Ukrainian Mathematical Journal, Vol. 54, No. 2, 2002

ELEMENTARY REPRESENTATIONS OF THE GROUP $B_0^{\mathbb{Z}}$ OF UPPER-TRIANGULAR MATRICES INFINITE IN BOTH DIRECTIONS. I

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UDC 519.46

We define so-called "elementary representations" $T_p^{R,\mu}$, $p \in \mathbb{Z}$, of the group $B_0^{\mathbb{Z}}$ of finite upper-triangular matrices infinite in both directions by using quasi-invariant measures on certain homogeneous spaces and give a criterion for the irreducibility and equivalence of the representations constructed. We also give a criterion for the irreducibility of the tensor product of finitely many and infinitely many elementary representations.

1. *G*-action, Quasiinvariant Measures, and Representations

The following construction of unitary representations of a topological group G is well known: Assume that we have a measurable space X with probability measure μ on which the group G acts, i.e., we have a group homomorphism $\alpha : G \to \operatorname{Aut}(X)$ satisfying the following conditions:

(i) $\alpha_e(x) = x \quad \forall x \in X$, where $e \in G$ is the identity element;

(ii) $\alpha_{t_1}\big(\alpha_{t_2}(x)\big) = \alpha_{t_1 t_2}(x) \quad \forall t_1, t_2 \in G,\ x \in X.$

Let μ^{α_t}, $t \in G$, be images of the measure μ with respect to the action α, i.e., $\mu^{\alpha_t}(\Delta) = \mu\big(\alpha_{t^{-1}}(\Delta)\big)$. If $\mu^{\alpha_t} \sim \mu \quad \forall t \in G$, one can define a unitary representation $\pi^{\alpha,\mu} : G \to U\big(L^2(X, d\mu)\big)$ of the group G as follows:

$$\left(\pi_t^{\alpha,\mu} f\right)(x) = \left(\frac{d\mu^{\alpha_t}(x)}{d\mu(x)}\right)^{1/2} f\big(\alpha_{t^{-1}}(x)\big), \qquad f \in L^2(X, d\mu). \tag{1}$$

2. Analog of Regular Representations of Infinite-Dimensional Groups

A regular representation of a locally compact group G is well known (see, e.g., [1]). It uses the existence of a G-invariant measure on the group G, the Haar measure, and is defined by formula (1), where $X = G$ and α is the right or the left action of the group G onto itself.

For a group G that is not locally compact, it is impossible to define a regular representation because there is no G-invariant measure on the group G [2], nor is there a G-quasiinvariant measure [3].

An analog of regular representations of some infinite-dimensional noncommutative groups (current groups) was first constructed and studied in [4–7].

Institute of Mathematics, Ukrainian Academy of Sciences, Kiev. Published in Ukrains'kyi Matematychnyi Zhurnal, Vol. 54, No. 2, pp. 205–215, February, 2002. Original article submitted October 25, 2001.

An analog of a regular representation for any infinite-dimensional group G, using G-quasiinvariant measures μ on some *completions* $\tilde{G}$ *of the group* G, was first defined in [8–10]. It uses formula (1), where $X = \tilde{G}$ and α is the right or the left action of the group G on $\tilde{G}$. More precisely, let $H_\mu = L^2(\tilde{G}, d\mu)$. We define *analogs of the right* $T^{R,\mu}$ *and the left* $T^{L,\mu}$ *regular representations of the group* G in the space H_μ, i.e.,

$$T^{R,\mu},\ T^{L,\mu}:\ G \to U(H_\mu),$$

in a natural way, namely,

$$\left(T_t^{R,\mu} f\right)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt), \tag{2}$$

$$\left(T_s^{L,\mu} f\right)(x) = \left(\frac{d\mu(s^{-1}x)}{d\mu(x)}\right)^{1/2} f\left(s^{-1}x\right). \tag{3}$$

It is obvious that $\left[T_t^{R,\mu}, T_s^{L,\mu}\right] = 0 \ \ \forall t, s \in G$. Hence, the right regular representation $T^{R,\mu}$ is reducible if $\mu^{L_s} \sim \mu$ for some $s \in G \backslash e$ or the measure μ is not G-right ergodic. Let μ be a G-right quasiinvariant measure on $\tilde{G}$, i.e., $\mu^{R_t} \sim \mu \ \ \forall t \in G$.

Conjecture 1. *The right regular representation* $T^{R,\mu}:\ G \to U(H_\mu)$ *is irreducible if and only if*

(i) $\mu^{L_s} \perp \mu \ \ \forall s \in G \backslash e$,

(ii) the measure μ *is* G*-right ergodic.*

Remark. This conjecture was formulated by Ismagilov in 1985 for the group $B_0^{\mathbb{N}}$ of finite real upper-triangular matrices infinite in one direction and having unities on the principal diagonal and any Gaussian centered product measure μ_b.

In this case, the conjecture was proved in [8, 9]. For the same group $B_0^{\mathbb{N}}$ and any product measure $\mu = \otimes_{k<n}\mu_{kn}$, it was proved in [11] under certain technical assumption. In [12], this conjecture was proved for the group $B_0^{\mathbb{Z}}$ of finite upper-triangular matrices infinite in both directions for some Gaussian centered product measures. In [10], a criterion was proved for groups of interval and circle diffeomorphisms and the Wiener measure.

3. Analog of Regular Representations of the Group $B_0^{\mathbb{Z}}$

Let $B_0^{\mathbb{Z}}$ be the group of finite upper-triangular matrices infinite in both directions and having unities on the principal diagonal and let $B^{\mathbb{Z}}$ be the group of all matrices of this type (not necessarily finite), i.e.,

$$B_0^{\mathbb{Z}} = \left\{ I + x = I + \sum_{k<n} x_{kn} E_{kn} \,|\, x \ \text{ is finite} \right\},$$

$$B^{\mathbb{Z}} = \left\{ I + x = I + \sum_{k<n} x_{kn} E_{kn} \mid x \text{ is arbitrary} \right\},$$

where E_{kn}, $k, n \in \mathbb{Z}$, are matrix units of infinite order. Let R and L denote the right and the left action of the group $B^{\mathbb{Z}}$ onto itself; $R_s(t) = ts^{-1}$, $L_s(t) = st$, $s, t \in B^{\mathbb{Z}}$. Let μ be a probability measure on the group $B^{\mathbb{Z}}$. If $\mu^{R_t} \sim \mu$ and $\mu^{L_t} \sim \mu$ $\forall t \in B_0^{\mathbb{Z}}$, then we can define by formulas (2) and (3) an analog of the right $T^{R,\mu}$ and the left $T^{L,\mu}$ regular representations of the group $B_0^{\mathbb{Z}}$ in the space $H_\mu = L^2\left(B^{\mathbb{Z}}, d\mu\right)$, $T^{R,\mu}$, $T^{L,\mu} : B_0^{\mathbb{Z}} \to U(H_\mu)$, as follows:

$$\left(T_t^{R,\mu} f\right)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt),$$

$$\left(T_t^{L,\mu} f\right)(x) = \left(\frac{d\mu(t^{-1}x)}{d\mu(x)}\right)^{1/2} f\left(t^{-1}x\right).$$

For the generators $A_{kn}^{R,\mu}$ $\left(A_{kn}^{L,\mu}\right)$ of the one-parameter groups $I + tE_{kn}$, $t \in \mathbb{R}^1$, $k < n$, corresponding to the right $T^{R,\mu}$ (respectively, the left $T^{L,\mu}$) regular representation, we have the following formulas:

$$A_{kn}^{R,\mu} = \frac{d}{dt} T_{I+tE_{kn}}^{R,\mu}\Big|_{t=0} = \sum_{r=-\infty}^{k-1} x_{rk} D_{rn}(\mu) + D_{kn}(\mu), \tag{4}$$

$$A_{kn}^{L,\mu} = \frac{d}{dt} T_{I+tE_{kn}}^{L,\mu}\Big|_{t=0} = -\left(D_{kn}(\mu) + \sum_{m=n+1}^{\infty} x_{nm} D_{km}(\mu)\right), \tag{5}$$

where

$$D_{kn}(\mu) = \frac{\partial}{\partial x_{kn}} + \frac{d}{dt}\left(\frac{d\mu(x(I+tE_{kn}))}{d\mu(x)}\right)^{1/2}\Big|_{t=0}.$$

For an arbitrary product measure $\mu = \otimes_{k<n} \mu_{kn}$, we have

$$D_{kn}(\mu) = \frac{\partial}{\partial x_{kn}} + \frac{\partial}{\partial x_{kn}}\left(\ln \mu_{kn}^{1/2}(x_{kn})\right),$$

where $d\mu_{kn}(x) = \mu_{kn}(x)\,dx$, $x \in \mathbb{R}^1$. Denote

$$M_{kn}(p) = \int_{\mathbb{R}^1} x^p \mu_{kn}(x)\,dx, \qquad \tilde{M}_{kn}(p) = \left(\left(i^{-1} D_{km}(\mu)\right)^p \mathbb{1}, \mathbb{1}\right)_{L^2(\mathbb{R}^1, d\mu_{kn})}, \qquad p \in \mathbb{N}.$$

We define a Gaussian measure μ_b on the group $B^{\mathbb{Z}}$ in the following way:

$$d\mu_b(x) = \otimes_{k<n}\left(b_{kn}/\pi\right)^{1/2}\exp\left(-b_{kn}x_{kn}^2\right)dx_{kn} = \otimes_{k<n}d\mu_{b_{kn}}(x_{kn}),$$

where $b = (b_{kn})_{k<n}$ is some set of positive numbers. In this case, we have (see, e.g., formulas (6) and (7) in [13])

$$D_{kn}(\mu_b) = \frac{\partial}{\partial x_{kn}} - b_{kn}x_{kn},$$

$$M_{kn}(2) = \frac{1}{2b_{kn}}, \qquad M_{kn}(4) = \frac{3}{\left(2b_{kn}\right)^2}, \qquad M_{kn}(2m) = \frac{(2m-1)!!}{\left(2b_{kn}\right)^m}, \tag{6}$$

$$\tilde{M}_{kn}(2) = \frac{b_{kn}}{2}, \qquad \tilde{M}_{kn}(4) = 3\left(\frac{b_{kn}}{2}\right)^2, \qquad \tilde{M}_{kn}(2m) = (2m-1)!!\left(\frac{b_{kn}}{2}\right)^m. \tag{7}$$

For an arbitrary Gaussian product measure $\mu_b = \otimes_{k<n}\mu_{b_{kn}}$, one can easily verify the equivalences $\mu_b^{R_t} \sim \mu_b$ and $\mu_b^{L_t} \sim \mu_b$ $\forall t \in B_0^{\mathbb{Z}}$. The following three lemmas were proved in [12]:

Lemma 1.

$$\mu_b^{R_t} \sim \mu_b \quad \forall t \in B_0^{\mathbb{Z}} \Leftrightarrow S_{kn}^R(\mu_b) = \sum_{r=-\infty}^{k-1} M_{rk}(2)\tilde{M}_{rn}(2) = \frac{1}{4}\sum_{r=-\infty}^{k-1}\frac{b_{rn}}{b_{rk}} < \infty \quad \forall k < n.$$

Lemma 2.

$$\mu_b^{L_t} \sim \mu_b \quad \forall t \in B_0^{\mathbb{Z}} \Leftrightarrow S_{kn}^L(\mu_b) = \sum_{m=n+1}^{\infty} \tilde{M}_{km}(2)M_{nm}(2) = \frac{1}{4}\sum_{m=n+1}^{\infty}\frac{b_{km}}{b_{nm}} < \infty \quad \forall k < n.$$

Lemma 3. *For* $k, n \in \mathbb{Z}$, $k < n$, *we have* $\mu_b^{L_{I+tE_{kn}}} \perp \mu_b$ $\forall t \in \mathbb{R}^1 \setminus 0 \Leftrightarrow S_{kn}^L(\mu_b) = \infty$.

4. Elementary Representations of the Group $B_0^{\mathbb{Z}}$

Consider the subgroups X_p, $p \in \mathbb{Z}$, and $X^{\{p\}}$ in the group $B^{\mathbb{Z}}$, where $\{p\}$ is a finite or infinite subset of $\mathbb{Z}$. For $\{p\}$ infinite in both directions, we have $\{p\} = (p_k)_{k\in\mathbb{Z}}$, $p_k < p_{k+1}$ $\forall k \in \mathbb{Z}$,

$$X_p = \left\{ I + x \in B^{\mathbb{Z}} \mid I + x = I + \sum_{n=p+1}^{\infty} x_{pn}E_{pn} \right\},$$

$$X^{\{p\}} = \prod\nolimits_{p_k \in \{p\}} X_{p_k} = \left\{ I + x \in B^{\mathbb{Z}} \mid I + x = I + \sum_{p_k \in \{p\}} \sum_{n=p_k+1}^{\infty} x_{p_k n} E_{p_k n} \right\}.$$

Obviously, the right action of the group $B_0^{\mathbb{Z}}$ is well defined on the groups X_p and $X^{\{p\}}$.

For a $B_0^{\mathbb{Z}}$-right quasiinvariant measure μ on X_p (respectively $X^{\{p\}}$), we define a representation $T_p^{R,\mu}$ (respectively, $T^{R,\mu,\{p\}}$) as follows:

$$\left(T_t^{R,\mu} f\right)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt), \quad f \in H_p(\mu) := L^2\left(X_p, d\mu\right),$$

$$\left(T_t^{R,\mu,\{p\}} f\right)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt), \quad f \in H^{\{p\}}(\mu) := L^2\left(X^{\{p\}}, d\mu\right).$$

In the particular case $\{p\} = (1, 2, \ldots, q)$, we denote

$$X^q = X^{(1,2,\ldots,q)}, \quad T^{R,\mu,q} = T^{R,\mu,(1,2,\ldots,q)}, \quad H^q(\mu) = L^2\left(X^{(1,2,\ldots,q)}, d\mu\right).$$

Definition 1. *The representations $T_p^{R,\mu}$, $p \in \mathbb{Z}$, are called elementary (see also [14]).*

5. Irreducibility and Equivalence of Elementary Representations

For the Gaussian measure $\mu = \mu_b$ and its projections $\mu_{b,p} = \otimes_{n=p+1}^{\infty} \mu_{b_{pn}}$, we have the following theorem:

Theorem 1.

1. *The representation $T_p^{R,\mu}$ is irreducible if and only if the measure μ on the space X_p is $B_0^{\mathbb{Z}}$-right-ergodic.*

2. *Two irreducible representations $T_{p_1}^{R,\mu_1}$ and $T_{p_2}^{R,\mu_2}$ are equivalent if and only if $p_1 = p_2$ and $\mu_1 \sim \mu_2$.*

Since $T_p^{R,\mu}$ (respectively, $T^{R,\mu,\{p\}}$) is the restriction of the representation $T^{R,\mu}$ to the subspace $H_p(\mu) = L^2\left(X_p, d\mu_p\right)$ (respectively, $H^{\{p\}}(\mu) = L^2(X^{\{p\}}, d\mu^{\{p\}})$) of the space $H_\mu = L^2(B^{\mathbb{Z}}, d\mu)$, we have

$$A_{p,kn}^{R,\mu} = \begin{cases} 0 & \text{if } k < p, \\ D_{pn}(\mu) & \text{if } p = k < n, \\ x_{pk} D_{pn}(\mu) & \text{if } p < k < n, \end{cases} \tag{8}$$

$$A_{kn}^{R,\mu,q} := A_{kn}^{R,\mu,(1,2,\ldots,q)} = \sum_{p=1}^{q} A_{p,kn}^{R,\mu} = \begin{cases} 0 & \text{if } k<1, \\ \sum_{r=1}^{k-1} x_{rk} D_{rn}(\mu) + D_{kn}(\mu) & \text{if } 1 \le k \le q,\ k<n, \\ \sum_{r=1}^{q} x_{rk} D_{rn}(\mu) & \text{if } q<k<n, \end{cases} \tag{9}$$

$$A_{kn}^{R,\mu,\{p\}} := \sum_{p_m \in \{p\},\, p_m \le k} A_{p_m,kn}^{R,\mu}$$

$$= \begin{cases} 0 & \text{if } k<p_{\min}, \\ \sum_{p_m \in \{p\},\, p_m<k} x_{p_m k} D_{p_m n}(\mu) + D_{kn}(\mu) & \text{if } k \in \{p\},\ k<n, \\ \sum_{p_m \in \{p\},\, p_m<k} x_{p_m k} D_{p_m n}(\mu) & \text{if } k \notin \{p\},\ p_{\min}<k<n, \end{cases} \tag{10}$$

where $p_{\min} = \min\{p_m \mid p_m \in \{p\}\} \in \mathbb{R}^1 \cup \{-\infty\}$.

Proof. See the proof of Theorem 5 in [14].

1. Assume that a bounded operator A on the Hilbert space $H_p(\mu)$ commutes with the representation $T_p^{R,\mu}$, i.e., $\left[A, T_{p,t}^{R,\mu}\right] = 0 \ \ \forall t \in B_0^{\mathbb{Z}}$. We prove that A is trivial, $A = \lambda I$, $\lambda \in \mathbb{C}^1$. To prove this, we consider the commutative set of generators $\left\{i^{-1} A_{p,pn}^{R,\mu}\right\}_{n=p+1}^{\infty}$. By formulas (8), we have $i^{-1} A_{p,pn}^{R,\mu} = i^{-1} D_{pn}(\mu)$. Since the family of operators $i^{-1}\mathbb{D}_p(\mu) = \left\{i^{-1} D_{pn}(\mu)\right\}_{n=p+1}^{\infty}$ has a common simple spectrum in the space $H_p(\mu) = L^2\big(X_p, d\mu\big)$, any bounded operator A on the space $H_p(\mu)$ that commutes with this family is an essentially bounded function of this family:

$$A = a\big(i^{-1}\mathbb{D}_p(\mu)\big) = a\big(i^{-1} D_{pp+1}(\mu), i^{-1} D_{pp+2}(\mu), \ldots, i^{-1} D_{pn}(\mu), \ldots\big).$$

To complete the proof, we use the Fourier–Wiener transform defined in [13]. Let F_{kn}^b denote the one-dimensional Fourier transform corresponding to the measure $d\mu_{b_{kn}}(x_{kn}) = (b_{kn}/\pi)^{1/2} \exp\big(-b_{kn} x_{kn}^2\big) dx_{kn}$,

$$F_{kn}^b : L^2\big(\mathbb{R}^1, d\mu_{b_{kn}}\big) \to L^2\big(\mathbb{R}^1, d\mu_{b_{kn}^{-1}}\big),$$

and given by the formula

$$\big(F_{kn}^b f\big)(y_{kn}) = \exp\left(\frac{y_{kn}^2}{2b_{kn}}\right) \sqrt{\frac{b_{kn}}{2\pi}} \int_{\mathbb{R}^1} f(x_{kn}) \exp(i y_{kn} x_{kn}) \exp\left(-\frac{b_{kn} x_{kn}^2}{2}\right) dx_{kn}.$$

It is obvious that $F_{kn}^b \mathbb{1} = \mathbb{1}$, where $\mathbb{1}(x) \equiv 1$.

For any $p \in \mathbb{Z}$, we define the Fourier–Wiener transform $F_p^b = \otimes_{n=p+1}^{\infty} F_{pn}^b$. The operator F_p^b is an isometry between two spaces, namely, $F_p^b \colon H_p(\mu_b) \to H_p(\mu_{b^{-1}})$, where $H_p(\mu_b) = L^2\big(X_p, d\mu_{b,\,p}\big)$ and $H_p(\mu_{b^{-1}}) = L^2\big(X_p, d\mu_{b^{-1},\,p}\big)$. We have (see [13])

$$F_p^b\big(i^{-1}D_{pn}(\mu_b)\big)\big(F_p^b\big)^{-1} = y_{pn}, \quad p < n, \tag{11}$$

$$F_p^b\big(x_{pn}i^{-1}D_{pm}(\mu_b)\big)\big(F_p^b\big)^{-1} = i^{-1}D_{pn}(\mu_{b^{-1}})y_{pm}, \quad p < n < m,$$

$$F_p^b A\big(F_p^b\big)^{-1} = F_p^b a\big(i^{-1}D_{pp+1}(\mu),\ldots, i^{-1}D_{pn}(\mu),\ldots\big)\big(F_p^b\big)^{-1} = a\big(y_{pp+1},\ldots, y_{pn},\ldots\big).$$

The one-parameter group $\tilde{T}^{R,\,\mu_b}_{I+tE_{nm}} = F_p^b T^{R,\,\mu_b}_{I+tE_{nm}}\big(F_p^b\big)^{-1}$ corresponds to the generator $i^{-1}D_{pn}(\mu_{b^{-1}})y_{pm}$ in the space $H_p(\mu_{b^{-1}})$ and, therefore, it acts according to the formula

$$\big(\tilde{T}^{R,\,\mu_b}_{I+tE_{nm}} f\big)\big(\ldots, y_{pn},\ldots, y_{pm},\ldots\big) = \left(\frac{d\mu_{b^{-1},\,p}\big(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots\big)}{d\mu_{b^{-1},\,p}\big(\ldots, y_{pn},\ldots, y_{pm},\ldots\big)}\right)^{1/2} f\big(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots\big).$$

Hence, the commutation $\big[\,\tilde{A}, \tilde{T}^{R,\,\mu_b}_{I+tE_{nm}}\,\big] = 0 \ \ \forall\, t \in \mathbb{R}^1$, where $\tilde{A} = F_p^b A\big(F_p^b\big)^{-1}$, yields

$$a\big(y_{pp+1},\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots\big) = a\big(y_{pp+1},\ldots, y_{pn},\ldots, y_{pm},\ldots\big) \qquad \forall\, t \in \mathbb{R}^1.$$

Indeed, it is sufficient to compare two equations, namely,

$$\begin{aligned}\big(\tilde{A}\tilde{T}^{R,\,\mu_b}_{I+tE_{nm}} f\big)\big(\ldots, y_{pn},\ldots, y_{pm},\ldots\big) &= a\big(\ldots, y_{pn},\ldots, y_{pm},\ldots\big)\\ &\times\left(\frac{d\mu_{b^{-1},p}\big(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots\big)}{d\mu_{b^{-1},p}\big(\ldots, y_{pn},\ldots, y_{pm},\ldots\big)}\right)^{1/2} f\big(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots\big),\end{aligned}$$

$$\begin{aligned}\big(\tilde{T}^{R,\,\mu_b}_{I+tE_{nm}} \tilde{A} f\big)\big(\ldots, y_{pn},\ldots, y_{pm},\ldots\big) &= \left(\frac{d\mu_{b^{-1},p}\big(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots\big)}{d\mu_{b^{-1},p}\big(\ldots, y_{pn},\ldots, y_{pm},\ldots\big)}\right)^{1/2}\\ &\times a\big(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots\big) f\big(\ldots, y_{pn}+ty_{pm},\ldots, y_{pm},\ldots\big).\end{aligned}$$

By virtue of the ergodicity of the measure $\mu_{b^{-1},\,p}$, the function

$$a = a\big(y_{pp+1},\ldots, y_{pn},\ldots\big)$$

is constant and the operator A is trivial, $A = \lambda I$.

2. The sufficiency is obvious. Let $T_p^{R,\mu} \sim T_{p'}^{R,\mu'}$. We prove that $p = p'$ and $\mu \sim \mu'$. We assume that $p \neq p'$, say, $p > p'$, and consider the restrictions $T|_G$ of the representations $T = T_p^{R,\mu}$ and $T_{p'}^{R,\mu'}$ to the subgroup $G = X_{p,0} = \left\{ I + x \in B_0^{\mathbb{Z}} \mid I + x \in X_p \right\}$. The spectral measure $\mathbb{E}_p^{\mu}$ of the restriction $T_p^{R,\mu}\big|_{X_{p,0}}$ is the spectral measure of the commutative family of self-adjoint operators $i^{-1}\mathbb{D}_p(\mu) = \left\{ i^{-1} D_{pn}(\mu) \right\}_{p=n+1}^{\infty}$, and the spectral measure $\mathbb{E}_p^{\mu'}$ of $T_{p'}^{R,\mu'}\big|_{X_{p,0}}$ is trivial [see (8)], whence $p = p'$. In this case, the spectral measures $\mathbb{E}_p^{\mu}$ and $\mathbb{E}_p^{\mu'}$ are equivalent and, therefore, $\mu \sim \mu'$.

Indeed, let us use the Fourier–Wiener transform F_p^b. Denote by $\mathbb{E}_p^{\mu_{b^{-1}}}(y)$ the spectral measure of the family of operators of multiplication by independent variables $(y_{pn})_{n=p+1}^{\infty}$ in the Hilbert space $H_p(\mu_{b^{-1}})$. Since the spectral measures $\mathbb{E}_p^{\mu}$ and $\mathbb{E}_p^{\mu'}$ are equivalent, by using (11) we establish that the spectral measures $\mathbb{E}_p^{\mu_{b^{-1}}}(y)$ and $\mathbb{E}_p^{\mu_{(b')^{-1}}}(y)$ are equivalent. Moreover, we have

$$\left(\mathbb{E}_p^{\mu_{b^{-1}}}(y)(\Delta)\mathbb{1}, \mathbb{1} \right)_{H_p(\mu_{b^{-1}})} = \mu_{b^{-1},p}(\Delta).$$

Finally,

$$\mathbb{E}_p^{\mu} \sim \mathbb{E}_p^{\mu'} \Leftrightarrow \mathbb{E}_p^{\mu_{b^{-1}}}(y) \sim \mathbb{E}_p^{\mu_{(b')^{-1}}}(y) \Leftrightarrow \mu_{b^{-1},p} \sim \mu_{(b')^{-1},p}$$

$$\Leftrightarrow \prod_{n=p+1}^{\infty} \frac{4(b_{pn})^{-1}(b'_{pn})^{-1}}{\left((b_{pn})^{-1} + (b'_{pn})^{-1}\right)^2} > 0 \Leftrightarrow \prod_{n=p+1}^{\infty} \frac{4 b_{pn} b'_{pn}}{\left(b_{pn} + b'_{pn}\right)^2} > 0 \Leftrightarrow \mu_{b,p} \sim \mu_{b',p}.$$

6. Tensor Product of Finitely Many Elementary Representations and Irreducibility

Let $\{p\} = (p_1, \ldots, p_m)$ be a finite subset of $\mathbb{Z}$.

Theorem 2.

1. *The representation* $T^{R,\mu,\{p\}}$ *is the tensor product of the representations* $T_{p_k}^{R,\mu_{p_k}}$, $1 \le k \le m$:

$$T^{R,\mu,\{p\}} = \otimes_{k=1}^{m} T_{p_k}^{R,\mu_{p_k}}. \tag{12}$$

2. *The representation* $T^{R,\mu,\{p\}}$ *is irreducible if and only if*

   (i) $S_{p_k p_n}^{L}(\mu) = \infty$, $1 \le k < n \le m$, *and*

   (ii) *the measure* μ *on the space* $X^{\{p\}}$ *is* $B_0^{\mathbb{Z}}$*-right-ergodic.*

Proof. We prove the theorem for $\{p\} = (1, 2, \ldots, q)$. For other finite $\{p\}$, the proof is the same. We show that, by using the generators $A_{kn}^{R,\mu,q} := A_{kn}^{R,\mu,(1,2,\ldots,q)}$, $k < n$, one can approximate the operators of multiplication by independent variables x_{kn}, $1 \le k < n \le q$, and the set of operators $D_{kn}(\mu)$, $k < n$, $k \le q$. Indeed, according to (9), we have

$$A_{1n}^{R,\mu,q} = D_{1n}(\mu), \quad 1 < n, \quad A_{2n}^{R,\mu,q} = x_{12}D_{1n}(\mu) + D_{2n}(\mu), \quad 2 < n,$$

$$A_{3n}^{R,\mu,q} = x_{13}D_{1n}(\mu) + x_{23}D_{2n}(\mu) + D_{3n}(\mu), \quad 3 < n,$$

$$A_{kn}^{R,\mu,q} = \sum_{r=1}^{k-1} x_{rk}D_{rn}(\mu) + D_{kn}(\mu), \quad k \le q, \quad k < n,$$

$$A_{kn}^{R,\mu,q} = \sum_{r=1}^{q} x_{rk}D_{rn}(\mu), \text{ if } q < k < n.$$

The proof of approximation is the same as in [9]. It is based on Lemma 6 in [14].

Denote by $\mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$ the von-Neumann algebra generated by the representation $T^{R,\mu,q}$, i.e., $\mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}}) = \left(T_t^{R,\mu,q} \,\middle|\, t \in B_0^{\mathbb{Z}}\right)''$. Also let $\langle f_n \mid n = 1, 2, \ldots \rangle$ be the closure of the linear space generated by the set of vectors $\{f_n\}_{n=1}^{\infty}$ in a Hilbert space H.

Definition 2. *Recall [15] that a (not necessarily bounded) self-adjoint operator A on a Hilbert space H is affiliated with the von-Neumann algebra M of operators on H (which is denoted by $A \,\eta\, M$) if* $\exp(itA) \in M \quad \forall\, t \in \mathbb{R}^1$.

Lemma 4 [14]. $\{x_{kn}\}_{1 \le k < n \le q} \,\eta\, \mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$ *if* $S_{kn}^L(\mu) = \infty$, $k < n \le q$. *In this case, we also have* $D_{kn}(\mu) \,\eta\, \mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$, $k < n$, $k \le q$.

Finally, we have $\{x_{kn}\}_{k < n \le q} \,\eta\, \mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$ and $\{D_{kn}(\mu)\}_{k < n, k \le q} \,\eta\, \mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$, and, therefore, the commutant $\left(\mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})\right)'$ of the von-Neumann algebra $\mathfrak{A}^{R,\mu,q}(B_0^{\mathbb{Z}})$ coincides with essentially bounded functions from the family of operators $i^{-1}\mathbb{D}^q(\mu) = \{i^{-1}D_{kn}(\mu)\}_{k \le q < n}$.

Now assume that a bounded operator $A \in L(H^q(\mu))$ commutes with $T_t^{R,\mu,q}$, $t \in B_0^{\mathbb{Z}}$. Then this operator A is an operator of multiplication in the space $H^q(\mu)$ by some essentially bounded function, i.e., $A = a\left(\{i^{-1}D_{kn}(\mu)\}_{k < n, k \le q}\right)$.

As in the proof of Theorem 1, we use an appropriate Fourier–Wiener transform to prove the irreducibility. Denote $F^{b,q} = \otimes_{p=1}^{q} F_p^b$. This operator is an isometry between $H^q(\mu_b)$ and $H^q(\mu_{b^{-1}})$. It is obvious that $\tilde{A}F^{b,q}A(F^{b,q})^{-1} = a\left(\{y_{kn}\}_{k \le q < n}\right)$ and the operator $\tilde{T}_{I+tE_{kn}}^{R,\mu,q} = F^{b,q}\tilde{T}_{I+tE_{kn}}^{R,\mu,q}(F^{b,q})^{-1}$ acts as follows:

$$\left(\tilde{T}^{R,\mu,q}_{I+tE_{kn}} f\right)\begin{pmatrix} y_{1q+1} & \dots & y_{1k} & \dots & y_{1n} & \dots \\ & \dots & & \dots & & \dots \\ y_{qq+1} & \dots & y_{qk} & \dots & y_{qn} & \dots \end{pmatrix}$$

$$= \left(\frac{d\mu^q_{b^{-1}}\left(\tilde{R}_{I+tE_{kn}}(y)\right)}{d\mu^q_{b^{-1}}(y)}\right)^{1/2} f\left(\tilde{R}_{I+tE_{kn}}(y)\right)$$

$$:= \left(\frac{d\mu^q_{b^{-1}}\left(\tilde{R}_{I+tE_{kn}}(y)\right)}{d\mu^q_{b^{-1}}(y)}\right)^{1/2} f\begin{pmatrix} y_{1q+1} & \dots & y_{1k}+ty_{1n} & \dots & y_{1n} & \dots \\ & \dots & & \dots & & \dots \\ y_{qq+1} & \dots & y_{qk}+ty_{qn} & \dots & y_{qn} & \dots \end{pmatrix}.$$

Hence, as in the proof of Theorem 1, the commutation $\left[\tilde{A}, \tilde{T}^{R,\mu,q}_{I+tE_{nm}}\right] = 0 \quad \forall\, t \in \mathbb{R}^1$ yields

$$a\begin{pmatrix} y_{1q+1} & \dots & y_{1k} & \dots & y_{1n} & \dots \\ & \dots & & \dots & & \dots \\ y_{qq+1} & \dots & y_{qk} & \dots & y_{qn} & \dots \end{pmatrix}$$

$$= a\begin{pmatrix} y_{1q+1} & \dots & y_{1k}+ty_{1n} & \dots & y_{1n} & \dots \\ & \dots & & \dots & & \dots \\ y_{qq+1} & \dots & y_{qk}+ty_{qn} & \dots & y_{qn} & \dots \end{pmatrix} \quad \forall\, t \in \mathbb{R}^1, \qquad \forall\, q < k < n.$$

By virtue of the ergodicity of the measure $\mu^q_{b^{-1}}$, this means that the function $a\left(\{ y_{kn} \}_{k \le q < n}\right)$ is constant, i.e., $a(y) = \text{const}$.

7. Regular Representations as Infinite Tensor Product of Elementary Representations.

Theorem 3.

1. *The representation* $T^{R,\mu}$ *is the infinite tensor product of the representations* T^{R,μ_p}_p, $p \in \mathbb{Z}$:

$$T^{R,\mu} = \otimes_{p \in \mathbb{Z}} T^{R,\mu_p}_p. \tag{13}$$

2. *The representation* $T^{R,\mu}$ *is irreducible if*

   (i) $S^L_{kn}(\mu) = \infty \quad \forall\, k < n$,

   (ii) *the measure* μ *on the group* $B^{\mathbb{Z}}$ *is* $B^{\mathbb{Z}}_0$*-right-ergodic.*

3. $\sup\limits_{n,\, n>k} \dfrac{S^R_{kn}(\mu)}{b_{kn}} = C_k < \infty \quad \forall\, k \in \mathbb{Z}.$

Proof. The irreducibility was proved in [12]. Representation (13) follows from (4) and (10).

8. Tensor Product of Infinitely Many Elementary Representations and Irreducibility

Let $\{p\}$ be an infinite subset of $\mathbb{Z}$ with finitely many negative integers.

Theorem 4.

1. The representation $\otimes_{p_k \in \{p\}} T^{R,\,\mu_{p_k}}_{p_k}$ *is irreducible if and only if*

(i) $S^L_{p_k p_n}(\mu) = \infty \quad \forall\, p_k < p_n, \ \ p_k, p_n \in \{p\},$

(ii) the measure $\otimes_{p_k \in \{p\}} \mu_{p_k}$ *is* $B_0^{\mathbb{Z}}$*-right-ergodic.*

2. In this case, $\otimes_{p_k \in \{p\}} T^{R,\,\mu_{p_k}}_{p_k} = T^{R,\,\mu,\,\{p\}}$, *where* $\mu = \otimes_{p_k \in \{p\}} \mu_{p_k}$.

3. $T^{R,\,\mu,\,\{p\}} \sim T^{R,\,\mu',\,\{p'\}}$ *if and only if* $\{p\} = \{p'\}$ *and* $\mu \sim \mu'$.

4. The tensor product of two irreducible representations $T^{R,\,\mu,\,\{p\}} \otimes T^{R,\,\mu',\,\{p'\}}$ *is irreducible if and only if* $\{p\} \cap \{p'\} = \{\varnothing\}$ *and* $S^L_{p_k p'_n}(\mu \otimes \mu') = \infty \quad \forall\, p_k \in \{p\}, \ \ p'_n \in \{p'\}$.

Proof. The irreducibility and equivalence for $\{p\} = \{p'\} = (p_n)_{n=1}^{\infty}$, $p_n = n$ follows from Theorem 1.1 and Theorem 3.1 in [9]. For other infinite $\{p\}$ with finitely many negative integers, the proof of assertions 1 and 2 is the same.

Let us prove assertion 3 for a general $\{p\}$. The sufficiency is obvious. The proof of necessity is based on Theorem 1 (assertion 2) and Theorem 3.1 in [9]. Let $T^{R,\,\mu,\,\{p\}} \sim T^{R,\,\mu',\,\{p'\}}$, where $\{p\} = (p_1, p_2, \ldots)$ and $\{p'\} = (p'_1, p'_2, \ldots)$. We prove that $\{p\} = \{p'\}$ and $\mu \sim \mu'$. We assume that $p_1 \neq p'_1$, say, $p_1 > p'_1$, and consider the spectral measures $\mathbb{E}^{\mu}_{p_1}$ and $\mathbb{E}^{\mu'}_{p_1}$ of the restrictions of the representations $T^{R,\,\mu,\,\{p\}}$ and $T^{R,\,\mu',\,\{p'\}}$ to the subgroup $X_{p_1,\,0}$. The spectral measure $\mathbb{E}^{\mu}_{p_1}$ is the spectral measure of the commutative family of self-adjoint operators $i^{-1}\mathbb{D}_{p_1}(\mu) = \{ i^{-1} D_{p_1 n}(\mu) \}_{n=p_1+1}^{\infty}$; it is not trivial, but the spectral measure $\mathbb{E}^{\mu'}_{p_1}$ is trivial [see (9), (10)]. This contradicts the assumption that $T^{R,\,\mu,\,\{p\}} \sim T^{R,\,\mu',\,\{p'\}}$ and, therefore, $p_1 = p'_1$. In this case, the spectral measures $\mathbb{E}^{\mu}_{p_1}$ and $\mathbb{E}^{\mu'}_{p_1}$ are equivalent, whence $\mu_{p_1} \sim \mu'_{p_1}$ and $T^{R,\,\mu_{p_1}}_{p_1} \sim T^{R,\,\mu'_{p_1}}_{p_1}$. Since, by virtue of (13), we have

$$T^{R,\mu,\{p\}} = T_{p_1}^{R,\mu_{p_1}} \otimes T^{R,\mu^{\{p_2\}},\{p_2\}}, \qquad T^{R,\mu',\{p'\}} = T_{p_1}^{R,\mu'_{p_1}} \otimes T^{R,\mu'^{\{p'_2\}},\{p'_2\}},$$

and $T^{R,\mu,\{p\}} \sim T^{R,\mu',\{p'\}}$, we conclude that $T^{R,\mu^{\{p_2\}},\{p_2\}} \sim T^{R,\mu'^{\{p'_2\}},\{p'_2\}}$, where $\{p_2\} = (p_2, p_3, \ldots)$, $\{p'_2\} = (p'_2, p'_3, \ldots)$, and

$$T^{R,\mu,\{p_2\}} = \otimes_{p_k \in \{p_2\}} T_{p_k}^{R,\mu_{p_k}}, \qquad T^{R,\mu',\{p'_2\}} = \otimes_{p_k \in \{p'_2\}} T_{p_k}^{R,\mu'_{p_k}}.$$

By analogy, we establish that $p_2 = p'_2$ and $\mu_{p_2} \sim \mu'_{p_2}$. Finally, $\{p\} = \{p'\}$ and $\mu_{p_k} \sim \mu'_{p_k}$ $\forall p_k \in \{p\} = \{p'\}$. For finite $\{p\}$ and $\{p'\}$, the proof is completed because, in this case, we have $\mu = \otimes_{p_k \in \{p\}} \mu_{p_k} \sim \mu' = \otimes_{p_k \in \{p'\}} \mu'_{p_k}$. In the general case (for infinite $\{p\}$ and $\{p'\}$), the equivalence $\mu_{p_k} \sim \mu'_{p_k}$ $\forall p_k \in \{p\} = \{p'\}$ does not yield $\mu = \otimes_{p_k \in \{p\}} \mu_{p_k} \sim \mu' = \otimes_{p_k \in \{p'\}} \mu'_{p_k}$. In the particular case $\{p\} = (p_k)_{k=1}^{\infty}$, $p_k = k$, $k \in \mathbb{N}$, the equivalence of the measures $\mu \sim \mu'$ follows from Theorem 3.1 in [9]. For general $\{p\}$, the proof is the same.

4. The sufficiency follows from assertions 1 and 2 because, in this case, we have

$$T^{R,\mu,\{p\}} \otimes T^{R,\mu',\{p'\}} = T^{R,\mu\otimes\mu',\{p\}\cup\{p'\}},$$

where $\{p\} \cup \{p'\} = \{p_k, p'_n \mid p_k \in \{p\},\ p'_n \in \{p'\}\}$. Now let $\{p\} \cap \{p'\} = \{p''\}$ be finite, $\{p''\} := (p_1, \ldots, p_k)$. For infinite $\{p''\}$, the proof is the same. In this case, we have $\{p\} = \{q\} \cup \{p''\}$ and $\{p'\} = \{q'\} \cup \{p''\}$, whence $\{p\} \cup \{p'\} = \{q\} \cup \{q'\} \cup \{p''\}$ and

$$T^{R,\mu,\{p\}} \otimes T^{R,\mu',\{p'\}} = T^{R,\mu^{\{q\}}\otimes\mu^{\{p''\}}\otimes\mu'^{\{q'\}},\{q\}\cup\{p''\}\cup\{q'\}} \otimes T^{R,\mu'^{\{p''\}},\{p''\}}.$$

Thus, the proof of the fact that the last tensor product is reducible is analogous to the proof of the fact that the tensor product

$$T^{R,\mu,q} \otimes T^{R,\mu',q+k}$$

is reducible.

Consider an essentially bounded function $a : X^q \ni x \mapsto a(x) \in \mathbb{C}^1$ and let A_0 be the operator of multiplication in the space

$$H^q(\mu) \otimes H^{q+k}(\mu') = L^2(X^q, d\mu) \otimes L^2(X^{q+k}, d\mu') = L^2(X^q \otimes X^{q+k}, d\mu \otimes \mu')$$

by a function $a_0 : X^q \times X^{q+k} \ni (x, y, z) \mapsto a_0(x, y, z) = a(yx^{-1}) \in \mathbb{C}^1$. We show that the representation $T^{R,\mu,q} \otimes T^{R,\mu',q+k}$ commutes with A_0. Indeed, for any function $f(x, y, z) \in L^2(X^q \otimes X^{q+k}, d\mu \otimes \mu')$, by using the property that $(y, z) = zy$ for any $(y, z) \in X^q \times X^k = X^{q+k}$ in $B^{\mathbb{Z}}$, we get

$$\begin{aligned}\left(T_t^{R,\mu,q}\otimes T_t^{R,\mu',q+k}A_0 f\right)(x,zy) &= \left(T_t^{R,\mu,q}\otimes T_t^{R,\mu',q+k}a_0 f\right)(x,zy)\\ &= \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2}\left(\frac{d\mu'(zyt)}{d'\mu(zy)}\right)^{1/2} a\big((yt)(xt)^{-1}\big)f(xt,zyt)\\ &= a(yx^{-1})\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2}\left(\frac{d\mu'(zyt)}{d\mu'(zy)}\right)^{1/2} f(xt,zyt)\\ &= \left(A_0\left(T_t^{R,\mu,q}\otimes T_t^{R,\mu',q+k}\right)f\right)(x,zy).\end{aligned}$$

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