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Journal of Functional Analysis 218 (2005) 445–474
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Quasiregular representations of the infinite-dimensional Borel group

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Received 10 February 2004; accepted 20 March 2004

Communicated by Paul Malliavin

Abstract

The notion of quasiregular (Representation of Lie groups, Nauka, Moscow, 1983) or geometric (Grundlehren der Mathematischen Wissenschaften, Band 220, Springer, Berlin, New York, 1976; Encyclopaedia of Mathematical Science, Vol. 22, Springer, Berlin, 1994, pp. 1–156) representation is well known for locally compact groups. In the present work an analog of the quasiregular representation for the solvable infinite-dimensional Borel group $G = Bor_0^{\mathbb{N}}$ is constructed and a criterion of irreducibility of the constructed representations is presented. This construction uses G-quasi-invariant Gaussian measures on some G-spaces X and extends the method used in Kosyak (Funktsional. Anal. i Priložhen 37 (2003) 78–81) for the construction of the quasiregular representations as applied to the nilpotent infinite-dimensional group $B_0^{\mathbb{N}}$.

MSC: 22E65; 28C20; 43A80; 58D20

Keywords: Infinite-dimensional groups; Borel group; Regular representations; Quasiregular (geometric) representations; Irreducibility; Quasi-invariant measures; Ergodic measures; Ismagilov conjecture; Solvable group

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0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.03.009

1. Introduction

1.1. The setting and the main results

With any action $\alpha : G \to Aut(X)$ ($Aut(\cdot)$ denoting the group of all measurable automorphisms) of a group G on a G-space X (i.e. a space on which G acts) and G-quasi-invariant measure μ on X one can associate a unitary representation $\pi^{\alpha,\mu,X} : G \to U(L^2(X,\mu))$, of the group G by the formula $(\pi_t^{\alpha,\mu,X} f)(x) = (d\mu(\alpha_{t^{-1}}(x))/d\mu(x))^{1/2} f(\alpha_{t^{-1}}(x)), f \in L^2(X,\mu)$. Let us set $\alpha(G) = \{\alpha_t \in Aut(X) | t \in G\}$. Let $\alpha(G)'$ be centralizer of the subgroup $\alpha(G)$ in $Aut(X)$: $\alpha(G)' = \{g \in Aut(X) | \{g, \alpha_t\} = g\alpha_t g^{-1} \alpha_t^{-1} = e \forall t \in G\}$.

Conjecture 1 (Kosyak [27,28]). *The representation $\pi^{\alpha,\mu,X} : G \to U(L^2(X,\mu))$ is irreducible if and only if*

(1) $\mu^g \perp \mu \forall g \in \alpha(G)' \backslash \{e\}$, (*where* $\perp$ *stands for singular*),
(2) *the measure μ is G-ergodic.*

We say that a measure μ is G-ergodic if $f(\alpha_t(x)) = f(x) \forall t \in G$ implies $f(x) = const$ for all functions $f \in L^1(X,\mu)$.

In this paper we shall prove Conjecture 1 in the case where G is the infinite-dimensional group, namely the Borel group $G = Bor_0^{\mathbb{N}}$, the space $X = X^m$ being the set of left cosets $G_m \backslash Bor^{\mathbb{N}}$, G_m suitable subgroups of the group $Bor^{\mathbb{N}}$ and μ any Gaussian product-measures on X^m. See below for explanation of the concepts used here.

1.2. Regular and quasiregular representations of locally compact groups

Let G be a locally compact group. The *right* ρ (respectively *left* λ) *regular representation* of the group G is a particular case of the representation $\pi^{\alpha,\mu,X}$ with the space $X = G$, the action α being the right action $\alpha = R$ (respectively the left action $\alpha = L$), and the measure μ being the right invariant Haar measure on the group G (see for example [9,21,22,45]).

A *quasiregular representation* of a locally compact group G is also a particular case of the representation $\pi^{\alpha,\mu,X}$ (see for example [45, p. 27]) with the space $X = H\backslash G$, the action α being the right action of the group G on the space X and the measure μ being some quasi-invariant measure on the space X (this measure is unique up to a scalar multiple). In [21,22] this representation is also called *geometric representation*.

1.3. An analog of the regular and quasiregular representations of infinite-dimensional groups and the Ismagilov conjecture

The work of Gel'fand played a decisive role in representation theory in general and in representation theory of infinite-dimensional groups, in particular see [11–13].

Using his orbit method, developed in [19], Kirillov described in [20] all unitary irreducible representations of the completion in strong operator topology of the group $U(\infty) = \varinjlim_n U(n)$ (where $\varinjlim$ stands for inductive limit).

This approach was generalized by Ol'shanskiĭ for the inductive limits of other classical groups $K(\infty) = \varinjlim_n K(n)$ where K is U, O or Sp. In [38] the complete classification of the so-called "tame" representations of the group $K(\infty)$ was obtained. The book [37] deals with the representation theory of the automorphism groups of infinite-dimensional Riemannian symmetric spaces.

The book of Ismagilov [16] is devoted to the representations of two classes of infinite-dimensional Lie groups: that of current groups and that of diffeomorphism group and some of their semidirect product.

The book of Neretin [34] is devoted to the representations theory of the following infinite-dimensional groups: groups of diffeomorphisms of manifolds, groups associated to Virasoro or Kac–Moody algebras, infinite groups of permutations S_∞, groups of operators in Hilbert spaces, groups of currents, and finally groups of automorphisms of measure spaces.

The book of Albeverio and coauthors [2] is devoted to representation theory of gauge groups and related topics.

Let $S_\infty = \bigcup_{n\geqslant 1} S_n$ be the group of finite permutation of the natural numbers. All indecomposable central positive definite functions on S_∞, which are related to factor representations of II_1, were given by Toma [42].

Later Vershik and Kerov obtained the same result by a different method in [43] and gave a realization of the representation of type II_1 in [44].

In [35,36] Obata construct and classifies a family $U^{\theta,\chi}$ of uncountably many irreducible representations of the group S_∞. This family consists of induced representations.

In [18] generalized regular representations $\{T_z : z \in \mathbb{C}\}$ of the group $S_\infty \times S_\infty$ were studied. These representations are deformations of the biregular representation of S_∞ in $l^2(S_\infty)$. A two-parameter family of the generalized regular representations $T_{z,z'}$ of the group S_∞ was mentioned also in [18]. In [8] the corresponding spectral measure $P_{z,z'}$ was studied. The correlation functions are of a determinantal form similar to those studied in random matrix theory.

In [7] the asymptotics of the Plancherel measures M_n for the symmetric groups S_n is studied. It is shown that M_n converge to the delta measure supported on a certain subset Ω of $\mathbb{R}^2$ closely connected to Wigner's semicircle law for distribution of eigenvalues of random matrices. In particular this gives a positive answer to a conjecture of Baik et al. [5].

In the present article we will consider only the approach which deals with the analog for infinite-dimensional groups of the regular and quasiregular representations of finite-dimensional groups. Let G be an infinite-dimensional topological group. To define an analog of the regular representation, let us consider some topological group $\tilde{G}$, containing the initial group G as a dense subgroup $\bar{G} = \tilde{G}$ ($\bar{G}$ being the closure of G). Suppose we have some quasi-invariant measure μ on $X = \tilde{G}$

with respect to the right action of the group G, i.e. $\alpha = R, R_t(x) = xt^{-1}$. In this case we shall call the representation $\pi^{\alpha,\mu,\tilde{G}}$ an *analog of the regular representation*. We shall denote this representation by $T^{R,\mu}$, and the Conjecture 1 is reduced to the following *Ismagilov conjecture*.

Conjecture 2. *The right regular representation $T^{R,\mu}: G \to U(L^2(\tilde{G},\mu))$ is irreducible if and only if*

(1) $\mu^{L_t} \perp \mu \forall t \in G \backslash \{e\}$,
(2) *the measure μ is G-ergodic.*

Remark 3. In the case of the right regular representation, the group $\alpha(G)' = R(G)' \subset Aut(\tilde{G})$ obviously contains the group $L(G)$, the image of the group G with respect to the left action.

The work [14] initiated the study of representations of current groups, i.e. groups $C(X,U)$ of continuous mappings $X \mapsto U$, where X is a finite-dimensional Riemannian manifold and U is a finite-dimensional Lie group.

The regular representation of infinite-dimensional groups, in the case of current groups, was studied firstly in [1,3,4,15] (see also the book [2]). An analog of the regular representation for an arbitrary infinite-dimensional group G, using a G-quasi-invariant measure on some completion $\tilde{G}$ of such a group, is defined in [23,25].

For $X = S^1$, U a compact or non-compact connected Lie group, a Wiener measures on the loop groups $\tilde{G} = C(X,U)$ were constructed and their quasi-invariance were proved in [31–33].

Conjecture 2 was formulated by Ismagilov for the group $G = B_0^{\mathbb{N}}$ and any Gaussian product measure on the group $\tilde{G} = B^{\mathbb{N}}$ and was proved for this case in [23,24]. Here $G = B_0^{\mathbb{N}}$ is the nilpotent group of finite upper triangular matrices of infinite order with unities on the principal diagonal and $B^{\mathbb{N}}$ is the group of all upper triangular matrices of infinite order with unities on the principal diagonal.

For any product measure on the group $B^{\mathbb{N}}$ Conjecture 2 was proved in [26] under some technical assumptions on the measure.

In [25] the criterion was proved for groups of the interval and circle diffeomorphisms. For the group of the interval diffeomorphisms the Shavgulidze measure [40] was used, the image of the classical Wiener measure with respect to some bijection. For the group of circle diffeomorphisms the Malliavin measure [32] was used.

Whether the Conjecture 2 holds in the general case is an open problem.

Let us consider the special case of G-spaces, namely the homogeneous space $X = H\backslash\tilde{G}$, where H is a subgroup of the group $\tilde{G}$ and the measure μ is some quasi-invariant measure on X (if it exists) with respect to the right action of the group G on the homogeneous space $H\backslash\tilde{G}$. In this case we call the corresponding representation $\pi^{\alpha,\mu,H\backslash\tilde{G}}$ an *analog of the quasiregular or geometric representation* of the group G (see [27]).

In [27,28] the Conjecture 1 was proved for the *nilpotent group* $G = B_0^{\mathbb{N}}$ and some G-spaces $X^m, m \in \mathbb{N}$, being the set of left cosets $G_m \backslash B^{\mathbb{N}}$, where G_m are some subgroups of the group $B^{\mathbb{N}}$. Here μ is an arbitrary Gaussian product-measure on X^m. In [29] it was shown that Conjecture 1 holds for the inductive limit $G = SL_0(2\infty, \mathbb{R}) = \varinjlim_n SL(2n-1, \mathbb{R})$, of the special linear groups (*simple groups*) acting on a strip of length $m \in \mathbb{N}$ in the space of real matrices infinite in both directions, and the measure μ being the product Gaussian measure.

In the present article we prove Conjecture 1 for the *solvable* infinite-dimensional Borel group $G = Bor_0^{\mathbb{N}}$ acting on G-spaces $X^m, m \in \mathbb{N}$, where X^m is the set of left cosets $G_m \backslash Bor^{\mathbb{N}}$, and G_m are some subgroups of the group $Bor^{\mathbb{N}}$. The measure μ can be any Gaussian product-measure on X^m.

2. The infinite-dimensional Borel group $Bor_0^{\mathbb{N}}$

Let E_{kn} be infinite-dimensional matrix units $k, n \in \mathbb{N}$. We define the infinite-dimensional group of upper triangular matrices

$$\tilde{G} = Bor^{\mathbb{N}} = \left\{ x = \sum_{1 \leqslant k \leqslant n} x_{kn} E_{kn} | x_{kn} \in \mathbb{R}, x_{kk} \neq 0, k, n \in \mathbb{N} \right\}$$

and the subgroup

$$G = Bor_0^{\mathbb{N}} = \{ x \in Bor^{\mathbb{N}} | x - I \text{ is finite} \},$$

where $I = \sum_{k \in \mathbb{N}} E_{kk}$ is a neutral element in the group $Bor^{\mathbb{N}}$.

We call $Bor_0^{\mathbb{N}}$ the *infinite-dimensional Borel group*. Obviously $Bor_0^{\mathbb{N}}$ is the inductive limit $Bor_0^{\mathbb{N}} = \varinjlim_m Bor(m, \mathbb{R})$, of the finite dimensional (classical) Borel group

$$Bor(m, \mathbb{R}) = \left\{ x = \sum_{1 \leqslant k \leqslant n \leqslant m} x_{kn} E_{kn} | x_{kn} \in \mathbb{R}, x_{kk} \neq 0, 1 \leqslant k \leqslant n \leqslant m \right\},$$

with respect to the imbedding $Bor(m, \mathbb{R}) \ni x \mapsto x + E_{m+1,m+1} \in Bor(m+1, \mathbb{R})$. For $m \in \mathbb{N}$ we also define the subgroups G_m resp. G^m of the group $Bor^{\mathbb{N}}$ as follows:

$$G_m = \left\{ x \in Bor^{\mathbb{N}} | x = \sum_{m < k \leqslant n} x_{kn} E_{kn} + \sum_{k=1}^{m} E_{kk} \right\},$$

$$G^m = \left\{ x \in Bor^{\mathbb{N}} | x = \sum_{1 \leqslant k \leqslant m, k \leqslant n} x_{kn} E_{kn} + \sum_{k=m+1}^{\infty} E_{kk} \right\}.$$

Since $Bor^{\mathbb{N}} = G_m \cdot G^m$ the space X^m of left cosets $X^m = G_m \backslash Bor^{\mathbb{N}}$ is isomorphic to the group G^m. By construction, the right action R of the group G is well defined on the space X^m. More precisely if we define the decomposition $x = x_m \cdot x^m$:

$$Bor^{\mathbb{N}} \ni x \mapsto x_m \cdot x^m \in G_m \cdot G^m,$$

the right action R_t will be defined as follows:

$$R_t(x^m) = (x^m t^{-1})^m, \quad x^m \in G^m, t \in Bor_0^{\mathbb{N}}.$$

Define the measure $\mu^m := \mu^m_{(b,a)}$ on the space $X^m \simeq G^m$ by the formula

$$d\mu^m_{(b,a)}(x) = \bigotimes_{1 \leqslant k \leqslant m, k \leqslant n} d\mu_{(b_{kn}, a_{kn})}(x_{kn}),$$

where $b = (b_{kn})_{k \leqslant n}, a = (a_{kn})_{k \leqslant n}, b_{kn} > 0, a_{kn} \in \mathbb{R}$ and the one-dimensional Gaussian measure $\mu_{(b,a)}$ is defined as follows:

$$d\mu_{(b,a)}(t) = (b/\pi)^{1/2} \exp(-b(t-a)^2)\, dt, \quad b > 0, \ a \in \mathbb{R}.$$

Lemma 4. *We have* $(\mu^m_{(b,a)})^{R_t} \sim \mu^m_{(b,a)}$ *for all* $t \in Bor_0^{\mathbb{N}}$, *where* $\sim$ *means equivalence.*

Proof. Let us fix some $t \in Bor_0^{\mathbb{N}}$. Since the group $Bor_0^{\mathbb{N}}$ is the inductive limit so $t \in Bor(p, \mathbb{R})$ for some $p \in \mathbb{N}$. Hence we are in the case of the right action of some locally compact group G on some finite-dimensional homogeneous space $X = H \backslash G$ with some quasi-invariant measure.

Let us suppose that $p > m$, if $p \leqslant m$ the proof will be even simpler. We define two subgroup $G^m(p)$ and $G_m(p)$ of the group $Bor(p, \mathbb{R})$

$$G^m(p) = G^m \cap Bor(p, \mathbb{R}), \quad G_m(p) = G_m \cap Bor(p, \mathbb{R}).$$

Then $Bor(p, \mathbb{R}) = G_m(p) \cdot G^m(p)$ for $m < p$. Since for $t \in Bor(p, \mathbb{R})$ the right action $x \mapsto R_t(x)$ changes only a finite number of coordinates of the point $x \in X^m$, so we are in some finite dimensional subgroup $X^m(p) = G_m(p) \backslash Bor(p, \mathbb{R}) \simeq G^m(p)$ of the group $X^m \simeq G^m$. The measure $\mu^m_{(b,a)}$ is a product Gaussian measure, hence the projection $\mu^m_{(b,a)}(p)$ of this measure onto this subgroup $G^m(p)$

$$d\mu^m_{(b,a)}(p)(x) = \bigotimes_{1 \leqslant k \leqslant m, k \leqslant n \leqslant p} d\mu_{(b_{kn}, a_{kn})}(x_{kn})$$

is equivalent with the corresponding Haar measure on this subgroup. We note that the Haar measure $dh(x)$ on the group $Bor(p,\mathbb{R})$ is equal

$$dh(x) = \frac{1}{|det(x)|} \prod_{1\leqslant k\leqslant n\leqslant p} dx_{kn} = \left(\prod_{1\leqslant k\leqslant p} |x_{kk}|\right)^{-1} \prod_{1\leqslant k\leqslant n\leqslant p} dx_{kn},$$

where dx is the Lebesgue measure on the real line $\mathbb{R}$. Hence the Haar measure $dh^m(p)(x)$ on the group $G^m(p)$ is equal

$$dh^m(p)(x) = \frac{1}{|det(x)|} \prod_{1\leqslant k\leqslant m, k\leqslant n\leqslant p} dx_{kn} = \left(\prod_{1\leqslant k\leqslant m} |x_{kk}|\right)^{-1} \prod_{1\leqslant k\leqslant m, k\leqslant n\leqslant p} dx_{kn}.$$

So for $t\in G^m(p)\subset Bor(p,\mathbb{R})$ the Haar measure $dh^m(p)$ is right invariant by definition of the Haar measure. It is easy to verify that for another $t\in G_m(p)\subset Bor(p,\mathbb{R})$ it is quasi-invariant. $\square$

Let us define the representation

$$T^{R,\mu,m} : Bor_0^{\mathbb{N}} \mapsto U(H_m = L^2(X^m, \mu^m_{(b,a)}))$$

by the following formula:

$$(T_t^{R,\mu,m} f)(x) = (d\mu^m_{b,a}(R_t^{-1}(x))/d\mu^m_{b,a}(x))^{1/2} f(R_t^{-1}(x)).$$

It is natural to call this representation *an analog of the quasiregular or geometric representation.*

Let us set for $t\in\mathbb{R}, k,n\in\mathbb{N}, 1\leqslant k\leqslant n\leqslant m$,

$$S^L_{kn}(\mu) = \sum_{m=n}^{\infty} \frac{b_{km}}{2}\left(\frac{1}{2b_{nm}} + a^2_{nm}\right), k<n, \quad S^L_{nn}(\mu) = 2\sum_{m=n}^{\infty} b_{nm}\, a^2_{nm},$$

$$S^{L,-}_{kn}(\mu,t) = \frac{t^2}{4}\sum_{m=n}^{\infty} \frac{b_{km}}{b_{nm}} + \sum_{m=n}^{\infty} \frac{b_{km}}{2}(-2a_{km} + ta_{nm})^2.$$

Let also $P_k = \sum_{n=1}^m E_{nn} - 2E_{kk}, 1\leqslant k\leqslant m$. In the case $m=2$ we have

$$P_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\exp(tE_{12}) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \exp(tE_{12})P_1 = \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}.$$

Theorem 5. *Four following conditions* (i)–(iv) *are equivalent for the measure* $\mu = \mu^m_{(b,a)}$:

(i) *the representation* $T^{R,\mu,m}$ *is irreducible*;
(ii) $\mu^{L_t} \perp \mu$ *for all* $t \in Bor(m,\mathbb{R})\backslash\{e\}$, *where* $L_t(x) = tx, x \in X^m$;
(iii)

$$\begin{cases} \text{(a)} & \mu^{L_{P_k}} \perp \mu, & 1 \leqslant k \leqslant m, \\ \text{(b)} & \mu^{L_{\exp(tE_{kn})}} \perp \mu \forall \in \mathbb{R}\backslash\{0\}, & 1 \leqslant k < n \leqslant m, \\ \text{(c)} & \mu^{L_{\exp(tE_{kn})P_k}} \perp \mu \forall t \in \mathbb{R}, & 1 \leqslant k < n \leqslant m, \end{cases}$$

(iv)

$$\begin{cases} \text{(a)} & S^L_{kk}(\mu) = \infty, & 1 \leqslant k \leqslant m, \\ \text{(b)} & S^L_{kn}(\mu) = \infty, & 1 \leqslant k < n \leqslant m, \\ \text{(c)} & S^{L,-}_{kn}(\mu, t) = \infty \forall t \in \mathbb{R}, & 1 \leqslant k < n \leqslant m. \end{cases}$$

Moreover (iii)(a) $\Leftrightarrow$ (iv)(a), (iii)(b) $\Leftrightarrow$ (iv)(b) and (iii)(c) $\Leftrightarrow$ (iv)(c).

Remark 6. We note that the measure $\mu^m_{(b,a)}$ on the space X^m is $Bor_0^{\mathbb{N}}$-right-ergodic since it is a product measure.

We note also that conditions (iii)(a) for $1 \leqslant k < m$ are the particular cases of conditions (iii)(c) for $t = 0$ and $1 \leqslant k < n = m$. Indeed $\exp(tE_{km})P_k|_{t=0} = P_k, 1 \leqslant k < m$.

Proof of Theorem 5. It is sufficient to prove the following implications:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \left\{ \begin{array}{ccc} \text{(iii)(a)} & \Rightarrow & \text{(iv)(a)} \\ \text{(iii)(b)} & \Rightarrow & \text{(iv)(b)} \\ \text{(iii)(c)} & \Rightarrow & \text{(iv)(c)} \end{array} \right\} \Rightarrow \text{(i)}.$$

Parts (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. To prove (iii) $\Rightarrow$ (iv) it is sufficient to consider the elements $P_k = \sum_{n=1}^m E_{nn} - 2E_{kk}, 1 \leqslant k \leqslant m$ in the group $Bor(m,\mathbb{R})$ (see Lemma 9), the one parameter subgroups $\exp(tE_{kn}) = I + tE_{kn}, 1 \leqslant k < n \leqslant m, t \in \mathbb{R}\backslash\{0\}$ of the group $Bor(m,\mathbb{R})$ (see Lemma 10) and the following images of these subgroups: $\exp(tE_{kn})P_k, t \in \mathbb{R}$ (see Lemma 12).

Remark 7. In [27,28] it was proved that in the case of the connected nilpotent group

$$G = B(m,\mathbb{R}) = \left\{ I + x = I + \sum_{1 \leqslant k < n \leqslant m} x_{kn}E_{kn} | x_{kn} \in \mathbb{R} \right\}$$

it is sufficient, for $(\mu^m_{(b,a)})^{L_t} \perp \mu^m_{(b,a)} \forall t \in B(m,\mathbb{R})\backslash\{e\}$ to verify only for one-parameter subgroups $\exp(tE_{kn}) = I + tE_{kn}, 1 \leqslant k < n \leqslant m, t \in \mathbb{R}\backslash\{0\}$ (which generate G).

However in the case of the solvable (non-connected) classical Borel group $G = Bor(m,\mathbb{R})$ it is not sufficient to verify the conditions $(\mu^m_{(b,a)})^{L_t} \perp \mu^m_{(b,a)} \forall t \in Bor(m,\mathbb{R})\backslash\{e\}$ for one-parameter subgroups $\exp(tE_{kn}), 1 \leqslant k \leqslant n \leqslant m, t \in \mathbb{R}\backslash\{0\}$. Even if we add all the elements $P_k, 1 \leqslant k \leqslant m$ it will still not be sufficient, in general. This can be seen by the following.

Counterexample. Let $m = 2$ and let the measure $\mu^2_{(b,a)}$ be defined by taking $a_{1n} = a_{2n+1} = b_{1n} = 1, b_{2n+1} = (n+1)^2, n \in \mathbb{N}$. In this case $S^L_{11}(\mu) = S^L_{22}(\mu) = S^L_{12}(\mu) = \infty$. By Lemmas 9 and 10, we conclude that $(\mu^2_{(b,a)})^{L_{tE_{kk}}} \perp \mu^2_{(b,a)} \forall t \in \mathbb{R}\backslash\{0\}, k = 1,2$ and $(\mu^2_{(b,a)})^{L_{\exp(tE_{12})}} \perp \mu^2_{(b,a)}, \forall t \in \mathbb{R}\backslash\{0\}$. But $S^{L,-}_{12}(\mu,2) = \frac{2^2}{4}\sum_{n=2}^{\infty}\frac{b_{1n}}{b_{2n}} + \sum_{n=2}^{\infty}\frac{b_{1n}}{2}(-2a_{1n} + 2a_{2n})^2 = \sum_{n=2}^{\infty}\frac{1}{n^2} < \infty$. Hence, by Lemma 12 we conclude that $(\mu^2_{(b,a)})^{L_{\exp(2E_{12})P_1}} \sim \mu^2_{(b,a)}$.

The idea of the proof of irreducibility (*i.e. part* (iv) $\Rightarrow$ (i)): Let us denote by $\mathfrak{A}^m$ the von Neumann algebra generated by the representation $T^{R,\mu,m}$

$$\mathfrak{A}^m = (T^{R,\mu,m}_t | t \in G)''.$$

We show that conditions (iv) imply $(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu^m)$. Using the ergodicity of the measure $\mu^m = \mu^m_{(b,a)}$ this proves the irreducibility. Indeed in this case an operator $A \in (\mathfrak{A}^m)'$ should be the operator of multiplication by some essentially bounded function $a \in L^\infty(X^m, \mu^m)$. The commutation relation $[A, T^{R,\mu,m}_t] = 0\ \forall t \in Bor^{\mathbb{N}}_0$ implies $a(xt) = a(x)(mod\,\mu^m)\ \forall t \in Bor^{\mathbb{N}}_0$, so by ergodicity of the measure μ^m on the space X^m we conclude that $A = a = const$. This then proves part (iv) $\Rightarrow$ (i) of Theorem 5.

The inclusion $(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu^m)$ is based on the fact that the operators of multiplication by independent variables $x_{kn}, 1 \leqslant k \leqslant m, k \leqslant n$, may be approximated (if conditions (iv) are valid) in the strong resolvent sense, by some polynomials in the generators $A^{R,m}_{kn} = \frac{d}{dt} T^{R,\mu,m}_{I+tE_{kn}}|_{t=0}, k,n \in \mathbb{N}, k \leqslant n$ i.e. that the operator x_{kn} are affiliated to the von Neumann algebra $\mathfrak{A}^m$.

Definition. Recall (see [10]) that a non necessarily bounded operator A in a Hilbert space H is said to be *affiliated* to a von Neumann algebra M of operators in this Hilbert space H, which is denoted by $A\eta M$, if $\exp(itA) \in M$ for all $t \in \mathbb{R}$.

Remark 8. We will prove the approximation of x_{kn} firstly for one vector $\mathbf{1} \in L^2(X^m, \mu^m_{(b,a)})$. Secondly, the approximation also holds for some dense (in the space $L^2(X^m, \mu^m_{(b,a)})$) set D of analytic vectors for the corresponding operators

$$D = l.s.(X^\alpha = \prod_{1 \leqslant k \leqslant m, k \leqslant n} x^{\alpha_{kn}}_{kn} | \alpha \in \Lambda),$$

where $\Lambda = \{\alpha = (\alpha_{kn})_{1 \leqslant k \leqslant m, k \leqslant n}\}$ are the set of finite (i.e. $\alpha_{kn} = 0$ for a large n) multi-indices $\alpha_{kn} = 0, 1, \ldots.$ and $l.s.(f_n)$ means linear space generated by the set of vectors (f_n). So using the [39, Theorem VIII, 25] we conclude that the convergence holds in the strong resolvent sense. The proof is the same as the proof of [24, Lemma 2.2, p. 250]. Since the generators $A_{kn}^{R,m}$ are affiliated to the von Neumann algebra $\mathfrak{A}^m$ so the limit x_{kn} is also affiliated.

We have for the generators the following expressions:

$$A_{kn}^{R,m} = \sum_{r=1}^{m} x_{rk} D_{rn}, \quad m < k \leqslant n, \quad A_{kn}^{R,m} = \sum_{r=1}^{k} x_{rk} D_{rn}, \quad 1 \leqslant k \leqslant m, k \leqslant n,$$

where $D_{kn} = \partial/\partial x_{kn} - b_{kn}(x_{kn} - a_{kn})$.

The approximation uses conditions (iv) and is based on the following estimation (see for example [6, Chapter I, Section 52])

$$\min_{x \in \mathbb{R}^n} \left(\sum_{k=1}^{n} a_k x_k^2 \middle| \sum_{k=1}^{n} x_k = 1 \right) = \left(\sum_{k=1}^{n} \frac{1}{a_k} \right)^{-1}. \tag{1}$$

We will also use the same estimation in a slightly different form

$$\min_{x \in \mathbb{R}^n} \left(\sum_{k=1}^{n} a_k x_k^2 \middle| \sum_{k=1}^{n} x_k b_k = 1 \right) = \left(\sum_{k=1}^{n} \frac{b_k^2}{a_k} \right)^{-1}. \tag{2}$$

The extremum in (2) is obtained for $x_k = \frac{b_k}{a_k} \left(\sum_{k=1}^{n} \frac{b_k^2}{a_k} \right)^{-1}$.

In what follows, we will say that *two series* $\sum_{k \in \mathbb{N}} a_k$ and $\sum_{k \in \mathbb{N}} b_k$ with positive elements $a_k, b_k > 0$ *are equivalent* and will use the notation $\sum_{k \in \mathbb{N}} a_k \sim \sum_{k \in \mathbb{N}} b_k$ if they are convergent or divergent simultaneously. It is easy to see that for positive $a_k, b_k > 0, k \in \mathbb{N}$ the following asymptotic holds:

$$\sum_{k \in \mathbb{N}} \frac{a_k}{a_k + b_k} \sim \sum_{k \in \mathbb{N}} \frac{a_k}{b_k}. \tag{3}$$

Here and in the following we shall use for two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ the notation $a_n \sim b_n$ as $n \to \infty$, if for some const $C_1, C_2 > 0$ and a large numbers $N \in \mathbb{N}$ holds

$$a_N \leqslant C_1 b_N \leqslant C_2 a_N.$$

In this case we say that the sequences are "equivalent at infinity".

Proof of part (iii) $\Rightarrow$ (iv): This follows from Lemmas 9–12. Let us denote by μ_k the measure

$$\mu_k = \bigotimes_{n=k}^{\infty} \mu_{(b_{kn}, a_{kn})}.$$

Then obviously $\mu^m_{(b,a)} = \otimes_{k=1}^m \mu_k$. We will prove (iii) $\Rightarrow$ (iv) only for $m = 2$. For another $m > 2$ the proof is similar. To prove (iii)(a) $\Leftrightarrow$ (iv)(a) it is sufficient to consider $m = 1$ and the measure $\mu^1 = \mu^1_{(b,a)}$. In this case $P_1 = -E_{11}$. We prove a little more, namely

Lemma 9. *The following three conditions are equivalent*:

(1) $(\mu^1_{(b,a)})^{L_t} \perp \mu^1_{(b,a)} \forall t \in Bor(1, \mathbb{R}) \backslash \{e\} = GL(1, \mathbb{R}) \backslash \{e\}$;
(2) $(\mu^1_{(b,a)})^{L_{-E_{11}}}(x) = \mu^1_{(b,a)}(-x) \perp \mu^1_{(b,a)}(x)$;
(3) $S^L_{11}(\mu) = 2\sum_{n \in \mathbb{N}} b_{1n} a^2_{1n} = \infty$.

Proof. Let us consider $t := tE_{11} \in Bor(1, \mathbb{R})$. We have $L_t(x) = tx = \sum_{n \in \mathbb{N}} tx_{1n} E_{1n}$, so

$$\begin{aligned} d(\mu^1_{(b,a)})^{L_t}(x) &= \bigotimes_{n \in \mathbb{N}} d\mu^{L_t}_{(b_{1n}, a_{1n})}(x_{1n}) = \bigotimes_{n \in \mathbb{N}} \sqrt{\frac{b_{1n}}{\pi}} \exp(-b_{1n}(tx_{1n} - a_{1n})^2)\, dtx_{1n} \\ &= \bigotimes_{n \in \mathbb{N}} \sqrt{\frac{t^2 b_{1n}}{\pi}} \exp(-t^2 b_{1n}(x_{1n} - a_{1n}/t)^2)\, dx_{1n} = \bigotimes_{n \in \mathbb{N}} d\mu_{(t^2 b_{1n}, a_{1n}/t)} \\ &= d\mu^1_{(t^2 b, a/t)}(x). \end{aligned}$$

It is known (see [17,41]) that two Gaussian product measures $\mu^1_{(b,a)}$ and $\mu^1_{(b',a')}$ on $X^1 \cong \mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$ are equivalent if and only if (1) $\mu^1_{(b,0)} \sim \mu^1_{(b',0)}$ and (2) $\mu^1_{(b,a)} \sim \mu^1_{(b,a')}$. Otherwise they are orthogonal. Condition (1) is equivalent with $\prod_{n \in \mathbb{N}} \frac{4 b_{1n} b_{1n}'}{(b_{1n} + b_{1n}')^2} > 0$ and condition (2) is equivalent with $\sum_{n \in \mathbb{N}} b_{1n}(a_{1n} - a_{1n}')^2 < \infty$. Obviously $\mu^1_{(t^2 b, 0)} \perp \mu^1_{(b,0)}$ since $\prod_{n \in \mathbb{N}} \frac{4t^2 b_{1n} b_{1n}}{(t^2 b_{1n} + b_{1n})^2} = \prod_{n \in \mathbb{N}} \frac{4t^2}{(t^2+1)^2} = 0$ if $|t| \neq 1$, so $(\mu^1_{(b,a)})^{L_t} = \mu^1_{(t^2 b, a/t)} \perp \mu^1_{(b,a)} \forall t \in \mathbb{R}, |t| \neq 1, t \neq 0$. In the case $t = -1$ we have $(\mu^1_{(b,a)})^{L_{-E_{11}}} = \mu^1_{(b,-a)}$ so $(\mu^1_{(b,a)})^{L_{-E_{11}}} \perp \mu^1_{(b,a)}$ if and only if $\sum_{n \in \mathbb{N}} b_{1n}(a_{1n} + a_{1n})^2 = 4\sum_{n \in \mathbb{N}} b_{1n} a^2_{1n} = \infty$. $\square$

Equivalence (iii)(b) $\Leftrightarrow$ (iv)(b) follows from the following

Lemma 10. *For the measure* $\mu^2_{(b,a)} = \mu_1 \otimes \mu_2$ *one has* $(\mu_1 \otimes \mu_2)^{L_{\exp(tE_{12})}} \sim (\mu_1 \otimes \mu_2) \forall t \in \mathbb{R}\backslash 0 \Leftrightarrow$

$$S^L_{12}(\mu) = \frac{1}{4} \sum_{n=2}^{\infty} \frac{b_{1n}}{b_{2n}} + \frac{1}{2} \sum_{n=2}^{\infty} b_{1n} a^2_{2n} < \infty. \tag{4}$$

Proof. We will use two obvious formulas (the second formula follows from the first one)

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-bx^2 + cx)\, dx = \frac{1}{\sqrt{b}} \exp\left(\frac{c^2}{4b}\right), \tag{5}$$

$$\sqrt{\frac{b}{\pi}} \int_{\mathbb{R}} \exp(-b/2[(x+s)^2 + x^2])\, dx = \exp\left(-\frac{bs^2}{4}\right). \tag{6}$$

Since

$$\exp(tE_{12}) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \exp(tE_{12}) \begin{pmatrix} x_{1n} \\ x_{2n} \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{1n} \\ x_{2n} \end{pmatrix} = \begin{pmatrix} x_{1n} + tx_{2n} \\ x_{2n} \end{pmatrix},$$

we have for the Hellinger integral $H(\mu, v)$ (see [30]):

$$\begin{aligned}
&H((\mu_1 \otimes \mu_2)^{L_{\exp(tE_{12})}}, \mu_1 \otimes \mu_2) \\
&= \prod_{n=2}^{\infty} H((\mu_{(b_{1n},a_{1n})} \otimes \mu_{(b_{2n},a_{2n})})^{L_{\exp(tE_{12})}}, \mu_{(b_{1n},a_{1n})} \otimes \mu_{(b_{2n},a_{2n})}) \\
&= \prod_{n=2}^{\infty} \int_{\mathbb{R}^2} \sqrt{\frac{b_{1n} b_{2n}}{\pi^2}} \exp\Bigg(-\frac{b_{1n}}{2} [(x_{1n} + tx_{2n} - a_{1n})^2 \\
&\quad + (x_{1n} - a_{1n})^2] - b_{2n}(x_{2n} - a_{2n})^2\Bigg)\, dx_{1n}\, dx_{2n} \\
&\overset{(6)}{=} \prod_{n=2}^{\infty} \int_{\mathbb{R}^1} \sqrt{\frac{b_{2n}}{\pi}} \exp\left(-\frac{t^2 b_{1n} x^2_{2n}}{4} - b_{2n}(x_{2n} - a_{2n})^2\right) dx_{2n} \\
&= \prod_{n=2}^{\infty} \int_{\mathbb{R}^1} \sqrt{\frac{b_{2n}}{\pi}} \exp\left(-x^2_{2n}\left(b_{2n} + \frac{t^2 b_{1n}}{4}\right) + 2b_{2n} a_{2n} x_{2n} - b_{2n} a^2_{2n}\right) dx_{2n} \\
&\overset{(5)}{=} \prod_{n=2}^{\infty} \sqrt{\frac{b_{2n}}{b_{2n} + \frac{t^2 b_{1n}}{4}}} \exp\left(\frac{b^2_{2n} a^2_{2n}}{b_{2n} + \frac{t^2 b_{1n}}{4}} - b_{2n} a^2_{2n}\right) \\
&= \prod_{n=2}^{\infty} \frac{1}{\sqrt{1 + \frac{t^2 b_{1n}}{4b_{2n}}}} \exp\left(-\frac{t^2 b_{1n} a^2_{2n}}{4\left(1 + \frac{t^2 b_{1n}}{4b_{2n}}\right)}\right) > 0
\end{aligned}$$

if and only if

$$S_{12}^{L}(\mu)=\frac{1}{4}\sum_{n=2}^{\infty}\frac{b_{1n}}{b_{2n}}+\frac{1}{2}\sum_{n=2}^{\infty}b_{1n}a_{2n}^{2}<\infty. \qquad \square$$

Remark 11. To obtain the same conditions we may use the generator corresponding to the left action of the group $\exp(tE_{12})$ on the space X^2.

Indeed if $(\mu^2_{(b,a)})^{L_{\exp(tE_{12})}}\sim\mu^2_{(b,a)}$ we can define a one-parameter unitary group $T^{L,\mu^2_{(b,a)}}_{\exp(tE_{12})}$ as follows:

$$(T^{L,\mu^2_{(b,a)}}_{\exp(tE_{12})}f)(x)=\sqrt{\frac{(d\mu^2_{(b,a)})^{L^{-1}_{\exp(tE_{12})}}(x)}{d\mu^2_{(b,a)}(x)}}f(L^{-1}_{\exp(tE_{12})}(x)).$$

A direct calculation gives us the generator

$$A_{12}^{L,\mu^2_{(b,a)}}=\frac{d}{dt}T^{L,\mu^2_{(b,a)}}_{I+tE_{12}}|_{t=0}=-\sum_{n=2}^{\infty}x_{2n}D_{1n},$$

where $D_{1n}=\partial/\partial x_{kn}-b_{1n}(x_{1n}-a_{1n})$. Finally we get

$$\begin{aligned}||A_{12}^{L,\mu^2_{(b,a)}}\mathbf{1}||^2&=\left|\left|\sum_{n=2}^{\infty}x_{2n}D_{1n}\mathbf{1}\right|\right|^2\\&=\sum_{n=2}^{\infty}||x_{2n}b_{1n}(x_{1n}-a_{1n})||^2=\sum_{n=2}^{\infty}\frac{b_{1n}}{2}\left(\frac{1}{2b_{2n}}+a_{2n}^2\right)=S_{12}^{L}(\mu).\end{aligned}$$

To prove (iii)(c)$\Leftrightarrow$(iv)(c) we use

Lemma 12. *For the measure* $\mu^2_{(b,a)}$ *we have* $(\mu^2_{(b,a)})^{L_{\exp(tE_{12})P_1}}\sim\mu^2_{(b,a)}\forall t\in\mathbb{R}\Leftrightarrow$

$$S_{12}^{L,-}(\mu,t)=\frac{t^2}{4}\sum_{n=2}^{\infty}\frac{b_{1n}}{b_{2n}}+\frac{1}{2}\sum_{n=2}^{\infty}b_{1n}(-2a_{1n}+ta_{2n})^2<\infty. \tag{7}$$

Proof. Since

$$\exp(tE_{12})P_1=\begin{pmatrix}1&t\\0&1\end{pmatrix}\begin{pmatrix}-1&0\\0&1\end{pmatrix}=\begin{pmatrix}-1&t\\0&1\end{pmatrix}$$

and

$$\exp(tE_{12})P_1\begin{pmatrix} x_{1n} \\ x_{2n} \end{pmatrix} = \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_{1n} \\ x_{2n} \end{pmatrix} = \begin{pmatrix} -x_{1n}+tx_{2n} \\ x_{2n} \end{pmatrix},$$

we have for the Hellinger integral $H(\mu,\nu)$ (see [30]):

$$\begin{aligned}
&H((\mu^2_{(b,a)})^{L_{\exp(tE_{12})P_1}}, \mu^2_{(b,a)}) \\
&= \prod_{n=2}^{\infty} H\Big((\mu_{(b_{1n},a_{1n})}\otimes\mu_{(b_{2n},a_{2n})})^{L_{\exp(tE_{12})P_1}}, (\mu_{(b_{1n},a_{1n})}\otimes\mu_{(b_{2n},a_{2n})})\Big) \\
&= \prod_{n=2}^{\infty} \int_{\mathbb{R}^2} \sqrt{\frac{b_{1n}b_{2n}}{\pi^2}}\exp\Big(-\frac{b_{1n}}{2}[(-x_{1n}+tx_{2n}-a_{1n})^2 \\
&\quad + (x_{1n}-a_{1n})^2] - b_{2n}(x_{2n}-a_{2n})^2\Big)\,dx_{1n}\,dx_{2n} \\
&\overset{(6)}{=} (\text{since } -x_{1n}+tx_{2n}-a_{1n} = -(x_{1n}-a_{1n}) + (-2a_{1n}+tx_{2n})) \\
&= \prod_{n=2}^{\infty} \int_{\mathbb{R}^1} \sqrt{\frac{b_{2n}}{\pi}}\exp\left(-\frac{b_{1n}(-2a_{1n}+tx_{2n})^2}{4} - b_{2n}(x_{2n}-a_{2n})^2\right)dx_{2n} \\
&= \prod_{n=2}^{\infty} \int_{\mathbb{R}^1} \sqrt{\frac{b_{2n}}{\pi}}\exp\Big(-x_{2n}^2\Big(b_{2n}+\frac{t^2b_{1n}}{4}\Big) + x_{2n}(2b_{2n}a_{2n}+tb_{1n}a_{1n}) \\
&\quad -b_{1n}a_{1n}^2 - b_{2n}a_{2n}^2\Big)\,dx_{2n} \\
&\overset{(5)}{=} \prod_{n=2}^{\infty} \sqrt{\frac{b_{2n}}{b_{2n}+\frac{t^2b_{1n}}{4}}}\exp\left(\frac{(2b_{2n}a_{2n}+tb_{1n}a_{1n})^2}{4b_{2n}+t^2b_{1n}} - (b_{1n}a_{1n}^2+b_{2n}a_{2n}^2)\right) \\
&= \prod_{n=2}^{\infty} \sqrt{\frac{1}{1+\frac{t^2b_{1n}}{4b_{2n}}}}\exp\left(-\frac{b_{1n}(-2a_{1n}+ta_{2n})^2}{4\Big(1+\frac{t^2b_{1n}}{4b_{2n}}\Big)}\right) > 0
\end{aligned}$$

if and only if

$$S_{12}^{L,-}(\mu,t) = \frac{t^2}{4}\sum_{n=2}^{\infty}\frac{b_{1n}}{b_{2n}} + \frac{1}{2}\sum_{n=2}^{\infty} b_{1n}(-2a_{1n}+ta_{2n})^2 < \infty. \qquad \square$$

This finishes the *proof of Theorem* 5(iii) $\Rightarrow$ (iv).

Proof of Theorem 5 (iv) $\Rightarrow$ (i): Let us denote by $\langle f_n | n\in\mathbb{N}\rangle$ the closure of the linear space generated by the set of vectors $(f_n)_{n\in\mathbb{N}}$ in a Hilbert space H. For the measure $d\mu_{(b,a)}(x) = (b/\pi)^{1/2}\exp(-b(x-a)^2)\,dx$ on $\mathbb{R}$ we shall consider the following expectations, using the notation Mf for $f \in L^1(\mathbb{R}, \mu_{(b,a)})$, with

$$Mf := \int_{\mathbb{R}} f(x)\, d\mu_{(b,a)}(x),$$

$$Mx = a, \quad Mx^2 = (2b)^{-1} + a^2 =: c, \quad Mx^3 = 3(2b)^{-1}a + a^3 \sim ac, \tag{8}$$

$$Mx^4 = 3(2b)^{-2} + 6(2b)^{-1}a^2 + a^4 \sim c^2, \tag{9}$$

$$M|x^2 - Mx^2|^2 = Mx^4 - (Mx^2)^2 = 2(2b)^{-2} + 4(2b)^{-1}a^2 \sim (2b)^{-1}c. \tag{10}$$

If $D = d/dx - b(x-a)$ and $D_{kn} = \partial/\partial x_{kn} - b_{kn}(x_{kn} - a_{kn})$ we have

$$MD^2\mathbf{1} = -b/2, \quad M|D\mathbf{1}|^2 = b/2, \quad M|D^2\mathbf{1}|^2 = 3(b/2)^2, \tag{11}$$

$$M|(D^2 - (MD^2\mathbf{1})\mathbf{1}|^2 = 2(b/2)^2, \quad (D_{kn}\mathbf{1}, \mathbf{1}) = 0, \tag{12}$$

$$(D_{kn}\mathbf{1}, D_{rs}\mathbf{1}) = 0, \quad ((D_{kn}^2 - MD^2\mathbf{1})\mathbf{1}, D_{kn}D_{rs}\mathbf{1}) = 0 \quad \text{for } (kn) \neq (rs). \tag{13}$$

For $m = 1$ we have

$$A_{1n}^{R,1} = x_{11}D_{1n}, 1 \leqslant n, \quad A_{kn}^{R,1} = x_{1k}D_{1n}, 2 \leqslant k \leqslant n.$$

Lemma 13. *We have for* $n \in \mathbb{N}$

$$x_{11}x_{1n} \in \langle A_{1k}^{R,1}A_{nk}^{R,1}\mathbf{1} = x_{11}x_{1n}D_{1k}^2\mathbf{1} | k \in \mathbb{N}, n < k \rangle.$$

Moreover $x_{11}x_{1n}\eta\mathfrak{A}^1$.

Proof. We have $A_{1k}^{R,1}A_{nk}^{R,1}\mathbf{1} = x_{11}x_{1n}D_{1k}^2\mathbf{1}$. Since $b_k = MD_{1k}^2\mathbf{1} = -\frac{b_{1k}}{2}$ and $\sum_k t_k MD_{1k}^2\mathbf{1} = 1$ hence

$$\left\| \left[\sum_{k=1}^N t_k A_{1k}^{R,1}A_{nk}^{R,1} - x_{11}x_{1n} \right] \mathbf{1} \right\|^2 = \|x_{11}x_{1n}\|^2 \sum_{k=1}^N t_k^2 \left\| \left(D_{1k}^2 + \frac{b_{1k}}{2} \right) \mathbf{1} \right\|^2.$$

Using (2) and $a_k = c_{11}c_{1n}||(D_{1k}^2 + \frac{b_{1k}}{2})\mathbf{1}||^2 = c_{11}c_{1n}2(\frac{b_{1k}}{2})^2 \sim 2(\frac{b_{1k}}{2})^2$ we have

$$\min_{\{t_k\}}\left\{\left|\left|\left[\sum_{k=1}^{N} t_k A_{1k}^{R,1}A_{nk}^{R,1} - x_{11}x_{1n}\right]\mathbf{1}\right|\right|^2 \Bigg| \sum_{k=1}^{N} t_k M D_{1k}^2 \mathbf{1} = 1\right\}$$
$$= \left(\sum_{k=1}^{N} \frac{b_k^2}{a_k}\right)^{-1} \xrightarrow{N\to\infty} 0 \Leftrightarrow \infty = \sum_{k\in\mathbb{N}} \frac{b_k^2}{a_k} = \sum_{k\in\mathbb{N}} \frac{(b_{1k}/2)^2}{2(b_{1k}/2)^2} = 1/2 \sum_{k\in\mathbb{N}} 1. \qquad \square$$

Lemma 14. *One has*

$$x_{11} \in \langle x_{11}x_{1k} | k\in\mathbb{N}\rangle \Leftrightarrow S_{11}^L(\mu) = \infty.$$

Moreover $x_{11}\eta\mathfrak{A}^1$.

Proof. Since $b_k = Mx_{1k} = a_{1k}$ and $\sum_k t_k Mx_{1k} = 1$, hence

$$\left|\left|\left[\sum_k t_k x_{11}x_{1k} - x_{11}\right]\mathbf{1}\right|\right|^2 = ||x_{11}||^2\left|\left|\left[\sum_k t_k x_{1k} - 1\right]\mathbf{1}\right|\right|^2$$
$$= c_{11}\left|\left|\sum_k t_k(x_{1k} - a_{1k})\right|\right|^2 = c_{11}\sum_k t_k^2||x_{1k} - a_{1k}||^2$$
$$= c_{11}\sum_k t_k^2\frac{1}{2b_{1k}} \sim \sum_k t_k^2\frac{1}{2b_{1k}}.$$

Using (2) we have

$$\min_{\{t_k\}}\left\{\left|\left|\left[\sum_{k=N_1}^{N_2} t_k x_{11}x_{1k} - x_{11}\right]\mathbf{1}\right|\right|^2 \Bigg| \sum_{k=N_1}^{N_2} t_k Mx_{1k} = 1\right\}$$
$$= \left(\sum_{k=N_1}^{N_2} \frac{b_k^2}{a_k}\right)^{-1} \xrightarrow{N_2\to\infty} 0 \Leftrightarrow \infty = \sum_{k\in\mathbb{N}} \frac{b_k^2}{a_k} \sim \sum_{k\in\mathbb{N}} 2b_{1k}a_{1k}^2 = S_{11}^L(\mu). \qquad \square$$

So x_{11} and $x_{11}x_{1k}$ are affiliated to the von Neumann algebra $\mathfrak{A}^1$ and hence $x_{1k}\eta\mathfrak{A}^1, k\in\mathbb{N}$. This completes the proof of the Theorem 5(iv) $\Rightarrow$ (i) for $m = 1$.

For $m = 2$ we have

$$A_{1n}^{R,2} = x_{11}D_{1n}, 1\leqslant n, \quad A_{kn}^{R,2} = x_{1k}D_{1n} + x_{2k}D_{2n}, 2\leqslant k\leqslant n.$$

We have 4 conditions:

$$S_{11}^{L}(\mu) = S_{12}^{L}(\mu) = S_{22}^{L}(\mu) = \infty, \quad S_{12}^{L,-}(\mu, t) = \infty \, \forall t \in \mathbb{R}.$$

Consider the two cases:
(a) $\sum_{m=2}^{\infty} b_{1m}/b_{2m} = \infty$,
(b) $\sum_{m=2}^{\infty} b_{1m}/b_{2m} < \infty$.
We use the expression $A_{1n}^{R,2} A_{kn}^{R,2} = x_{11}x_{1k}D_{1n}^2 + x_{11}x_{2k}D_{1n}D_{2n}$

Lemma 15. *We get for* $n \in \mathbb{N}$

$$x_{11}x_{1n} \in \langle A_{1k}^{R,2} A_{nk}^{R,2}\mathbf{1} | k \in \mathbb{N}, n < k \rangle \Leftrightarrow \sum_{n=2}^{\infty} b_{1k}/b_{2k} = \infty.$$

Proof. We have

$$A_{1k}^{R,2} A_{nk}^{R,2} = x_{11}D_{1k}(x_{1n}D_{1k} + x_{2n}D_{2k}) = x_{11}x_{1n}D_{1k}^2 + x_{11}x_{2n}D_{1k}D_{2k}.$$

Since $b_k = MD_{1k}^2\mathbf{1} = -\frac{b_{1k}}{2}$ and $\sum_k t_k MD_{1k}^2\mathbf{1} = 1$ hence

$$\left|\left|\left[\sum_{k=2}^{N} t_k A_{1k}^{R,2} A_{nk}^{R,2} + x_{11}x_{1n}\right]\mathbf{1}\right|\right|^2$$
$$= ||x_{11}||^2 \sum_{k=2}^{N} t_k^2 \left|\left|\left[x_{1n}\left(D_{1k}^2 + \frac{b_{1k}}{2}\right) + x_{2n}D_{1k}D_{2k}\right]\mathbf{1}\right|\right|^2.$$

Using (8), (11) and (12) we have

$$a_k = c_{11}\left|\left|\left[x_{1n}\left(D_{1k}^2 + \frac{b_{1k}}{2}\right) + x_{2n}D_{1k}D_{2k}\right]\mathbf{1}\right|\right|^2 = c_{11}\left(c_{1n}2\left(\frac{b_{1k}}{2}\right)^2 + c_{2n}\frac{b_{1k}}{2}\frac{b_{2k}}{2}\right)$$

$\sim (b_{1k}^2 + b_{1k}b_{2k})$ so, using (2) we get

$$\min_{\{t_k\}}\left\{\left|\left|\left[\sum_{k=2}^{N} t_k A_{1k}^{R,1} A_{nk}^{R,1} + x_{11}x_{1n}\right]\mathbf{1}\right|\right|^2 \middle| \sum_{k=2}^{N} t_k MD_{1k}^2\mathbf{1} = 1\right\}$$
$$= \left(\sum_{k=2}^{N} \frac{b_k^2}{a_k}\right)^{-1} \xrightarrow{N\to\infty} 0 \Leftrightarrow \infty = \sum_{k=2}^{\infty} \frac{b_k^2}{a_k} \sim \sum_{k=2}^{\infty} \frac{b_{1k}^2}{b_{1k}^2 + b_{1k}b_{2k}} \overset{(3)}{\sim} \sum_{k=2}^{\infty} \frac{b_{1k}}{b_{2k}}. \qquad \square$$

So, in case (a) we have $x_{11}x_{1k}\eta\mathfrak{A}^2, k \in \mathbb{N}$. By Lemma 14 we have $x_{11} \in \langle x_{11}x_{1k} | 1 < k \rangle \Leftrightarrow S_{11}^{L}(\mu) = \infty$. Since $x_{11}, x_{11}x_{1k}$ and $x_{11}D_{1k}$ are affiliated to the algebra $\mathfrak{A}^2$ we conclude that x_{1k} and D_{1k} are also affiliated to the algebra $\mathfrak{A}^2$. Hence $A^{R,2}_{nk} - x_{1n}D_{1k} = x_{2n}D_{2k}\eta\mathfrak{A}^2, 2\leqslant n\leqslant k$ and $x_{2n}x_{2p}D^2_{2k}\eta\mathfrak{A}^2, 2\leqslant n\leqslant p\leqslant k$. By analogy with Lemma 13 we conclude that $x_{2n}x_{2p}, 2\leqslant n\leqslant p$ are affiliated to the algebra $\mathfrak{A}^2$.

Lemma 16. *We have for* $n\geqslant 2$

$$x_{2n}\in\langle x_{2n}x_{2k}|k\in\mathbb{N}, n<k\rangle \Leftrightarrow S^L_{22}(\mu) = \infty.$$

Proof. The proof is similar to the one of Lemma 14. $\square$

So $x_{2k}\eta\mathfrak{A}^2, k\geqslant 2$. This proves the irreducibility for $m=2$ in case (a).
Now we consider case (b). For $2\leqslant p\leqslant n<k$ we use the expression

$$\begin{aligned}A^{R,2}_{pk}A^{R,2}_{nk} &= (x_{1p}D_{1k} + x_{2p}D_{2k})(x_{1n}D_{1k} + x_{2n}D_{2k})\\ &= x_{1p}x_{1n}D^2_{1k} + (x_{1p}x_{2n} + x_{2p}x_{1n})D_{1k}D_{2k} + x_{2p}x_{2n}D^2_{2k}.\end{aligned}$$

Lemma 17. *We get for* $p,n\in\mathbb{N}, 2\leqslant p\leqslant n$

$$x_{2p}x_{2n} + \beta(p,n)x_{1p}x_{1n}\in\langle A^{R,2}_{pk}A^{R,2}_{nk}\mathbf{1}|k\in\mathbb{N}, n<k\rangle \quad \text{if } \Sigma_1 = \sum_{k=n+1}^{\infty} b^2_{2k}/(b_{1k}+b_{2k})^2 = \infty$$

and when the limit exists $\lim_m \beta_m(p,n) = \beta(p,n)\in\mathbb{R}$, *where*

$$\beta_m(p,n) = -\sum_{k=n+1}^{m}\frac{b_{1k}b_{2k}}{a_k(p,n)}\left(\sum_{k=n+1}^{m}\frac{b^2_{2k}}{a_k(p,n)}\right)^{-1} \tag{14}$$

and

$$\begin{aligned}&a_k(p,n)\\ &\quad = c_{1p}c_{1n}2\left(\frac{b_{1k}}{2}\right)^2 + \left(c_{1p}c_{2n} + c_{2p}c_{1n} + 2a_{1p}a_{2n}a_{2p}a_{1n}\right)\frac{b_{1k}}{2}\frac{b_{2k}}{2} + c_{2p}c_{2n}2\left(\frac{b_{2k}}{2}\right)^2.\end{aligned}$$

Proof. Since $b_k = MD_{2k}^2\mathbf{1} = -\frac{b_{2k}}{2}$ and $\sum_k t_k MD_{2k}^2\mathbf{1} = 1$ hence

$$\begin{aligned}
&\left\|\left[\sum_{k=n+1}^{N} t_k A_{pk}^{R,2} A_{nk}^{R,2} + (x_{2p}x_{2n} + \beta(p,n)x_{1p}x_{1n})\right]\mathbf{1}\right\| \\
&= \left\|\sum_{k=n+1}^{N} t_k \left[x_{1p}x_{1n}\left(D_{1k}^2 + \frac{b_{1k}}{2}\right) + (x_{1p}x_{2n} + x_{2p}x_{1n})D_{1k}D_{2k}\right.\right. \\
&\quad \left.\left. + x_{2p}x_{2n}\left(D_{2k}^2 + \frac{b_{2k}}{2}\right)\right]\mathbf{1} + x_{1p}x_{1n}\left(\beta(p,n) - \sum_{k=n+1}^{N} t_k \frac{b_{1k}}{2}\right)\mathbf{1}\right\| \\
&\leqslant \left\|\sum_{k=n+1}^{N} t_k \left[x_{1p}x_{1n}\left(D_{1k}^2 + \frac{b_{1k}}{2}\right) + (x_{1p}x_{2n} + x_{2p}x_{1n})D_{1k}D_{2k}\right.\right. \\
&\quad \left.\left. + x_{2p}x_{2n}\left(D_{2k}^2 + \frac{b_{2k}}{2}\right)\right]\mathbf{1}\right\| + \left|\beta(p,n) - \sum_{k=n+1}^{N} t_k\frac{b_{1k}}{2}\right| \|x_{1p}x_{1n}\mathbf{1}\|. \qquad (15)
\end{aligned}$$

Since

$$\begin{aligned}
&\left\|\sum_{k=n+1}^{N} t_k \left[x_{1p}x_{1n}\left(D_{1k}^2 + \frac{b_{1k}}{2}\right) + (x_{1p}x_{2n} + x_{2p}x_{1n})D_{1k}D_{2k}\right.\right. \\
&\quad \left.\left. + x_{2p}x_{2n}\left(D_{2k}^2 + \frac{b_{2k}}{2}\right)\right]\mathbf{1}\right\|^2 = \sum_{k=n+1}^{N} t_k^2 a_k(p,n),
\end{aligned}$$

where (we will use (8), (11) and (12))

$$\begin{aligned}
&a_k(p,n) \\
&:= \left\|\left[x_{1p}x_{1n}\left(D_{1k}^2 + \frac{b_{1k}}{2}\right) + (x_{1p}x_{2n} + x_{2p}x_{1n})D_{1k}D_{2k} + x_{2p}x_{2n}\left(D_{2k}^2 + \frac{b_{2k}}{2}\right)\right]\mathbf{1}\right\|^2 \\
&= c_{1p}c_{1n}2\left(\frac{b_{1k}}{2}\right)^2 + (c_{1p}c_{2n} + c_{2p}c_{1n} + 2a_{1p}a_{2n}a_{2p}a_{1n})\frac{b_{1k}}{2}\frac{b_{2k}}{2} + c_{2p}c_{2n}2\left(\frac{b_{2k}}{2}\right)^2 \\
&\sim a_k := (b_{1k} + b_{2k})^2, \text{ using (2) we have}
\end{aligned}$$

$$\begin{aligned}
&\min_{\{t_k\}}\left\{\left\|\sum_{k=n+1}^{N} t_k \left[x_{1p}x_{1n}\left(D_{1k}^2 + \frac{b_{1k}}{2}\right) + (x_{1p}x_{2n} + x_{2p}x_{1n})D_{1k}D_{2k}\right.\right.\right. \\
&\quad \left.\left.\left. + x_{2p}x_{2n}\left(D_{2k}^2 + \frac{b_{2k}}{2}\right)\right]\mathbf{1}\right\|^2 \middle| \sum_{k=n+1}^{N} t_k MD_{2k}^2\mathbf{1} = 1\right\} = \left(\sum_{k=n+1}^{N} \frac{b_{2k}^2}{4a_k(p,n)}\right)^{-1} \xrightarrow{N\to\infty} 0 \\
&\Leftrightarrow \infty = \sum_{k\in\mathbb{N}} \frac{b_{2k}^2}{4a_k(p,n)} \sim \sum_{k\in\mathbb{N}} \frac{b_{2k}^2}{(b_{1k} + b_{2k})^2}.
\end{aligned}$$

Using (2) we get $t_k = -\frac{b_{2k}}{2a_k(p,n)}\left(\sum_{k=n+1}^{N} \frac{b_{2k}^2}{4a_k(p,n)}\right)^{-1}$ so

$$\sum_{k=n+1}^{N} t_k \frac{b_{1k}}{2} = -\sum_{k=n+1}^{N} \frac{b_{1k}b_{2k}}{4a_k(p,n)}\left(\sum_{k=n+1}^{N} \frac{b_{2k}^2}{4a_k(p,n)}\right)^{-1} = \beta_N(p,n).$$

To complete the proof of the lemma it is sufficient to use (15). $\square$

Lemma 18. *Let we have three sequences of real numbers* $(a_n), (b_n)$ *and* (α_n) *with* $(a_n) > 0, \sum_{n\in\mathbb{N}} a_n = \infty, \sum_{k=1}^{m} |b_k| (\sum_{k=1}^{m} a_k)^{-1} \leqslant C, m \in \mathbb{N}$, *for some* $C > 0$ *and* $\lim_n \alpha_n = \alpha \neq 0$. *If the limit exists* $\lim_m \beta_m = \beta \in \mathbb{R}$, *where*

$$\beta_m = \sum_{k=1}^{m} b_k \left(\sum_{k=1}^{m} a_k\right)^{-1}, \tag{16}$$

then the limit also exists $\lim_m \beta_m(\alpha) = \beta(\alpha) \in \mathbb{R}$, *and* $\beta(\alpha) = \beta$, *where*

$$\beta_m(\alpha) = \sum_{k=1}^{m} \alpha_k b_k \left(\sum_{k=1}^{m} \alpha_k a_k\right)^{-1}. \tag{17}$$

Proof. Let us put $\varepsilon_n = \alpha_n - \alpha$, then $\lim_n \varepsilon_n = 0$ and we have

$$\beta_m(\alpha) = \frac{\sum_{k=1}^{m} \alpha_k b_k}{\sum_{k=1}^{m} \alpha_k a_k} = \frac{\beta_m + \alpha^{-1} \sum_{k=1}^{m} \varepsilon_k b_k \left(\sum_{k=1}^{m} a_k\right)^{-1}}{1 + \alpha^{-1} \sum_{k=1}^{m} \varepsilon_k a_k \left(\sum_{k=1}^{m} a_k\right)^{-1}}.$$

It is sufficient to prove that $\lim_m \beta_{1,m} = 0$ and $\lim_m \beta_{2,m} = 0$, where

$$\beta_{1,m} = \sum_{k=1}^{m} \varepsilon_k b_k \left(\sum_{k=1}^{m} a_k\right)^{-1} \quad \text{and} \quad \beta_{2,m} = \sum_{k=1}^{m} \varepsilon_k a_k \left(\sum_{k=1}^{m} a_k\right)^{-1}.$$

We prove that $\lim_m \beta_{2,m} = 0$. Indeed, let us fix some $\delta > 0$. We can find a number $N \in \mathbb{N}$ such that $|\varepsilon_k| < \delta, k > N$ and for this number N we can find another number M such that $|\sum_{k=1}^{N} \varepsilon_k a_k| (\sum_{k=1}^{N+M} a_k)^{-1} < \delta$. Finally, we have

$$|\beta_{2,N+M}| \leqslant \frac{|\sum_{k=1}^{N} \varepsilon_k a_k| + \delta \sum_{k=N+1}^{N+M} a_k}{\sum_{k=1}^{N+M} a_k} \leqslant \delta + \delta = 2\delta.$$

To prove that $\lim_m \beta_{1,m} = 0$ we have

$$|\beta_{1,N+M}| = \frac{|\sum_{k=1}^{N+M} \varepsilon_k b_k|}{\sum_{k=1}^{N+M} a_k} \leqslant \frac{|\sum_{k=1}^{N} \varepsilon_k b_k| + \delta \sum_{k=N+1}^{N+M} |b_k|}{\sum_{k=1}^{N+M} a_k} \leqslant \delta + \delta C,$$

if we chose N like before and M such that $|\sum_{k=1}^{N} \varepsilon_k b_k| (\sum_{k=1}^{N+M} a_k)^{-1} < \delta$. $\square$

To prove that $\beta(p,n)$ in Lemma 17 does not depend on p and n let us denote by

$$b_k = \frac{b_{1k} b_{2k}}{(b_{1k} + b_{2k})^2}, \quad a_k = \frac{b_{2k}^2}{(b_{1k} + b_{2k})^2}, \quad \alpha_k = \frac{(b_{1k} + b_{2k})^2}{a_k(p,n)}.$$

In case (b) we have

$$\sum_{k \in \mathbb{N}} a_k = \sum_{k \in \mathbb{N}} \frac{b_{2k}^2}{(b_{1k} + b_{2k})^2} = \infty \quad \text{and} \quad \lim_k \alpha_k \neq 0.$$

Indeed

$$\begin{aligned} \alpha_k^{-1} = \frac{a_k(p,n)}{(b_{1k} + b_{2k})^2} &= \frac{c_{1n} c_{1p}}{2} \left(\frac{b_{1k}}{b_{1k} + b_{2k}} \right)^2 \\ &+ (c_{1n} c_{2p} + c_{2n} c_{1p} + 2 a_{1n} a_{2p} a_{2n} a_{1p}) \frac{b_{1k} b_{2k}}{4(b_{1k} + b_{2k})^2} \\ &+ \frac{c_{2n} c_{2p}}{2} \left(\frac{b_{2k}}{b_{1k} + b_{2k}} \right)^2, \end{aligned}$$

so $\lim_k \alpha_k^{-1} = \frac{c_{2n} c_{2p}}{2}$, hence $\lim_k \alpha_k = \frac{2}{c_{2n} c_{2p}} \neq 0$. Then using (14), (16) and (17) we conclude that $\beta_m(p,n) = \beta_m(\alpha)$. By Lemma 18 we have $\beta(p,n) = \beta(\alpha) = \beta$. We show that $\beta = 0$ in case (b). Indeed, in this case we have

$$\sum_{n=2}^{\infty} \frac{b_{1n}^2}{(b_{1n} + b_{2n})^2} < \infty \quad \text{since} \quad \infty > \sum_{n=2}^{\infty} \frac{b_{1n}}{b_{2n}} \sim \sum_{n=2}^{\infty} \frac{b_{1n}}{b_{1n} + b_{2n}}.$$

Using Cauchy–Schwarz–Bunyakovskii inequality we have

$$\sum_{n=2}^{N} \frac{|b_{1n} b_{2n}|}{(b_{1n} + b_{2n})^2} \leqslant \left(\sum_{n=2}^{N} \frac{b_{1n}^2}{(b_{1n} + b_{2n})^2} \right)^{1/2} \left(\sum_{n=2}^{N} \frac{b_{2n}^2}{(b_{1n} + b_{2n})^2} \right)^{1/2},$$

hence

$$|\beta_N| \leqslant \frac{1}{2} \left(\sum_{n=2}^{N} \frac{b_{1n}^2}{(b_{1n} + b_{2n})^2} \right)^{1/2} \left(\sum_{n=2}^{N} \frac{b_{2n}^2}{(b_{1n} + b_{2n})^2} \right)^{-1/2},$$

so $\beta = \lim_N \beta_N = 0$.

Hence by Lemma 17 we have in case (b) $x_{2p}x_{2n}\eta\mathfrak{A}^2, 2\leqslant p\leqslant n$. By Lemma 16 we conclude that $x_{2n}\eta\mathfrak{A}^2, 2\leqslant n$.

Now we use the combination for $2\leqslant p\leqslant k\leqslant n$

$$\begin{aligned} det\begin{bmatrix} A^{R,2}_{pn} & A^{R,2}_{kn} \\ x_{2p} & x_{2k} \end{bmatrix} &= det\begin{bmatrix} x_{1p}D_{1n}+x_{2p}D_{2n} & x_{1k}D_{1n}+x_{2k}D_{2n} \\ x_{2p} & x_{2k} \end{bmatrix} \\ &= det\begin{bmatrix} x_{1p}D_{1n} & x_{1k}D_{1n} \\ x_{2p} & x_{2k} \end{bmatrix} \\ &= det\begin{bmatrix} x_{1p} & x_{1k} \\ x_{2p} & x_{2k} \end{bmatrix} D_{1n} = (x_{1p}x_{2k}-x_{1k}x_{2p})D_{1n}. \end{aligned}$$

Multiplying the latter expression by $A^{R,2}_{1n} = x_{11}D_{1n}$ we get $x_{11}(x_{1p}x_{2k}-x_{1k}x_{2p})D^2_{1n}$. Using the same argument as in Lemma 13 we get

$$x_{11}(x_{1p}x_{2k}-x_{1k}x_{2p})\eta\mathfrak{A}^2, \quad 2\leqslant p\leqslant k. \tag{18}$$

By Lemma 13 we have $x_{11}^2\eta\mathfrak{A}^2$ and since $x_{2n}\eta\mathfrak{A}^2, 2\leqslant n$, using (18) we get $(x^{-1})_{1k} = (x_{11}x_{22})^{-1}(x_{12}x_{2k}-x_{22}x_{1k})\eta\mathfrak{A}^2, 2<k$, and $(x^{-1})_{2k} = -x_{2k}(x_{22})^{-1}, k>2$ (see Remark 19 for details).

Remark 19. In the case of the group $B_0^{\mathbb{N}}$ acting on the space X^2 with the measure $\mu_{(b,a)}$ under the conditions $\sum_{k=1}^{\infty} b_{1k}/b_{2k}<\infty$ it was possible to approximate (see [27]) firstly the elements $(x^{-1})_{2n}, n>2$, further $(x^{-1})_{1n}, n>2$, of the inverse matrix $\mathbb{X}^{-1}$ and only then the element $(x^{-1})_{12} = -x_{12}$, where

$$\begin{aligned} \mathbb{X}^{-1} &= \begin{pmatrix} 1 & x_{12} & x_{13} & \dots & x_{1n} & \dots \\ 0 & 1 & x_{23} & \dots & x_{2n} & \dots \\ 0 & 0 & 1 & \dots & 0 & \dots \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & -x_{12} & -x_{13}+x_{12}x_{23} & \dots & -x_{1n}+x_{12}x_{2n} & \dots \\ 0 & 1 & -x_{23} & \dots & -x_{2n} & \dots \\ 0 & 0 & 1 & \dots & 0 & \dots \end{pmatrix}. \end{aligned}$$

In the case of the group $Bor_0^{\mathbb{N}}$ acting on the similar space X^2 with the measure $\mu_{(b,a)}$ (we use the same notation for the space and the measure) under the conditions $\sum_{k=1}^{\infty} b_{1k}/b_{2k}<\infty$ we will have a similar sequence of action. We note that for the matrix $\mathbb{X}$ in this case we have

$$\mathbb{X}^{-1} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} & \dots \\ 0 & x_{22} & x_{23} & \dots & x_{2n} & \dots \\ 0 & 0 & 1 & \dots & 0 & \dots \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{x_{11}} & -\frac{x_{12}}{x_{11}x_{22}} & \frac{1}{x_{11}x_{22}}(x_{12}x_{23} - x_{22}x_{13}) & \dots & \frac{1}{x_{11}x_{22}}(x_{12}x_{2n} - x_{22}x_{1n}) & \dots \\ 0 & \frac{1}{x_{22}} & -\frac{x_{23}}{x_{22}} & \dots & -\frac{x_{2n}}{x_{22}} & \dots \\ 0 & 0 & 1 & \dots & 0 & \dots \end{pmatrix}.$$

As before we approximate firstly the elements $(x^{-1})_{2n} = -x_{2n}(x_{22})^{-1}, n>2$, further $(x^{-1})_{1n} = (x_{11}x_{22})^{-1}(x_{12}x_{2n} - x_{22}x_{1n}), n>2$, of the inverse matrix $\mathbb{X}^{-1}$, then the element $(x^{-1})_{11} = (x_{11})^{-1}$ (Lemma 22) and at last the element $(x^{-1})_{12} = -\frac{x_{12}}{x_{11}x_{22}}$.

Lemma 20. *We have for* $2\leqslant p$

$$x_{1p} - \beta_{12}(p)x_{2p} \in \langle (x_{1p}x_{2k} - x_{2p}x_{1k}) | 1<p<k \rangle \text{ if } \sum_{k=p+1}^{\infty} \frac{a_{2k}^2}{\frac{c_{2p}}{2b_{1k}} + \frac{c_{1p}}{2b_{2k}}} = \infty$$

and when the limit exists $\lim_m \beta_{12,m}(p) = \beta_{12}(p) \in \mathbb{R}$, *where*

$$\beta_{12,m}(p) = \sum_{k=p+1}^{m} \frac{a_{1k}a_{2k}}{\frac{c_{2p}}{2b_{1k}} + \frac{c_{1p}}{2b_{2k}}} \left(\sum_{k=p+1}^{m} \frac{a_{2k}^2}{\frac{c_{2p}}{2b_{1k}} + \frac{c_{1p}}{2b_{2k}}} \right)^{-1}.$$

Proof. Since $a_k = Mx_{2k} = a_{2k}$ and $\sum_k t_k Mx_{2k} = 1$ hence

$$\begin{aligned} &\left\| \left[\sum_{k=p+1}^{N} t_k(x_{1p}x_{2k} - x_{1k}x_{2p}) - (x_{1p} - \beta_{12}(p)x_{2p}) \right] \mathbf{1} \right\| \\ &= \left\| \left[x_{1p} \sum_{k=p+1}^{N} t_k(x_{2k} - a_{2k}) - x_{2p} \sum_{k=p+1}^{N} t_k(x_{1k} - a_{1k}) \right.\right. \\ &\quad \left.\left. - x_{2p} \left(\sum_{k=p+1}^{N} t_k a_{1k} - \beta_{12}(p) \right) \right] \mathbf{1} \right\| \\ &\leqslant \left\| x_{1p} \sum_{k=p+1}^{N} t_k(x_{2k} - a_{2k}) - x_{2p} \sum_{k=p+1}^{N} t_k(x_{1k} - a_{1k}) \right\| \\ &\quad + \left| \sum_{k=p+1}^{N} t_k a_{1k} - \beta_{12}(p) \right| \|x_{2p}\|. \end{aligned} \tag{19}$$

Since

$$\left\| x_{1p} \sum_{k=p+1}^{N} t_k(x_{2k} - a_{2k}) - x_{2p} \sum_{k=p+1}^{N} t_k(x_{1k} - a_{1k}) \right\|^2$$
$$= \|x_{1p}\|^2 \left\| \sum_{k=p+1}^{N} t_k(x_{2k} - a_{2k}) \right\|^2 + \|x_{2p}\|^2 \left\| \sum_{k=p+1}^{N} t_k(x_{1k} - a_{1k}) \right\|^2$$
$$= \sum_{k=p+1}^{N} t_k^2 \left(\frac{c_{1p}}{2b_{2k}} + \frac{c_{2p}}{2b_{1k}} \right) = \sum_{k=p+1}^{N} t_k^2 a_k(p),$$

where $a_k(p) = \frac{c_{2p}}{2b_{1k}} + \frac{c_{1p}}{2b_{2k}}$, using (2) we have

$$\min_{\{t_k\}} \left\{ \left\| x_{1p} \sum_{k=N_1}^{N_2} t_k(x_{2k} - a_{2k}) - x_{2p} \sum_{k=N_1}^{N_2} t_k(x_{1k} - a_{1k}) \right\|^2 \middle| \sum_{k=N_1}^{N_2} t_k M x_{2k} = 1 \right\}$$
$$= \left(\sum_{k=N_1}^{N_2} \frac{b_k^2}{a_k(p)} \right)^{-1} \xrightarrow{N_2 \to \infty} 0 \Leftrightarrow \infty = \sum_{k \in \mathbb{N}} \frac{b_k^2}{a_k(p)} = \sum_{k \in \mathbb{N}} \frac{a_{2k}^2}{\frac{c_{2p}}{2b_{1k}} + \frac{c_{1p}}{2b_{2k}}}.$$

Using (2) we get $t_k = \frac{a_{2k}}{\frac{c_{2p}}{2b_{1k}} + \frac{c_{1p}}{2b_{2k}}} \left(\sum_{k=p+1}^{m} \frac{a_{2k}^2}{\frac{c_{2p}}{2b_{1k}} + \frac{c_{1p}}{2b_{2k}}} \right)^{-1}$, so

$$\sum_{k=p+1}^{N} t_k a_{1k} = \sum_{k=p+1}^{N} \frac{a_{1k} a_{2k}}{\frac{c_{2p}}{2b_{1k}} + \frac{c_{1p}}{2b_{2k}}} \left(\sum_{k=p+1}^{N} \frac{a_{2k}^2}{\frac{c_{2p}}{2b_{1k}} + \frac{c_{1p}}{2b_{2k}}} \right)^{-1} = \beta_{12,N}(p).$$

To complete the proof of the lemma it is sufficient to use (19). $\square$

Using Lemma 18 we prove that $\beta_{12}(p)$ does not depend on p. Let us denote $b_k = b_{1k} a_{1k} a_{2k}, a_k = b_{1k} a_{2k}^2$,

$$\alpha_k = \frac{a_{1k} a_{2k}}{\frac{c_{2p}}{2b_{1k}} + \frac{c_{1p}}{2b_{2k}}} (b_{1k} a_{1k} a_{2k})^{-1}, \quad \beta_m = \sum_{k=p+1}^{m} b_{1k} a_{1k} a_{2k} \left(\sum_{k=p+1}^{m} b_{1k} a_{2k}^2 \right)^{-1}.$$

Since $\lim_k \alpha_k = 2c_{2p}^{-1} > 0$, and $\lim_m \beta_m(p) = \beta(p) \in \mathbb{R}$ so by Lemma 18 we conclude that $\beta_{12}(p) = \beta = \lim_m \beta_m$.

Lemma 21. *One has for* $2 \leqslant p$

$$\beta_{21} x_{1p} - x_{2p} \in \langle (x_{1p} x_{2k} - x_{1k} x_{2p}) | 1 < p < k \rangle \text{ if } \sum_{k=p+1}^{\infty} \frac{a_{1k}^2}{\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}} = \infty$$

and when the limit exists $\lim_m \beta_{21,m} = \beta_{21} \in \mathbb{R}$, *where*

$$\beta_{21,m} = \sum_{k=p+1}^{m} \frac{a_{1k}a_{2k}}{\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}} \left(\sum_{k=p+1}^{m} \frac{a_{1k}^2}{\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}} \right)^{-1}.$$

Proof. The proof is the same as the proof of the previous lemma. $\square$

Now we consider two subsets of the set $\mathbb{N}$ of all natural numbers:

$$\mathbb{N}_1 = \{n \in \mathbb{N} | a_{2n}^2 \leqslant a_{1n}^2\}, \quad \mathbb{N}_2 = \{n \in \mathbb{N} | a_{1n}^2 < a_{2n}^2\}.$$

By definition we have

$$\begin{aligned}\sum_{n \in \mathbb{N}} \frac{a_{1n}^2 + a_{2n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}} &= \sum_{n \in \mathbb{N}_1} \frac{a_{1n}^2 + a_{2n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}} + \sum_{n \in \mathbb{N}_2} \frac{a_{1n}^2 + a_{2n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}} \\ &\leqslant \sum_{n \in \mathbb{N}_1} \frac{2a_{1n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}} + \sum_{n \in \mathbb{N}_2} \frac{2a_{2n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}} = \sigma_1 + \sigma_2.\end{aligned}$$

with

$$\sigma_i = \sum_{n \in \mathbb{N}_i} \frac{2a_{in}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}}, \quad i = 1, 2.$$

In the case (b) the left part of the latter inequality is infinity. Indeed,

$$\begin{aligned}\sum_{n \in \mathbb{N}} \frac{a_{1n}^2 + a_{2n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}} &= \sum_{n \in \mathbb{N}} \frac{2b_{1n}(a_{1n}^2 + a_{2n}^2)}{1 + \frac{b_{1n}}{b_{2n}}} \sim \sum_{n \in \mathbb{N}} 2b_{1n}(a_{1n}^2 + a_{2n}^2) \\ &= S_{11}^L(\mu) + \sum_{n \in \mathbb{N}} 2b_{1n}a_{2n}^2 \sim S_{11}^L(\mu) \\ &\quad + \sum_{n \in \mathbb{N}} 2b_{1n}\left(\frac{1}{2b_{2n}} + a_{2n}^2\right) = S_{11}^L(\mu) + 4S_{12}^L(\mu) = \infty.\end{aligned}$$

So $\sigma_1 = \infty$ or $\sigma_2 = \infty$. Let, for example, $\sigma_2 = \infty$, then we conclude that

$$\sum_{n \in \mathbb{N}_2} \frac{a_{2n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{12n}}} = \infty, \quad \sum_{k \in [p+1,m] \cap \mathbb{N}_2} \frac{|a_{1n}a_{2n}|}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}} \leqslant \sum_{k \in [p+1,m] \cap \mathbb{N}_2} \frac{a_{2n}^2}{\frac{1}{2b_{1n}} + \frac{1}{2b_{2n}}}.$$

So for some subsequence $(m_r)_{r\in\mathbb{N}}\subset\mathbb{N}_2$ the limit exists $\lim_r \beta^{\mathbb{N}_2}_{12,m_r}=\beta^{\mathbb{N}_2}_{12}$ with $|\beta^{\mathbb{N}_2}_{12}|\leqslant 1$, where

$$\beta^{\mathbb{N}_2}_{12,m}=\sum_{k\in[p+1,m]\cap\mathbb{N}_2}\frac{a_{1k}a_{2k}}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}}\left(\sum_{k\in[p+1,m]\cap\mathbb{N}_2}\frac{a^2_{2k}}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}}\right)^{-1}.$$

hence $x_{1p}-\beta^{\mathbb{N}_2}_{12}x_{2p}\in\langle(x_{1p}x_{2k}-x_{1k}x_{2p})|1<p<k\rangle$ so $x_{11}(x_{1p}-\beta^{\mathbb{N}_2}_{12}x_{2p})\eta\mathfrak{A}^2$. If $\beta^{\mathbb{N}_2}_{12}=0$ we have $x_{11}x_{1p}\eta\mathfrak{A}^2$ hence $x_{11}\eta\mathfrak{A}^2$ by Lemma 14 and so x_{1p} also. In this case the proof of Theorem 5(iv) $\Rightarrow$ (i) is finished.

Let us suppose that $\beta^{\mathbb{N}_2}_{12}\neq 0$.

Lemma 22. *We have for $\beta\in\mathbb{R}$*

$$x_{11}\in\langle x_{11}(x_{1k}-\beta x_{2k})|2<k\rangle\Leftrightarrow\Sigma_2=\sum_{k=p+1}^{\infty}\frac{(a_{1k}-\beta a_{2k})^2}{\frac{1}{2b_{1k}}+\beta^2\frac{1}{2b_{2k}}}=\infty.$$

Proof. Since $M(x_{1k}-\beta x_{2k})=(a_{1k}-\beta a_{2k})$ and $\sum_k t_kM(x_{1k}-\beta x_{2k})=1$ we have

$$\begin{aligned}&\left\|\left[\sum_{k=p+1}^{N}t_kx_{11}(x_{1k}-\beta x_{2k})-x_{11}\right]\mathbf{1}\right\|^2\\&=\|x_{11}\|^2\left\|\sum_{k=p+1}^{N}t_k[(x_{1k}-a_{1k})-\beta(x_{2k}-a_{2k})]\right\|^2\\&=c_{11}\sum_{k=p+1}^{N}t_k^2\left(\frac{1}{2b_{1k}}+\beta^2\frac{1}{2b_{2k}}\right)\sim\sum_{k=p+1}^{N}t_k^2\left(\frac{1}{2b_{1k}}+\beta^2\frac{1}{2b_{2k}}\right).\end{aligned}$$

At last estimation (2) completes the proof of the lemma. $\square$

Using (b) and $S^{L,-}_{12}(\mu,t)=\infty$ we conclude that $\Sigma_2=\infty$. Indeed

$$\begin{aligned}\Sigma_2&=\sum_{k=p+1}^{\infty}\frac{(a_{1k}-\beta a_{2k})^2}{\frac{1}{2b_{1k}}+\beta^2\frac{1}{2b_{2k}}}=2\sum_{k=p+1}^{\infty}\frac{b_{1k}(a_{1k}-\beta a_{2k})^2}{1+\beta^2\frac{b_{1k}}{b_{2k}}}\\&\sim\frac{(2\beta)^2}{4}\sum_{k=p+1}^{\infty}\frac{b_{1k}}{b_{2k}}+\sum_{k=p+1}^{\infty}\frac{b_{1k}}{2}(-2a_{1k}+2\beta a_{2k})^2=S^{L,-}_{kn}(\mu,2\beta)=\infty.\end{aligned}$$

At last $x_{11}(x_{1p}-\beta x_{2p}),2\leqslant p$ and x_{11} are affiliated to $\mathfrak{A}^2$, so $(x_{1p}-\beta x_{2p})\eta\mathfrak{A}^2$ and finally x_{1p} is also affiliated to $\mathfrak{A}^2$ (since $x_{2p}\eta\mathfrak{A}^2$). Thus we have

$x_{1k}, x_{2n}\eta\mathfrak{A}^2, 1\leqslant k, 2\leqslant n$. If now $\sigma_1 = \infty$, we conclude that

$$\sum_{n\in\mathbb{N}_1} \frac{a_{1n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}} = \infty, \qquad \sum_{k\in[p+1,m]\cap\mathbb{N}_1} \frac{|a_{1n}a_{2n}|}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}}$$
$$\leqslant \sum_{k\in[p+1,m]\cap\mathbb{N}_1} \frac{a_{1n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}}.$$

So for some subsequence $(m_r)_{r\in\mathbb{N}}\subset\mathbb{N}_1$ the limit exists $\lim_r \beta_{21,m_r}^{\mathbb{N}_1} = \beta_{21}^{\mathbb{N}_1}, |\beta_{21}^{\mathbb{N}_1}|\leqslant 1$, where

$$\beta_{21,m}^{\mathbb{N}_1} = \sum_{k\in[p+1,m]\cap\mathbb{N}_1} \frac{a_{1k}a_{2k}}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}}\left(\sum_{k\in[p+1,m]\cap\mathbb{N}_1} \frac{a_{1k}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}}\right)^{-1}.$$

hence $\beta_{21}^{\mathbb{N}_2 1}x_{1p} - x_{2p} \in \langle (x_{1p}x_{2k} - x_{1k}x_{2p}) | 1<p<k\rangle$ so $x_{11}(\beta_{21}^{\mathbb{N}_1}x_{1p} - x_{2p})\eta\mathfrak{A}^2$.

If $\beta_{21}^{\mathbb{N}_1}\neq 0$ we have $\beta_{21}^{\mathbb{N}_1}x_{1p} - x_{2p} = \beta_{21}^{\mathbb{N}_1}(x_{1p} - \beta x_{2p})$ with $\beta = (\beta_{21}^{\mathbb{N}_1})^{-1}$. Lemma 22 finishes the proof in this case. If $\beta_{12}^{\mathbb{N}_2} = 0$ we have $x_{11}x_{2p}\eta\mathfrak{A}^2, 2\leqslant p$ hence $x_{11}\eta\mathfrak{A}^2$ by Lemma 16. We have by (18) $x_{11}(x_{1p}x_{2k} - x_{1k}x_{2p})\eta\mathfrak{A}^2$ so $x_{1p}x_{2k} - x_{1k}x_{2p}\eta\mathfrak{A}^2$. We use now the following expression $x_{1p}x_{2k} - x_{2p}x_{1k} + x_{2p}a_{1k} = x_{1p}x_{2k} - x_{2p}(x_{1k} - a_{1k}), 2\leqslant p\leqslant k$.

Lemma 23. *We have for* $2\leqslant p$

$$x_{1p}\in\langle x_{1p}x_{2k} - x_{2p}(x_{1k} - a_{1k}) | 1<p<k\rangle \Leftrightarrow \Sigma_3 = \sum_{k=p+1}^{\infty} \frac{a_{2k}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}} = \infty.$$

Proof. Since $Mx_{2k} = a_{2k}$ and $\sum_k t_k M x_{2k} = 1$ we have

$$\begin{aligned}
&\left\|\left[\sum_{k=p+1}^{N} t_k(x_{1p}x_{2k} - x_{2p}(x_{1k} - a_{1k})) - x_{1p}\right]\mathbf{1}\right\|^2 \\
&= \left\|\sum_{k=p+1}^{N} t_k[x_{1p}(x_{2k} - a_{2k}) - x_{2p}(x_{1k} - a_{1k})]\mathbf{1}\right\|^2 \\
&= \sum_{k=p+1}^{N} t_k^2\|[x_{1p}(x_{2k} - a_{2k}) - x_{2p}(x_{1k} - a_{1k})]\|^2 \\
&= \sum_{k=p+1}^{N} t_k^2\left(\frac{c_{1p}}{2b_{2k}} + \frac{c_{2p}}{2b_{1k}}\right) \sim \sum_{k=p+1}^{N} t_k^2\left(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\right).
\end{aligned}$$

At last estimation (2) complete the proof of the lemma. $\square$

But in case (b) as before $\Sigma_3 \sim 4S_{12}^L(\mu) = \infty$. This proves the irreducibility in case (b) for $m = 2$.

The proof of Theorem 5(iv) $\Rightarrow$ (i) for $m > 2$ is similar. It follows the schema used in [27] (see also Ref. [10] in [27]). $\square$

Acknowledgments

A.K. thank the Institute of Applied Mathematics, University of Bonn for the hospitality. The financial support by the DFG project 436 UKR 113/72 is gratefully acknowledged.

References

[1] S. Albeverio, R. Höegh-Krohn, The energy representation of Sobolev–Lie group, Composito Math. 36 (1978) 37–52.
[2] S. Albeverio, R. Höegh-Krohn, J. Marion, D. Testard, B. Torésani, Noncommutative distributions, Unitary representation of gauge groups and algebras, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 175. Marcel Dekker, New York, 1993.
[3] S. Albeverio, R. Höegh-Krohn, D. Testard, Irreducibility and reducibility for the energy representation of a group of mapping of a Riemannian manifold into a compact Lie group, J. Funct. Anal. 41 (1981) 378–396.
[4] S. Albeverio, R. Höegh-Krohn, D. Testard, A. Vershik, Factorial representations of path groups, J. Funct. Anal. 51 (1983) 115–131.
[5] J. Baik, P. Deift, K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, J. Amer. Math. Soc. 12 (1999) 1119–1178.
[6] E.F. Beckenbach, R. Bellmann, Inequalities, Springer, Berlin, Göttingen, Heidelberg, 1961.
[7] A. Borodin, A. Okounkov, G. Ol'shanskiĭ, Asymptotics of Plancherel measures for symmetric groups, J. Amer. Math. Soc. 13 (2000) 481–515 (electronic).
[8] A. Borodin, G. Ol'shanskiĭ, Point processes and the infinite symmetric group, Math. Res. Lett. 5 (1998) 799–816.
[9] J. Dixmier, Les C^*-algèbres et leur représentation, Gautier-Villars, Paris, 1969.
[10] J. Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien, 2nd Edition, Gauthier-Villars, Paris, 1969.
[11] I.M. Gel'fand, in Collected Papers, Vol. I, Springer, Berlin, 1987.
[12] I.M. Gel'fand, in: S.G. Gindikin, V.W. Guillemin, A.A. Kirillov, B. Kostant, S. Sternberg (Eds.), Collected Papers, Vol. II, Springer, Berlin, 1988.
[13] I.M. Gel'fand, in: S.G. Gindikin, V.W. Guillemin, A.A. Kirillov, B. Kostant, S. Sternberg (Eds.), Collected Papers, Vol. III, Springer, Berlin, 1989.
[14] I.M. Gel'fand, A.M. Vershik, M.I. Graev, Representations of $SL_2(R)$, where R is a ring of functions, Uspehi Mat. Nauk. 28 (1973) 83–128.
[15] R.S. Ismagilov, Representations of the group of smooth mappings in a compact Lie group, Funktsional. Anal. i Priložhen 15 (1981) 73–74.
[16] R.S. Ismagilov, Representations of infinite-dimensional groups, in: Translations of Mathematical Monographs, Vol. 152, American Mathematical Society, Providence, RI, 1996.
[17] S. Kakutani, On equivalence of infinite product measures, Ann. Math. 4 (1948) 214–224.
[18] S. Kerov, G. Ol'shanskiĭ, A. Vershik, Harmonic analysis on the infinite symmetric group, A deformation of the regular representation, C.R. Acad. Sci. Paris Sér. 1 316 (1993) 773–778.

[19] A.A. Kirillov, Unitary representations of nilpotent Lie groups, Uspehi Mat. Nauk. 17 (106) (1962) 57–110 (in Russian).
[20] A.A. Kirillov, Representations of the infinite-dimensional unitary group, Dokl. Akad. Nauk. SSSR. 212 (1973) 288–290 (in Russian).
[21] A.A. Kirillov, Elements of the theory of representations Grundlehren der Mathematischen Wissenschaften, Band 220. Springer, Berlin, New York, 1976 (translated from the Russian).
[22] A.A. Kirillov, Introduction to the theory of representations and noncommutative harmonic analysis, Representation theory and noncommutative harmonic analysis, I, Encyclopaedia of Mathematical Sciences, Vol. 22, Springer, Berlin, 1994, pp. 1–156.
[23] A.V. Kosyak, Irreducibility criterion for regular Gaussian representations of group of finite upper triangular matrices, Funktsional. Anal. i Priložhen 24 (1990) 82–83.
[24] A.V. Kosyak, Criteria for irreducibility and equivalence of regular Gaussian representations of group of finite upper triangular matrices of infinite order, Selecta Math. Soviet. 11 (1992) 241–291.
[25] A.V. Kosyak, Irreducible regular Gaussian representations of the group of the interval and the circle diffeomorphisms, J. Funct. Anal. 125 (1994) 493–547.
[26] A.V. Kosyak, Regular representations of the group of finite upper-triangular matrices, corresponding to product measures, and criteria for their irreducibility, Methods Funct. Anal. Topol. 6 (2000) 43–65.
[27] A.V. Kosyak, The generalized Ismagilov conjecture for the group $B_0^{\mathbb{N}}$. II, Methods Funct. Anal. Topol. 8 (2002) 27–45.
[28] A.V. Kosyak, Irreducibility criterion for quasiregular representations of the group of finite upper-triangular matrices, Funktsional. Anal. i Priložhen 37 (2003) 78–81.
[29] A.V. Kosyak, Quasi-invariant measures and irreducible representations of the inductive limit of the special linear groups, Funktsional. Anal. i Priložhen 38 (2004) 82–84.
[30] H.H. Kuo, Gaussian measures in Banach Spaces, in: Lecture Notes in Mathematics, Vol. 463, Springer, Berlin, 1975.
[31] M.P. Malliavin, Naturality of quasi-invariance of some measures, in: A.B. Cruzeiro, J.-C. Zambrini (Eds.), Stochastic Analysis and Applications (Lisbon, 1989), Progress in Probability, Vol. 26, Birkhaüser, Boston, Boston, MA, 1991, pp. 144–154.
[32] M.P. Malliavin, P. Malliavin, Measures quasi invariantes sur certain groupes de dimension infinie, C.R. Acad. Sci. Paris Sér. 1 311 (1990) 765–768.
[33] M.P. Malliavin, P. Malliavin, Integration on loop groups, I, J. Funct. Anal. 93 (1990) 207–237.
[34] Yu.A. Neretin, Categories of symmetries and infinite-dimensional groups, London Mathematical Society Monographs. New Series, 16, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.
[35] N. Obata, Certain unitary representations of the infinite symmetric group, I, Nagoya Math. J. 105 (1987) 109–119.
[36] N. Obata, Certain unitary representations of the infinite symmetric group, II, Nagoya Math. J. 106 (1987) 143–162.
[37] G.I. Ol'shanskiĭ, Unitary representations of infinite-dimensional pairs (G, K) and the formalism of R. Howe, in: A.M. Vershik, D.P. Zhelobenko (Eds.), Representation of Lie Groups and Related Topics, Advance Studies Contemporary Mathematics, Vol. 7, Gordon and Breach, New York, 1990, pp. 269–463.
[38] G.I. Ol'shanskiĭ, Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians, in: A.A. Kirillov (Ed.), Topics in Representation Theory, Advances in Soviet Mathematics, Vol. 2, Amer. Math. Soc., Providence, RI, 1991, pp. 1–66.
[39] M. Reed, B. Simon, Methods of Modern Mathematical Physics, Vol. I, Academic Press, New York, London, 1972.
[40] E.T. Shavgulidze, Distributions on infinite-dimensional spaces and second quantization in string theories, II, in: V International Vilnius Conference on Probability Theory and Math. Statistics, Abstracts of Comm., Vilnius, June 26–July 1, 1989, pp. 359–360.

[41] A.V. Skorokhod, Integration in Hilbert Space, Springer, Berlin, 1974.
[42] E. Toma, Die unzerlegbaren, positive-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe, Math. Z. 85 (1964) 40–61.
[43] A.M. Vershik, S.V. Kerov, Asymptotic theory of characters of the symmetric group, Funct. Anal. Appl. 15 (1981) 246–255.
[44] A.M. Vershik, S.V. Kerov, Characters and factor representations of the infinite symmetric group, Soviet Math. Dokl. 23 (1981) 389–392.
[45] D.P. Zhelobenko, A.I. Shtern, Representations of Lie Groups, Nauka, Moscow, 1983.