ca, bc \in \mathcal{I}$ and $\mathbf{A}$ is isomorphic to the algebra (21) from Table 2.

(c) $e_1\mathbf{A}e_1$ is finite-dimensional and $e_2\mathbf{A}e_2$ is infinite-dimensional. Then it follows from Theorem A and Lemma 3.1 that $a^2 \in \mathcal{I}$ and $e_2\mathbf{A}e_2 \cong \mathbf{L}_2$. Then we conclude from Lemma 3.3 that $ca \notin \mathcal{I}$. If $bc \notin \mathcal{I}$ or $ca - bc \notin \mathcal{I}$ then it follows from [27] that $\mathbf{A}/\left\langle b^5\right\rangle$ is wild, therefore $\mathbf{A}$ is wild and hence $\mathbf{A}$ is derived wild. Hence we obtain that either $bc \in \mathcal{I}$ and $\mathbf{A}$ is isomorphic to the algebra (19) from Table 2 or $ca - bc \in \mathcal{I}$ and $\mathbf{A}$ is isomorphic to the algebra $\mathbf{D}_1$.

(d) $e_1\mathbf{A}e_1$ is infinite-dimensional and $e_2\mathbf{A}e_2$ is finite-dimensional. This case is dual to the previous case. Then we obtain in this case that $\mathbf{A}$ is isomorphic to the algebra (20) from Table 2 or to the algebra $\mathbf{D}_2$.

Case 10. $\mathcal{Q} = \mathcal{Q}10$. Suppose first that $\mathbf{A}$ is pure noetherian. Since $\mathbf{A}$ is derived tame, it is tame and hence nodal by [15]. Then it follows from Proposition 2.2 that $\mathbf{A}$ is isomorphic to one of the algebras $(22) - (24)$ from Table 2 or to the algebra (9) from Table 1.

Suppose finally that $\mathbf{A}$ has some minimal ideal $\mathcal{J}$ and consider $0 \neq z \in \mathcal{J}$. Assume that $\mathfrak{t}(z) = \mathfrak{s}(b)$ (the case $\mathfrak{t}(z) = \mathfrak{s}(a)$ is similar). Let $M = M(\mathbf{W}_2)$ (see Section 3.1) and denote by $N_\bullet$ the following complex of $\mathbf{A}$-$\Bbbk\langle x, y\rangle$-bimodules.

$$\begin{tikzcd}
d_1\widetilde{A}_1 \arrow[r, "aM(p)"] \arrow[dr, "dM(t)"'] & d_2\widetilde{A}_1 & \\
d_3\widetilde{A}_2 \arrow[r, "bM(q)"] \arrow[ur, "cM(s)"'] & d_4\widetilde{A}_2 \arrow[r, "zM(r)"] & d_5\widetilde{A}_{\mathfrak{s}(z)}
\end{tikzcd}$$

Again we immediately conclude that $\mathbf{A}$ is derived wild.

$(iii) \Rightarrow (ii)$. This follows from Lemma 2.2 and Lemma 2.6.

$(ii) \Rightarrow (i)$. It follows from [10] that nodal algebras are derived tame. The derived tameness of gentle algebras follows from Theorem 2.7 while the derived tameness of algebras $\mathbf{D}_1$ and $\mathbf{D}_2$ follows from Lemma 2.8.

Statements (2) and (3) follow from [10] and [9].

□

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