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# Quasiregular representations of the infinite-dimensional nilpotent group

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**Abstract**

In the present work an analog of the quasiregular representation which is well known for locally-compact groups is constructed for the nilpotent infinite-dimensional group $B_0^{\mathbb{N}}$ and a criterion for its irreducibility is presented. This construction uses the infinite tensor product of arbitrary Gaussian measures in the spaces $\mathbb{R}^m$ with $m > 1$ extending in a rather subtle way previous work of the second author for the infinite tensor product of one-dimensional Gaussian measures.


*Keywords:* Infinite-dimensional groups; Nilpotent groups; Quasiregular representations; Irreducibility; Infinite tensor products; Gaussian measures; Ismagilov conjecture

## 1. Introduction

### 1.1. The setting and the main results

Let $(X, \mathfrak{B})$ be a measurable space and let $\mathrm{Aut}(X)$ denote the group of all measurable automorphisms of the space $X$. With any measurable action $\alpha : G \to \mathrm{Aut}(X)$ of a group

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$G$ on the space $X$ and a $G$-quasi-invariant measure $\mu$ on $X$ one can associate a unitary representation $\pi^{\alpha,\mu,X}: G \to U(L^2(X,\mu))$, of the group $G$ by the formula $(\pi_t^{\alpha,\mu,X} f)(x) = (d\mu(\alpha_{t^{-1}}(x))/d\mu(x))^{1/2} f(\alpha_{t^{-1}}(x))$, $f \in L^2(X,\mu)$. Let us set $\alpha(G) = \{\alpha_t \in \mathrm{Aut}(X) \mid t \in G\}$. Let $\alpha(G)'$ be the centralizer of the subgroup $\alpha(G)$ in $\mathrm{Aut}(X)$: $\alpha(G)' = \{g \in \mathrm{Aut}(X) \mid \{g, \alpha_t\} = g\alpha_t g^{-1}\alpha_t^{-1} = e\ \forall t \in G\}$. The following conjecture has been discussed in [23–25].

**Conjecture 1.** *The representation $\pi^{\alpha,\mu,X}: G \to U(L^2(X,\mu))$ is irreducible if and only if*:

(1) $\mu^g \perp \mu\ \forall g \in \alpha(G)' \setminus \{e\}$ *(where $\perp$ stands for singular),*
(2) *the measure $\mu$ is $G$-ergodic.*

We recall that a measure $\mu$ is $G$-ergodic if $f(\alpha_t(x)) = f(x)\ \forall t \in G$ implies $f(x) = const\ \mu$ a.e. for all functions $f \in L^1(X,\mu)$.

In this paper we shall prove Conjecture 1 in the case where $G$ is the infinite-dimensional nilpotent group $G = B_0^{\mathbb{N}}$ of finite upper-triangular matrices of infinite order with unities on the diagonal, the space $X = X^m$ being the set of left cosets $G_m \setminus B^{\mathbb{N}}$, $G_m$ being suitable subgroups of the group $B^{\mathbb{N}}$ of all upper-triangular matrices of infinite order with unities on the diagonal, and $\mu$ an infinite tensor product of Gaussian measures on the spaces $\mathbb{R}^m$ with some fixed $m > 1$. A more detailed explanation of the concepts used here is given in the following sections.

### *1.2. Regular and quasiregular representations of locally compact groups*

Let $G$ be a locally compact group. The *right* $\rho$ (respectively *left* $\lambda$) *regular representation* of the group $G$ is a particular case of the representation $\pi^{\alpha,\mu,X}$ with the space $X = G$, the action $\alpha$ being the right action $\alpha = R$ (respectively the left action $\alpha = L$), and the measure $\mu$ being the right invariant Haar measure on the group $G$ (see, for example, [8,16,17,37]).

A *quasiregular representation* of a locally compact group $G$ is also a particular case of the representation $\pi^{\alpha,\mu,X}$ (see, for example, [37, p. 27]) with the space $X = H \setminus G$, where $H$ is a subgroup of the group $G$, the action $\alpha$ being the right action of the group $G$ on the space $X$ and the measure $\mu$ being some quasi-invariant measure on the space $X$ (this measure is unique up to a scalar multiple). We remark that in [16,17] this representation has also been called *geometric representation*.

### *1.3. Analogs of the regular and quasiregular representations of infinite-dimensional groups and the Ismagilov conjecture*

In the present article we will consider the approach which deals with analogs for infinite-dimensional groups of the regular and quasiregular representations of finite-dimensional groups. Let $G$ be an infinite-dimensional topological group. To define an analog of the regular representation, let us consider some topological group $\widetilde{G}$, containing the initial group $G$ as a dense subgroup, i.e. $\overline{G} = \widetilde{G}$ ($\overline{G}$ being the closure of $G$). Suppose we have some quasi-invariant measure $\mu$ on $X = \widetilde{G}$ with respect to the right action of the group $G$, i.e. $\alpha = R$, $R_t(x) = xt^{-1}$. In this case we shall call the representation $\pi^{\alpha,\mu,\widetilde{G}}$ an *analog of the regular representation*. We shall denote this representation by $T^{R,\mu}$, and the Conjecture 1 is reduced to the following *Ismagilov conjecture*.

**Conjecture 2.** (Ismagilov, 1985) *The right regular representation* $T^{R,\mu}: G \to U(L^2(\widetilde{G}, \mu))$ *is irreducible if and only if*:

(1) $\mu^{L_t} \perp \mu\ \forall t \in G \setminus \{e\}$,
(2) *the measure* $\mu$ *is* $G$*-ergodic.*

**Remark 3.** In the case of the right regular representation, the group $\alpha(G)' = R(G)' \subset \mathrm{Aut}(\widetilde{G})$ obviously contains the group $L(G)$, the image of the group $G$ with respect to the left action.

The work [11] initiated the study of representations of current groups, i.e. groups $C(X, U)$ of continuous mappings $X \mapsto U$, where $X$ is a finite-dimensional Riemannian manifold and $U$ is a finite-dimensional Lie group.

The regular representation of infinite-dimensional groups, in the case of current groups, was studied firstly in [1,4,5,14] (see also the book [6]). An analog of the regular representation for an arbitrary infinite-dimensional group $G$, using a $G$-quasi-invariant measure on some completion $\widetilde{G}$ of such a group, is defined in [18,20].

For $X = S^1$, $U$ a compact or non-compact connected Lie group, Wiener measures on the loop groups $\widetilde{G} = C(X, U)$ were constructed and their quasi-invariance were proved in [1,4–6,28–32].

Conjecture 2 was formulated by R.S. Ismagilov for the group $G = B_0^{\mathbb{N}}$ and the measure $\mu$ being the product of arbitrary one-dimensional centered Gaussian measures on the group $\widetilde{G} = B^{\mathbb{N}}$ and was proved for this case in [18,19].

The first result in this direction was proved in [33]. For the complex infinite-dimensional Borel group $Bor_0^{c,\mathbb{N}}$ and the standard Gaussian measure on its completion $Bor^{c,\mathbb{N}}$ the irreducibility of the corresponding regular representation was proved there. Here $Bor_0^{c,\mathbb{N}}$ (respectively $Bor^{c,\mathbb{N}}$) is the group of matrices of the form $x = \exp t + s$ where $t$ is a diagonal matrix with a finite number of nonzero real elements (respectively arbitrary real elements) and $s$ is a finite (respectively arbitrary) complex strictly upper-triangular matrix.

For the product of arbitrary one-dimensional measures on the group $B^{\mathbb{N}}$ Conjecture 2 was proved in [21] under some technical assumptions on the measure.

In [20] Conjecture 2 was proved for the groups of the interval and circle diffeomorphisms. For the group of the interval diffeomorphisms the Shavgulidze measure [35] was used, the image of the classical Wiener measure with respect to some bijection. For the group of circle diffeomorphisms the Malliavin measure [30] was used.

Whether Conjecture 2 holds in the general case is an open problem.

In [25] it was shown that Conjecture 1 holds for the inductive limit $G = \mathrm{SL}_0(2\infty, \mathbb{R}) = \varinjlim_n \mathrm{SL}(2n-1, \mathbb{R})$, of the special linear groups (*simple groups*) acting on a strip of length $m \in \mathbb{N}$ in the space of real matrices which are infinite in both directions, the measure $\mu$ being a product Gaussian measure.

Let us consider the special case of a $G$-space, namely the homogeneous space $X = H \setminus \widetilde{G}$, where $H$ is a subgroup of the group $\widetilde{G}$ and $\mu$ is some quasi-invariant measure on $X$ (if it exists) with respect to the right action $R$ of the group $G$ on the homogeneous space $H \setminus \widetilde{G}$. In this case we call the corresponding representation $\pi^{R,\mu,H\setminus\widetilde{G}}$ an *analog of the quasiregular or geometric representation* of the group $G$ (see [22]).

In [2] Conjecture 1 was proved for the *solvable* infinite-dimensional real Borel group $G = Bor_0^{\mathbb{N}}$ acting on $G$-spaces $X^m$, $m \in \mathbb{N}$, where $X^m$ is the set of left cosets $G_m \setminus Bor^{\mathbb{N}}$, and $G_m$ is some subgroups of the group $Bor^{\mathbb{N}}$ of all upper-triangular matrices of infinite order with nonzero elements on the diagonal. The measure $\mu$ on $X^m$ is the product of infinitely many one-dimensional Gaussian measures on $\mathbb{R}$.

In [23,24] Conjecture 1 was proved for the *nilpotent group* $G = B_0^{\mathbb{N}}$ and some $G$-spaces $X^m$, $m \in \mathbb{N}$, being the set of left cosets $G_m \setminus B^{\mathbb{N}}$, where $G_m$ are some subgroups of the group $B^{\mathbb{N}}$. Here the measure $\mu$ on $X^m$ is the infinite product of arbitrary one-dimensional Gaussian measures on $\mathbb{R}$. In this case the variables $x_{pq}$, $1 \leqslant p < q \leqslant m$, can be approximated by linear combinations of the expressions $A_{pn}A_{qn}$, $q < n$, where $A_{kn}$ are generators of one-parameter groups $\exp(tE_{kn})$, $k < n$, $t \in \mathbb{R}$.

In [3], using results of [21], we extended the results of [22–24] to the case of an infinite tensor product of one-dimensional non-Gaussian (general) measures.

In the present article we generalize results of [22–24] in another direction. Namely we prove Conjecture 1 for the same nilpotent infinite-dimensional group $G = B_0^{\mathbb{N}}$ and the same $G$-spaces $X^m$, $m \in \mathbb{N}$, but with a measure $\mu$ which is the infinite tensor product of arbitrary centered Gaussian measures on $\mathbb{R}^m$, for any arbitrary fixed $m \in \mathbb{N}$. More precisely, the measure $\mu$ on $X^m \simeq \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^{m-1} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$ is the infinite tensor product of arbitrary Gaussian centered measures:

$$\mu = \mu_B^m = \bigotimes_{n=2}^{\infty} \mu_{B^{(n)}},$$

where $\mu_{B^{(n)}}$ is a Gaussian measure on the space $\mathbb{R}^{n-1}$ for $2 \leqslant n \leqslant m$ and $\mu_{B^{(n)}}$ is a Gaussian measure on the space $\mathbb{R}^m$ for $n > m$. In this case for the approximation of the variables $x_{pq}$, $1 \leqslant p < q \leqslant m$, we also use the commutative family of the generators $A_{kn}$, $1 \leqslant k \leqslant m < n$, but the corresponding expressions are much more complicated. In fact the extensions of [22–24] to the present case are not at all simple, the above expressions are no longer polynomials in the generators $A_{kn}$ they rather involve, next to the generators, also the one-parameter groups

$$T^{R,\mu_B^m}_{\exp(tE_{kn})} = \exp(tA_{kn}), \quad t \in \mathbb{R},$$

their derivatives and very special suitable chosen combinations that allow to approximate in an appropriate way the variables involved (see Lemmas 12 and 15).

## 2. Main objects

Let us consider the group $\widetilde{G} = B^{\mathbb{N}}$ of all upper-triangular real matrices of infinite order with unities on the diagonal

$$\widetilde{G} = B^{\mathbb{N}} = \left\{ I + x \;\middle|\; x = \sum_{1 \leqslant k < n} x_{kn} E_{kn} \right\},$$

and its subgroup

$$G = B_0^{\mathbb{N}} = \left\{ I + x \in B^{\mathbb{N}} \mid x \text{ is finite} \right\},$$

where $E_{kn}$ is an infinite-dimensional matrix with 1 at the place $k, n \in \mathbb{N}$ and zeros elsewhere, $x = (x_{kn})_{k<n}$ is *finite* means that $x_{kn} = 0$ for all $(k, n)$ except for a finite number of indices $k, n$.

Obviously, $B_0^{\mathbb{N}} = \varinjlim_n B(n, \mathbb{R})$ is the inductive limit of the group $B(n, \mathbb{R})$ of real upper-triangular matrices with units on the principal diagonal

$$B(n, \mathbb{R}) = \left\{ I + \sum_{1 \leqslant k < r \leqslant n} x_{kr} E_{kr} \;\middle|\; x_{kr} \in \mathbb{R} \right\}$$

with respect to the natural imbedding $B(n, \mathbb{R}) \subset B(n+1, \mathbb{R})$. For $m \in \mathbb{N}$ we also define the subgroups $G_m$, respectively $G^m$, of the group $B^{\mathbb{N}}$ as follows:

$$G_m = \left\{ I + x \in B^{\mathbb{N}} \;\middle|\; x = \sum_{m < k < n} x_{kn} E_{kn} \right\},$$

$$G^m = \left\{ I + x \in B^{\mathbb{N}} \;\middle|\; x = \sum_{1 \leqslant k \leqslant m,\, k < n} x_{kn} E_{kn} \right\}.$$

Since $B^{\mathbb{N}} = G_m \cdot G^m$ the space $X^m$ of left cosets $X^m = G_m \setminus B^{\mathbb{N}}$ is isomorphic to the group $G^m$. We use the notation $X^m \simeq G^m$. By construction, the right action $R$ of the group $G$ is well defined on the space $X^m$. More precisely if we define the decomposition $x = x_m \cdot x^m$:

$$B^{\mathbb{N}} \ni x \mapsto x_m \cdot x^m \in G_m \cdot G^m,$$

the right action $R$ of the group $B_0^{\mathbb{N}}$ on the space $X^m$ is defined as follows:

$$R_t\big(x^m\big) = \big(x^m t^{-1}\big)^m, \quad x^m \in G^m,\ t \in B_0^{\mathbb{N}}.$$

Define the measure $\mu^m := \mu_B^m$ on the space $X^m \simeq G^m$

$$X^m \simeq \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^{m-1} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$$

by the formula $\mu_B^m = \bigotimes_{n=2}^{\infty} \mu_{B^{(n)}}$, where $\mu_{B^{(n)}}$ is the Gaussian measure on the space $\mathbb{R}^m$ for $n > m$ (respectively on the space $\mathbb{R}^{n-1}$ for $2 \leqslant n \leqslant m$) defined by

$$\begin{aligned} d\mu_{B^{(n)}}(x) &= \frac{1}{\sqrt{(2\pi)^m \det B^{(n)}}} \exp\left( -\frac{1}{2} \left( \big(B^{(n)}\big)^{-1} x, x \right) \right) dx \\ &= \sqrt{\frac{\det C^{(n)}}{(2\pi)^m}} \exp\left( -\frac{1}{2} \big( C^{(n)} x, x \big) \right) dx, \end{aligned} \tag{1}$$

where $B^{(n)}$ are positive-definite operators in the space $\mathbb{R}^m$ (or $\mathbb{R}^{n-1}$), $x = (x_{1n}, x_{2n}, \ldots, x_{mn})$, $dx$ is a Lebesgue measure on $\mathbb{R}^m$ and $C^{(n)} = (B^{(n)})^{-1}$.

**Lemma 4.** *For the measure $\mu_B^m$ we have*

$$\big(\mu_B^m\big)^{R_t} \sim \mu_B^m \quad \forall t \in B_0^{\mathbb{N}}$$

(*with $\sim$ meaning equivalence*).

**Proof.** The right action $R_t$ for $t \in B_0^{\mathbb{N}}$ changes linearly only a finite number of coordinates of the point $x \in X^m$. $\square$

Now we can define the representation associated with the right action

$$T^{R,\mu_B^m} : B_0^{\mathbb{N}} \to U\big(L^2\big(X^m, \mu_B^m\big)\big)$$

in the natural way, i.e.

$$\big(T_t^{R,\mu_B^m} f\big)(x) = \big(d\mu_B^m\big(R_t^{-1}(x)\big)/d\mu_B^m(x)\big)^{1/2} f\big(R_t^{-1}(x)\big).$$

**Theorem 5.** *For the measure $\mu_B^m$ the following four statements are equivalent*:

(i) *the representation $T^{R,\mu_B^m}$ is irreducible*;
(ii) $(\mu_B^m)^{L_t} \perp \mu_B^m$ $\forall t \in B(m,\mathbb{R}) \setminus \{e\}$;
(iii) $(\mu_B^m)^{L_{\exp(tE_{pq})}} \perp \mu_B^m$ $\forall t \in \mathbb{R} \setminus \{0\}$ $\forall 1 \leqslant p < q \leqslant m$;
(iv) $S_{pq}^L(\mu_B^m) = \sum_{n=q+1}^{\infty} c_{pp}^{(n)} b_{qq}^{(n)} = \infty$ $\forall 1 \leqslant p < q \leqslant m$,

*where* $B^{(n)} = (b_{kr}^{(n)})_{k,r=1}^m$, $C^{(n)} = (c_{kr}^{(n)})_{k,r=1}^m$ *and* $C^{(n)} = (B^{(n)})^{-1}$.

The proof of Theorem 5 is given in Sections 3–5 and Appendices A–C.

**Lemma 6.** *The measure $\mu_B^m$ on the space $X^m$ is ergodic with respect to the right action $R$ of the group $B_0^{\mathbb{N}}$ on the space $X^m$.*

**Proof.** It is well known that any measurable function on $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$ with the standard Gaussian measure $\mu_{\mathbb{I}} = \bigotimes_{n=1}^{\infty} \mu_{I_n}$, where $I_n \equiv I$ (see (1)) which is invariant under any change of the first coordinates (i.e. with respect to the additive action of the group $\mathbb{R}_0^\infty$) coincides almost everywhere with a constant function (see [36, Section 3, Corollary 1]). The proof works also in the case where we replace $\mathbb{R}$ by $\mathbb{R}^m$, $m > 1$, and the standard Gaussian measure $\mu_I$ on $\mathbb{R}$ with any probability measure $\mu_{B^{(n)}}$ on $\mathbb{R}^m$ equivalent with the Lebesgue measure on $\mathbb{R}^m$. To prove this it is sufficient to see that any function $f \in L^1((\mathbb{R}^m)^\infty, \bigotimes_{n=1}^{\infty} \mu_{B^{(n)}})$ is the limit of $\mu_k$-a.e. constant functions $f^k$: $f = \lim_k f^k$, where $\mu_k = \bigotimes_{n=1}^{k} \mu_{B^{(n)}}$,

$$f^k = \int_{(\mathbb{R}^m)^\infty} f(x)\, d\mu^k(x) \quad \text{and} \quad \mu^k = \bigotimes_{n=k+1}^{\infty} \mu_{B^{(n)}}.$$

Therefore the proof follows from the fact that the measure $\mu_B^m = \bigotimes_{n=2}^{\infty} \mu_{B^{(n)}}$ on the space $X^m = \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^{m-1} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$ is the infinite tensor product of Gaussian measures $\mu_{B^{(n)}}$ on the space $\mathbb{R}^m$ (for $n > m$), from the fact that the right action $R_t$ for $t \in B_0^{\mathbb{N}}$ changes only a finite number of coordinates of the point $x \in X^m$, and that the group $G_0^m = G^m \cap B_0^{\mathbb{N}} \subset X^m$ acts transitively on itself. In fact it is shown that the measure is ergodic with respect to the action of the subgroup $G_0^m \subset B_0^{\mathbb{N}}$. $\square$

## 3. Idea of the proof of irreducibility

**Proof of Theorem 5.** The proof of Theorem 5 is organized as follows:

$$\text{(i)} \quad \Rightarrow \quad \text{(ii)} \quad \Rightarrow \quad \text{(iii)} \quad \Rightarrow \quad \text{(iv)} \quad \Rightarrow \quad \text{(i)}.$$

The parts (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are evident. The part (iii) $\Leftrightarrow$ (iv) follows from Lemma 8, which is based on the Kakutani criterion [15].

*The idea of the proof of irreducibility, i.e. the part* (iv) $\Rightarrow$ (i). Let us denote by $\mathfrak{A}^m$ the von Neumann algebra generated by the representation $T^{R,\mu_B^m}$

$$\mathfrak{A}^m = \big(T_t^{R,\mu_B^m} \mid t \in G\big)''.$$

We show that (iv) $\Rightarrow [(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)] \Rightarrow$ (i). Let the inclusion $(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)$ holds. Using the ergodicity of the measure $\mu_B^m$ (Lemma 6) this proves the irreducibility. Indeed in this case an operator $A \in (\mathfrak{A}^m)'$ should be the operator of multiplication (since $(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)$) by some essentially bounded function $a \in L^\infty(X^m, \mu_B^m)$. The commutation relation $[A, T_t^{R,\mu_B^m}] = 0$ $\forall t \in B_0^{\mathbb{N}}$ implies $a(R_t^{-1}(x)) = a(x)$ (mod $\mu_B^m$) $\forall t \in B_0^{\mathbb{N}}$, so by ergodicity of the measure $\mu_B^m$ with respect to the right action of the group $B_0^{\mathbb{N}}$ on the space $X^m$ we conclude that $A = a = const$ (mod $\mu_B^m$). This then proves the irreducibility in Theorem 5, i.e. the part $[(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)] \Rightarrow$ (i).

The proof of the remaining part, i.e. the implication (iv) $\Rightarrow [(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)]$ is based on the fact that the operators of multiplication by independent variables $x_{pq}$, $1 \leqslant p \leqslant m$, $p < q$, may be approximated in the strong resolvent sense by some functions of the generators

$$A_{kn}^{R,m} = \frac{d}{dt} T_{I+tE_{kn}}^{R,\mu_B^m}\bigg|_{t=0}, \quad k, n \in \mathbb{N},\ k < n,$$

i.e. that the operators $x_{pq}$ are affiliated with the von-Neumann algebra $\mathfrak{A}^m$. See Lemma 15 and Corollary 17.

**Definition.** Recall (cf., e.g., [9]) that a non-necessarily bounded self-adjoint operator $A$ in a Hilbert space $H$ is said to be *affiliated* with a von Neumann algebra $M$ of operators in this Hilbert space $H$, if $\exp(itA) \in M$ for all $t \in \mathbb{R}$. One then writes $A \,\eta\, M$.

Since the algebra $(\exp(itx_{pq}) \mid t \in \mathbb{R},\ 1 \leqslant p \leqslant m,\ p < q)''$ is the maximal abelian subalgebra in the von Neumann algebra $B(H)$ of all bounded operator in the Hilbert space $H = L^2(X^m, \mu_B^m)$ we conclude that $(\exp(itx_{pq}) \mid t \in \mathbb{R},\ 1 \leqslant p \leqslant m,\ p < q)'' = L^\infty(X^m, \mu_B^m)$. The inclusion $(\exp(itx_{pq}),\ 1 \leqslant p \leqslant m,\ p < q) \subset \mathfrak{A}^m$ implies $(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)$.

To finish the proof of Theorem 5 it remains to prove the implication

$$\text{(iv)} \quad \Rightarrow \quad \big(x_{pq} \,\eta\, \mathfrak{A}^m,\ 1 \leqslant p \leqslant m,\ p < q\big) \quad \Leftrightarrow \quad \big(\exp(itx_{pq}) \in \mathfrak{A}^m,\ 1 \leqslant p \leqslant m,\ p < q\big)$$

(see Section 5). It is sufficient to prove that $\Sigma_m > C S_m$, for some $C > 0$, where

$$S_m := \sum_{1 \leqslant p < q \leqslant m} S_{pq}^L\big(\mu_B^m\big), \qquad \Sigma_m := \sum_{1 \leqslant r \leqslant p < q \leqslant m} \Sigma_{pq}^r(m),$$

and the series $S_{pq}^{L}(\mu^m)$ and $\Sigma_{pq}^{r}(m)$ are defined in Lemmas 8 and 15 (see also (18)). This is done in Appendices A–C.

In Appendix A we define the *generalization of the characteristic polynomial* for matrix $C$ and establish some its properties. These properties are used then in Appendices B and C. For a matrix $C \in \mathrm{Mat}(k, \mathbb{C})$ we set

$$G_k(\lambda) = \det C_k(\lambda), \quad \text{where } C_k(\lambda) = C + \sum_{r=1}^{k} \lambda_r E_{rr},\ \lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k.$$

**Lemma A.** (See Appendix A, Lemma A.7) *For a positive definite matrix $C \in \mathrm{Mat}(k, \mathbb{C})$, $\lambda \in \mathbb{R}^k$ with $\lambda_r \geqslant 0$, $r = 1, \dots, k$, we have*

$$\frac{\partial}{\partial \lambda_p} \frac{G_k(\lambda)}{G_l(\lambda)} \geqslant 0,$$

*where $G_l(\lambda) = M_{12\dots l}^{12\dots l}(C_k(\lambda))$ and $1 \leqslant p \leqslant l \leqslant k$.*

The proof of Lemma A is based on the following inequality (see Lemma A.6).

**Lemma B.** (Hadamard–Ficher's inequality [12,13], see also [27]) *Let $C \in \mathrm{Mat}(m, \mathbb{R})$ be a positive definite matrix and $\emptyset \subseteq \alpha, \beta \subseteq \{1, \dots, m\}$. Then*

$$\begin{vmatrix} \det C_\alpha & \det C_{\alpha\cap\beta} \\ \det C_{\alpha\cup\beta} & \det C_\beta \end{vmatrix} = \begin{vmatrix} M(\alpha) & M(\alpha\cap\beta) \\ M(\alpha\cup\beta) & M(\beta) \end{vmatrix} \geqslant 0,$$

*where $C_\alpha$ for $\alpha = \{\alpha_1, \dots, \alpha_s\}$ denotes the matrix which entries lie on the intersection of $\alpha_1, \dots, \alpha_s$ rows and $\alpha_1, \dots, \alpha_s$ columns of the matrix $C$ and $M(\alpha) = M_\alpha^\alpha(C) = \det C_\alpha$ are corresponding minors of the matrix $C$.*

The "best" approximation of $x_{pq}$ by the generators $A_{kn}^{R,m}$ is based on the exact computation of the matrix elements

$$\phi_p(t) = \left(T_t^{R,\mu_B^m} \mathbf{1}, \mathbf{1}\right), \quad t = I + \sum_{r=1}^{p} t_r E_{rn},\ (t_r)_{r=1}^{p} \in \mathbb{R}^p,$$

of the representation $T^{R,\mu_B^m}$ and their generalization (see Appendix B, Lemma B.1), and on the finding the appropriate combinations of operator functions of the generators $A_{kn}^{R,m}$ (see Remark 13) to approximate the operators of multiplication by $x_{pq}$.

Finally the proof of the inequality $\Sigma_m > C S_m$, is based on Lemmas A, B and 16 dealing with some inequalities involving the generalized characteristic polynomials. Lemma 16 is proved in Appendix C.

**Remark 7.** We shall firstly prove the approximation of $x_{kn}$ in the above sense for the one vector $\mathbf{1} \in L^2(X^m, \mu_B^m)$. Secondly, the approximation also holds for some dense set $D$ of analytic vectors in the space $L^2(X^m, \mu_B^m)$

$$D = \left\langle X^\alpha = \prod_{1 \leqslant k \leqslant m,\, k<n} x_{kn}^{\alpha_{kn}} \;\Big|\; \alpha \in \Lambda \right\rangle,$$

for the corresponding operators, where $\Lambda = \{\alpha = (\alpha_{kn})_{1\leqslant k \leqslant m,\, k<n}\}$ is the set of finite (i.e. $\alpha_{kn} = 0$ for a large $n$) multi-indices $\alpha$ with $\alpha_{kn} = 0, 1, \dots$ and $\langle f_n \mid n \in \mathbb{N}\rangle$ means the closure of the linear space generated by the set of vectors $(f_n \mid n \in \mathbb{N})$. So using [34, Theorem VIII.25] we conclude that the convergence holds in the strong resolvent sense. (We observe that the proof of approximation in the strong resolvent sense is the same as the one given in [19, Lemma 2.2, p. 250].) Since the generators $A_{kn}^{R,m}$ are affiliated with the von Neumann algebra $\mathfrak{A}^m$ the limit $x_{kn}$ is also affiliated with $\mathfrak{A}^m$.

We prove the part (iii) $\Leftrightarrow$ (iv). The proof is an immediate consequence of the following:

**Lemma 8.** *For the measure $\mu_B^m$ we have the equivalence of*

$$\text{(iii)} \qquad \left(\mu_B^m\right)^{L_{\exp(tE_{pq})}} \perp \mu_B^m \quad \forall t \in \mathbb{R} \setminus \{0\}\ \forall 1 \leqslant p < q \leqslant m \quad \textit{and}$$

$$\text{(iv)} \qquad S_{pq}^L\left(\mu_B^m\right) = \sum_{n=q+1}^{\infty} c_{pp}^{(n)} b_{qq}^{(n)} = \sum_{n=q+1}^{\infty} \frac{c_{pp}^{(n)} A_q^q(C^{(n)})}{\det(C^{(n)})} = \infty \quad \forall 1 \leqslant p < q \leqslant m,$$

*where $B^{(n)} = (b_{kr}^{(n)})_{k,r=1}^m$, $C^{(n)} = (c_{kr}^{(n)})_{k,r=1}^m$ and $C^{(n)} = (B^{(n)})^{-1}$.*

**Proof.** The proof is based on the Kakutani criterion [15] and on the exact formula for the Hellinger integral

$$H(\mu, \nu) = \int_X \sqrt{\frac{d\mu}{d\rho} \frac{d\nu}{d\rho}}\, d\rho,$$

for two Gaussian measure $\mu = \mu_{B_1}$ and $\nu = \mu_{B_2}$ (see [26]):

$$H(\mu_{B_1}, \mu_{B_2}) = \left(\frac{\det B_1 \det B_2}{\det^2 \frac{B_1 + B_2}{2}}\right)^{-1/4} = \left(\frac{\det C_1 \det C_2}{\det^2 \frac{C_1 + C_2}{2}}\right)^{1/4}, \tag{2}$$

where $C_i = (B_i)^{-1}$, $i = 1, 2$.

Let us consider the one-parameter subgroup $\exp(tE_{pq}) = I + tE_{pq} \in B(m, \mathbb{R})$, $1 \leqslant p < q \leqslant m$, $t \in \mathbb{R}$. Using (1) we have for the positive definite operator $B = B^{(n)}$ in $\mathbb{R}^m$:

$$d\mu_B^{L_{I+tE_{pq}}}(x) = \sqrt{\frac{\det C}{(2\pi)^m}} \exp\left(-\frac{1}{2}\left(C \exp(tE_{pq})x, \exp(tE_{pq})x\right)\right) d \exp(tE_{pq})x$$

$$= \sqrt{\frac{\det C}{(2\pi)^m}} \exp\left(-\frac{1}{2}\left(\exp(tE_{pq})^* C \exp(tE_{pq})x, x\right)\right) dx = d\mu_{B_{pq}(t)}(x),$$

where $(B_{pq}(t))^{-1} = C_{pq}(t) = \exp(tE_{pq})^* C \exp(tE_{pq})$ (we note that $\det C = \det C_{pq}(t)$). Hence, using (2) we get

$$H\left(\mu_B^{L_{I+tE_{pq}}}, \mu_B\right) = \left(\frac{\det C_{pq}(t) \det C}{\det^2 \frac{C_{pq}(t)+C}{2}}\right)^{1/4} = \left(\frac{\det C}{\det \frac{C_{pq}(t)+C}{2}}\right)^{1/2}. \tag{3}$$

We shall prove that

$$\det \frac{C_{pq}(t) + C}{2} = \det C + \frac{t^2}{4} c_{pp} A_q^q(C), \tag{4}$$

where $A_q^p(C)$, $1 \leqslant p, q \leqslant m$, denote the cofactors of the matrix $C$ corresponding to the row $p$ and the column $q$. We have

$$\frac{\det \frac{C_{pq}(t)+C}{2}}{\det C} = \frac{\det C + \frac{t^2}{4} c_{pp} A_q^q(C)}{\det C} = 1 + \frac{t^2}{4} c_{pp} b_{qq},$$

hence

$$\left(\frac{\det C}{\det \frac{C_{pq}(t)+C}{2}}\right)^{1/2} = \left(1 + \frac{t^2}{4} c_{pp} b_{qq}\right)^{-1/2}$$

and finally, using (3) we get

$$H\left(\left(\mu_B^m\right)^{L_{I+tE_{pq}}}, \mu_B^m\right) = \prod_{n=q+1}^{\infty} H\left(\mu_{B^{(n)}}^{L_{I+tE_{pq}}}, \mu_{B^{(n)}}\right) = \prod_{n=q+1}^{\infty} \left(1 + \frac{t^2}{4} c_{pp}^{(n)} b_{qq}^{(n)}\right)^{-1/2},$$

where

$$B^{(n)} = \sum_{1 \leqslant r,s \leqslant m} b_{rs}^{(n)} E_{rs} \quad \text{and} \quad C^{(n)} := \left(B^{(n)}\right)^{-1} = \sum_{1 \leqslant r,s \leqslant m} c_{rs}^{(n)} E_{rs}.$$

So using the properties of the Hellinger integral for two Gaussian measures we conclude that

$$\begin{aligned}
\left(\mu_B^m\right)^{L_{I+tE_{pq}}} \perp \mu_B^m \quad \forall t \in \mathbb{R} \setminus \{0\} \quad &\Leftrightarrow \quad \prod_{n=q+1}^{\infty} \left(1 + \frac{t^2}{4} c_{pp}^{(n)} b_{qq}^{(n)}\right)^{-1/2} = 0 \\
&\Leftrightarrow \quad S_{pq}^L\left(\mu_B^m\right) = \infty.
\end{aligned}$$

To prove (4) we set $C_{pq}(t)=\exp(tE_{pq})^*C\exp(tE_{pq})$. We have for $m\in\mathbb{N}$ and $1\leqslant p<q\leqslant m$ using the identity $\exp(tE_{pq})=I+tE_{pq}$, $t\in\mathbb{R}$,

$$C_{pq}(t)=\begin{vmatrix} c_{11} & \dots & c_{1p} & \dots & c_{1q}+tc_{1p} & \dots & c_{1m}\\ & \dots & & \dots & & \dots & \\ c_{1p} & \dots & c_{pp} & \dots & c_{pq}+tc_{pp} & \dots & c_{pm}\\ & \dots & & \dots & & \dots & \\ c_{1q}+tc_{1p} & \dots & c_{pq}+tc_{pp} & \dots & c_{qq}+2tc_{pq}+t^2c_{pp} & \dots & c_{qm}+tc_{pm}\\ & \dots & & \dots & & \dots & \\ c_{1m} & \dots & c_{pm} & \dots & c_{qm}+tc_{pm} & \dots & c_{mm}\end{vmatrix},$$

hence

$$\det\frac{C_{pq}(t)+C}{2}=\begin{vmatrix} c_{11} & \dots & c_{1p} & \dots & c_{1q}+\frac{t}{2}c_{1p} & \dots & c_{1m}\\ & \dots & & \dots & & \dots & \\ c_{1p} & \dots & c_{pp} & \dots & c_{pq}+\frac{t}{2}c_{pp} & \dots & c_{pm}\\ & \dots & & \dots & & \dots & \\ c_{1q}+tc_{1p} & \dots & c_{pq}+\frac{t}{2}c_{pp} & \dots & c_{qq}+tc_{pq}+\frac{t^2}{2}c_{pp} & \dots & c_{qm}+\frac{t}{2}c_{pm}\\ & \dots & & \dots & & \dots & \\ c_{1m} & \dots & c_{pm} & \dots & c_{qm}+\frac{t}{2}c_{pm} & \dots & c_{mm}\end{vmatrix}$$

$$=\begin{vmatrix} c_{11} & \dots & c_{1p} & \dots & c_{1q} & \dots & c_{1m}\\ & \dots & & \dots & & \dots & \\ c_{1p} & \dots & c_{pp} & \dots & c_{pq} & \dots & c_{pm}\\ & \dots & & \dots & & \dots & \\ c_{1q} & \dots & c_{pq} & \dots & c_{qq}+\frac{t^2}{4}c_{pp} & \dots & c_{qm}\\ & \dots & & \dots & & \dots & \\ c_{1m} & \dots & c_{pm} & \dots & c_{qm} & \dots & c_{mm}\end{vmatrix}=\det C+\frac{t^2}{4}c_{pp}A_q^q(C).$$

This ends the proof of Lemma 8, and thus also of (iii) $\Leftrightarrow$ (iv). □

## 4. Approximation of the variables $x_{pq}$

**Remark 9.** In what follows we shall omit the upper and lower index $n\in\mathbb{N}$ in all the expressions $c_{kr}^{(n)}$, $b_{kr}^{(n)}$, $B^{(n)}$, $C^{(n)}$, $\Xi_n^{pq}$, $g_{kn}$, $\lambda_k^{(n)}$, etc.

We first prove Lemmas 12 and 15, which give a suitable approximation of $x_{pq}$ only on the vector $f=\mathbf{1}\in L^2(X^m,\mu_B^m)$ (cf. Remark 7).

We shall also use the well-known result (see, for example, [7, Chapter I, Section 52])

$$\min_{x\in\mathbb{R}^n}\left(\sum_{k=1}^n a_kx_k^2\,\middle|\,\sum_{k=1}^n x_k=1\right)=\left(\sum_{k=1}^n\frac{1}{a_k}\right)^{-1},\quad a_k>0,\ k=1,2,\dots,n.$$

We use the same result in a slightly different form with $b_k\neq 0$, $k=1,2,\dots,n$,

$$\min_{x\in\mathbb{R}^n}\left(\sum_{k=1}^n a_kx_k^2\,\middle|\,\sum_{k=1}^n x_kb_k=1\right)=\left(\sum_{k=1}^n\frac{b_k^2}{a_k}\right)^{-1}.\tag{5}$$

The minimum is realized for

$$x_k = \frac{b_k}{a_k}\left(\sum_{k=1}^{n}\frac{b_k^2}{a_k}\right)^{-1}.$$

For any subset $I \subset \mathbb{N}$ let us denote as before by $\langle f_n \mid n \in I\rangle$ the closure of the linear space generated by the set of vectors $(f_n \mid n \in I)$ in a Hilbert space $H$.

We note that the distance $d(f_{n+1}; \langle f_1, \dots, f_n\rangle)$ of the vector $f_{n+1}$ in $H$ from the hyperplane $\langle f_1, \dots, f_n\rangle$ may be calculated in terms of the *Gram determinants* $\Gamma(f_1, f_2, \dots, f_k)$ corresponding to the set of vectors $f_1, f_2, \dots, f_k$ (see [10]):

$$d\big(f_{n+1}; \langle f_1, \dots, f_n\rangle\big) = \min_{t=(t_k)\in\mathbb{R}^n}\left\| f_{n+1} + \sum_{k=1}^{n} t_k f_k\right\|^2 = \frac{\Gamma(f_1, f_2, \dots, f_{n+1})}{\Gamma(f_1, f_2, \dots, f_n)}, \tag{6}$$

where the Gram determinant is defined by $\Gamma(f_1, f_2, \dots, f_n) = \det\gamma(f_1, f_2, \dots, f_n)$ and $\gamma(f_1, f_2, \dots, f_n) =: \gamma_n$ is the *Gram matrix*

$$\gamma(f_1, f_2, \dots, f_n) = \begin{pmatrix} (f_1, f_1) & (f_1, f_2) & \dots & (f_1, f_n)\\ (f_2, f_1) & (f_2, f_2) & \dots & (f_2, f_n)\\ & & \ddots & \\ (f_n, f_1) & (f_n, f_2) & \dots & (f_n, f_n)\end{pmatrix}.$$

**Lemma 10.** *We have*

$$d\big(f_{n+1}; \langle f_1, \dots, f_n\rangle\big) = \frac{\det\gamma_{n+1}}{\det\gamma_n} = (f_{n+1}, f_{n+1}) - \big(\gamma_n^{-1} d_{n+1}, d_{n+1}\big),$$

*where* $d_{n+1} = ((f_1, f_{n+1}), (f_2, f_{n+1}), \dots, (f_n, f_{n+1})) \in \mathbb{R}^n$.

**Proof.** We may write

$$\begin{aligned}\left\|\sum_{k=1}^{n} t_k f_k - f_{n+1}\right\|^2 &= \sum_{k,m=1}^{n} t_k t_m (f_k, f_k) - 2\sum_{k=1}^{n} t_k (f_k, f_{n+1}) + (f_{n+1}, f_{n+1})\\ &= (\gamma_n t, t) - 2(t, d_{n+1}) + (f_{n+1}, f_{n+1}),\end{aligned}$$

where $t = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$. Using (58) for $A_n = \gamma_n$ we get

$$(\gamma_n t, t) - 2(t, d_{n+1}) = \big(\gamma_n(t - t_0), (t - t_0)\big) - \big(\gamma_n^{-1} d_{n+1}, d_{n+1}\big),$$

where $t_0 = \gamma_n^{-1} d_n$. Hence we get (see (6))

$$\min_{t=(t_k)\in\mathbb{R}^n}\left\| f_{n+1} - \sum_{k=1}^{n} t_k f_k\right\|^2 = \min_{t=(t_k)\in\mathbb{R}^n}\big((\gamma_n t, t) - 2(t, d_{n+1}) + (f_{n+1}, f_{n+1})\big)$$

$$= (f_{n+1}, f_{n+1}) - \left(\gamma_n^{-1} d_{n+1}, d_{n+1}\right) + \min_{t=(t_k)\in\mathbb{R}^n} \left(\gamma_n (t - t_0), (t - t_0)\right)$$
$$= (f_{n+1}, f_{n+1}) - \left(\gamma_n^{-1} d_{n+1}, d_{n+1}\right). \qquad \square$$

**Remark 11.** In fact a more general result holds. Let us denote by $A_{n+1}$ the real non-necessarily symmetric matrix in $\mathbb{R}^{n+1}$ and by $A_n$ its $n \times n$ block after crossing the element in the last column and row, by $v_{n+1} = (a_{1n+1}, a_{2n+1}, \dots, a_{nn+1})$, $h_{n+1} = (a_{n+11}, a_{n+12}, \dots, a_{n+1n})$ vectors $v_{n+1}$, $h_{n+1} \in \mathbb{R}^n$. If $\det A_n \neq 0$ then we have

$$a_{n+1n+1} - \left(A_n^{-1} v_{n+1}, h_{n+1}\right) = \frac{\det A_{n+1}}{\det A_n}. \tag{7}$$

**Proof.** It is sufficient to use the identity (Schur–Frobenius decomposition)

$$A_{n+1} = \begin{pmatrix} A_n & v_{n+1}^t \\ h_{n+1} & a_{n+1n+1} \end{pmatrix} = \begin{pmatrix} A_n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Id & A_n^{-1} v_{n+1}^t \\ h_{n+1} & a_{n+1n+1} \end{pmatrix}. \qquad \square$$

The generators

$$A_{kn} := A_{kn}^{R,m} = \frac{d}{dt} T_{I+tE_{kn}}^{R,\mu_B^m} \Big|_{t=0}$$

of the one-parameter groups $I + tE_{kn}$ have the following form (on smooth functions of compact support):

$$A_{kn} = \sum_{r=1}^{k-1} x_{rk} D_{rn} + D_{kn}, \quad 1 \leqslant k \leqslant m,\ k < n, \qquad A_{kn} = \sum_{r=1}^{m} x_{rk} D_{rn}, \quad m < k < n,$$

where

$$D_{kn} = \partial/\partial x_{kn} - \frac{1}{2}\left(x, \left(B^{(n)}\right)^{-1} E_{kn}\right), \quad 1 \leqslant k < n. \tag{8}$$

To simplify the further computations let us consider the corresponding Fourier transforms $F_m$ in the variables $x_{kn}$, $1 \leqslant k \leqslant m$, $m < n$,

$$F_m : L^2\left(X^m, \mu_B^m\right) \to L^2\left(X^m, \mu_C^m\right).$$

We have

$$F_m D_{kn} F_m^{-1} = i y_{kn} \quad \text{for } (k,n),\ 1 \leqslant k \leqslant m,\ m < n, \quad \text{and} \quad F_m \mathbf{1} = \mathbf{1}.$$

Let us set $\mu_C = \bigotimes_{n=2}^{\infty} \mu_{C^{(n)}}$ with $C^{(n)} = B^{(n)}$ for $2 \leqslant n \leqslant m$ and $C^{(n)} = (B^{(n)})^{-1}$ for $n > m$. We define the Fourier transform $F_m$ as the infinite tensor product $F_m = \bigotimes_{n=m+1}^{\infty} F_{mn}$ where

$$F_{mn} : L^2\left(\mathbb{R}^m, \mu_{B^{(n)}}\right) \to L^2\left(\mathbb{R}^m, \mu_{C^{(n)}}\right)$$

is the image of the standard Fourier transform $F^m$ in the space $L^2(\mathbb{R}^m, dx)$, i.e. $F_{mn} = U(C^{(n)})^{-1} F^m U(B^{(n)})$, where

$$U\big(B^{(n)}\big) = \left(\frac{d\mu_{B^{(n)}}(x)}{dx}\right)^{1/2} \quad \begin{array}{ccc} L^2(\mathbb{R}^m, \mu_{B^{(n)}}) & \xrightarrow{F_{mn}} & L^2(\mathbb{R}^m, \mu_{C^{(n)}}) \\ \big\downarrow & & \big\downarrow \\ L^2(\mathbb{R}^m, dx) & \xrightarrow{F^m} & L^2(\mathbb{R}^m, dx). \end{array} \quad U\big(C^{(n)}\big) = \left(\frac{d\mu_{C^{(n)}}(x)}{dx}\right)^{1/2}$$

Since the standard Fourier transform $F^m$ is defined as follows:

$$\big(F^m f\big)(y) = \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} \exp i(y, x) f(x)\, dx,$$

and, for $D = B^{(n)}$ respectively $D = C^{(n)}$

$$U(D) = \left(\frac{d\mu_D(x)}{dx}\right)^{1/2} = \frac{1}{((2\pi)^m \det D)^{1/4}} \exp\left(-\frac{1}{4}\big(D^{-1}x, x\big)\right),$$

we have finally for $F_{mn}$:

$$\begin{aligned}
(F_{mn} f)(y) &= \big(U\big(C^{(n)}\big)^{-1} F^m U\big(B^{(n)}\big) f\big)(y) \\
&= \frac{1}{((2\pi)^m \det C^{(n)})^{1/4}} \exp\left(\frac{1}{4}\big(\big(C^{(n)}\big)^{-1} y, y\big)\right) \frac{1}{\sqrt{(2\pi)^m}} \\
&\quad \times \int_{\mathbb{R}^m} \exp i(y, x) f(x) \big((2\pi)^m \det B^{(n)}\big)^{1/4} \exp\left(-\frac{1}{4}\big(\big(B^{(n)}\big)^{-1} x, x\big)\right) dx \\
&= \frac{\exp(\frac{1}{4}((C^{(n)})^{-1} y, y))}{\sqrt{(2\pi)^m \det C^{(n)}}} \int_{\mathbb{R}^m} \exp\left(i(y, x) - \frac{1}{4}\big(\big(B^{(n)}\big)^{-1} x, x\big)\right) f(x)\, dx.
\end{aligned}$$

Using Fourier transform $F_m$ we obtain for $\widetilde{A_{kn}} = F_m A_{kn} (F_m)^{-1}$:

$$\widetilde{A_{kn}} = i\left(\sum_{r=1}^{k-1} x_{rk} y_{rn} + y_{kn}\right), \quad 1 \leqslant k \leqslant m < n, \qquad \widetilde{A_{kn}} = \sum_{r=1}^{m} D_{rk}(y) y_{rn}, \quad m < k < n, \tag{9}$$

where

$$D_{kn}(y) = \frac{\partial}{\partial y_{kn}} - \frac{1}{2}\big(x, \big(C^{(n)}\big)^{-1} E_{kn}\big), \quad 1 \leqslant k < n.$$

Let us set for $s = (s_1, \ldots, s_r) \in \mathbb{R}^r$ and $1 \leqslant r \leqslant p < q \leqslant m$

$$\xi_n^{rp}(s) = F_m\left(D_{pn} \exp\left(\sum_{l=1}^{r} s_l A_{ln}\right)\right)\mathbf{1} = i y_{pn} \exp\left(\sum_{l=1}^{r} s_l \widetilde{A_{ln}}\right)\mathbf{1}. \tag{10}$$

For a function $f : X^m \to \mathbb{C}$ we set

$$Mf = \int\limits_{X^m} f(x)\, d\mu_B^m(x).$$

To approximate the variables $x_{pq}$, $1 \leqslant p < q \leqslant m$, we use

**Lemma 12.** *Let* $1 \leqslant r \leqslant p < q \leqslant m$. *For any* $s^{(n)} = (s_1^{(n)}, \ldots, s_r^{(n)}) \in \mathbb{R}^r$, *and for any* $\alpha^{(n)} = (\alpha_1^{(n)}, \ldots, \alpha_m^{(n)}) \in \mathbb{R}^m$, $n \in \mathbb{N}$, *we have*

$$x_{pq} \in \left\langle \exp\left(\sum_{l=1}^{r} s_l^{(n)} A_{ln}\right)\left(\sum_{k=1}^{m} \alpha_k^{(n)} A_{kn}\right)\mathbf{1} \;\middle|\; n \in \mathbb{N},\ m < n \right\rangle \quad \Leftrightarrow \quad \Sigma_{pq}^r(s, \alpha, m) = \infty,$$

*where* $s = (s^{(n)})_{n=m+1}^{\infty}$, $\alpha = (\alpha^{(n)})_{n=m+1}^{\infty}$, $\alpha_q^{(n)} = 1$ *and*

$$\Sigma_{pq}^r(s, \alpha, m) = \sum_{n=m+1}^{\infty} \frac{|M\xi_n^{rp}(s^{(n)})|^2}{c_{pp}^{(n)} - |M\xi_n^{rp}(s^{(n)})|^2 + \|(A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k\neq p}^{m} \alpha_k^{(n)} A_{kn})\mathbf{1}\|^2}. \tag{11}$$

Before proving Lemma 12 let us make some comments about the procedure for arriving at the expressions used for the approximation of the variables $x_{pq}$ on the left-hand side of the equivalence in Lemma 12.

**Remark 13.** 1. The operator $A_{qn} = \sum_{r=1}^{q-1} x_{rq} D_{rn} + D_{qn}$ contains $x_{pq}$ for $r = p$.

2. Since $MD_{pn}\mathbf{1} = 0$ and $MD_{pn}\exp(sA_{pn})\mathbf{1} \neq 0$ we may first think of considering $\exp(sA_{pn})A_{qn}\mathbf{1}$, $1 \leqslant p < q \leqslant m$ (similarly as in [23,24] where the linear combinations of $A_{pn}A_{qn}$ were used). But this is not sufficient for the approximation. We might then try to consider the expression

$$\exp(sA_{pn})\left(\sum_{k=1}^{m} \alpha_k A_{kn}\right), \quad 1 \leqslant p < m < n,$$

with $\alpha_q = 1$. The calculations show again that these combinations are still not sufficient to approximate $x_{pq}$. We arrive then at the suggestion to take

$$\exp\left(\sum_{l=1}^{r} s_l A_{ln}\right)\left(\sum_{k=1}^{m} \alpha_k A_{kn}\right), \quad 1 \leqslant r \leqslant p < q \leqslant m < n,$$

which is the choice made in Lemma 12.

3. For approximation of the variable $x_{pq}$ we use $p$ different combinations, corresponding to $\Sigma_{pq}^r(s, \alpha, m)$, $1 \leqslant r \leqslant p$. All these combinations are essential, i.e. none of them can be omitted. This can be seen by constructing corresponding counterexamples and is in a contrast to the previous cases considered in [23,24] where only one combination of $A_{pn}A_{qn}$ were used to approximate $x_{pq}$.

4. To make the expression $\Sigma_{pq}^{r}(s,\alpha,m)$ in (11) larger (to apply then the criterium in Lemma 12) we chose $s^{(n)}\in\mathbb{R}^r$ such that

$$\left|M\xi_n^{rp}\left(s^{(n)}\right)\right|^2=\max_{s\in\mathbb{R}^r}\left|M\xi_n^{rp}(s)\right|^2$$

(which is possible, $|M\xi_n^{rp}(s)|^2$ being continuous and bounded).

5. With the same aim we chose $\alpha_k^{(n)}$ in such a way that

$$\left\|\left(A_{qn}-x_{pq}D_{pn}+\sum_{k=1,\,k\neq q}^{m}\alpha_k^{(n)}A_{kn}\right)\mathbf{1}\right\|^2=\min_{(t_k)\in\mathbb{R}^{m-1}}\left\|\left(A_{qn}-x_{pq}D_{pn}+\sum_{k=1,\,k\neq q}^{m}t_kA_{kn}\right)\mathbf{1}\right\|^2.$$

6. The right-hand side of the previous expression is equal (see (6)) to

$$\frac{\Gamma(g_1,g_2,\ldots,g_q^p,\ldots,g_m)}{\Gamma(g_1,g_2,\ldots,g_{q-1},g_{q-1},\ldots,g_m)},$$

where

$$g_k:=g_{kn}:=A_{kn}\mathbf{1},\quad 1\leqslant k\leqslant m,\ k\neq q,\qquad g_q^p:=g_{qn}^p:=(A_{qn}-x_{pq}D_{pn})\mathbf{1}. \tag{12}$$

**Proof of Lemma 12.** If we put $\sum_n t_nM\xi_n^{rp}(s^{(n)})=1$ we get

$$\begin{aligned}
&\left\|\left[\sum_n t_n\exp\left(\sum_{l=1}^{r}s_l^{(n)}A_{ln}\right)\left(\sum_{k=1}^{m}\alpha_k^{(n)}A_{kn}\right)-x_{pq}\right]\mathbf{1}\right\|^2\\
&=\left\|\left[\sum_n t_n\exp\left(\sum_{l=1}^{r}s_l^{(n)}A_{ln}\right)\left(A_{qn}-x_{pq}D_{pn}+x_{pq}D_{pn}+\sum_{k=1,\,k\neq q}^{m}\alpha_k^{(n)}A_{kn}\right)-x_{pq}\right]\mathbf{1}\right\|^2\\
&=\left\|\sum_n t_n\left[x_{pq}\left(D_{pn}\exp\left(\sum_{l=1}^{r}s_l^{(n)}A_{ln}\right)-M\xi_n^{rp}\left(s^{(n)}\right)\right)\right.\right.\\
&\qquad\left.\left.+\exp\left(\sum_{l=1}^{r}s_l^{(n)}A_{ln}\right)\left(A_{qn}-x_{pq}D_{pn}+\sum_{k=1,\,k\neq q}^{m}\alpha_k^{(n)}A_{kn}\right)\right]\mathbf{1}\right\|^2\\
&=\sum_n t_n^2\left[\|x_{pq}\|^2\left\|\left(D_{pn}\exp\left(\sum_{l=1}^{r}s_l^{(n)}A_{ln}\right)-M\xi_n^{rp}\left(s^{(n)}\right)\right)\mathbf{1}\right\|^2\right.\\
&\qquad\left.+\left\|\exp\left(\sum_{l=1}^{r}s_l^{(n)}A_{ln}\right)\left(A_{qn}-x_{pq}D_{pn}+\sum_{k=1,\,k\neq q}^{m}\alpha_k^{(n)}A_{kn}\right)\mathbf{1}\right\|^2\right]\\
&=\sum_n t_n^2\left[\|x_{pq}\|^2\left(c_{pp}^{(n)}-\left|M\xi_n^{rp}\left(s^{(n)}\right)\right|^2\right)\right.
\end{aligned}$$

$$+\left\|\exp\left(\sum_{l=1}^{r}s_l^{(n)}A_{ln}\right)\left(A_{qn}-x_{pq}D_{pn}+\sum_{k=1,\,k\neq q}^{m}\alpha_k^{(n)}A_{kn}\right)\mathbf{1}\right\|^2\Bigg],$$

where we have used the equality $\|\xi - M\xi\|^2 = \|\xi\|^2 - |M\xi|^2$:

$$\left\|\left[D_{pn}\exp\left(\sum_{l=1}^{r}s_l^{(n)}A_{ln}\right) - M\xi_n^{rp}\left(s^{(n)}\right)\right]\mathbf{1}\right\|^2$$
$$= \|D_{pn}\mathbf{1}\|^2 - \left|M\xi_n^{rp}\left(s^{(n)}\right)\right|^2 = c_{pp}^{(n)} - \left|M\xi_n^{rp}\left(s^{(n)}\right)\right|^2.$$

**Definition.** We shall say that two series $\sum_n a_n$ and $\sum_n b_n$ with positive members are *equivalent* and shall denote this by $\sum_n a_n \sim \sum_n b_n$ if they are convergent or divergent simultaneously. We note that if $a_n > 0$, $b_n > 0$, $n \in \mathbb{N}$, then we have

$$\sum_{n\in\mathbb{N}}\frac{a_n}{a_n+b_n} \sim \sum_{n\in\mathbb{N}}\frac{a_n}{b_n}. \tag{13}$$

Using (5) we get, setting $b = (M\xi_n^{rp}(s^{(n)}))_{n=m+1}^{m+1+N} \in \mathbb{R}^N$, $N \in \mathbb{N}$,

$$\min_{t\in\mathbb{R}^N}\left(\left\|\left[\sum_{n=m+1}^{m+1+N}t_n\exp\left(\sum_{l=1}^{r}s_l^{(n)}A_{ln}\right)\left(\sum_{k=1}^{m}\alpha_k^{(n)}A_{kn}\right)-x_{pq}\right]\mathbf{1}\right\|^2 \,\Big|\, (t,b)=-1\right)$$
$$\sim\left(\sum_{n=m+1}^{m+1+N}\frac{|M\xi_n^{rp}(s^{(n)})|^2}{c_{pp}^{(n)}-|M\xi_n^{rp}(s^{(n)})|^2+\|(A_{qn}-x_{pq}D_{pn}+\sum_{k=1,k\neq p}^{m}\alpha_k^{(n)}A_{kn})\mathbf{1}\|^2}\right)^{-1}. \qquad \square$$

Due to Remark 9 we shall write $C$ (respectively $\hat{C}$) instead of $C^{(n)}$ (respectively $\hat{C}^{(n)}$), where

$$C^{(n)} = \begin{pmatrix} c_{11}^{(n)} & c_{12}^{(n)} & \dots & c_{1m}^{(n)} \\ c_{12}^{(n)} & c_{22}^{(n)} & \dots & c_{2m}^{(n)} \\ & & \ddots & \\ c_{1m}^{(n)} & c_{2m}^{(n)} & \dots & c_{mm}^{(n)} \end{pmatrix},$$

$$\hat{C}^{(n)} = \begin{pmatrix} c_{11}^{(n)} & c_{12}^{(n)} & \dots & c_{1m}^{(n)} \\ c_{12}^{(n)} & c_{11}^{(n)}+c_{22}^{(n)} & \dots & c_{2m}^{(n)} \\ & & \ddots & \\ c_{1m}^{(n)} & c_{2m}^{(n)} & \dots & c_{11}^{(n)}+c_{22}^{(n)}+\dots+c_{mm}^{(n)} \end{pmatrix}.$$

**Remark 14.** To simplify the further computations we assume that the measures $\mu_{B^{(n)}}$ for $2 \leqslant n \leqslant m$ are standard: $B^{(n)} = I$. Since $\mu_B^m = \bigotimes_{n=2}^{\infty}\mu_{B^{(n)}}$ this assumption, which only concerns finitely many of the $\mu_{B^{(n)}}$'s, does not change the equivalence class of the initial measure $\mu_B^m$ and the equivalence class of the corresponding representation $T^{R,\mu_B^m}$.

Using this remark, notation (12) and Fourier transforms we conclude that

$$\Gamma(g_1, g_2, \ldots, g_m) = \det \hat{C}, \quad \text{i.e.} \quad \Gamma(g_{1n}, g_{2n}, \ldots, g_{mn}) = \det \hat{C}^{(n)}, \tag{14}$$

since $(g_q, g_p) = (\hat{C})_{pq}$, $1 \leqslant p, q \leqslant m$. Indeed for $p \neq q$ we have

$$(g_{qn}, g_{pn}) = (g_{pn}, g_{qn}) = \left(\sum_{r=1}^{p-1} x_{rp} y_{rn} + y_{pn}, \sum_{s=1}^{q-1} x_{sq} y_{sn} + y_{qn}\right) = (y_{pn}, y_{qn}) = c_{pq}^{(n)},$$

$$(g_{pn}, g_{pn}) = \left\| \sum_{r=1}^{p-1} x_{rp} y_{rn} + y_{pn} \right\|^2 = \sum_{r=1}^{p-1} \|x_{rp}\|^2 \|y_{rn}\|^2 + \|y_{pn}\|^2 = \sum_{r=1}^{p} c_{rr}^{(n)} = \big(\hat{C}^{(n)}\big)_{pp}$$

(we reinserted here the upper index $n$ in $c_{pq}^{(n)}$ for clarity).

In the following we shall need a variant of Lemma 12 using Remark 13 replacing the $|M\xi_n^{rp}(s)|$ by its maximum $\Xi_n^{rp}$. Let us set (see (10) for definition of $\xi_n^{rp}(s)$)

$$\Xi_n^{rp} = \max_{s \in \mathbb{R}^r} \big| M\xi_n^{rp}(s) \big|^2. \tag{15}$$

Now we see that using $s$ and $\alpha$ as in parts 4 and 5 of Remark 13 we have

$$\begin{aligned}
&\Sigma_{pq}^r(s, \alpha, m) \\
&\quad = \sum_n \frac{\max_{s^{(n)} \in \mathbb{R}^r} |M\xi_n^{rp}(s^{(n)})|^2}{c_{pp}^{(n)} - \max_{s^{(n)} \in \mathbb{R}^r} |M\xi_n^{rp}(s^{(n)})|^2 + \|(A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k\neq p}^m \alpha_k^{(n)} A_{kn}) \mathbf{1}\|^2} \\
&\quad \overset{(13)}{\sim} \sum_n \frac{\max_{s^{(n)} \in \mathbb{R}^r} |M\xi_n^{rp}(s^{(n)})|^2}{c_{pp}^{(n)} + \|(A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k\neq p}^m \alpha_k^{(n)} A_{kn}) \mathbf{1}\|^2} \\
&\quad \overset{(15)}{=} \sum_n \frac{\Xi_n^{rp}}{c_{pp}^{(n)} + \frac{\Gamma(g_{1n}, g_{2n}, \ldots, g_{qn}^p, \ldots, g_{mn})}{\Gamma(g_{1n}, g_{2n}, \ldots, g_{q-1n}, g_{q+1n}, \ldots, g_{mn})}} \\
&\quad \overset{\text{Remark 9}}{=} \sum_n \frac{\Xi^{rp} \Gamma(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m)}{c_{pp} \Gamma(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m) + \Gamma(g_1, g_2, \ldots, g_q^p, \ldots, g_m)} \\
&\quad = \Sigma_{pq}^r(m) := \sum_n \frac{\Xi^{rp} \Gamma(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m)}{\Gamma(g_1, g_2, \ldots, g_m)} \overset{(14)}{=} \sum_n \frac{\Xi_n^{rp} A_q^q(\hat{C}^{(n)})}{\det \hat{C}^{(n)}}.
\end{aligned}$$

For the latter equality we have used the fact that

$$c_{pp} \Gamma(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m) + \Gamma\big(g_1, g_2, \ldots, g_q^p, \ldots, g_m\big) = \Gamma(g_1, g_2, \ldots, g_m),$$

which follows from (26). Indeed it is sufficient to take in (26) $C = \hat{C} - c_{pp} E_{qq}$ and $\lambda_q = c_{pp}$. Then we have

$$\begin{aligned}\Gamma(g_1, g_2, \dots, g_m) &= \det\hat{C} = \det(\hat{C} - c_{pp}E_{qq} + c_{pp}E_{qq})\\ &= \det(\hat{C} - c_{pp}E_{qq}) + c_{pp}A_q^q(\hat{C} - c_{pp}E_{qq})\\ &= \Gamma\big(g_1, g_2, \dots, g_q^p, \dots, g_m\big) + c_{pp}\Gamma(g_1, g_2, \dots, g_{q-1}, g_{q+1}, \dots, g_m).\end{aligned}$$

So we have proved the following lemma.

**Lemma 15.** *Let $1 \leqslant r \leqslant p < q \leqslant m$. Then for some $s_l = (s_l^{(n)})_{n=m+1}^{\infty}$, $\alpha_k = (\alpha_k^{(n)})_{n=m+1}^{\infty}$, where $s_l^{(n)}, \alpha_k^{(n)} \in \mathbb{R}$, $1 \leqslant l \leqslant r$, $1 \leqslant k \leqslant m$, we have*

$$\begin{aligned} x_{pq} &\in \left\langle \exp\left(\sum_{l=1}^{r} s_l^{(n)} A_{ln}\right)\left(\sum_{k=1}^{m} \alpha_k^{(n)} A_{kn}\right)\mathbf{1} \;\middle|\; n \in \mathbb{N},\ m < n\right\rangle \\ \Leftrightarrow \quad & \Sigma_{pq}^r(m) = \sum_n \frac{\Xi_n^{rp} A_q^q(\hat{C}^{(n)})}{\det\hat{C}^{(n)}} = \infty. \end{aligned} \tag{16}$$

## 5. The proof of (iv) $\Rightarrow$ ($x_{pq}\ \eta\ \mathfrak{A}^m$, $1 \leqslant p \leqslant m$, $p < q$) in Theorem 5

**Idea.** We prove firstly that $x_{pq}\ \eta\ \mathfrak{A}^m$ for some $(p, q)$: $1 \leqslant p < q \leqslant m$ if conditions (iv) are valid. Further we prove that this also holds for all such $(p, q)$. For this it is sufficient to prove that

$$\Sigma_m > C S_m \quad \text{for some } C > 0, \tag{17}$$

where

$$S_m := \sum_{1\leqslant p<q\leqslant m} S_{pq}^L\big(\mu^m\big), \quad \text{and} \quad \Sigma_m := \sum_{1\leqslant r\leqslant p<q\leqslant m} \Sigma_{pq}^r(m) \tag{18}$$

(see (16) for the definition of $\Sigma_{pq}^r(m)$). Indeed, in this case $S_m = \infty$ since $S_{pq}^L(\mu^m) = \infty$ $\forall p, q$: $1 \leqslant p < q \leqslant m$ by Lemma 8 hence $\Sigma_m = \infty$ by (17) and finally we conclude that $\Sigma_{pq}^r(m) = \infty$ for some $r, p, q$: $1 \leqslant r \leqslant p < q \leqslant m$. By Lemma 15 we get that $x_{pq}\ \eta\ \mathfrak{A}^m$.

The proof of (17) is based on Appendices A–C. In Appendix A we define the *generalization of the characteristic polynomial* for matrix $C \in \mathrm{Mat}(m, \mathbb{C})$ and establish some its properties. These properties are used then in Appendices B and C.

In Appendix B we estimate $\Xi_n^{pq} = \max_{t\in\mathbb{R}^p} |M\xi_n^{pq}(t)|^2$. This estimation is based on Lemma B.1 which gives us the exact formula for

$$M\xi_n^{pq}(t) = \big(D_{qn} T^{R,\mu_B^m}_{\exp(\sum_{r=1}^p t_r E_{rn})}\mathbf{1}, \mathbf{1}\big), \quad t = (t_1, t_2, \dots, t_p) \in \mathbb{R}^p,\ 1 \leqslant p \leqslant m$$

(see (44)), where $D_{kn}$ is defined in (8). The latter formula is based of the exact formulas for the matrix elements

$$\phi_p(t) := \phi_p^{(n)}(t) = \big(T^{R,\mu_B^m}_{\exp(\sum_{r=1}^p t_r E_{rn})}\mathbf{1}, \mathbf{1}\big), \quad t = (t_r)_{r=1}^p \in \mathbb{R}^p,\ 1 \leqslant p \leqslant m$$

(see (40)) and theirs generalizations (see (42)). We cannot calculate explicitly

$$\Xi_n^{pq} = \max_{t\in\mathbb{R}^p}\left|M\xi_n^{pq}(t)\right|^2,$$

but we are able by Lemmas B.1 and B.2 to obtain the estimation $\Xi_n^{pq} > \Psi_n^{pq}$,

$$\Psi_n^{pq} := \frac{(M^{12\dots p-1p}_{12\dots p-1q}(C^{(n)}_{p,q}))^2\exp(-1)}{M^{12\dots p-1}_{12\dots p-1}(C^{(n)}_p)M^{12\dots p}_{12\dots p}(C^{(n)}_p) + \sum_{k=2}^p \hat\lambda_k(A^p_k(C^{(n)}_p))^2}$$

(see (47) and (48)). The *crucial* for proving (17) is Lemma 16 dealing with some inequalities involving the generalized characteristic polynomials. This lemma is proved in Appendix C.

We use the notations of Lemma 8 (see Remark 9):

$$S^L_{pq}\left(\mu^m_B\right) = \sum_{n=q+1}^{\infty} c^{(n)}_{pp}b^{(n)}_{qq} = \sum_{n=q+1}^{\infty}\frac{c^{(n)}_{pp}A^q_q(C^{(n)}_m)}{\det C^{(n)}_m} = \sum_{n=q+1}^{\infty}\frac{c_{pp}A^q_q(C_m)}{\det C_m}.$$

Let

$$\lambda = (\lambda_k)^m_{k=1}\in\mathbb{R}^m,\qquad \hat\lambda = (\hat\lambda_k)^m_{k=1},\qquad \hat\lambda_1 = 0,\qquad \hat\lambda_k = \sum_{r=1}^{k-1}c_{rr},\quad 2\leqslant k\leqslant m, \tag{19}$$

$$f_q = e\sum_{1\leqslant r\leqslant p<q}\Psi^{rp},\quad 2\leqslant q\leqslant m,\qquad f_2 = e\Psi^{11} = c_{11},$$

$$f_3 = e\left(\Psi^{11}+\Psi^{12}+\Psi^{22}\right),\quad \dots, \tag{20}$$

$$C_m = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m}\\ c_{12} & c_{22} & \dots & c_{2m}\\ & & \ddots & \\ c_{1m} & c_{2m} & \dots & c_{mm}\end{pmatrix},$$

$$\hat C_m = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m}\\ c_{12} & c_{11}+c_{22} & \dots & c_{2m}\\ & & \ddots & \\ c_{1m} & c_{2m} & \dots & c_{11}+\dots+c_{mm}\end{pmatrix}. \tag{21}$$

Obviously, we have $\hat C_m = C_m(\hat\lambda)$, where $\hat\lambda\in\mathbb{C}^m$, is defined in (19) and we use the notation $C_m(\lambda) := C_m + \sum_{k=1}^m\lambda_k E_{kk}$.

We have the following expressions for $S_m$ and $\Sigma_m$:

$$S_m := \sum_{1\leqslant r<k\leqslant m} S^L_{rk}\left(\mu^m\right)\sim\sum_{n=m+1}^{\infty}\frac{\sum_{k=2}^m(\sum_{r=1}^{k-1}c_{rr})A^k_k(C_m)}{\det C_m} = \sum_{n=m+1}^{\infty}\frac{\sum_{k=2}^m\hat\lambda_k A^k_k(C_m)}{\det C_m}.$$

We have replaced the series

$$S^L_{pq}(\mu^m_B)=\sum_{n=q+1}^{\infty}c^{(n)}_{pp}b^{(n)}_{qq}$$

with the equivalent one

$$S^L_{pq}(\mu^m_B)\sim\sum_{n=m+1}^{\infty}c^{(n)}_{pp}b^{(n)}_{qq}.$$

If we use the equality $\hat{C}_m=C_m(\hat{\lambda})$, we get

$$\begin{aligned}\Sigma_m &:=\sum_{1\leqslant r\leqslant p<q\leqslant m}\Sigma^r_{pq}(m)=\sum_{2\leqslant q\leqslant m}\sum_{1\leqslant r\leqslant p<q}\Sigma^r_{pq}(m)=\sum_{2\leqslant q\leqslant m}\sum_{1\leqslant r\leqslant p<q}\sum_n\frac{\Xi^{rp}_n A^q_q(\hat{C}^{(n)}_m)}{\det\hat{C}^{(n)}_m}\\ &=\sum_n\frac{\sum_{q=2}^m(\sum_{1\leqslant r\leqslant p<q}\Xi^{rp})A^q_q(C_m(\hat{\lambda}))}{\det C_m(\hat{\lambda})}\overset{(47)}{>}\sum_n\frac{\sum_{q=2}^m(\sum_{1\leqslant r\leqslant p<q}\Psi^{rp})A^q_q(C_m(\hat{\lambda}))}{\det C_m(\hat{\lambda})}\\ &\overset{(28)}{=}\sum_n\frac{e^{-1}\sum_{q=2}^m f_q A^q_q(C_m(\hat{\lambda}))}{\det C_m+\sum_{q=2}^m\hat{\lambda}_q A^q_q(C_m(\hat{\lambda}^{[q]}))}.\end{aligned}\tag{22}$$

The implications $S_m=\infty\Rightarrow\Sigma_m=\infty$ is based on the equality (see (26))

$$A^k_k(C_m(\lambda^{[k]}))=A^k_k(C_m)+\sum_{r=1}^{m-k}\sum_{k<i_1<i_2<\cdots<i_r\leqslant m}\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}A^{ki_1i_2\ldots i_r}_{ki_1i_2\ldots i_r}(C_m)\tag{23}$$

and on the following lemma.

**Lemma 16.** *For $\hat{\lambda}=(\hat{\lambda}_r)_{r=1}^m\in\mathbb{R}^m$, $\hat{\lambda}_1=0$, $\hat{\lambda}_k=\sum_{r=1}^{k-1}c_{rr}$, $2\leqslant k\leqslant m$, we have*

$$I^k_m:=f_kA^k_k(C_m(\hat{\lambda}))-\hat{\lambda}_kA^k_k(C_m(\hat{\lambda}^{[k]}))\geqslant 0,\quad 2\leqslant k\leqslant m.\tag{24}$$

Let us suppose that Lemma 16 holds. Using (13), (22)–(24) we have

$$\begin{aligned}\Sigma_m &\overset{(22)}{>}\sum_n\frac{e^{-1}\sum_{q=2}^m f_qA^q_q(C_m(\hat{\lambda}))}{\det C_m+\sum_{q=2}^m\hat{\lambda}_qA^q_q(C_m(\hat{\lambda}^{[q]}))}\overset{(24)}{\geqslant}\sum_n\frac{e^{-1}\sum_{q=2}^m f_qA^q_q(C_m(\hat{\lambda}))}{\det C_m+\sum_{q=2}^m f_qA^q_q(C_m(\hat{\lambda}))}\\ &\overset{(13)}{\sim}\sum_n\frac{\sum_{q=2}^m f_qA^q_q(C_m(\hat{\lambda}))}{\det C_m}\overset{(24)}{>}\sum_n\frac{\sum_{q=2}^m\hat{\lambda}_qA^q_q(C_m(\hat{\lambda}^{[q]}))}{\det C_m}\\ &\overset{(23)}{>}\sum_n\frac{\sum_{q=2}^m\hat{\lambda}_qA^q_q(C_m)}{\det C_m}=S_m.\end{aligned}$$

Finally we have $\Sigma_m>S_m$.

**Corollary 17.** *If $S_{kn}^{L}(\mu_B^m)=\infty$ for some $1\leqslant k<n\leqslant m$ then one of the series $\Sigma_{pq}^{r}(m)$, $1\leqslant r\leqslant p<q\leqslant m$, is divergent and hence by Lemma* 15 *we can approximate the corresponding variable $x_{pq}$.*

**Remark 18.** The approximation of other variables $x_{pq}$, $1\leqslant p<q\leqslant m$, follows the schema used in [23]. For the particular case $1\leqslant m\leqslant 4$ see also the schema used in [22].

Further we can approximate the remaining variables $x_{kn}$, $1\leqslant k\leqslant m<n$, as in [23]. This implies the inclusion $(\mathfrak{A}^m)'\subset L^\infty(X^m,\mu_B^m)$ and so the irreducibility of the representation (see "The idea of the proof of irreducibility" at the beginning of Section 3). □

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## Appendix A. The generalized characteristic polynomial and its properties

We define $G_m(\lambda)$ *the generalization of the characteristic polynomial* $p_C(t)=\det(tI-C)$, $t\in\mathbb{C}$, of the matrix $C\in\mathrm{Mat}(m,\mathbb{C})$:

$$G_m(\lambda)=\det C_m(\lambda),\quad \lambda\in\mathbb{C}^m,\quad \text{where } C_m(\lambda)=C+\sum_{k=1}^{m}\lambda_k E_{kk}. \tag{25}$$

We denote by

$$M^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_r}(C)\quad \bigl(\text{respectively } A^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_r}(C)\bigr),\quad 1\leqslant i_1<\cdots<i_r\leqslant m,\ 1\leqslant j_1<\cdots<j_r\leqslant m,$$

the minors (respectively the cofactors) of the matrix $C$ with $i_1,i_2,\ldots,i_r$ rows and $j_1,j_2,\ldots,j_r$ columns. By definition

$$A^{12\ldots m}_{12\ldots m}(C)=M^{\emptyset}_{\emptyset}(C)=1\quad\text{and}\quad M^{12\ldots m}_{12\ldots m}(C)=A^{\emptyset}_{\emptyset}(C)=\det C.$$

**Lemma A.1.** *For the generalized characteristic polynomial $G_m(\lambda)$ of $C\in\mathrm{Mat}(m,\mathbb{C})$ and $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m)\in\mathbb{C}^m$ we have*:

$$G_m(\lambda)=\det\left(C+\sum_{k=1}^{m}\lambda_k E_{kk}\right)=\det C+\sum_{r=1}^{m}\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant m}\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}A^{i_1i_2\ldots i_r}_{i_1i_2\ldots i_r}(C). \tag{26}$$

**Remark A.2.** If we set $\lambda_\alpha=\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}$ where $\alpha=(i_1,i_2,\ldots,i_r)$ and $A^{\alpha}_{\alpha}(C)=A^{i_1i_2\ldots i_r}_{i_1i_2\ldots i_r}(C)$, $\lambda_\emptyset=1$, $A^{\emptyset}_{\emptyset}(C)=\det C$ we may write (26) as follows:

$$G_m(\lambda)=\det C_m(\lambda)=\sum_{\emptyset\subseteq\alpha\subseteq\{1,2,\ldots,m\}}\lambda_\alpha A^{\alpha}_{\alpha}(C). \tag{27}$$

**Proof.** Probably Lemma A.1 is known in the literature, but since we do not know any precise reference, we provide here a direct proof. Obviously $G_m(\lambda)$ is a polynomial in the variables $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^m$ of order $m$. A direct calculation gives us

$$\frac{\partial^r G_m(\lambda)}{\partial\lambda_1\partial\lambda_2\ldots\partial\lambda_r} = \begin{vmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ & & \ddots & & & \ddots & \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ c_{1r+1} & c_{2r+1} & \ldots & c_{rr+1} & c_{r+1r+1} + \lambda_{r+1} & \ldots & c_{r+1m} \\ & & \ddots & & & \ddots & \\ c_{1m} & c_{2m} & \ldots & c_{rm} & c_{r+1m} & \ldots & c_{mm} + \lambda_m \end{vmatrix},$$

hence

$$\left.\frac{\partial^r G_m}{\partial\lambda_1\partial\lambda_2\ldots\partial\lambda_r}\right|_{\lambda=0} = A^{12\ldots r}_{12\ldots r}(C).$$

Similarly we have for $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant m$

$$\left.\frac{\partial^r G_m}{\partial\lambda_{i_1}\partial\lambda_{i_2}\ldots\partial\lambda_{i_r}}\right|_{\lambda=0} = A^{i_1 i_2\ldots i_r}_{i_1 i_2\ldots i_r}(C). \qquad \square$$

**Lemma A.3.** *For $C \in \mathrm{Mat}(m, \mathbb{C})$ and $\lambda \in \mathbb{C}^m$ we have*

$$G_m(\lambda) = A^{\emptyset}_{\emptyset}\big(C_m(\lambda)\big) = \det C_m(\lambda) = \det C_m + \sum_{r=1}^{m} \lambda_r A^r_r\big(C_m\big(\lambda^{[r]}\big)\big), \tag{28}$$

$$A^k_k\big(C_m(\lambda)\big) = A^k_k(C_m) + \sum_{r=1,\, r\neq k}^{m} \lambda_r A^{rk}_{rk}\big(C_m\big(\lambda^{[r]}\big)\big), \tag{29}$$

$$G_m(\lambda) = A^{\emptyset}_{\emptyset}\big(C_m(\lambda)\big) = \det C_m(\lambda) = \det C_m + \sum_{r=1}^{m} \lambda_r A^r_r\big(C_m\big(\lambda^{\{r\}}\big)\big), \tag{30}$$

$$A^k_k\big(C_m(\lambda)\big) = A^k_k(C_m) + \sum_{r=1,\, r\neq k}^{m} \lambda_r A^{rk}_{rk}\big(C_m\big(\lambda^{\{r\}}\big)\big), \tag{31}$$

*where for $\lambda \in \mathbb{C}^m$ and $1 \leqslant k \leqslant m$ we have set*

$$\lambda^{[k]} = (0, \ldots, 0, \lambda_{k+1}, \ldots, \lambda_m), \qquad \lambda^{\{k\}} = (\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0). \tag{32}$$

**Proof.** We have for $m = 2$ using (26)

$$\begin{aligned} G_2(\lambda) &= \det C_2 + \lambda_1 A^1_1(C_2) + \lambda_2 A^2_2(C_2) + \lambda_1\lambda_2 A^{12}_{12}(C_2) \\ &= \det C_2 + \lambda_1\big[A^1_1(C_2) + \lambda_2 A^{12}_{12}(C_2)\big] + \lambda_2 A^2_2(C_2) \\ &= \det C_2 + \lambda_1 A^1_1\big(C_2\big(\lambda^{[1]}\big)\big) + \lambda_2 A^2_2\big(C_2\big(\lambda^{[2]}\big)\big), \end{aligned}$$

$$\begin{aligned} G_2(\lambda) &= \det C_2 + \lambda_1 A_1^1(C_2) + \lambda_2\big[A_2^2(C_2) + \lambda_1 A_{12}^{12}(C_2)\big] \\ &= \det C_2 + \lambda_1 A_1^1\big(C_2\big(\lambda^{\{1\}}\big)\big) + \lambda_2 A_2^2\big(C_2\big(\lambda^{\{2\}}\big)\big). \end{aligned}$$

For $m = 3$ we have

$$\begin{aligned} G_3(\lambda) &= \det C_3 + \lambda_1 A_1^1(C_3) + \lambda_2 A_2^2(C_3) + \lambda_3 A_3^3(C_3) + \lambda_1\lambda_2 A_{12}^{12}(C_3) + \lambda_1\lambda_3 A_{13}^{13}(C_3) \\ &\quad + \lambda_2\lambda_3 A_{23}^{23}(C_3) + \lambda_1\lambda_2\lambda_3 A_{123}^{123}(C_3) \\ &= \det C_2 + \lambda_1\big[A_1^1(C_3) + \lambda_2 A_{12}^{12}(C_3) + \lambda_3 A_{13}^{13}(C_3) + \lambda_2\lambda_3 A_{123}^{123}(C_3)\big] \\ &\quad + \lambda_2\big[A_2^2(C_3) + \lambda_3 A_{23}^{23}(C_3)\big] + \lambda_3 A_3^3(C_3) \\ &= \det C_3 + \lambda_1 A_1^1\big(C_3\big(\lambda^{[1]}\big)\big) + \lambda_2 A_2^2\big(C_3\big(\lambda^{[2]}\big)\big) + \lambda_3 A_3^3\big(C_3\big(\lambda^{[3]}\big)\big), \end{aligned}$$

$$\begin{aligned} G_3(\lambda) &= \det C_3 + \lambda_1 A_1^1(C_3) + \lambda_2\big[A_2^2(C_3) + \lambda_1 A_{12}^{12}(C_3)\big] \\ &\quad + \lambda_3\big[A_1^1(C_3) + \lambda_1 A_{13}^{12}(C_3) + \lambda_2 A_{23}^{23}(C_3) + \lambda_1\lambda_2 A_{123}^{123}(C_3)\big] \\ &= \det C_3 + \lambda_1 A_1^1\big(C_3\big(\lambda^{\{1\}}\big)\big) + \lambda_2 A_2^2\big(C_3\big(\lambda^{\{2\}}\big)\big) + \lambda_3 A_3^3\big(C_3\big(\lambda^{\{3\}}\big)\big). \end{aligned}$$

For $m > 3$ the proof of (28) and (30) is the same. The identity (29) follows from (28) and (31) follows from (30). □

The proof of Lemma 16 is based on Lemmas A.4, A.6 and A.7 concerning the properties of a positive matrices.

**Lemma A.4.** (Sylvester [10, Chapter II, Section 3]) *Let $C \in \mathrm{Mat}(n,\mathbb{R})$ and $1 \leqslant p < n$. We consider a matrix $B = (b_{ik})_{p+1}^n$ defined by $b_{ik} = M_{12\ldots pk}^{12\ldots pi}(C)$. Then the following Sylvester determinant identity holds*:

$$\det B = \big[M_{12\ldots p}^{12\ldots p}(C)\big]^{n-p-1} \det C.$$

**Corollary A.5.** *If $p = n-2$ we have in particular*

$$\begin{vmatrix} A_n^n(C) & A_{n-1}^n(C) \\ A_n^{n-1}(C) & A_{n-1}^{n-1}(C) \end{vmatrix} = A_{n-1n}^{n-1n}(C) A_\emptyset^\emptyset(C).$$

*For arbitrary $1 \leqslant p < q \leqslant n$ we have*

$$\begin{vmatrix} A_p^p(C) & A_q^p(C) \\ A_p^q(C) & A_q^q(C) \end{vmatrix} = A_\emptyset^\emptyset(C) A_{pq}^{pq}(C) \quad \text{or} \quad \begin{vmatrix} A_p^p(C) & A_{pq}^{pq}(C) \\ A_\emptyset^\emptyset(C) & A_q^q(C) \end{vmatrix} = A_q^p(C) A_p^q(C). \tag{33}$$

**Lemma A.6.** (Hadamard–Ficher's inequality [12,13], see also [27]) *For any positive definite matrix $C \in \mathrm{Mat}(m,\mathbb{R})$, $m \in \mathbb{N}$, and any two subsets $\alpha$ and $\beta$ with $\emptyset \subseteq \alpha, \beta \subseteq \{1,\ldots,m\}$ the following inequality holds*:

$$\begin{vmatrix} M(\alpha) & M(\alpha\cap\beta) \\ M(\alpha\cup\beta) & M(\beta) \end{vmatrix} = \begin{vmatrix} A(\hat\alpha) & A(\hat\alpha\cup\hat\beta) \\ A(\hat\alpha\cap\hat\beta) & A(\hat\beta) \end{vmatrix} \geqslant 0, \tag{34}$$

*where* $M(\alpha)=M^{\alpha}_{\alpha}(C)$, $A(\alpha)=A^{\alpha}_{\alpha}(C)$ *and* $\hat{\alpha}=\{1,\dots,m\}\setminus\alpha$.

More precisely, see [12, p. 573]; [13, Chapter 2.5, Problem 36]. See also [27, Corollary 3.2, p. 34].

Let us set as before (see (25)) for $\lambda=(\lambda_1,\dots,\lambda_k)\in\mathbb{C}^k$ and $C\in\mathrm{Mat}(k,\mathbb{C})$

$$G_k(\lambda)=\det C_k(\lambda),\quad \text{where } C_k(\lambda)=C+\sum_{r=1}^{k}\lambda_r E_{rr}.$$

In the following lemma we use the notation for $\lambda=(\lambda_1,\dots,\lambda_k)\in\mathbb{C}^k$:

$$\lambda^{]l[}=(\lambda_1,\dots,\lambda_{l-1},0,\lambda_{l+1},\dots,\lambda_k),\quad 1\leqslant l\leqslant k,$$

and $G_l(\lambda)=M^{12\dots l}_{12\dots l}(C_k(\lambda))$, $1\leqslant l\leqslant k$. For $\alpha$ and $\beta$ such that $\emptyset\subseteq\alpha\subseteq\{1,2,\dots,l\}$ and $\emptyset\subseteq\beta\subseteq\{l+1,\dots,k\}$, with $l<k$, $C\in\mathrm{Mat}(k,\mathbb{C})$ we set

$$\bigl(A^{\alpha}_{\alpha}(C)\bigr)^{\beta}_{\beta}:=A^{\alpha\cup\beta}_{\alpha\cup\beta}(C),\quad \text{and}\quad G_l(\lambda)^{\beta}_{\beta}:=\sum_{\emptyset\subseteq\alpha\subseteq\{1,2,\dots,l\}}\lambda_\alpha A^{\alpha\cup\beta}_{\alpha\cup\beta}(C).$$

By definition we have

$$G_l(\lambda)=A^{l+1\dots k}_{l+1\dots k}\bigl(C_k(\lambda)\bigr)=\bigl(A^{\emptyset}_{\emptyset}\bigl(C_k(\lambda)\bigr)\bigr)^{l+1\dots k}_{l+1\dots k}=G_k(\lambda)^{l+1\dots k}_{l+1\dots k}. \tag{35}$$

**Lemma A.7.** *We have for* $1\leqslant p\leqslant l\leqslant k$ *and* $C\in\mathrm{Mat}(k,\mathbb{C})$

$$\frac{G_k(\lambda)}{G_l(\lambda)}=\frac{G_k(\lambda^{]p[})+\lambda_p G_k(\lambda^{]p[})^{p}_{p}}{G_k(\lambda^{]p[})^{l+1\dots k}_{l+1\dots k}+\lambda_p G_k(\lambda^{]p[})^{pl+1\dots k}_{pl+1\dots k}}. \tag{36}$$

*For the positive definite matrix* $C$ *and* $\lambda=(\lambda_1,\dots,\lambda_k)\in\mathbb{R}^k$ *with* $\lambda_r\geqslant 0$, $r=1,\dots,k$, *we have*

$$\bigl(G_l(\lambda)\bigr)^2\frac{\partial}{\partial\lambda_p}\frac{G_k(\lambda)}{G_l(\lambda)}=\begin{vmatrix} G_k(\lambda^{]p[})^{p}_{p} & G_k(\lambda^{]p[}) \\ G_k(\lambda^{]p[})^{pl+1\dots k}_{pl+1\dots k} & G_k(\lambda^{]p[})^{l+1\dots k}_{l+1\dots k}\end{vmatrix}\geqslant 0. \tag{37}$$

**Proof.** We have for $1\leqslant p\leqslant l\leqslant k$

$$\frac{\partial G_k(\lambda)}{\partial\lambda_p}=\frac{\partial}{\partial\lambda_p}\det\left(C+\sum_{r=1}^{k}\lambda_r E_{rr}\right)=A^{p}_{p}\bigl(C\bigl(\lambda^{]p[}\bigr)\bigr)=G_k\bigl(\lambda^{]p[}\bigr)^{p}_{p},\quad\text{so} \tag{38}$$

$$G_k(\lambda)-\lambda_p G_k\bigl(\lambda^{]p[}\bigr)^{p}_{p}=G_k(\lambda)\big|_{\lambda_p=0}=G_k\bigl(\lambda^{]p[}\bigr),$$

hence

$$G_k(\lambda)=G_k\bigl(\lambda^{]p[}\bigr)+\lambda_p G_k\bigl(\lambda^{]p[}\bigr)^{p}_{p},\quad 1\leqslant p\leqslant k.$$

Similarly, we have

$$G_l(\lambda) = G_l\big(\lambda^{]p[}\big) + \lambda_p G_l\big(\lambda^{]p[}\big)_p^p \overset{(35)}{=} G_k\big(\lambda^{]p[}\big)_{l+1\ldots k}^{l+1\ldots k} + \lambda_p G_k\big(\lambda^{]p[}\big)_{pl+1\ldots k}^{pl+1\ldots k}, \quad 1 \leqslant p \leqslant l.$$

Finally we get (36). Using the following formula:

$$\left(\frac{a+bx}{c+dx}\right)' = \frac{bc-ad}{(c+dx)^2}$$

we conclude that (36) implies the identity in (37).

To prove the inequality in (36) we get

$$\begin{vmatrix} G_k(\lambda^{]p[})_p^p & G_k(\lambda^{]p[}) \\ G_k(\lambda^{]p[})_{pl+1\ldots k}^{pl+1\ldots k} & G_k(\lambda^{]p[})_{l+1\ldots k}^{l+1\ldots k} \end{vmatrix} = \begin{vmatrix} A_p^p(C_k(\lambda^{]p[})) & A_\emptyset^\emptyset(C_k(\lambda^{]p[})) \\ A_{pl+1\ldots k}^{pl+1\ldots k}(C_k(\lambda^{]p[})) & A_{l+1\ldots k}^{l+1\ldots k}(C_k(\lambda^{]p[})) \end{vmatrix}$$
$$= \begin{vmatrix} A_\alpha^\alpha(C) & A_{\alpha\cap\beta}^{\alpha\cap\beta}(C) \\ A_{\alpha\cup\beta}^{\alpha\cup\beta}(C) & A_\beta^\beta(C) \end{vmatrix} \overset{(34)}{\geqslant} 0,$$

where $C = C_k(\lambda^{]p[})$, $\alpha = \{p\}$ and $\beta = \{l+1, l+2, \ldots, k\}$. □

## Appendix B. Calculation of the matrix elements $\phi_p(t)$ for $t \in \mathbb{R}^p$, their generalizations and $\Xi_n^{pq}$

Let us recall (see (10) and (19)) that $\hat{\lambda}_k = \sum_{r=1}^{k-1} c_{rr}$, $2 \leqslant r \leqslant m$, $\hat{\lambda}_1 = 0$ and

$$\Xi_n^{pq} = \max_{t\in\mathbb{R}^p} \big|M\xi_n^{pq}(t)\big|^2, \quad 1 \leqslant p \leqslant q \leqslant m. \tag{39}$$

To estimate

$$\max_{t\in\mathbb{R}^p} \big|M\xi_n^{pq}(t)\big|^2 = \max_{t\in\mathbb{R}^p} \big|\big(\xi_n^{pq}(t)\mathbf{1}, \mathbf{1}\big)\big|^2,$$

where $\xi_n^{pq}(t) = iy_{qn}\exp(\sum_{r=1}^p t_r \widetilde{A_{rn}})$ we shall find the exact formulas for the matrix elements

$$\phi_p(t) = \phi_p^{(n)}(t) = \big(T^{R,\mu_B^m}_{\exp(\sum_{r=1}^p t_r E_{rn})}\mathbf{1}, \mathbf{1}\big), \quad t = (t_r)_{r=1}^p \in \mathbb{R}^p,\ 1 \leqslant p \leqslant m, \tag{40}$$

of the restriction of the representation $T^{R,\mu_B^m}$ on the commutative subgroup $(\exp(\sum_{r=1}^p t_r E_{rn}) \mid t \in \mathbb{R}^p) \simeq \mathbb{R}^p$ of the group $B_0^{\mathbb{N}}$ and theirs generalization defined below. We note that $\exp(\sum_{r=1}^p t_r E_{rn}) = I + \sum_{r=1}^p t_r E_{rn}$.

For $1 \leqslant p \leqslant q$, $p, q \in \mathbb{N}$ we get

$$\xi_n^{pq}(t) = iy_{qn}\exp\left(\sum_{r=1}^p t_r \widetilde{A_{rn}}\right) = iy_{qn}\exp i\left[\sum_{r=1}^p t_r\left(\sum_{k=1}^{r-1} x_{kr}y_{kn} + y_{rn}\right)\right]; \tag{41}$$

we have used the expression $\widetilde{A_{rn}} = \sum_{k=1}^{r-1} x_{kr}y_{kn} + y_{rn} = \sum_{k=1}^{r} x_{kr}y_{kn}$ (see (9)). We have

$$\widetilde{T}^{R,\mu_B^m}_{\exp(\sum_{r=1}^p t_r E_{rn})} = \exp\left(\sum_{r=1}^p t_r \widetilde{A_{rn}}\right) = \exp i\left[\sum_{r=1}^p t_r\left(\sum_{k=1}^r x_{kr} y_{kn}\right)\right]$$
$$= \exp i\left[\sum_{k=1}^p\left(\sum_{r=k}^p x_{kr} t_r\right) y_{kn}\right].$$

To obtain $\xi^{pp}(t)$ we generalize the function

$$\widetilde{T}^{R,\mu_B^m}_{\exp(\sum_{r=1}^p t_r E_{rn})}$$

in the following way. We replace in the latter identity the vectors $(t_r,\dots,t_r)\in\mathbb{R}^{p-k+1}$ by $(t_{rk})_{r=k}^p\in\mathbb{R}^{p-k+1}$ and denote the result by $\xi_{pp}(t)$:

$$\xi_{pp}(t) = \xi_{pp}\begin{pmatrix} t_{11} & & & \\ t_{21} & t_{22} & & \\ t_{31} & t_{32} & \dots & \\ t_{p1} & t_{p2} & \dots & t_{pp}\end{pmatrix} := \exp i\left[\sum_{k=1}^p\left(\sum_{r=k+1}^p x_{kr} t_{rk} + t_{kk}\right) y_{kn}\right]. \tag{42}$$

To obtain $\xi^{pq}(t)$ we consider the function $\xi_{pq}(t;t_{qq}) = \xi_{pp}(t)\exp(it_{qq}y_{qn})$. We have

$$\xi_{pq}(t;t_{qq}) = \xi_{pq}\begin{pmatrix} t_{11} & & & & \\ t_{21} & t_{22} & & & \\ t_{31} & t_{32} & \dots & & \\ t_{p1} & t_{p2} & \dots & t_{pp}; & t_{qq}\end{pmatrix}$$
$$:= \exp i\left[\sum_{k=1}^p\left(\sum_{r=k+1}^p x_{kr} t_{rk} + t_{kk}\right) y_{kn} + t_{qq} y_{qn}\right].$$

Finally we have

$$\xi^{pp}(t) = \left.\frac{\partial \xi_{pp}(t)}{\partial t_{pp}}\right|_{t_{kr}=t_k,\,1\leqslant r\leqslant k\leqslant p} \quad\text{and}\quad \xi^{pq}(t) = \left.\frac{\partial \xi_{pq}(t;t_{qq})}{\partial t_{qq}}\right|_{t_{qq}=0,\,t_{kr}=t_k,\,1\leqslant r\leqslant k\leqslant p}.$$

Let us define $\phi_p(t) = \int \xi_{pp}(t)\,d\mu(x,y)$, $\phi_{pq}(t) = \int \xi_{pq}(t)\,d\mu(x,y)$, where $\mu(x,y) = \mu_I(x)\otimes(\bigotimes_{n=m+1}^\infty \mu_{C^{(n)}}(y))$ and $\mu_I(x)$ is the standard Gaussian measure in $\mathbb{R}\times\mathbb{R}^2\times\cdots\times\mathbb{R}^m$.

Using definition (39) and the previous equalities we have finally

$$\Xi^{pp} = \max_{t\in\mathbb{R}^p}\left|\frac{\partial\phi_p(t)}{\partial t_{pp}}\right|^2_{t_{kr}=t_k,\,1\leqslant r\leqslant k\leqslant p},$$
$$\Xi^{pq} = \max_{t\in\mathbb{R}^p}\left|\frac{\partial\phi_{pq}(t)}{\partial t_{qq}}\right|^2_{t_{qq}=0,\,t_{kr}=t_k,\,1\leqslant r\leqslant k\leqslant p}. \tag{43}$$

Our aim is to estimate $\Xi^{pq}$. We shall use the notation $C_k := C_{\{1,2,\dots,k\}}$ for $\mathrm{Mat}(m,\mathbb{C})$ and $1\leqslant k\leqslant m$ (see notation $C_\alpha$ for $\emptyset\subseteq\alpha\subseteq\{1,\dots,m\}$ in Lemma B of Section 3).

**Lemma B.1.** *For $1 \leqslant p \leqslant q \leqslant m$ and $\phi_{pq}(t) = \int \xi_{pq}(t)\,d\mu(x,y)$ we have*

$$\phi_{pq}\begin{pmatrix} t_{11} & & & & & \\ t_{21} & t_{22} & & & & \\ t_{31} & t_{32} & t_{33} & & & \\ & & & \dots & & \\ t_{p1} & t_{p2} & t_{p3} & \dots & t_{pp}; & t_{qq} \end{pmatrix}$$
$$= \int\limits_{\mathbb{R}^{\frac{(p-1)(p-2)}{2}+p}} \exp i\left[\sum_{k=1}^{p}\left(\sum_{r=k}^{p} x_{kr}t_{rk}\right)y_{kn} + t_{qq}y_{qn}\right] d\mu(x,y)$$
$$= \frac{1}{\sqrt{\det C_1(t)}}\exp\left(-\frac{1}{2}\left[(CT,T) - \left(C_1(t)^{-1}d,d\right)\right]\right), \tag{44}$$

*where we set $T = (t_{11}, t_{22}, t_{33}, \dots, t_{pp}; t_{qq}) \in \mathbb{R}^{p+1}$, $C \in \mathrm{Mat}(p+1,\mathbb{C})$ is defined by*

$$C := C_{p,q} := C_{\{1,2,\dots,p,q\}} := \begin{pmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1p} & c_{1q} \\ c_{12} & c_{22} & c_{23} & \dots & c_{2p} & c_{2q} \\ c_{13} & c_{23} & c_{33} & \dots & c_{3p} & c_{3q} \\ & & & \ddots & & \\ c_{1p} & c_{2p} & c_{3p} & \dots & c_{pp} & c_{pq} \\ c_{1q} & c_{2q} & c_{3q} & \dots & c_{pq} & c_{qq} \end{pmatrix},$$

$$d = \left(d_{21}(t), d_{31}(t), \dots, d_{p1}(t); d_{32}(t), d_{42}(t), \dots, d_{p2}(t); \dots; d_{pp-1}(t)\right) \in \mathbb{R}^{\frac{(p-1)(p-2)}{2}},$$

$$d_{rs}(t) = t_{rs}e_s(t), \quad 1 \leqslant s < r < p, \qquad e_s(t) = (CT)_s = \sum_{k=1}^{p} c_{sk}t_{kk} + c_{sq}t_{qq}, \quad 1 \leqslant s \leqslant p,$$

*the operator*

$$C_1(t) = 1 + C(t) \in \mathrm{Mat}\left(\frac{(p-1)(p-2)}{2}, \mathbb{C}\right)$$

*being defined by*

$$D(t)^{-1}C_1(t)D(t)^{-1}$$
$$= \begin{pmatrix} c_{11}+t_{21}^{-2} & \dots & c_{11} & c_{12} & \dots & c_{12} & \dots & c_{1p-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{11} & \dots & c_{11}+t_{p1}^{-2} & c_{12} & \dots & c_{12} & \dots & c_{1p-1} \\ c_{12} & \dots & c_{12} & c_{22}+t_{32}^{-2} & \dots & c_{22} & \dots & c_{2p-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{12} & \dots & c_{12} & c_{22} & \dots & c_{22}+t_{p2}^{-2} & \dots & c_{2p-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{1p-1} & \dots & c_{1p-1} & c_{2p-1} & \dots & c_{2p-1} & \dots & c_{p-1p-1}+t_{pp-1}^{-2} \end{pmatrix}, \tag{45}$$

*where* $D(t) = \mathrm{diag}(t_{21}, \ldots, t_{p1}; t_{32}, \ldots, t_{p2}; t_{43}, \ldots, t_{p3}; \ldots; t_{pp-1})$. *We have*

$$\det C_1(t) = 1 + \sum_{r=1}^{p} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant p} \alpha_{i_1}^2 \alpha_{i_2}^2 \ldots \alpha_{i_r}^2 M_{i_1 i_2 \ldots i_r}^{i_1 i_2 \ldots i_r}(C_p), \quad \alpha_k^2 := \sum_{s=k+1}^{p} t_{sk}^2. \tag{46}$$

**Lemma B.2.** *For* $1 \leqslant p \leqslant q \leqslant m$ *we have*

$$\Xi^{pq} \geqslant \Psi^{pq}, \tag{47}$$

*where*

$$\Psi^{pq} = \frac{(M_{12\ldots p-1q}^{12\ldots p-1p}(C_{p,q}))^2 \exp(-1)}{M_{12\ldots p-1}^{12\ldots p-1}(C_p) M_{12\ldots p}^{12\ldots p}(C_p) + \sum_{k=2}^{p} \hat{\lambda}_k (A_k^p(C_p))^2}. \tag{48}$$

**List of formulas for $\Psi^{pq}$ for small $p$ and $p < q$.**

$$\Psi^{11} = c_{11} \exp(-1), \qquad \Psi^{1q} = \frac{c_{1q}^2 \exp(-1)}{c_{11}}, \quad 1 \leqslant q, \tag{49}$$

$$\Psi^{22} = \frac{(M_{12}^{12})^2 \exp(-1)}{c_{11}(M_{12}^{12} + c_{11}^2)}, \qquad \Psi^{2q} = \frac{(M_{1q}^{12})^2 \exp(-1)}{c_{11}(M_{12}^{12} + c_{11}^2)}, \quad 2 \leqslant q, \tag{50}$$

$$\Psi^{3q} = \frac{(M_{12q}^{123})^2 \exp(-1)}{M_{12}^{12} M_{123}^{123} + c_{11}(M_{13}^{12})^2 + (c_{11} + c_{22})(M_{12}^{12})^2}, \quad 3 \leqslant q, \tag{51}$$

$$\Psi^{4q} = \frac{(M_{123q}^{1234})^2 \exp(-1)}{M_{123}^{123} M_{1234}^{1234} + c_{11}(M_{134}^{123})^2 + (c_{11} + c_{22})(M_{124}^{123})^2 + (c_{11} + c_{22} + c_{33})(M_{123}^{123})^2}. \tag{52}$$

**Remark B.3.** We have $\Psi^{pp} > \Psi_0^{pp}$ where

$$\Psi_0^{pp} := \frac{(M_{12\ldots p-1p}^{12\ldots p-1p}(C_p))^2 e^{-1}}{A_p^p(C_p) G_p(\hat{\lambda})} = \frac{(A_\emptyset^\emptyset(C_p))^2 e^{-1}}{A_p^p(C_p) G_p(\hat{\lambda})} = \frac{(G_p(0))^2 e^{-1}}{A_p^p(C_p) G_p(\hat{\lambda})}, \tag{53}$$

and

$$\Psi^{pq} > \Psi_0^{pq} := \frac{(M_{12\ldots p-1q}^{12\ldots p-1p}(C_{p,q}))^2 e^{-1}}{A_p^p(C_p) G_p(\hat{\lambda})}. \tag{54}$$

**Proof.** For positive definite matrix $C_p$ we conclude by Sylvester lemma (see Lemma A.4 and (33) of Corollary A.5) that

$$\begin{vmatrix} A_k^k(C_p) & A_{kp}^{kp}(C_p) \\ A_\emptyset^\emptyset(C_p) & A_p^p(C_p) \end{vmatrix} = A_k^k(C_p) A_p^p(C_p) - A_{kp}^{kp}(C_p) A_\emptyset^\emptyset(C_p) = \left(A_k^p(C_p)\right)^2,$$

hence

$$\big(A_k^p(C_p)\big)^2 < A_k^k(C_p)A_p^p(C_p), \quad 1 \leqslant k \leqslant p.$$

Using the latter inequality we get (see (48))

$$\begin{aligned} &M^{12\ldots p-1}_{12\ldots p-1}(C_p)M^{12\ldots p}_{12\ldots p}(C_p) + \sum_{k=2}^{p} \hat{\lambda}_k\big(A_k^p(C_p)\big)^2 \\ &= A_p^p(C_p)A_\emptyset^\emptyset(C_p) + \sum_{k=2}^{p} \hat{\lambda}_k\big(A_k^p(C_p)\big)^2 < A_p^p(C_p)\left[A_\emptyset^\emptyset(C_p) + \sum_{k=2}^{p} \hat{\lambda}_k A_k^k(C_p)\right] \\ &\overset{(26)}{<} A_p^p(C_p)G_p(\hat{\lambda}). \qquad \square \end{aligned}$$

**Proof of Lemmas B.1 and B.2.** For a positive definite operator $C$ in the space $\mathbb{R}^m$ we have the well-known formulas:

$$\frac{1}{\sqrt{(2\pi)^m}}\int_{\mathbb{R}^m} \exp\left(-\frac{1}{2}(Cx,x)\right)dx = \frac{1}{\sqrt{\det C}}. \tag{55}$$

Using formula (55) we obtain the following formula for $d \in \mathbb{R}^m$:

$$\frac{1}{\sqrt{(2\pi)^m}}\int_{\mathbb{R}^m} \exp\left(-\frac{1}{2}(Cx,x) + (d,x)\right)dx = \frac{1}{\sqrt{\det C}}\exp\left(\frac{(C^{-1}d,d)}{2}\right), \tag{56}$$

and as a particular case for $m = 1$ we have

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(-\frac{1}{2}cx^2 + dx\right)dx = \frac{1}{\sqrt{c}}\exp\left(\frac{d^2}{2c}\right). \tag{57}$$

To obtain (56) from (55) we use the following formula:

$$(Cx,x) - 2(d,x) = \big(C(x-x_0),(x-x_0)\big) - \big(C^{-1}d,d\big), \quad \text{where } x_0 = C^{-1}d. \tag{58}$$

Indeed we find $x_0 \in \mathbb{R}^m$ and $d_0 \in \mathbb{R}$ such that

$$(Cx,x) - 2(d,x) = \big(C(x-x_0),(x-x_0)\big) + d_0.$$

We have

$$(Cx,x) - 2(d,x) = \big(C(x-x_0),(x-x_0)\big) + d_0 = (Cx,x) - 2(Cx_0,x) + (Cx_0,x_0) + d_0,$$

so $Cx_0 = d$ hence $x_0 = C^{-1}d$ and since $(Cx_0,x_0) + d_0 = 0$ we conclude that $d_0 = -(Cx_0,x_0) = -(CC^{-1}d, C^{-1}d) = -(C^{-1}d,d)$.

Fourier transform for the Gaussian measure $\mu_C$ in the space $\mathbb{R}^m$ is:

$$\frac{1}{\sqrt{(2\pi)^m}}\int_{\mathbb{R}^m}\exp i(y,x)\,d\mu_C(x)=\exp\left(-\frac{1}{2}(Cy,y)\right),\quad y\in\mathbb{R}^m.$$

Let $p=1$. Using (55)–(57) we have

$$\phi_1(t_{11})=\int_{\mathbb{R}}\exp(it_{11}y_{1n})\,d\mu(y)=\exp\left(-\frac{1}{2}c_{11}t_{11}^2\right);$$

$$\phi_{1q}(t_{11};t_{qq})=\int_{\mathbb{R}^2}\exp i(t_{11}y_{1n}+t_{qq}y_{qn})\,d\mu(y)=\exp\left(-\frac{1}{2}\left(c_{11}t_{11}^2+2c_{1q}t_{11}t_{qq}+c_{qq}t_{qq}^2\right)\right);$$

$$M\xi^{1q}(t_{11})=\int_{\mathbb{R}}iy_{qn}\exp(it_{11}y_{1n})\,d\mu(y)=\left.\frac{\partial\phi_{1q}(t_{11};t_{qq})}{\partial t_{qq}}\right|_{t_{qq}=0}=-c_{1q}t_{11}\exp\left(-\frac{1}{2}c_{11}^2t_{11}^2\right),$$

$$\left|M\xi^{1,q}(t_{11})\right|^2=c_{1q}^2t_{11}^2\exp\left(-c_{11}t_{11}^2\right);$$

$$\Xi^{1q}=\max_{t_{11}\in\mathbb{R}}\left|M\xi^{1q}(t_{11})\right|^2=\frac{c_{1q}^2\exp(-1)}{c_{11}}=\Psi^{1q},$$

we have used the obvious result

$$\max_{x\in\mathbb{R}}f(x)=f\left(\frac{1}{a}\right)=\frac{1}{ea},\quad\text{where } f(x)=x\exp(-ax),\ a>0. \tag{59}$$

This proves (47) for $(p,q)=(1,q)$.

To prove (44) in the general case we note that

$$\sum_{k=1}^{p}\left(\sum_{r=k+1}^{p}x_{kr}t_{rk}+t_{kk}\right)y_{kn}+t_{qq}y_{qn}=\left(a(x)+T,y\right)_{\mathbb{R}^{p+1}},$$

where

$$y=(y_{1n},y_{2n},\ldots,y_{pn};y_{qn}),\qquad T=(t_{11},t_{22},\ldots,t_{pp};t_{qq})\in\mathbb{R}^{p+1},$$

$$a(x)=\left(a_1(x),a_2(x),\ldots,a_p(x);0\right)\in\mathbb{R}^{p+1},\quad a_k(x)=\sum_{r=k+1}^{p}x_{kr}t_{rk}=(xt)_{kk},$$

$$x=\sum_{1<k<r\leqslant m}x_{kr}E_{kr},\quad t=\sum_{1<r<k\leqslant m}t_{kr}E_{kr},\quad 1\leqslant k\leqslant p.$$

Using the definition of the Fourier transform we have

$$\phi_{pq}(t;t_{qq}) = \int\int_{\mathbb{R}^{p+1}} \exp i\left[\sum_{k=1}^{p}\left(\sum_{r=k}^{p} x_{kr}t_{rk}\right)y_{kn} + t_{qq}y_{qn}\right] d\mu(x,y)$$
$$= \int \exp i\big(a(x)+T, y\big)\, d\mu(x,y) = \int \exp\left[-\frac{1}{2}\big(C\big(a(x)+T\big), a(x)+T\big)\right] d\mu_I(x).$$

Since

$$\big(C\big(a(x)+T\big), a(x)+T\big) = \big(Ca(x), a(x)\big) + 2\big(a(x), CT\big) + (CT, T),$$

we have

$$\phi_{pq}(t;t_{qq}) = \exp\left[-\frac{1}{2}(CT,T)\right]\int \exp\left(-\frac{1}{2}\big[\big(Ca(x),a(x)\big)+2\big(a(x),CT\big)\big]\right) d\mu_I(x). \quad (60)$$

To calculate the latter integral we use (56). Let us introduce the notation

$$X = (x_{12}; x_{13}, x_{23}; \ldots; x_{1p}, \ldots; x_{p-1p}) \in \mathbb{R}^{\frac{(p-1)(p-2)}{2}}.$$

We show that

$$\big(Ca(x), a(x)\big) + 2\big(a(x), CT\big) = \big(C(t)X, X\big) + 2\big(d(t), X\big)$$

for some

$$d(t) \in \mathbb{R}^{\frac{(p-1)(p-2)}{2}} \quad \text{and} \quad C(t) \in \mathrm{Mat}\left(\frac{(p-1)(p-2)}{2}, \mathbb{R}\right).$$

We have

$$\big(a(x), CT\big) = \sum_{k=1}^{p} a_k(x)(CT)_k = \sum_{k=1}^{p}\sum_{r=k+1}^{p} x_{kr}t_{rk}e_k(t) = \sum_{1\leqslant k<r\leqslant p} x_{kr}t_{rk}e_k(t)$$
$$= \sum_{1\leqslant k<r\leqslant p} x_{kr}d_{rk}(t) = \big(X, d(t)\big),$$

where

$$d(t) = \big(d_{rk}(t)\big)_{1\leqslant k<r\leqslant p} \in \mathbb{R}^{\frac{(p-1)(p-2)}{2}},$$

$$d_{rk}(t) = t_{rk}e_k(t) \text{ and } e_k(t) = (CT)_k = \sum_{r=1}^{p} c_{kr}t_{rr} + c_{kq}t_{qq},\ 1 \leqslant k \leqslant p-1.$$

Further

$$\begin{aligned}\big(Ca(x),a(x)\big) &= \sum_{1\leqslant k,n\leqslant p} c_{kn}a_k(x)a_n(x) = \sum_{1\leqslant k,n\leqslant p} c_{kn}\sum_{r=k+1}^{p} x_{kr}t_{rk}\sum_{s=n+1}^{p} x_{ns}t_{sn}\\ &= \sum_{1\leqslant k<r\leqslant p}\ \sum_{1\leqslant n<s\leqslant p} c_{kn}t_{rk}t_{sn}x_{kr}x_{ns} = \big(C(t)X,X\big),\end{aligned}$$

where the operator $C(t)$ is defined by its entries:

$$\big(C(t)\big)_{kr,ns} = c_{kn}t_{rk}t_{sn} \quad \text{for } 1\leqslant k<r\leqslant p \text{ and } 1\leqslant n<s\leqslant p. \tag{61}$$

This prove the representation (45) for the operator $C_1(t)$. Finally we have

$$\big(Ca(x),a(x)\big) = \big(C(t)X,X\big) \quad\text{and}\quad \big(a(x),CT\big) = \big(X,d(t)\big).$$

Putting the latter equalities in (60) we get using (56)

$$\begin{aligned}\phi_{pq}(t;t_{qq}) &= \exp\left[-\frac{1}{2}(CT,T)\right]\int \exp\left(-\frac{1}{2}\big[\big(C(t)X,X\big)+2\big(X,d(t)\big)\big]\right)d\mu_I(x)\\ &= \frac{1}{\sqrt{\det C_1(t)}}\exp\left(-\frac{1}{2}\big[(CT,T)-\big(C_1(t)^{-1}d(t),d(t)\big)\big]\right),\end{aligned}$$

where $C_1(t)=I+C(t)$. This proves (44) of Lemma B.1.

We estimate now $\Xi^{pq}$. For $(p,q)=(2,2)$ we get

$$\begin{aligned}\phi_2(t) &= \frac{1}{\sqrt{\det C_1(t)}}\exp\left(-\frac{1}{2}\big[(CT,T)-\big(C_1(t)^{-1}d(t),d(t)\big)\big]\right)\\ &= \frac{1}{\sqrt{1+c_{11}t_{21}^2}}\exp\left(-\frac{1}{2}\left[c_{11}t_{11}^2+2c_{12}t_{11}t_{22}+c_{22}t_{22}^2-\frac{(c_{11}t_{11}+c_{12}t_{22})^2t_{21}^2}{1+c_{11}t_{21}^2}\right]\right),\end{aligned}$$

where

$$T=(t_{11},t_{22}),\qquad d(t)=d_{21}(t)=t_{21}e_1(t)=t_{21}(c_{11}t_{11}+c_{12}t_{22}),$$

$$e_1(t)=c_{11}t_{11}+c_{12}t_{22},\qquad e_2(t)=c_{21}t_{11}+c_{22}t_{22},$$

$$C=C_2=\begin{pmatrix}c_{11}&c_{12}\\c_{12}&c_{22}\end{pmatrix},\qquad C(t)\overset{(61)}{=}c_{11}t_{21}^2,\qquad C_1(t)=1+c_{11}t_{21}^2,$$

$$\begin{aligned}\frac{\partial\phi_2(t)}{\partial t_{11}} &= \left[-(c_{11}t_{11}+c_{12}t_{22})+\frac{(c_{11}t_{11}+c_{12}t_{22})c_{11}t_{21}^2}{1+c_{11}t_{21}^2}\right]\\ &\quad\times\frac{\exp(-\frac{1}{2}[(CT,T)-(C_1(t)^{-1}d(t),d(t))])}{\sqrt{\det C_1(t)}},\\ \frac{\partial\phi_2(t)}{\partial t_{22}} &= \left[-(c_{21}t_{11}+c_{22}t_{22})+\frac{(c_{11}t_{11}+c_{12}t_{22})c_{12}t_{21}^2}{1+c_{11}t_{21}^2}\right]\\ &\quad\times\frac{\exp(-\frac{1}{2}[(CT,T)-(C_1(t)^{-1}d(t),d(t))])}{\sqrt{\det C_1(t)}}.\end{aligned}$$

Let $e_1(t) = c_{11}t_{11} + c_{12}t_{22} = 0$ so $t_{11} = -c_{12}t_{22}/c_{11}$. In this case

$$(CT,T) = c_{11}t_{11}^2 + 2c_{12}t_{11}t_{22} + c_{22}t_{22}^2 = \left(\frac{c_{12}^2}{c_{11}} - 2\frac{c_{12}^2}{c_{11}} + c_{22}\right)t_{22}^2 = \frac{M_{12}^{12}}{c_{11}}t_{22}^2,$$

$$c_{12}t_{11} + c_{22}t_{22} = \left(-\frac{c_{12}c_{12}}{c_{11}} + c_{22}\right)t_{22} = \frac{c_{11}c_{22} - c_{12}^2}{c_{11}} = \frac{M_{12}^{12}}{c_{11}}.$$

Finally

$$\left|M\xi^{22}(t)\right|^2 = \left|Miy_{2n}\exp\bigl(it_{11} + it_{22}(x_{12}y_{1n} + y_{2n})\bigr)\right|^2 = \left|\frac{\partial\phi_2(t)}{\partial t_{22}}\right|^2_{e_1(t)=0,\,t_{21}=t_{22}}$$

$$= \frac{\bigl(\frac{M_{12}^{12}}{c_{11}}t_{22}\bigr)^2\exp\bigl(-\frac{M_{12}^{12}}{c_{11}}t_{22}^2\bigr)}{1 + c_{11}t_{22}^2} \geqslant \frac{M_{12}^{12}}{c_{11}}t_{22}^2\exp\left[-\left(\frac{M_{12}^{12}}{c_{11}} + c_{11}\right)t_{22}^2\right].$$

We have used the inequality

$$1 + x \leqslant \exp x, \quad x \in \mathbb{R}. \tag{62}$$

Hence if we denote $t = (t_{11}, t_{22}) \in \mathbb{R}^2$ we have using (43)

$$\varXi^{22} = \max_{t\in\mathbb{R}^2}\left|M\xi^{22}(t)\right|^2 > \varPsi^{22} := \frac{(M_{12}^{12})^2\exp(-1)}{c_{11}(M_{12}^{12} + c_{11}^2)}.$$

This proves (47) for $(p,q) = (2,2)$. For $(2,q)$, $2 < q$, we have

$$\phi_{2q}\begin{pmatrix} t_{11} & & \\ t_{21} & t_{22}; & t_{qq}\end{pmatrix} = \int_{\mathbb{R}^{1+3}} \exp i\bigl[t_{11}y_{1n} + (t_{21}x_{12}y_{1n} + t_{22}y_{2n}) + t_{qq}y_{qn}\bigr]\,d\mu(x,y)$$

$$= \frac{1}{\sqrt{1 + c_{11}t_{21}^2}}\exp\biggl(-\frac{1}{2}\biggl[c_{11}t_{11}^2 + c_{22}t_{22}^2 + c_{qq}t_{qq}^2 + 2c_{12}t_{11}t_{22}$$

$$+ 2c_{1q}t_{11}t_{qq} + 2c_{2q}t_{22}t_{qq} - \frac{(c_{11}t_{11} + c_{12}t_{22} + c_{1q}t_{qq})^2t_{21}^2}{1 + c_{11}t_{21}^2}\biggr]\biggr)$$

$$= \frac{1}{\sqrt{\det C_1(t)}}\exp\left(-\frac{1}{2}\left[(CT,T) - \bigl(C_1(t)^{-1}d(t), d(t)\bigr)\right]\right),$$

where

$$T = (t_{11}, t_{22}; t_{qq}) \in \mathbb{R}^3, \qquad d(t) = t_{21}(c_{11}t_{11} + c_{12}t_{22} + c_{1q}t_{qq}) =: t_{21}e_1(t) \in \mathbb{R},$$

$$e_1(t) = c_{11}t_{11} + c_{12}t_{22} + c_{1q}t_{qq}, \qquad e_2(t) = c_{21}t_{11} + c_{22}t_{22} + c_{2q}t_{qq},$$

$$C = C_{2,q} = \begin{pmatrix} c_{11} & c_{12} & c_{1q} \\ c_{12} & c_{22} & c_{2q} \\ c_{1q} & c_{2q} & c_{qq}\end{pmatrix}, \qquad C_1(t) = \det C_1(t) = 1 + c_{11}t_{21}^2,$$

$$\frac{\partial\phi_{2q}(t;t_{qq})}{\partial t_{qq}}\bigg|_{t_{qq}=0} = \left[-(c_{1q}t_{11}+c_{2q}t_{22}+c_{qq}t_{qq}) + \frac{(c_{11}t_{11}+c_{12}t_{22}+c_{1q}t_{qq})c_{1q}t_{21}^2}{1+c_{11}t_{21}^2}\right]$$
$$\times\exp\biggl(-\frac{1}{2}\bigl[(CT,T)-\bigl(C_1(t)^{-1}d,d\bigr)\bigr]\biggr)\frac{1}{\sqrt{\det C_1(t)}},$$
$$\frac{\partial\phi_{2q}(t;t_{qq})}{\partial t_{qq}}\bigg|_{t_{qq}=0} = \left[-(c_{1q}t_{11}+c_{2q}t_{22}) + \frac{(c_{11}t_{11}+c_{12}t_{22})c_{1q}t_{21}^2}{1+c_{11}t_{21}^2}\right]$$
$$\times\exp\biggl(-\frac{1}{2}(CT,T)\biggr)\frac{1}{\sqrt{\det C_1(t)}}\bigg|_{t_{qq}=0}.$$

Let $t_{qq}=0$. We chose $d(t)=0$ so we have $c_{11}t_{11}+c_{12}t_{22}=0$ and $t_{11}=\frac{-c_{12}t_{22}}{c_{11}}$. In this case

$$(CT,T)=c_{11}t_{11}^2+2c_{12}t_{11}t_{22}+c_{22}t_{22}^2=\biggl(\frac{c_{12}^2}{c_{11}}-2\frac{c_{12}^2}{c_{11}}+c_{22}\biggr)t_{22}^2=\frac{M_{12}^{12}}{c_{11}}t_{22}^2,$$
$$c_{1q}t_{11}+c_{2q}t_{22}=\biggl(-\frac{c_{12}c_{1q}}{c_{11}}+c_{2q}\biggr)t_{22}=\frac{c_{11}c_{2q}-c_{12}c_{1q}}{c_{11}}t_{22}=\frac{M_{1q}^{12}}{c_{11}}t_{22}.$$

Finally, if we denote $t=(t_{11},t_{22})\in\mathbb{R}^2$, we have

$$\bigl|M\xi^{2q}(t)\bigr|^2=\bigl|Miy_{qn}\exp\bigl(it_{11}+it_{22}(x_{12}y_{1n}+y_{2n})\bigr)\bigr|^2=\left|\frac{\partial\phi_{2q}(t;t_{qq})}{\partial t_{qq}}\right|^2_{t_{qq}=0,\,e_1(t)=0}$$
$$=\frac{\bigl(\frac{M_{1q}^{12}}{c_{11}}t_{22}\bigr)^2\exp\bigl(-\frac{M_{12}^{12}}{c_{11}}t_{22}^2\bigr)}{1+c_{11}t_{22}^2}\overset{(62)}{>}\biggl(\frac{M_{1q}^{12}}{c_{11}}t_{22}\biggr)^2\exp\biggl(-\biggl(\frac{M_{12}^{12}}{c_{11}}+c_{11}\biggr)t_{22}^2\biggr).$$

By (59) we conclude using (43) that

$$\Xi^{2q}=\max_{t\in\mathbb{R}^2}\bigl|M\xi^{2q}(t)\bigr|^2\geqslant\max_{t_{22}\in\mathbb{R}}\left|\frac{\partial\phi_{2q}(t;t_{qq})}{\partial t_{qq}}\right|^2_{t_{qq}=0,\,e_1(t)=0}\geqslant\frac{(M_{1q}^{12})^2\exp(-1)}{c_{11}(M_{12}^{12}+c_{11}^2)}=\Psi^{2q}.$$

This proves (47) for $(p,q)=(2,q)$, $2<q$.

For $n=3$ we have

$$\phi_3\begin{pmatrix}t_{11}&&\\t_{21}&t_{22}&\\t_{31}&t_{32}&t_{33}\end{pmatrix}=\frac{1}{\sqrt{\det C_1(t)}}\exp\biggl(-\frac{1}{2}\bigl[(CT,T)-\bigl(C_1(t)^{-1}d,d\bigr)\bigr]\biggr),$$

where

$$T=(t_{11},t_{22},t_{33}),\qquad d(t)=\bigl(d_{21}(t),d_{31}(t),d_{32}(t)\bigr),$$
$$d_{21}(t)=t_{21}e_1(t),\qquad d_{31}(t)=t_{31}e_1(t),\qquad d_{32}(t)=t_{32}e_2(t),$$
$$e_1(t)=c_{11}t_{11}+c_{12}t_{22}+c_{13}t_{33},\qquad e_2(t)=c_{21}t_{11}+c_{22}t_{22}+c_{23}t_{33},$$

$$C = C_3 = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}, \qquad C(t) \overset{(61)}{=} \begin{pmatrix} c_{11}t_{21}^2 & c_{11}t_{21}t_{31} & c_{12}t_{21}t_{32} \\ c_{11}t_{21}t_{31} & c_{11}t_{31}^2 & c_{12}t_{31}t_{32} \\ c_{12}t_{21}t_{32} & c_{12}t_{31}t_{32} & c_{22}t_{32}^2 \end{pmatrix},$$

hence

$$\begin{aligned} C_1(t) = I + C(t) &= \begin{pmatrix} 1 + c_{11}t_{21}^2 & c_{11}t_{21}t_{31} & c_{12}t_{21}t_{32} \\ c_{11}t_{21}t_{31} & 1 + c_{11}t_{31}^2 & c_{12}t_{31}t_{32} \\ c_{12}t_{21}t_{32} & c_{12}t_{31}t_{32} & 1 + c_{22}t_{32}^2 \end{pmatrix} \\ &= \operatorname{diag}(t_{21}, t_{31}, t_{32}) \begin{pmatrix} c_{11} + t_{21}^{-2} & c_{11} & c_{12} \\ c_{11} & c_{11} + t_{31}^{-2} & c_{12} \\ c_{12} & c_{12} & c_{22} + t_{32}^{-2} \end{pmatrix} \operatorname{diag}(t_{21}, t_{31}, t_{32}). \end{aligned}$$

We prove the following inequality for an operator $C$ of order $n$ such that $I + C > 0$:

$$\det(I + C) \leqslant \exp \operatorname{tr} C. \tag{63}$$

Indeed by Hadamard inequality (see [7] or [13, Section 2.5.4]) we have for positive operator $C$ of order $n$

$$\det C \leqslant \prod_{i=1}^{n} c_{ii}.$$

Using the Hadamard inequality and (62) we have for an operator $C$ such that $I + C > 0$

$$\det(I + C) \leqslant \prod_{i=1}^{n} (1 + c_{ii}) \overset{(62)}{\leqslant} \prod_{i=1}^{n} \exp c_{ii} = \exp\left(\sum_{i=1}^{n} c_{ii}\right) = \exp(\operatorname{tr} C),$$

where we denote by $\operatorname{tr} C$ the trace of an operator $C$ in the space $\mathbb{C}^n$. Using (63) and (61) we conclude that

$$\det\bigl(I + C(t)\bigr) \leqslant \operatorname{tr} C(t) = \exp\left[\sum_{k=1}^{p-1} c_{kk} \left(\sum_{r=k+1}^{p} t_{rk}^2\right)\right] = \exp\left(\sum_{k=1}^{p-1} c_{kk} \alpha_k^2\right), \tag{64}$$

where $\alpha_k^2 = \sum_{r=k+1}^{p} t_{rk}^2$ since by (61) we have

$$\operatorname{tr} C(t) = \sum_{1 \leqslant k < r \leqslant p} C(t)_{kr,kr} = \sum_{1 \leqslant k < r \leqslant p} c_{kk} t_{rk}^2 = \sum_{k=1}^{p-1} c_{kk} \left(\sum_{r=k+1}^{p} t_{rk}^2\right). \tag{65}$$

Using (26) we get

$$\begin{aligned} \det C_1(t) = t_{21}^2 t_{31}^2 t_{32}^2 \bigl(&\det B + \lambda_1 A_1^1 + \lambda_1 A_2^2 + \lambda_3 A_3^3 + \lambda_1 \lambda_2 A_{12}^{12} \\ &+ \lambda_1 \lambda_3 A_{13}^{13} + \lambda_2 \lambda_3 A_{23}^{23} + \lambda_1 \lambda_2 \lambda_3 A_{123}^{123}\bigr) \end{aligned}$$

$$= t_{21}^2 t_{31}^2 t_{32}^2 \left[ \begin{vmatrix} c_{11} & c_{11} & c_{12} \\ c_{11} & c_{11} & c_{22} \\ c_{12} & c_{12} & c_{22} \end{vmatrix} + \left( \frac{1}{t_{21}^2} + \frac{1}{t_{31}^2} \right) \begin{vmatrix} c_{11} & c_{22} \\ c_{12} & c_{22} \end{vmatrix} \right.$$

$$\left. + \frac{1}{t_{21}^2 t_{31}^2} c_{22} + \left( \frac{1}{t_{21}^2 t_{32}^2} + \frac{1}{t_{31}^2 t_{32}^2} \right) c_{11} + \frac{1}{t_{21}^2 t_{31}^2 t_{32}^2} \right]$$

$$= 1 + c_{11}\left(t_{21}^2 + t_{31}^2\right) + c_{22} t_{32}^2 + M_{12}^{12}\left(t_{21}^2 + t_{31}^2\right) t_{32}^2.$$

Finally we have

$$\det C_1(t) = 1 + c_{11}\alpha_1^2 + c_{22}\alpha_2^2 + M_{12}^{12}\alpha_1^2\alpha_2^2, \quad \text{where } \alpha_1^2 = t_{21}^2 + t_{31}^2,\ \alpha_2^2 = t_{32}^2.$$

For general $n$ we have by analogy (it proves thus (46))

$$\det C_1(t) = 1 + \sum_{r=1}^{n-1} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1} \alpha_{i_1}^2 \alpha_{i_2}^2 \ldots \alpha_{i_r}^2 M_{i_1 i_2 \ldots i_r}^{i_1 i_2 \ldots i_r}(C_n), \quad \text{where } \alpha_k^2 = \sum_{s=k+1}^{n} t_{sk}^2.$$

For $n = 3$ we have

$$\frac{\partial \phi_3(t)}{\partial t_{33}} = \left[ -\frac{1}{2} \frac{\partial (CT, T)}{\partial t_{33}} + \frac{\partial (C_1(t)^{-1} d(t), d(t))}{\partial t_{33}} \right] \frac{\exp(-\frac{1}{2}[(CT, T) - (C_1(t)^{-1} d(t), d(t))])}{\sqrt{\det C_1(t)}},$$

$$\frac{\partial \phi_3(t)}{\partial t_{33}} = \left[ -e_3(t) + \frac{\partial (C_1(t)^{-1} d(t), d(t))}{\partial t_{33}} \right] \frac{\exp(-\frac{1}{2}[(CT, T) - (C_1(t)^{-1} d(t), d(t))])}{\sqrt{\det C_1(t)}}.$$

We calculate $|\partial \phi_3(t)/\partial t_{33}|^2$ under the conditions $e_1(t) = e_2(t) = 0$ on the variables $t = (t_{11}, t_{22}, t_{33}) \in \mathbb{R}^3$. It gives us

$$\begin{cases} c_{11}t_{11} + c_{12}t_{22} + c_{13}t_{33} = 0, \\ c_{21}t_{11} + c_{22}t_{22} + c_{23}t_{33} = 0. \end{cases}$$

The solutions are

$$t_{11} = \frac{M_{23}^{12}(C_3)}{M_{12}^{12}(C_3)} t_{33} = \frac{A_1^3(C_3)}{A_3^3(C_3)} t_{33}, \qquad t_{22} = -\frac{M_{13}^{12}(C_3)}{M_{12}^{12}(C_3)} t_{33} = \frac{A_2^3(C_3)}{A_3^3(C_3)} t_{33}. \tag{66}$$

In general, for the matrix $C_n$ conditions $e_1(t) = e_2(t) = \cdots = e_{n-1}(t) = 0$ gives us the system

$$\begin{cases} c_{11}t_{11} + c_{12}t_{22} + \cdots + c_{1n}t_{nn} = 0, \\ c_{21}t_{11} + c_{22}t_{22} + \cdots + c_{2n}t_{nn} = 0, \\ \vdots \\ c_{n-11}t_{11} + c_{n-12}t_{22} + \cdots + c_{n-1n}t_{nn} = 0 \end{cases} \tag{67}$$

and the following solutions:

$$t_{kk} = (-1)^{k+n} \frac{M_{12\ldots k-1k+1\ldots n}^{12\ldots k-1kk+1\ldots n-1}(C_n)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)} t_{nn} = \frac{A_k^n(C_n)}{A_n^n(C_n)} t_{nn}, \quad 1 \leqslant k \leqslant n-1. \tag{68}$$

If we denote $e_k(t)=\sum_{r=1}^{n} c_{kr}t_{rr}$ we get

$$(CT,T)=\sum_{1\leqslant k,r\leqslant n} c_{kr}t_{rr}t_{kk}=\sum_{k=1}^{n} e_k(t)t_{kk},\qquad \frac{1}{2}\frac{\partial(CT,T)}{\partial t_{nn}}=e_n(t). \tag{69}$$

Under conditions (67) we have

$$e_n(t)=\sum_{r=1}^{n} c_{nr}\frac{A_r^n(C_n)}{A_n^n(C_n)}t_{nn}=\frac{M_{12\ldots n}^{12\ldots n}(C_n)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)}t_{nn},\qquad \frac{\partial(C_1(t)^{-1}d(t),d(t))}{\partial t_{nn}}=0 \tag{70}$$

and

$$(CT,T)\overset{(69)}{=}\sum_{k=1}^{n} e_k(t)t_{kk}=e_n(t)t_{nn}=\frac{M_{12\ldots n}^{12\ldots n}(C_n)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)}t_{nn}^2. \tag{71}$$

For $n=3$ using (70) and (71) we can calculate

$$e_3(t)=\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}t_{33},\qquad (CT,T)=\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}t_{33}^2,\qquad \frac{\partial(C_1(t)^{-1}d(t),d(t))}{\partial t_{33}}=0.$$

If, in addition, $e_1(t)=e_2(t)=0$, we have (see (66))

$$\operatorname{tr}C(t)=c_{11}\big(t_{22}^2+t_{33}^2\big)+c_{22}t_{33}^2=\left[c_{11}\left(\left(\frac{M_{13}^{12}(C_3)}{M_{23}^{12}(C_3)}\right)^2+1\right)+c_{22}\right]t_{33}^2.$$

For $n=3$ we have if $e_1(t)=e_2(t)=0$, using the values for $t_{22}$, $e_3(t)$ and $(CT,T)$

$$\begin{aligned}\left|\frac{\partial\phi_3(t)}{\partial t_{33}}\right|^2&=\frac{e_3^2(t)\exp(-(CT,T))}{\det C_1(t)}\overset{(64)}{\geqslant} e_3^2(t)\exp\big(-(CT,T)-\operatorname{tr}C(t)\big)\\&=\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}\right)^2 t_{33}^2\exp\left[-t_{33}^2\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}+(c_{11}+c_{22})+c_{11}\left(\frac{M_{13}^{12}(C_3)}{M_{12}^{12}(C_3)}\right)^2\right)\right].\end{aligned}$$

We get by (59)

$$\begin{aligned}&\max_{t_{33}\in\mathbb{R}}\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}\right)^2 t_{33}^2\exp\left[-t_{33}^2\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}+(c_{11}+c_{22})+c_{11}\left(\frac{M_{13}^{12}(C_3)}{M_{12}^{12}(C_3)}\right)^2\right)\right]\\&\quad=\frac{\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}\right)^2\exp(-1)}{\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}+(c_{11}+c_{22})+c_{11}\left(\frac{M_{13}^{12}(C_3)}{M_{12}^{12}(C_3)}\right)^2}\\&\quad=\frac{(M_{123}^{123}(C_3))^2\exp(-1)}{M_{12}^{12}(C_3)M_{123}^{123}(C_3)+c_{11}(M_{13}^{12}(C_3))^2+(c_{11}+c_{22})(M_{12}^{12}(C_3))^2}=\Psi^{33}.\end{aligned}$$

Finally we have (see (43))

$$\Xi^{33}=\max_{t\in\mathbb{R}^2}\left|M\xi^{33}(t)\right|^2\geqslant\max_{t_{33}\in\mathbb{R}}\left|\frac{\partial\phi_3(t)}{\partial t_{33}}\right|^2_{e_1(t)=e_2(t)=0}\geqslant\Psi^{33}.$$

This proves (47) for $(p,q)=(3,3)$.

By analogy we have for general $n$:

$$\frac{\partial\phi_n(t)}{\partial t_{nn}}=\left[-\frac{1}{2}\frac{\partial(CT,T)}{\partial t_{nn}}+\frac{\partial(C_1(t)^{-1}d(t),d(t))}{\partial t_{nn}}\right]\frac{\exp(-\frac{1}{2}[(CT,T)-(C_1(t)^{-1}d(t),d(t))])}{\sqrt{\det C_1(t)}},$$

$$\frac{\partial\phi_n(t)}{\partial t_{nn}}=\left[-e_n(t)+\frac{\partial(C_1(t)^{-1}d(t),d(t))}{\partial t_{nn}}\right]\frac{\exp(-\frac{1}{2}[(CT,T)-(C_1(t)^{-1}d(t),d(t))])}{\sqrt{\det C_1(t)}}.$$

When $t_{rk}=t_{rr}$, $n\geqslant r\geqslant k\geqslant 2$, we have by (65)

$$\operatorname{tr}C(t)=\sum_{1\leqslant k<r\leqslant n}c_{kk}t_{rk}^2=\sum_{k=1}^{n-1}c_{kk}\left(\sum_{r=k+1}^{n}t_{rk}^2\right)=\sum_{k=1}^{n-1}c_{kk}\left(\sum_{r=k+1}^{n}t_{rr}^2\right).$$

When, in addition, $e_1(t)=\cdots=e_{n-1}(t)=0$ we get (see (68) and definition (19) of $\hat{\lambda}_k$)

$$\operatorname{tr}C(t)=\sum_{r=1}^{n-1}c_{rr}\left(\sum_{k=r+1}^{n}t_{kk}^2\right)=\sum_{k=2}^{n}\sum_{r=1}^{k-1}c_{rr}t_{kk}^2=\sum_{k=2}^{n}\hat{\lambda}_k t_{kk}^2=\sum_{k=2}^{n}\hat{\lambda}_k\left(\frac{A_k^n(C_n)}{A_n^n(C_n)}\right)^2t_{nn}^2.$$

Finally for general $n$ we have if $e_1(t)=\cdots=e_{n-1}(t)=0$

$$\left|\frac{\partial\phi_n(t)}{\partial t_{nn}}\right|^2=\frac{e_n^2(t)\exp(-(CT,T))}{\det C_1(t)}\overset{(64)}{\geqslant}e_n^2(t)\exp\bigl(-(CT,T)-\operatorname{tr}C(t)\bigr)$$
$$=\left(\frac{M_{12\ldots n}^{12\ldots n}(C_n)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)}\right)^2t_{nn}^2\exp\left(-t_{nn}^2\left(\frac{M_{12\ldots n}^{12\ldots n}(C_n)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)}+\frac{\sum_{k=2}^n\hat{\lambda}_k(A_k^n(C_n))^2}{(A_n^n(C_n))^2}\right)\right).$$

Using (59) we get

$$\Xi^{nn}\overset{(43)}{\geqslant}\max_{t_{nn}\in\mathbb{R}}\left|\frac{\partial\phi_n(t)}{\partial t_{nn}}\right|^2_{e_1(t)=\cdots=e_{n-1}(t)=0}\geqslant\frac{\left(\frac{M_{12\ldots n}^{12\ldots n}(C_n)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)}\right)^2\exp(-1)}{\frac{M_{12\ldots n}^{12\ldots n}(C_n)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)}+\frac{\sum_{k=2}^n\hat{\lambda}_k(A_k^n(C_n))^2}{(A_n^n(C_n))^2}}$$
$$=\frac{(M_{12\ldots n}^{12\ldots n}(C_n))^2\exp(-1)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)M_{12\ldots n}^{12\ldots n}(C_n)+\sum_{k=2}^n\hat{\lambda}_k(A_k^n(C_n))^2}=\Psi^{nn}.$$

Finally for general $(n,q)$, $n\leqslant q$, we have if $e_1(t)=\cdots=e_{n-1}(t)=0$, $t_{qq}=0$,

$$\left|\frac{\partial\phi_{nq}(t)}{\partial t_{qq}}\right|^2=\frac{e_q^2(t)\exp(-(CT,T))}{\det C_1(t)}\overset{(64)}{\geqslant}e_q^2(t)\exp\bigl(-(CT,T)-\operatorname{tr}C(t)\bigr),$$

where $C = C_{n,q}$ and $T$ are defined in Lemma B.1. Moreover, the above conditions gives us the same solutions (68) as before, hence using the decomposition of the minor $M^{12\ldots n-1n}_{12\ldots n-1q}(C_{n,q})$ we have

$$e_q(t) = (C_{n,q}T)_q = \sum_{r=1}^{n} c_{qr}t_{rr} = \sum_{r=1}^{n} c_{qr}\frac{A_r^n(C_n)}{A_n^n(C_n)}t_{nn} = \frac{M^{12\ldots n-1n}_{12\ldots n-1q}(C_{n,q})t_{nn}}{A_n^n(C_n)}.$$

Finally we get if $e_1(t) = \cdots = e_{n-1}(t) = 0$ and $t_{qq} = 0$

$$\begin{aligned}
\Xi^{nq} &\geqslant \max_{t_{nn}\in\mathbb{R}}\left|\frac{\partial\phi_{nq}(t;t_{qq})}{\partial t_{qq}}\right|^2_{t_{qq}=0} \geqslant \max_{t_{nn}\in\mathbb{R}} e_q^2(t)\exp\bigl(-(CT,T) - \operatorname{tr}C(t)\bigr)\\
&= \max_{t_{nn}\in\mathbb{R}}\left(\frac{M^{12\ldots n-1n}_{12\ldots n-1q}(C_{n,q})}{M^{12\ldots n-1}_{12\ldots n-1}(C_n)}\right)^2 t_{nn}^2\exp\left(-t_{nn}^2\left(\frac{M^{12\ldots n}_{12\ldots n}(C_n)}{M^{12\ldots n-1}_{12\ldots n-1}(C_n)} + \frac{\sum_{k=2}^n\hat\lambda_k(A_k^n(C_n))^2}{(A_n^n(C_n))^2}\right)\right)\\
&= \frac{(M^{12\ldots n-1n}_{12\ldots n-1q}(C_{n,q}))^2\exp(-1)}{M^{12\ldots n-1}_{12\ldots n-1}(C_n)M^{12\ldots n}_{12\ldots n}(C_n) + \sum_{k=2}^n\hat\lambda_k(A_k^n(C_n))^2} = \Psi^{nq}. \qquad \square
\end{aligned}$$

## Appendix C. Proof of Lemma 16

**Proof.** Firstly, we prove by induction the inequalities $I_k^k \geqslant 0$ for $k \geqslant 2$. Secondly, we show that inequality $I_k^k \geqslant 0$ and Lemma A.6 imply the inequality $I_m^k \geqslant 0$ for $m \geqslant k$ where (see (24)):

$$I_m^k := f_k A_k^k\bigl(C_m(\hat\lambda)\bigr) - \hat\lambda_k A_k^k\bigl(C_m\bigl(\hat\lambda^{[k]}\bigr)\bigr) \geqslant 0, \quad 2 \leqslant k \leqslant m.$$

We shall show also that $I_m^2 = 0$. In the case $m = 2$ we have

$$I_2^2 = f_2 A_2^2\bigl(C_2(\hat\lambda)\bigr) - \hat\lambda_2 A_2^2\bigl(C_2\bigl(\hat\lambda^{[2]}\bigr)\bigr) = 0$$

since $f_2 = \hat\lambda_2 = c_{11}$ by (19), (20) and (49), and

$$A_2^2\bigl(C_2(\hat\lambda)\bigr) = A_2^2\bigl(C_2\bigl(\hat\lambda^{[2]}\bigr)\bigr) = A_2^2(C_2) = c_{11},$$

where

$$C_2(\hat\lambda) = \begin{pmatrix} c_{11} & c_{12}\\ c_{12} & c_{11}+c_{22}\end{pmatrix}, \qquad C_2\bigl(\hat\lambda^{[2]}\bigr) = C_2 = \begin{pmatrix} c_{11} & c_{12}\\ c_{12} & c_{22}\end{pmatrix}.$$

In the case $m = 3$ we prove the following inequalities:

$$I_3^2 := f_2 A_2^2\bigl(C_3(\hat\lambda)\bigr) - \hat\lambda_2 A_2^2\bigl(C_3\bigl(\hat\lambda^{[2]}\bigr)\bigr) \geqslant 0, \tag{72}$$

$$I_3^3 := f_3 A_3^3\bigl(C_3(\hat\lambda)\bigr) - \hat\lambda_3 A_3^3\bigl(C_3\bigl(\hat\lambda^{[3]}\bigr)\bigr) \geqslant 0. \tag{73}$$

Since (see (21))

$$C_3(\hat\lambda)=\begin{pmatrix}c_{11}&c_{12}&c_{13}\\c_{12}&c_{11}+c_{22}&c_{23}\\c_{13}&c_{23}&c_{11}+c_{22}+c_{33}\end{pmatrix},\qquad C_3\big(\hat\lambda^{[2]}\big)=\begin{pmatrix}c_{11}&c_{12}&c_{13}\\c_{12}&c_{22}&c_{23}\\c_{13}&c_{23}&c_{11}+c_{22}+c_{33}\end{pmatrix},$$

and $C_3(\hat\lambda^{[3]})=C_3$ we have by (26)

$$A_2^2\big(C_3(\hat\lambda)\big)=A_2^2\big(C_3\big(\hat\lambda^{[2]}\big)\big)=A_2^2(C_3)+\hat\lambda_3A_{23}^{23}(C_3),\qquad A_3^3\big(C_3\big(\hat\lambda^{[3]}\big)\big)=A_3^3(C_3).$$

The latter equalities give us $I_3^2=0$. This proves (72). Indeed we have

$$I_3^2=\hat\lambda_2\big(A_2^2(C_3)+\hat\lambda_3A_{23}^{23}(C_3)\big)-\hat\lambda_2\big(A_2^2(C_3)+\hat\lambda_3A_{23}^{23}(C_3)\big)\equiv 0.$$

Since $f_2=\hat\lambda_2=c_{11}$ and $\hat\lambda_1=0$ we have $A_2^2(C_m(\hat\lambda))=A_2^2(C_m(\hat\lambda^{[2]}))$ hence

$$I_m^2:=f_2A_2^2\big(C_m(\hat\lambda)\big)-\hat\lambda_2A_2^2\big(C_m\big(\hat\lambda^{[2]}\big)\big)\equiv 0,\quad 2\leqslant m. \tag{74}$$

**Remark C.1.** In what follows we take $\lambda=(\lambda_r)_1^k\in\mathbb{R}^k$ with $\lambda_1=0$.

To prove (73) for $k=3$ we use the identity for $\lambda=(0,\lambda_2)\in\mathbb{R}^2$, $\hat\lambda=(0,c_{11})$,

$$A_3^3\big(C_3(\hat\lambda)\big)=M_{12}^{12}\big(C_3(\hat\lambda)\big)=M_{12}^{12}(C_3)+c_{11}^2=M_{12}^{12}(C_3)+\hat\lambda_2c_{11},$$
$$A_3^3\big(C_3(\lambda)\big)=M_{12}^{12}\big(C_3(\lambda)\big)=M_{12}^{12}(C_3)+\lambda_2c_{11},\qquad \frac{\partial M_{12}^{12}(C_3(\lambda))}{\partial\lambda_2}=c_{11}.$$

We have

$$\begin{aligned}I_3^3&:=f_3A_3^3\big(C_3(\hat\lambda)\big)-\hat\lambda_3A_3^3\big(C_3\big(\hat\lambda^{[3]}\big)\big)\\&=\left(c_{11}+\frac{c_{12}^2}{c_{11}}+\frac{(M_{12}^{12}(C_3))^2}{c_{11}(M_{12}^{12}(C_3)+c_{11}^2)}\right)\big(M_{12}^{12}(C_3)+c_{11}^2\big)-(c_{11}+c_{22})M_{12}^{12}(C_3)\\&=\left(c_{11}+\frac{c_{12}^2}{c_{11}}+\frac{(M_{12}^{12}(C_3))^2}{c_{11}M_{12}^{12}(C_3(\hat\lambda))}\right)M_{12}^{12}\big(C_3(\hat\lambda)\big)-(c_{11}+c_{22})M_{12}^{12}(C_3),\end{aligned}$$

we use here the definition of $f_q=e\sum_{1\leqslant r\leqslant p<q}\Psi^{rp}$ and $\Psi^{pq}$ (see (20), (48)–(50)),

$$f_3=e\big(\Psi^{11}+\Psi^{12}+\Psi^{22}\big)=c_{11}+\frac{c_{12}^2}{c_{11}}+\frac{(M_{12}^{12}(C_3))^2}{c_{11}(M_{12}^{12}(C_3)+c_{11}^2)}.$$

We define the function $I_3^3(\lambda)$ for $\lambda=(0,\lambda_2)$ by

$$\begin{aligned}I_3^3(\lambda)&:=\left(c_{11}+\frac{c_{12}^2}{c_{11}}+\frac{(M_{12}^{12}(C_3))^2}{c_{11}M_{12}^{12}(C_3(\lambda))}\right)M_{12}^{12}\big(C_3(\lambda)\big)-(c_{11}+c_{22})M_{12}^{12}(C_3)\\&=\left(c_{11}+\frac{c_{12}^2}{c_{11}}\right)\big(M_{12}^{12}(C_3)+\lambda_2c_{11}\big)+\frac{(M_{12}^{12}(C_3))^2}{c_{11}}-(c_{11}+c_{22})M_{12}^{12}(C_3).\end{aligned}$$

Since $I_3^3 = I_3^3(\hat{\lambda})$ it is sufficient to prove that $I_3^3(\lambda) > 0$ for $\lambda_2 > 0$. We show that

$$I_3^3(0) = 0 \quad \text{and} \quad \frac{\partial I_3^3(\lambda)}{\partial \lambda_2} > 0.$$

Indeed we have $M_{12}^{12}(C_3(0)) = M_{12}^{12}(C_3)$ hence

$$\begin{aligned} I_3^3(0) &= \left(c_{11} + \frac{c_{12}^2}{c_{11}} + \frac{M_{12}^{12}(C_3)}{c_{11}}\right) M_{12}^{12}(C_3) - (c_{11} + c_{22}) M_{12}^{12}(C_3) \\ &= M_{12}^{12}(C_3)\left(\frac{c_{12}^2 + M_{12}^{12}(C_3)}{c_{11}} - c_{22}\right) = 0 \quad \text{and} \\ &\frac{\partial I_3^3(\lambda)}{\partial \lambda_2} = \left(c_{11} + \frac{c_{12}^2}{c_{11}}\right) c_{11} > 0. \end{aligned}$$

Finally $I_3^3(\lambda) > 0$ for $\lambda_2 > 0$ so $I_3^3 = I_3^3(\hat{\lambda}) = I_3^3(0, c_{11}) > 0$ and (73) is proved. To prove that $I_k^k \geqslant 0$ let us denote $f^q = e \sum_{r=1}^{q-1} \Psi^{rq-1}$. Using (20) we have

$$f_q = e \sum_{1 \leqslant r \leqslant p < q} \Psi^{rp} = e \sum_{1 \leqslant r \leqslant p < q-1} \Psi^{rp} + e \sum_{r=1}^{q-1} \Psi^{rq-1} = f_{q-1} + f^q, \quad f_1 := 0, \tag{75}$$

for $2 \leqslant q \leqslant m$. We prove by induction that

$$I_k^k = f_k A_k^k\big(C_k(\hat{\lambda})\big) - \hat{\lambda}_k A_k^k(C_k) \geqslant 0, \quad 2 \leqslant k. \tag{76}$$

For $k = 2$ and $k = 3$ it is proved. Let us suppose that it holds for $k$. To find the general formula for $I_k^k(\lambda)$ with $I_k^k \geqslant I_k^k(\hat{\lambda})$ we consider the cases $m = 4$.

$$\begin{aligned} I_4^4 \quad &= \quad f_4 A_4^4\big(C_4(\hat{\lambda})\big) - \hat{\lambda}_4 A_4^4(C_4) = \big(f_3 + f^4\big) A_4^4\big(C_4(\lambda)\big) - \hat{\lambda}_4 A_4^4(C_4)\big|_{\lambda=\hat{\lambda}} \\ &\overset{(73)}{\geqslant} \left(\frac{\hat{\lambda}_3 A_{34}^{34}(C_4)}{A_{34}^{34}(C_4(\lambda))} + f^4\right) A_4^4\big(C_4(\lambda)\big) - \hat{\lambda}_4 A_4^4(C_4)\bigg|_{\lambda=\hat{\lambda}} \\ &\overset{(49)\text{–}(51)}{=} \left(\frac{(c_{11} + c_{22}) M_{12}^{12}(C_4)}{M_{12}^{12}(C_4(\lambda))} + \frac{c_{13}^2}{c_{11}} + \frac{(M_{13}^{12}(C_4))^2}{c_{11} M_{12}^{12}(C_4(\lambda))}\right. \\ &\quad \left. + \frac{(M_{123}^{123}(C_4))^2}{M_{12}^{12}(C_4) M_{123}^{123}(C_4) + c_{11}(M_{13}^{12}(C_4))^2 + (c_{11} + c_{22})(M_{12}^{12}(C_4))^2}\right) \\ &\quad \times M_{123}^{123}\big(C_4(\lambda)\big) - (c_{11} + c_{22} + c_{33}) M_{123}^{123}(C_4)\big|_{\lambda=\hat{\lambda}} \\ &\overset{(54)}{>} \left(\frac{(c_{11} + c_{22}) M_{12}^{12}(C_4)}{M_{12}^{12}(C_4(\lambda))} + \frac{c_{13}^2}{c_{11}} + \frac{(M_{13}^{12}(C_4))^2}{c_{11} M_{12}^{12}(C_4(\lambda))} + \frac{(M_{123}^{123}(C_4))^2}{M_{12}^{12}(C_4) M_{123}^{123}(C_4(\lambda))}\right) \\ &\quad \times M_{123}^{123}\big(C_4(\lambda)\big) - (c_{11} + c_{22} + c_{33}) M_{123}^{123}(C_4)\big|_{\lambda=\hat{\lambda}}. \end{aligned}$$

So we have $I_4^4 > I_4^4(\lambda)|_{\lambda=\hat{\lambda}}$ where $I_4^4(\lambda)$ is defined by the formula

$$I_4^4(\lambda) := \left(\frac{(c_{11}+c_{22})M_{12}^{12}(C_4)}{M_{12}^{12}(C_4(\lambda))} + \frac{c_{13}^2}{c_{11}} + \frac{(M_{13}^{12}(C_4))^2}{c_{11}M_{12}^{12}(C_4(\lambda))} + \frac{(M_{123}^{123}(C_4))^2}{M_{12}^{12}(C_4)M_{123}^{123}(C_4(\lambda))}\right)$$
$$\times M_{123}^{123}\big(C_4(\lambda)\big) - (c_{11}+c_{22}+c_{33})M_{123}^{123}(C_4)$$
$$= \left(a_1 + \frac{a_2}{M_{12}^{12}(C_4(\lambda))}\right)M_{123}^{123}\big(C_4(\lambda)\big) + b_1 = a_1 M_{123}^{123}\big(C_4(\lambda)\big) + a_2\frac{M_{123}^{123}(C_4(\lambda))}{M_{12}^{12}(C_4(\lambda))} + b_1,$$

where

$$a_1 = \frac{c_{13}^2}{c_{11}} > 0, \qquad a_2 = (c_{11}+c_{22})M_{12}^{12}(C_4) + \frac{(M_{13}^{12}(C_4))^2}{c_{11}} > 0,$$
$$b_1 = \frac{(M_{123}^{123}(C_4))^2}{M_{12}^{12}(C_4)} - (c_{11}+c_{22}+c_{33})M_{123}^{123}(C_4).$$

We prove that $I_4^4(\lambda) \geqslant 0$ for $\lambda = (0, \lambda_2, \lambda_3)$, when $\lambda_2 \geqslant 0$, $\lambda_3 \geqslant 0$. It then gives us $I_4^4 \geqslant I_4^4(\hat{\lambda}) \geqslant 0$. We have (see below the proof of $I_k^k(0) = 0$, $k \geqslant 3$)

$$I_4^4(0) = \left(\frac{c_{13}^2}{c_{11}} + \frac{(M_{13}^{12}(C_4))^2}{c_{11}M_{12}^{12}(C_4)} + \frac{M_{123}^{123}(C_4)}{M_{12}^{12}(C_4)} - c_{33}\right)M_{123}^{123}(C_4) = 0.$$

Moreover, by inequality (37) of Lemma A.7 we have for $\lambda_2 \geqslant 0$, $\lambda_3 \geqslant 0$

$$\frac{\partial I_4^4(\lambda)}{\partial \lambda_2} = a_1\frac{\partial M_{123}^{123}(C_4(\lambda))}{\partial \lambda_2} + a_2\frac{\partial}{\partial \lambda_2}\frac{M_{123}^{123}(C_4(\lambda))}{M_{12}^{12}(C_4(\lambda))} \geqslant 0,$$
$$\frac{\partial I_4^4(\lambda)}{\partial \lambda_3} = \left(a_1 + \frac{a_2}{M_{12}^{12}(C_4(\lambda))}\right)\frac{\partial M_{123}^{123}(C_4(\lambda))}{\partial \lambda_3} \geqslant 0.$$

Let us consider the function

$$i_4^4(t) = I_4^4(t\hat{\lambda}) = I_4^4(0, t\hat{\lambda}_2, t\hat{\lambda}_3), \quad t \in \mathbb{R}.$$

We have

$$i_4^4(0) = I_4^4(0) = 0 \quad \text{and} \quad \frac{di_4^4(t)}{dt} = \frac{\partial I_4^4(\lambda)}{\partial \lambda_2}\hat{\lambda}_2 + \frac{\partial I_4^4(\lambda)}{\partial \lambda_3}\hat{\lambda}_3 \geqslant 0$$

hence $i_4^4(t) \geqslant 0$ by the previous inequalities for $t > 0$. So

$$I_4^4 > I_4^4(0, \hat{\lambda}_2, \hat{\lambda}_3) = i_4^4(t)\big|_{t=1} \geqslant 0.$$

To prove that $I_k^k(\hat{\lambda}) \geqslant 0$ we show that

$$I_k^k(0) = 0, \quad 2 \leqslant k \quad \text{and} \quad \frac{\partial I_k^k(\lambda)}{\partial \lambda_p} \geqslant 0, \quad 2 \leqslant p < k. \tag{77}$$

To define the function $I_{k+1}^{k+1}(\lambda)$ with $I_{k+1}^{k+1} \geqslant I_{k+1}^{k+1}(\hat{\lambda})$ we have

$$\begin{aligned}
I_{k+1}^{k+1} &= f_{k+1} A_{k+1}^{k+1}\big(C_{k+1}(\hat{\lambda})\big) - \hat{\lambda}_{k+1} A_{k+1}^{k+1}(C_{k+1}) \\
&\overset{(75)}{=} \big(f_k + f^{k+1}\big) A_{k+1}^{k+1}\big(C_{k+1}(\lambda)\big) - \hat{\lambda}_{k+1} A_{k+1}^{k+1}(C_{k+1})\big|_{\lambda=\hat{\lambda}} \\
&\overset{(76)}{\geqslant} \left(\frac{\hat{\lambda}_k A_{kk+1}^{kk+1}(C_{k+1})}{A_{kk+1}^{kk+1}(C_{k+1}(\lambda))} + e\sum_{r=1}^{k} \Psi^{rk}\right) A_{k+1}^{k+1}\big(C_{k+1}(\lambda)\big) - \hat{\lambda}_{k+1} A_{k+1}^{k+1}(C_{k+1})\Bigg|_{\lambda=\hat{\lambda}} \\
&\overset{(54)}{\geqslant} \left(\frac{\hat{\lambda}_k A_{kk+1}^{kk+1}(C_{k+1})}{A_{kk+1}^{kk+1}(C_{k+1}(\lambda))} + e\sum_{r=1}^{k} \Psi_0^{rk}\right) A_{k+1}^{k+1}\big(C_{k+1}(\lambda)\big) - \hat{\lambda}_{k+1} A_{k+1}^{k+1}(C_{k+1})\Bigg|_{\lambda=\hat{\lambda}} \\
&:= I_{k+1}^{k+1}(\hat{\lambda}),
\end{aligned}$$

where the function $I_{k+1}^{k+1}(\lambda)$ is defined by (see definition (54) of $\Psi_0^{pq}$):

$$\begin{aligned}
I_{k+1}^{k+1}(\lambda) &= \left(\frac{\hat{\lambda}_k M_{12\ldots k-1}^{12\ldots k-1}(C_{k+1})}{M_{12\ldots k-1}^{12\ldots k-1}(C_{k+1}(\lambda))} + \frac{c_{1k}^2}{c_{11}} + \sum_{r=2}^{k} \frac{(M_{12\ldots r-1k}^{12\ldots r-1r}(C_{k+1}))^2}{M_{12\ldots r-1}^{12\ldots r-1}(C_{k+1}) M_{12\ldots r}^{12\ldots r}(C_{k+1}(\lambda))}\right) \\
&\quad \times M_{12\ldots k}^{12\ldots k}\big(C_{k+1}(\lambda)\big) - \hat{\lambda}_{k+1} M_{12\ldots k}^{12\ldots k}(C_{k+1}) \\
&= \left(\frac{\hat{\lambda}_k M_{12\ldots k-1}^{12\ldots k-1}(C_{k+1})}{M_{12\ldots k-1}^{12\ldots k-1}(C_{k+1}(\lambda))} + \frac{c_{1k}^2}{c_{11}} + \sum_{r=2}^{k-1} \frac{(M_{12\ldots r-1k}^{12\ldots r-1r}(C_{k+1}))^2}{M_{12\ldots r-1}^{12\ldots r-1}(C_{k+1}) M_{12\ldots r}^{12\ldots r}(C_{k+1}(\lambda))}\right) \\
&\quad \times M_{12\ldots k}^{12\ldots k}\big(C_{k+1}(\lambda)\big) + \frac{(M_{12\ldots k}^{12\ldots k}(C_{k+1}))^2}{M_{12\ldots k-1}^{12\ldots k-1}(C_{k+1})} - \hat{\lambda}_{k+1} M_{12\ldots k}^{12\ldots k}(C_{k+1}).
\end{aligned}$$

Finally we have the following expression for $I_{k+1}^{k+1}(\lambda)$ with corresponding positive constants $a_r$, $2 \leqslant r \leqslant k-1$ (depending on $k$) and $b_1 \in \mathbb{R}$:

$$\begin{aligned}
I_{k+1}^{k+1}(\lambda) &= \left(a_1 + \sum_{r=2}^{k-1} \frac{a_r}{M_{12\ldots r}^{12\ldots r}(C_{k+1}(\lambda))}\right) M_{12\ldots k}^{12\ldots k}\big(C_{k+1}(\lambda)\big) + b_1 \\
&= \left(a_1 + \sum_{r=2}^{k-1} \frac{a_r}{G_r(\lambda)}\right) G_k(\lambda) + b_1.
\end{aligned}$$

By (37) of Lemma A.7 we conclude that for $\lambda_r \geqslant 0$, $2 \leqslant r \leqslant k$, holds

$$\frac{\partial I_{k+1}^{k+1}(\lambda)}{\partial \lambda_k} = \left(a_1 + \sum_{r=2}^{k-1} \frac{a_r}{G_r(\lambda)}\right) \frac{\partial G_k(\lambda)}{\partial \lambda_k} \geqslant 0,$$

$$\frac{\partial I_{k+1}^{k+1}(\lambda)}{\partial \lambda_p} = a_1 \frac{\partial G_k(\lambda)}{\partial \lambda_p} + \sum_{r=2}^{k-1} a_r \frac{\partial}{\partial \lambda_p} \frac{G_k(\lambda)}{G_r(\lambda)} \geqslant 0, \quad 2 \leqslant p \leqslant k. \tag{78}$$

**Remark C.2.** In fact $\partial I_{k+1}^{k+1}(\lambda)/\partial\lambda_p > 0$, $2 \leqslant p \leqslant k$, for $\lambda = (\lambda_r)_{r=1}^{k+1} \in \mathbb{R}^{k+1}$, $\lambda_r \geqslant 0$, $1 \leqslant r \leqslant k+1$, since by (38) we have $\partial G_k(\lambda)/\partial\lambda_p = A_p^p(C(\lambda^{]p[})) > 0$.

Let us suppose that $I_k^k(0) = 0$, i.e.

$$0 = I_k^k(0) = M_{12\ldots k-1}^{12\ldots k-1}\left(\frac{c_{1k-1}^2}{c_{11}} + \frac{(M_{1k-1}^{12})^2}{c_{11}M_{12}^{12}} + \frac{(M_{12k-1}^{123})^2}{M_{12}^{12}M_{123}^{123}} + \cdots \right.$$
$$\left. + \frac{(M_{12\ldots k-3k-1}^{12\ldots k-3k-2})^2}{M_{12\ldots k-3}^{12\ldots k-3}M_{12\ldots k-2}^{12\ldots k-2}} + \frac{M_{12\ldots k-1}^{12\ldots k-1}}{M_{12\ldots k-2}^{12\ldots k-2}} - c_{k-1k-1}\right).$$

For $k=3$, $k=4$ and $k=5$ we have

$$I_3^3(0) = M_{12}^{12}\left(\frac{c_{12}^2}{c_{11}} + \frac{M_{12}^{12}}{c_{11}} - c_{22}\right) = 0,$$
$$I_4^4(0) = M_{123}^{123}\left(\frac{c_{13}^2}{c_{11}} + \frac{(M_{13}^{12})^2}{c_{11}M_{12}^{12}} + \frac{M_{123}^{123}}{M_{12}^{12}} - c_{33}\right),$$
$$I_5^5(0) = M_{1234}^{1234}\left(\frac{c_{14}^2}{c_{11}} + \frac{(M_{14}^{12})^2}{c_{11}M_{12}^{12}} + \frac{(M_{124}^{123})^2}{M_{12}^{12}M_{123}^{123}} + \frac{M_{1234}^{1234}}{M_{123}^{123}} - c_{44}\right).$$

We prove that $I_{k+1}^{k+1}(0) = 0$. Indeed, we get

$$I_{k+1}^{k+1}(0) = M_{12\ldots k}^{12\ldots k}\left(\frac{c_{1k}^2}{c_{11}} + \frac{(M_{1k}^{12})^2}{c_{11}M_{12}^{12}} + \frac{(M_{12k}^{123})^2}{M_{12}^{12}M_{123}^{123}} + \cdots \right.$$
$$\left. + \frac{(M_{12\ldots k-2k}^{12\ldots k-2k-1})^2}{M_{12\ldots k-2}^{12\ldots k-2}M_{12\ldots k-1}^{12\ldots k-1}} + \frac{M_{12\ldots k}^{12\ldots k}}{M_{12\ldots k-1}^{12\ldots k-1}} - c_{kk}\right).$$

Since by Corollary A.5 we have

$$\begin{vmatrix} A_{k-1}^{k-1}(C_k) & A_k^{k-1}(C_k) \\ A_{k-1}^k(C_k) & A_k^k(C_k) \end{vmatrix} = A_\emptyset^\emptyset(C_k)A_{k-1k}^{k-1k}(C_k) \quad \text{or}$$
$$\begin{vmatrix} A_{k-1}^{k-1}(C_k) & A_{k-1k}^{k-1k}(C_k) \\ A_\emptyset^\emptyset(C_k) & A_k^k(C_k) \end{vmatrix} = \big(A_{k-1}^k(C_k)\big)^2,$$

we conclude that

$$\begin{vmatrix} M_{12\ldots k-1}^{12\ldots k-1}(C_k) & M_{12\ldots k-2}^{12\ldots k-2}(C_k) \\ M_{12\ldots k}^{12\ldots k}(C_k) & M_{12\ldots k-2k}^{12\ldots k-2k}(C_k) \end{vmatrix} = \big(M_{12\ldots k-2k}^{12\ldots k-2k-1}(C_k)\big)^2.$$

Hence

$$\frac{(M_{12\ldots k-2k}^{12\ldots k-2k-1}(C_k))^2}{M_{12\ldots k-2}^{12\ldots k-2}(C_k)M_{12\ldots k-1}^{12\ldots k-1}(C_k)} + \frac{M_{12\ldots k}^{12\ldots k}(C_k)}{M_{12\ldots k-1}^{12\ldots k-1}(C_k)} = \frac{M_{12\ldots k-2k}^{12\ldots k-2k}(C_k)}{M_{12\ldots k-2}^{12\ldots k-2}(C_k)},$$

and

$$I_{k+1}^{k+1}(0)=M_{12\ldots k}^{12\ldots k}\left(\frac{c_{1k}^2}{c_{11}}+\frac{(M_{1k}^{12})^2}{c_{11}M_{12}^{12}}+\frac{(M_{12k}^{123})^2}{M_{12}^{12}M_{123}^{123}}+\cdots\right.$$
$$\left.+\frac{(M_{12\ldots k-3k}^{12\ldots k-3k-2})^2}{M_{12\ldots k-3}^{12\ldots k-3}M_{12\ldots k-2}^{12\ldots k-2}}+\frac{M_{12\ldots k-2k}^{12\ldots k-2k}}{M_{12\ldots k-2}^{12\ldots k-2}}-c_{kk}\right).$$

If we change $k$ with $k-1$ in the last expression we obtain the right-hand part (up to a positive factor) of the expression for $I_k^k(0)$.

Finally we have proved (77) for $I_{k+1}^{k+1}(\lambda)$. Let us consider the function

$$i_{k+1}^{k+1}(t)=I_{k+1}^{k+1}(t\hat{\lambda}),\quad t\in\mathbb{R}.$$

We have

$$i_{k+1}^{k+1}(0)=I_{k+1}^{k+1}(0)=0\quad\text{and}\quad\frac{di_{k+1}^{k+1}(t)}{dt}=\sum_{p=2}^{k}\frac{\partial I_{k+1}^{k+1}(\lambda)}{\partial\lambda_p}\hat{\lambda}_p>0$$

by (78) and Remark C.2. So

$$I_k^k>I_k^k(\hat{\lambda})=i_k^k(t)\big|_{t=1}\geqslant 0.$$

We recall (see (32)) that for $\lambda=(\lambda_1,\ldots,\lambda_m)\in\mathbb{C}^m$ and $1\leqslant k\leqslant m$ we denote

$$\lambda^{[k]}=(0,\ldots,0,\lambda_{k+1},\ldots,\lambda_m),\qquad\lambda^{\{k\}}=(\lambda_1,\ldots,\lambda_k,0,\ldots,0).$$

Using (27)

$$G_m(\lambda)=A_\emptyset^\emptyset\big(C_m(\lambda)\big)=\sum_{\emptyset\subseteq\delta\subseteq\{1,2,\ldots,m\}}\lambda_\delta A_\delta^\delta(C),$$

we get

$$A_k^k\big(C_m(\lambda)\big)=\sum_{\emptyset\subseteq\delta\subseteq\{1,2,\ldots,k-1,k+1,\ldots m\}}\lambda_\delta A_{k\cup\delta}^{k\cup\delta}(C_m).\tag{79}$$

If we put $C_m(\lambda^{[k]})=C_m+\sum_{r=k+1}^m\lambda_r E_{rr}$ in (79) we get

$$A_k^k\big(C_m\big(\lambda^{[k]}\big)\big)=\sum_{\emptyset\subseteq\delta\subseteq\{k+1,k+2,\ldots,m\}}\lambda_\delta A_{k\cup\delta}^{k\cup\delta}(C_m).\tag{80}$$

Similarly, if we put $C_m(\lambda)=C_m(\lambda^{\{k\}})+\sum_{r=k+1}^m\lambda_r E_{rr}$ we get

$$A_k^k\big(C_m(\lambda)\big)=\sum_{\emptyset\subseteq\delta\subseteq\{k+1,k+2,\ldots,m\}}\lambda_\delta A_{k\cup\delta}^{k\cup\delta}\big(C_m\big(\lambda^{\{k\}}\big)\big).\tag{81}$$

Using (76) we have

$$f_k \geqslant \hat{\lambda}_k A_k^k(C_k)\big(A_k^k\big(C_k(\hat{\lambda})\big)\big)^{-1} = \hat{\lambda}_k A_{kk+1\ldots m}^{kk+1\ldots m}(C_m)\big(A_{kk+1\ldots m}^{kk+1\ldots m}\big(C_m(\hat{\lambda})\big)\big)^{-1}$$

hence $I_m^k = f_k A_k^k(C_m(\hat{\lambda})) - \hat{\lambda}_k A_k^k(C_m(\lambda^{[k]})) \geqslant I_m^k(\hat{\lambda})$, where the function $I_m^k(\hat{\lambda})$ is defined by

$$\begin{aligned}
I_m^k(\hat{\lambda}) \quad &:= \quad \hat{\lambda}_k\big(A_{kk+1\ldots m}^{kk+1\ldots m}\big(C_m(\hat{\lambda})\big)\big)^{-1} A_{kk+1\ldots m}^{kk+1\ldots m}(C_m) A_k^k\big(C_m(\hat{\lambda})\big) - \hat{\lambda}_k A_k^k\big(C_m\big(\hat{\lambda}^{[k]}\big)\big) \\
&= \quad \hat{\lambda}_k\big(A_{kk+1\ldots m}^{kk+1\ldots m}\big(C_m(\hat{\lambda})\big)\big)^{-1} \begin{vmatrix} A_{kk+1\ldots m}^{kk+1\ldots m}(C_m) & A_k^k(C_m(\hat{\lambda}^{[k]})) \\ A_{kk+1\ldots m}^{kk+1\ldots m}(C_m(\hat{\lambda})) & A_k^k(C_m(\hat{\lambda})) \end{vmatrix} \\
&\overset{(80),(81)}{=} \hat{\lambda}_k\big(A_{kk+1\ldots m}^{kk+1\ldots m}\big(C_m(\hat{\lambda})\big)\big)^{-1} \\
&\qquad \times \sum_{\emptyset\subseteq\delta\subseteq\{k+1,k+2,\ldots,m\}} \hat{\lambda}_\delta \begin{vmatrix} A_{kk+1\ldots m}^{kk+1\ldots m}(C_m) & A_{k\cup\delta}^{k\cup\delta}(C_m) \\ A_{kk+1\ldots m}^{kk+1\ldots m}(C_m(\hat{\lambda})) & A_{k\cup\delta}^{k\cup\delta}(C_m(\hat{\lambda}^{\{k\}})) \end{vmatrix}.
\end{aligned}$$

Using (26) or (27) we conclude for $\lambda = (0, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^m$

$$A_{kk+1\ldots m}^{kk+1\ldots m}\big(C_m(\lambda)\big) = \sum_{\emptyset\subseteq\gamma\subseteq\{2,3,\ldots,k-1\}} \lambda_\gamma A_{\gamma\cup\{k,k+1,\ldots m\}}^{\gamma\cup\{k,k+1,\ldots m\}}(C_m),$$

$$A_{k\cup\delta}^{k\cup\delta}\big(C_m\big(\lambda^{\{k\}}\big)\big) = \sum_{\emptyset\subseteq\gamma\subseteq\{2,3,\ldots,k-1\}} \lambda_\gamma A_{\gamma\cup\{k\}\cup\delta}^{\gamma\cup\{k\}\cup\delta}(C_m).$$

Finally we obtain

$$\begin{aligned}
I_m^k(\hat{\lambda}) = \hat{\lambda}_k\big(A_{kk+1\ldots m}^{kk+1\ldots m}\big(C_m(\hat{\lambda})\big)\big)^{-1} &\sum_{\emptyset\subseteq\delta\subseteq\{k+1,k+2,\ldots,m\}} \hat{\lambda}_\delta \\
\times \sum_{\emptyset\subseteq\gamma\subseteq\{2,3,\ldots,k-1\}} \hat{\lambda}_\gamma &\begin{vmatrix} A_{kk+1\ldots m}^{kk+1\ldots m}(C_m) & A_{\gamma\cup\{k,k+1,\ldots m\}}^{\gamma\cup\{k,k+1,\ldots m\}}(C_m) \\ A_{k\cup\delta}^{k\cup\delta}(C_m) & A_{\gamma\cup\{k\}\cup\delta}^{\gamma\cup\{k\}\cup\delta}(C_m) \end{vmatrix} \geqslant 0
\end{aligned}$$

due to the Hadamard–Fisher's inequality (Lemma A.6), for $\alpha = \{k, k+1, \ldots, m\}$ and $\beta = \gamma \cup \{k\} \cup \delta$. This completes the proof of Lemma 16. □

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