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Inversion Quasi-Invariant Gaussian Measures on the Group of Upper-Triangular Matrices of Infinite Order

A. V. Kosyak UDC 519.46

The following commutation theorem holds for locally compact groups [1].

The commutant of the right regular representation ρ of a locally compact group G is generated by the operators of the left regular representation λ. Moreover, there exists an intertwining operator J such that $J\rho_t J = \lambda_t$, $t \in G$. It is given by $(Jf)(x) = (dh(x^{-1})/dh(x))^{1/2}\overline{f(x^{-1})}$, where dh is a left Haar measure.

Apparently, an analog of the regular representation for infinite-dimensional groups (current groups) appeared for the first time in [2, 3]. In [4], the commutation theorem was proved for an analog of the regular representation of the current group in the case of the Wiener measure. An analog of the regular representation for an arbitrary infinite-dimensional topological group G was defined with the use of G-quasi-invariant measures μ on an appropriate completion $\widetilde{G}$ of G in [5, 6].

Consider the group $B^\infty = \{x = I + x' = I + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} x_{kn}E_{kn}\}$ of real upper-triangular matrices of infinite order and the subgroup $B_0^\infty = \{I + x' \in B^\infty \mid x' \text{ is finite}\}$ of "finite" matrices. We define a Gaussian measure μ_b on B^∞ by setting

$$d\mu_b(x) = \bigotimes_{k<n} (b_{kn}/\pi)^{1/2}\exp(-b_{kn}x_{kn}^2)\,dx_{kn} = \bigotimes_{k<n} d\mu_{b_{kn}}(x_{kn}),$$

where $b = (b_{kn})_{k<n}$ is a double sequence of positive numbers. Let $R_t(x) = xt$ be the right action and $L_s(x) = s^{-1}x$ the left action of B^∞ on itself. By $\Phi(x) = x^{-1}$ we denote the inversion on B^∞. The following lemmas were proved in [5, 7].

Lemma 1. $\mu_b^{R_t} \sim \mu_b$ $\forall t \in B_0^\infty$.

Lemma 2. $\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^\infty \iff S_{kn}^L(b) = \sum_{m=n+1}^{\infty} b_{km}b_{nm}^{-1} < \infty$ $\forall k < n$.

We shall prove the following theorem.

Theorem 1. *If $E(b) = \frac{1}{8}\sum_{k<n} S_{kn}^L(b)b_{kn}^{-1} < \infty$, then $\mu_b^\Phi \sim \mu_b$. In this case the right and left regular representations T^{R,μ_b} and T^{L,μ_b} of B_0^∞ are well defined, and the commutation theorem holds. Moreover, the operator J_{μ_b} given by $(J_{\mu_b}f)(x) = (d\mu_b(x^{-1})/d\mu_b(x))^{1/2}\overline{f(x^{-1})}$ is an intertwining operator: $T_t^{L,\mu_b} = J_{\mu_b}T_t^{R,\mu_b}J_{\mu_b}$, $t \in B_0^\infty$.*

Remark. Apparently, the condition $E(b) < \infty$ is also necessary for the equivalence $\mu_b^\Phi \sim \mu_b$.

Sketch of proof. We define subgroups B_n, $n = 2, 3, \ldots$, of B^∞ by the formula $B_n = \{x \in B^\infty \mid x = I + \sum_{r<k,\, r\leqslant n} x_{rk}E_{rk}\}$. Let μ_n be the projection of μ_b on B_n. Then $d\mu_n(x) = \bigotimes_{r<k,\, r\leqslant n} d\mu_{b_{rk}}(x_{rk})$. We replace Φ by $\widetilde{\Phi} = \Phi \circ \theta$, where $\theta(I + x') = I - x'$, $I + x' \in B^\infty$. Since the measures $\mu_{b_{kn}}$ are centered, it follows that $\mu_b^{\widetilde{\Phi}} \sim \mu_b \iff \mu_b^\Phi \sim \mu_b$. We define one-parameter transformation groups $\widetilde{\Phi}^t$, $\widetilde{\Phi}_n^t$, and $\tilde{\phi}_n^t$, $t \in \mathbb{R}$, on B^∞ by the formulas

$$\widetilde{\Phi}^t(I + x') = I + (I - tx')^{-1}x', \qquad \widetilde{\Phi}_n^t(I + x') = I + (I - tx'e_n)^{-1}x',$$
$$\tilde{\phi}_n^t(I + x') = I + (I - tx'e(n))^{-1}x', \tag{1}$$

where $e_n = \mathrm{diag}(1, \ldots, 1, 0, \ldots)$ and $e(n) = \mathrm{diag}(0, \ldots, 0, 1, 0, \ldots)$. Then $\widetilde{\Phi} = \widetilde{\Phi}^1$, $\widetilde{\Phi}^t = \cdots \circ \tilde{\phi}_{n+1}^t \circ \tilde{\phi}_n^t \circ \tilde{\phi}_{n-1}^t \circ \cdots \circ \tilde{\phi}_2^t$, and $\widetilde{\Phi}_n^t = \tilde{\phi}_n^t \circ \tilde{\phi}_{n-1}^t \circ \cdots \circ \tilde{\phi}_2^t$. Using the groups $\tilde{\phi}_n^t$, we construct one-parameter unitary

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groups $U_n(t)$ in $L_2(B^\infty, d\mu_b)$ under the condition $\mu_n^{\tilde{\phi}_n^t} \sim \mu_n$ (see Lemma 3) by setting $(U_n(t)f)(x) = (d\mu_b(\tilde{\phi}_n^t(x))/d\mu_b(x))^{1/2} f(\tilde{\phi}_n^t(x))$. The generators A_n of these groups are given by

$$iA_n = \sum_{r=1}^{n-1} x_{rn} \sum_{m=n+1}^{\infty} x_{nm} D_{rm} = \sum_{r=1}^{n-1} x_{rn}(-iA_{rn}^L - D_{rn}),$$

where $D_{rn} = \partial/\partial x_{rn} - b_{rn} x_{rn}$ and the A_{rn}^L are the generators of the one-parameter groups corresponding to the left regular representation: $iA_{rn}^L = dT_{I+tE_{rn}}^{L,\mu_b}/dt\big|_{t=0}$. The groups $U_n(t)$ commute, $[U_n(t), U_m(s)] = 0$, and hence $V_n(t) = U_n(t)U_{n-1}(t)\cdots U_2(t)$ is the one-parameter group with generator $\mathscr{A}_n = A_n + A_{n-1} + \cdots + A_2$ and is given by the formula

$$(V_n(t)f)(x) = \left(\frac{d\mu_b(\widetilde{\Phi}_n^t(x))}{d\mu_b(x)}\right)^{1/2} f(\widetilde{\Phi}_n^t(x)).$$

Let $U(t) = \exp(itA)$ be an arbitrary one-parameter unitary group with generator $A = A^*$, and let $f \in D(A)$ be an element such that $\|f\| = 1$. We set $F_A(t) = (U(t)f, f)$. Then

$$|F_A(t) - 1|^2 \leqslant t^2 \|Af\|^2, \qquad t \in \mathbb{R}. \tag{2}$$

By $H(\mu, \nu)$ we denote the Hellinger integral for measures μ and ν (see [8, p. 99 of the Russian translation]).

Lemma 3. $\mu_n^{\tilde{\phi}_n} \sim \mu_n \iff H(\mu_n^{\tilde{\phi}_n^t}, \mu_n) > 0$, $t \in \mathbb{R} \setminus \{0\} \iff \|A_n \mathbf{1}\|^2 = \frac{1}{8} \sum_{r=1}^{n-1} S_{rn}^L(b) b_{rn}^{-1} < \infty$.

Lemma 4. *If $E(b) < \infty$, then there exists a strong limit $V(t) = \text{s-lim}_n V_n(t) = \prod_{n=2}^\infty U_n(t)$, which is a one-parameter unitary group. Moreover, $\mu_b^{\tilde{\Phi}} \sim \mu_b$.*

Proof of Lemma 3. By Theorem 1 in [9, §18] and the representations

$$\mu_n^{\tilde{\phi}_n^t}(\Delta) = \int_{\mathbb{R}^{n-1}} \bigotimes_{m=n+1}^{\infty} \mu_{(n,m)}^{\tilde{\phi}_{n,x}^t} \otimes \mu_{(n-1)}(\Delta_x)\, d\mu_{(n-1,n)}(x),$$

$$\mu_n(\Delta) = \int_{\mathbb{R}^{n-1}} \bigotimes_{m=n+1}^{\infty} \mu_{(n,m)} \otimes \mu_{(n-1)}(\Delta_x)\, d\mu_{(n-1,n)}(x),$$

where

$$\mu_{(n,m)} = \bigotimes_{r=1}^{n} \mu_{b_{rm}}, \qquad \mu_{(n-1)} = \bigotimes_{r<k<n} \mu_{b_{rk}},$$

$$\mu_{(n,m)}^{\tilde{\phi}_{n,x}^t}(x_{1m}, x_{2m}, \dots, x_{nm}) = \bigotimes_{r=1}^{n-1} \mu_{b_{rm}}(x_{rm} + t x_{rn} x_{nm}) \otimes \mu_{b_{nm}}(x_{nm}),$$

$$x = (x_{1n}, x_{2n}, \dots, x_{n-1,n}), \qquad \Delta = \bigcup_{x \in \mathbb{R}^{n-1}} \Delta_x,$$

we have

$$\mu_n^{\tilde{\phi}_n^t} \sim \mu_n \iff \bigotimes_{m=n+1}^{\infty} \mu_{(n,m)}^{\tilde{\phi}_{n,x}^t} \sim \bigotimes_{m=n+1}^{\infty} \mu_{(n,m)} \quad \text{for } \mu_{(n-1,n)}\text{-almost all } x$$

$$\iff H\left(\bigotimes_{m=n+1}^{\infty} \mu_{(nm)}^{\tilde{\phi}_{n,x}^t}, \bigotimes_{m=n+1}^{\infty} \mu_{(nm)}\right) > 0 \quad \text{for } \mu_{(n-1,n)}\text{-almost all } x.$$

Finally, for $\mu_{(n-1,n)}$-almost all x, the inequality $\sum_{r=1}^{n-1} S_{rn}^L(b) b_{rn}^{-1} < \infty$ implies that

$$H\left(\bigotimes_{m=n+1}^{\infty} \mu_{(nm)}^{\tilde{\phi}_{n,x}^t}, \bigotimes_{m=n+1}^{\infty} \mu_{(nm)}\right) = \prod_{m=n+1}^{\infty} \left(1 + \frac{t^2}{4} \sum_{r=1}^{n-1} \frac{b_{rm}}{b_{nm}} x_{rn}^2\right)^{-1/2} > 0.$$

To prove Lemma 4, let us show that $\lim_{n+m,\,n\to\infty}(V_{n+m}(t)-V_n(t))f=0$ for $f\in D$, where $D\subset H=L^2(B^\infty,\,d\mu_b)$ is the dense set of polynomials in the variables $(x_{kn})_{k<n}$. Since the function $f\in D$ is finite, for some p we have $f=\mathbf{1}\otimes f_{(p)}\in H^{(p)}\otimes H_{(p)}=H$, where

$$H_{(p)}=L^2(B_{(p)},\mu_{(p)}),\qquad H^{(p)}=L^2(B^{(p)},\mu^{(p)}),$$

$$B_{(p)}=\Big\{x\in B^\infty\;\Big|\;x=I+\sum_{r<k\leqslant p}x_{rk}E_{rk}\Big\},$$

$$B^{(p)}=\Big\{x\in B^\infty\;\Big|\;x=I+\sum_{r<k,\,p<k}x_{rk}E_{rk}\Big\},$$

$$\mu_{(p)}=\bigotimes_{r<k\leqslant p}\mu_{b_{rk}},\qquad \mu^{(p)}=\bigotimes_{r<k,\,p<k}\mu_{b_{rk}}.$$

Since $(V_n(t))^{-1}V_{n+m}(t)=[V_n(t)^{-1}V_{n+m}(t)]^{(p)}\otimes I_{(p)}$ for $n\geqslant p$, where $[V]^{(p)}$ is the projection of an operator $V\in L(H)$ on the subspace $H^{(p)}\otimes\mathbf{1}\subset H=H^{(p)}\otimes H_{(p)}$, we can use (2) to obtain

$$\begin{aligned}\|(V_{n+m}(t)-V_n(t))f\|^2&=\|([V_n(t)^{-1}V_{n+m}(t)]^{(p)}\otimes I_{(p)}-I^{(p)}\otimes I_{(p)})\mathbf{1}\otimes f_{(p)}\|^2\\&=\|([V_n(t)^{-1}V_{n+m}(t)]^{(p)}-I^{(p)})\mathbf{1}\|^2_{H^{(p)}}\|f_{(p)}\|^2_{H_{(p)}}\\&\leqslant\|f_{(p)}\|^2_{H_{(p)}}t^2\|\mathscr{A}_{n,n+m}\mathbf{1}\|^2=\|f_{(p)}\|^2_{H_{(p)}}t^2\sum_{k=n+1}^{n+m}\|A_k\mathbf{1}\|^2\\&=\|f_{(p)}\|^2_{H_{(p)}}\frac{t^2}{8}\sum_{k=n+1}^{n+m}\sum_{r=1}^{k-1}\frac{S^L_{rk}(b)}{b_{rk}}\to 0,\qquad n\to\infty,\end{aligned}$$

where $\mathscr{A}_{n,n+m}$ is the generator of the one-parameter unitary group $(V_n(t))^{-1}V_{n+m}(t)$. Thus

$$V(t)=\operatorname*{s\text{-}lim}_n V_n(t),\qquad (V(t)f)(x)=\left(\frac{d\mu_b(\widetilde{\Phi}^t(x))}{d\mu_b(x)}\right)^{1/2}f(\widetilde{\Phi}^t(x))$$

is a unitary operator, and therefore, $\int(d\mu_b(\widetilde{\Phi}^t(x))/d\mu_b(x))\,d\mu_b(x)=\|V(t)\mathbf{1}\|^2=\|\mathbf{1}\|^2=1$. Hence $\mu_b^{\tilde{\Phi}^t}\sim\mu_b$ and $\mu_b^{\tilde{\Phi}}\sim\mu_b$ (see Theorem 4 in [9, §15] and the remark to Theorem 1 in [9, §16]).

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