On modern applications of Lie algebras

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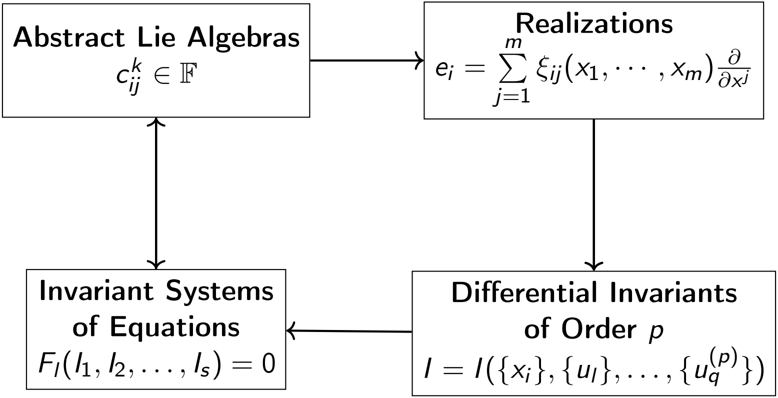
Abstract

This work reviews modern applications of Lie algebras, such as limiting transitions between different models, new classes of orthogonal special functions, and the construction of quasicrystals using root systems. We also provide a list of mathematical areas which Lie algebras theory has had a profound impact.

Construction of Invariant Models

Let g be a Lie algebra, i.e. $g = (V, [\cdot,\cdot])$, where V is a vector space over a field $\mathbb{F}$ with a basis $e_1, e_2, \ldots, e_n$ and $[\cdot,\cdot]$ is the Lie product.



- *Roman O. Popovych et al., 2003, J. Phys. A: Math. Gen. 36, 733.*
- *Peter J. Olver, Applications of Lie Groups to Differential Equations, 1986.*
- *Овсянников Л.В., Групповой анализ дифференциальных уравнений, 1978.*

Limiting transitions between models

Model with its invariance algebra

$$F_k\left(\{x_i\},\{u_j\},\ \dots\ \{u_k^{(p)}\}\right)=0 \Leftrightarrow \forall i:\ e_i^{(p)}F_k=0 \Leftrightarrow$$

$$\Leftrightarrow e_i=\sum_{j=1}^{n}\xi_{ij}\partial_j \Leftrightarrow [e_i,e_j]=C_{ij}^k e_k$$

Contraction matrix parameter

Let we have $g \xrightarrow{\epsilon\to 0} g_0 \Leftrightarrow C_{ij}^k \xrightarrow{\epsilon\to 0} \tilde{C}_{ij}^k(\epsilon) \Leftrightarrow e_i(\epsilon) \xrightarrow{\epsilon\to 0} \tilde{e}_i$
As a result one can obtain a limiting invariant model

- *Nesterenko M., Popovych R., Contractions of Low-dimensional Lie Algebras J. Math. Phys. 47 (2006) 123515, 45 pp.; arXiv:math-ph/0608018.*
- *Nesterenko M., Posta S., Contractions of realizations, Lie Theory and Its Applications in Physics, Springer Proc. Math. Stat. 335, Springer, Singapore, 2020, 447-453.*

Orbit functions

- Simple Lie algebras with their root systems:

$$A_{n\geq 1},\ B_{n\geq 2},\ C_{n\geq 3},\ D_{n\geq 4},\ E_6,\ E_7,\ E_8,\ F_4,\ G_2.$$

- $W(\lambda)$ — the Weyl group (or it's even subgroup), $\sigma\colon W \to \{\pm 1\}$ — homomorphism of Dynkin diagram.
- For each point of the weight lattice λ we define the following orbit functions:

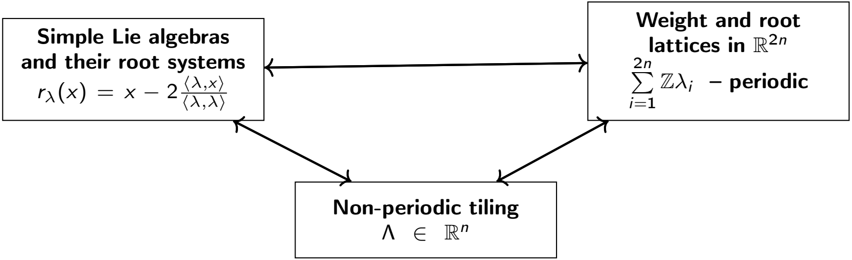
$$\phi_\lambda(x) = \sum_{w\in W} \sigma(w)\mathrm{e}^{2\pi\mathrm{i}\langle w(\lambda),x\rangle},$$

- *Nesterenko M., Patera J., Three-dimensional C-, S- and E-transforms, J. Phys. A: Math. Theor.* ***41*** *(2008), 475205, 31 pp*
- *Nesterenko M., Patera J., Szajewska M., Tereszkiewicz A., Orthogonal polynomials of compact simple Lie groups: branching rules for polynomials, J. Phys. A* ***43*** *(2010), 495207, 27 pp.*

Some properties of $\phi_\lambda(x)$

1. Orbit functions are finite sums of exponents, so they are continuous and have continuous derivatives of all orders.
2. The identity mappings from the affine Weyl group $\langle \lambda, r_i x \rangle = \langle r_i \lambda, x \rangle$ and sign homomorphisms give symmetry properties of orbit functions.
3. Any pair of orbit functions from the same class is orthogonal.
4. Orbit functions approximate the solutions well when the symmetry algebra includes the respective simple algebra.
5. Orbit functions given in an orthonormal basis are eigenfunctions of the Laplace operator and satisfy Dirichlet or Neumann boundary conditions.
6. For some roots orbit functions consist of pairs that generate complex conjugate terms. In general, for working with real-valued functions, Hartley kernel functions are used instead of exponents
7. Orbit functions generate orthogonal polynomials.

Quasicrystals



Quasicrystal is a point set $\Lambda \subset \mathbb{R}^d$, given by projection of the lattice $\sum_i \mathbb{Z}\alpha_i \in \mathbb{R}^{2n}$:

$$\begin{array}{ccccc}
\mathbb{R}^n & \xleftarrow{\,||\,} & \mathbb{R}^n \times \mathbb{R}^n & \xrightarrow{\perp} & \mathbb{R}^n \\
 & & \cup & & \\
L & \xleftarrow{1-1} & \widetilde{L} & \xrightarrow{\text{dense image}} & L'
\end{array}$$

Projections of $\widetilde{L}$ on $\mathbb{R}^n$ are oriented in such a way that they will be correspondingly dense and one-to-one.

The space $\mathbb{R}^n$ on the left is *physical space* and the space $\mathbb{R}^n$ on the right controls the projection of the lattice.

Other applications of Lie algebras

- Modular forms $\leftrightarrow$ Eichler–Shimura theory:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} \Leftrightarrow L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}$$

$$L(s,\chi) = L(s,E) = \prod_{p\mid\Delta} \left(\frac{1}{1-a_p p^{-s}}\right) \prod_{p\nmid\Delta} \left(\frac{1}{1-a_p p^{-s} + p^{1-2s}}\right)$$

Knapp, A. W., Elliptic Curves, Princeton University Press, 1992.

Diamond F., Shurman J., A First Course in Modular Forms, Springer, 2005.

- Monstrous Moonshine: Borcherds' denominator identity:

$$p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1-p^m q^n)^{c_{mn}} = j(p) - j(q),$$

where $j\colon H \to \mathbb{C}$ has the canonical isomorphism

$$j(\tau) - 744 = q^{-1} + \sum_{n\geq 1} c_n q^n.$$

Borcherds R., Generalised Kac-Moody Algebras, J. of Alg. **115** (1988) 501–512.

Kac V., Vertex Algebras for Beginners, University Lecture Series, **10** AMS, 1997.

Some promising problems

- Find the necessary conditions for the existence of transformations connecting realizations and matrix representations of Lie algebras, and construct them in an explicit form.
- Determine which classes of Lie algebras can be obtained from simple algebras using contractions.
- Develop a method for constructing realizations and representations of the final algebras resulting from contraction based on known representations of the original structures.