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Anti-Wick Symbols for Infinite Products in K-Homology

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Abstract. We consider infinite products in K-homology. We study these products in relation with operators on filtered Hilbert spaces, and infinite iterations of universal constructions on C^*-algebras. In particular, infinite tensor power of extensions of pseudodifferential operators on $\mathbf{R}$ are considered. We extend anti-Wick pseudodifferential operators to infinite tensor products of spaces of the type $L^2(\mathbf{R})$, and compare our infinite tensor power construction with an extension of pseudodifferential operators on $\mathbf{R}^\infty$. We show that the K-theory connecting maps coincide. We propose a natural definition of ellipticity for anti-Wick operators on $\mathbf{R}^\infty$, compute the corresponding index, and draw some consequences concerning these operators.

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1. Introduction and Notations

Differential operators on infinite-dimensional spaces arise as natural objects in several contexts. Representations of infinite dimensional groups entail considering partial derivations with respect to an infinite number of variables (see [12]). Infinite tensor product representations of the Canonical Commutation Relations are studied in [15]. In a recent paper, N. Higson, G. Kasparov and J. Trout construct an inductive limit Hilbert space, on which a C^*-algebra, and an infinite dimensional analog of the harmonic oscillator act in a natural way (see [8, 18]). The purpose of this work is the study of infinite products in K-theory, and their relations with pseudodifferential operators on infinite product spaces. We construct an inductive limit C^*-algebra $\mathcal{B}$, and a filtered Hilbert space. We shall study different kinds of extension of $\mathcal{B}$ by an ideal of operators, associated to the filtration of this Hilbert space. The algebra $\mathcal{B}$ can be viewed as an algebra of symbols of pseudodifferential operators, and is closely related to the C^*-algebra which appears in [8]. However, our construction depends on the choice of coordinates. For this reason, $\mathcal{B}$ does not carry the action of a locally compact group. (This algebra is rather adapted to the setting of [12]. This will be discussed in a subsequent work.) The definition of $\mathcal{B}$ rests on infinite iterations of Kasparov product by the Bott element ([9–11]). We use the description, due to J. Cuntz ([6]), of K-homology by homomorphisms from universal constructions qA, or ϵA, to the compact operators. The Bott periodicity theorem in K-homology can be described as a KK-equivalence of the C^*-algebras $\epsilon\mathbf{C}$, $\epsilon\mathbf{C} \otimes q\mathbf{C}$, and $\epsilon\mathbf{C} \otimes \mathbf{C}_0(\mathbf{R}^2)$. We shall define a homomorphism from $\epsilon\mathbf{C}$ to an algebra of matrices over $\epsilon\mathbf{C} \otimes \mathbf{C}_0(\mathbf{R}^2)$, which realizes the equivalence. This map will be constructed in a way compatible with the projection at the origin, of $\mathbf{R} \times \mathbf{R}^2$ onto $\mathbf{R}$. Iterating the homomorphism will give rise to our inductive limit C^*-algebra, $\mathcal{B}$, with a natural representation by local operators on a filtered Hilbert space $\mathcal{N}^\infty$. The subspaces $\mathcal{N}^n$ of the filtration are of the type $L^2(\mathbf{R}^{2n})$. We relate $\mathcal{B}$ to inductive limits of operators of pointwise multiplication by continuous functions on the odd-dimensional cospheres S^{2n+1}, hence to symbols of anti-Wick pseudodifferential operators. The natural frame for these inductive limit operators is a Fredholm theory relative to the sequence of the commutants of the projections on the subspaces $\mathcal{N}^n$. Local operators relative to the filtration are bounded linear operators on $\mathcal{N}^\infty$, which fulfill an approximate compatibility condition with the filtration. Compact operators relative to the filtration are defined accordingly and fit into an exact sequence analogous to the usual Calkin exact sequence. Fredholm operators relative to the filtration are classified by the map Λ-ind, which is defined in [13]. Our construction yields an extension of $\mathcal{B}$ by $\mathcal{K}_{\mathrm{loc}}(\mathcal{N}^\infty)$, entailing pseudodifferential operators on $\mathbf{R}^\infty$. This extension can be viewed as an internal product of an infinite number of KK-elements. Infinite external products will correspond to another type of extension of $\mathcal{B}$ by $\mathcal{K}_{\mathrm{loc}}(\mathcal{N}^\infty)$. One does not obtain, from external product, operators which can be interpreted as pseudodifferential operators. However, the corresponding extension is homotopic, in a suitable sense, to an extension of $\mathcal{B}$ by the usual compact operators on $\mathcal{N}^\infty$. This makes the connecting map in K-theory computable in a simple way. To relate these two types of constructions, we shall appeal to a notion of local homotopy which is weaker than homotopy but under which the index map Λ-ind remains invariant. This enables us to show that the K-theory connecting maps coincide and to derive some consequences concerning elliptic operators on the space $\mathbf{R}^\infty$. The universal construction qA is introduced in [6]. Its odd analogue ϵA is studied in [21]. We refer the reader to [1, 19] for details concerning infinite tensor products of Hilbert spaces, and to [13] for Fredholm theory on filtered Hilbert spaces. Let us now introduce some notations.

We denote by $C_b(Y), C_0(Y)$ the C^*-algebras of continuous bounded, respectively, vanishing at infinity, functions on a locally compact space Y. We shall denote by S the C^*-algebra $C_0(\mathbf{R})$, and, for any C^*-algebra A, we denote by $SA = S \otimes A$ the suspension of A. We let $\mathcal{M}(A)$ be the C^*-algebra of multipliers of A, and $\tilde{A}$ be the C^*-algebra generated by A, and the unit element of $\mathcal{M}(A)$. We identify $\tilde{S}$ with the C^*-algebra of continuous functions on the circle $\mathbf{T}$. We shall denote by $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ the $n \times n$ diagonal matrix whose entries are $\operatorname{diag}(a_1, a_2, \ldots, a_n)_{j,j} = a_j$, $1 \leqslant j \leqslant n$. As usual, $M_n(A)$ will denote the algebra of $n \times n$ matrices over the C^*-algebra A. Given a morphism ϕ, between two C^*-algebras A, and B, we shall call amplification of ϕ the morphism $\phi \otimes \mathrm{Id}_{M_n}$ from $M_n(A)$ to $M_n(B)$. When no confusion can arise, the morphism ϕ and its amplification to matrices will be denoted in the same way. If $D = A/J$ is an exact sequence of C^*-algebras and a is an element of A, the image of a in D will be denoted by $a + J$.

Given an infinite-dimensional separable Hilbert space $\mathcal{H}$, we denote by $\mathcal{K}(\mathcal{H})$ the C^*-algebra of compact operators on $\mathcal{H}$, by $\mathcal{L}(\mathcal{H})$ the C^*-algebra of bounded linear operators on $\mathcal{H}$. The norm of a bounded operator $T \in \mathcal{L}(\mathcal{H})$, will be denoted, as usual, by $\|T\|$. Given an operator $T \in \mathcal{L}(\mathcal{H})$, whose norm is less or equal to 1, we set

$$\mathcal{U}(T) = \begin{pmatrix} T & \sqrt{1 - TT^*} \\ -\sqrt{1 - T^*T} & T^* \end{pmatrix}.$$

Given a vector $\xi \in \mathcal{H}$, we denote by $[\xi]$ the projection on the subspace $\mathbf{C}\xi \subset \mathcal{H}$. Inductive limits of spaces will be denoted by

$$\mathcal{H}^\infty = \varinjlim_n (\mathcal{H}^n), \qquad \mathcal{N}^\infty = \varinjlim_n (\mathcal{N}^n), \text{ etc.}$$

Given a C^*-algebra A, we shall denote by $QA = A * A$ the free product of A by itself, and by qA the kernel of the map which identifies the two copies of A (see [6]). The algebra $Q\mathbf{C}$ is generated by two projections that we shall denote by p, and $\overline{p}$. The morphism of $Q\mathbf{C}$ onto $\mathbf{C}$ which sends p to 1, and $\overline{p}$ to 0 will be denoted by π_p. The morphism $\pi_{\overline{p}}$ is defined accordingly. We shall denote by EA the universal C^*-algebra generated by A, and a self adjoint unitary F, by E_1A the ideal of EA generated by $(1 + F)A$, and by ϵA the ideal of E_1A generated by the commutators $[F, A]$. The algebra $E\tilde{S}$ is generated by the symmetry F, and a unitary, of $\tilde{S}$, that we shall denote by ζ. Given A and B two C^*-algebras, we let π_ϵ be the canonical projection from $\epsilon M_n(A \otimes B)$ to $M_n(\epsilon A \otimes B)$, which sends the symmetry F to $1_{M_n} \otimes F \otimes 1_B$. (Here, F denotes the symmetry of the multipliers of ϵA, and, for any C^*-algebra D, 1_D denotes the unit element of the C^*-algebra $\tilde{D}$.)

When a tensor product of C^*-algebras $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ is considered, the elements of each tensorial factors are considered as generators in $\mathcal{M}(A_1 \otimes A_2 \otimes \cdots \otimes A_n)$. We shall use subscripts $1, 2, \ldots, n$ to refer to elements of the tensorial factors $A_1, A_2, \ldots, A_n$. Similar notations hold when considering products of spaces, and coordinates. We shall use the notations z_k, z'_k, ω_k, etc., to refer to complex coordinates in the unit ball of $\mathbf{C}^n$. We shall use the notations X_k for complex coordinates in $\mathbf{C}^n \equiv T^*\mathbf{R}^n$. Complex conjugates will be denoted by $\bar{X}_k$, $\bar{z}_k$, etc... Derivation with respect to a variable t will be denoted by ∂_t.

2. Iterated Constructions with Universal C^*-algebras

In this section, we construct an algebra which will serve as a replacement of continuous functions, when considering the infinite-dimensional Euclidean space $\mathbf{R}^\infty$ instead of $\mathbf{R}^n$. The Bott element is described as a homomorphism from the C^*-algebra $\epsilon\mathbf{C}$ to $M_4(\epsilon\mathbf{C} \otimes \epsilon S)$. This homomorphism extends, in a natural way, to matrices over $\epsilon\mathbf{C} \otimes (\epsilon S)^{\otimes n}$. We form the corresponding inductive limit C^*-algebra. At the level of KK-theory, this amounts to considering the Kasparov product of an infinite number of successive suspensions of the Bott element. Each subalgebra $\epsilon\mathbf{C} \otimes (\epsilon S)^{\otimes n}$ corresponds to operators of pointwise multiplication on the space $L^2(\mathbf{R} \times \mathbf{C}^n)$. In view of representing our algebra in a way similar to an algebra of symbols, the inductive limit homomorphisms are required to be compatible with evaluation at the origin, from $C_0(\mathbf{R} \times \mathbf{C}^n \times \mathbf{C})$, onto $C_0(\mathbf{R} \times \mathbf{C}^n)$. We recall that analog constructions were already set up in the work of Higson, et al. ([8]), in a different manner. A slightly different version of the inductive limit C^*-algebra constructed here has also been studied in [13].

2.1. STRICTLY POSITIVE ELEMENTS IN $q\mathbf{C}$, AND IN $\epsilon\mathbf{C}$

The product in KK-theory is based on the existence of suitable positive multipliers M, and N, which is established by the Kasparov Technical Theorem ([10]). In order to carry over the product to an infinite number of elements, a special choice of M and N has to be made. The construction of these operators entails considering some strictly positive elements of the C^*-algebra $q\mathbf{C}$, and of $\epsilon\mathbf{C}$. (We shall follow in this subsection, the general method of [10].)

DEFINITION 2.1.1. Let A be a C^*-algebra, and k be an element of A. Then k is said to be strictly positive in A, if, for any self-adjoint element a of A, and for any real $\epsilon > 0$, there exists $C > 0$, such that $a < Ck + \epsilon$ in $\tilde{A}$.

Remark 2.1.2. Suppose that there exists a sequence (x_n) of elements of A, whose linear span is dense in A. Then, for k be strictly positive in A, it suffices that for any integer n, there exists $C_n > 0$, such that $x_n^* x_n < C_n k$.

Indeed, for any positive element $a \in A$, and any $\epsilon > 0$, there exists a finite linear combination $S = \sum_{n<n_0} c_n x_n$, $c_n \in \mathbf{C}$, such that $\|S^*S - a\| < \epsilon$. Since, by Cauchy–Schwarz inequality $S^*S \leqslant n_0 \sum_{n<n_0} |c_n|^2 x_n^* x_n$, we obtain the desired majoration for positive elements. Assume now that a is self-adjoint. There is a unique pair (a_+, a_-) of positive elements of A, such that $a_+a_- = 0$, and that $a = a_+ - a_-$. The inequality $a \leqslant a_+ + a_-$ shows that the desired majoration extends to self-adjoint elements. The following criterion will be applied to the definition of M and N.

LEMMA 2.1.3 (Kasparov, [10]). *Let A be a unital C^*-algebra and let x, y be two elements of A, with x positive. The existence of the norm limit* $\lim_{\alpha\to 0^+}(x+\alpha)^{-1}y$ is equivalent to the condition: $\forall\epsilon > 0\,,\ \exists C > 0\,/\,yy^* \leqslant Cx^4 + \epsilon x^2$. □

PROPOSITION 2.1.4. (i) *Let p and $\overline{p}$ be universal projections generating the algebra $Q\mathbf{C}$. Let e and F be the universal projection and symmetry generating $E\mathbf{C}$. Set $q = p - \overline{p}$, and $de = i[F, e]$. Then q^2 (resp. de^2) is a strictly positive central element of $q\mathbf{C}$ (resp. $\epsilon\mathbf{C}$.)*

(ii) *The element*

$$M = (de^2 \otimes 1 + 1 \otimes q^2)^{-1}(1 \otimes q^2) + (de^2 \otimes q^2)(1 + de^2 \otimes 1)^{-1}$$

is well defined as a positive central multiplier of $\epsilon\mathbf{C} \otimes q\mathbf{C}$.

Proof. (i) Since $pq = p(p - \overline{p}) = q - q\overline{p}$, we have $pq^2 = q^2 p$. Similarly, $\overline{p}q^2 = q^2\overline{p}$. It follows that q^2 is a central element in $Q\mathbf{C}$. The C^*-algebra $q\mathbf{C}$ is generated, as a linear space, by the set $G = \{q^n, pq^n / n > 0\}$. For any $x \in G$, there exists a positive real number C_x, such that $x^*x \leqslant C_x q^2$.

Remark 2.1.2 shows that q^2 is a strictly positive element of $q\mathbf{C}$. In a similar way, one shows that de^2 is central and strictly positive. A generating set of $\epsilon\mathbf{C}$ is: $G = \{de^n, ede^n, Fde^n, Fede^n / n > 0\}$.

(ii) To show that M is a multiplier of $\epsilon\mathbf{C} \otimes q\mathbf{C}$, it suffices to show that for any $z \in \epsilon\mathbf{C} \otimes q\mathbf{C}$, the norm limit $\lim_{\alpha\to 0^+}(de^2 \otimes 1 + 1 \otimes q^2 + \alpha)^{-1}(1 \otimes q^2)z$ exists. Since $(de^2 \otimes 1 + 1 \otimes q^2 + \alpha)^{-1}(1 \otimes q^2) \leqslant 1, \forall\alpha > 0$, we can restrict our attention to elements of the form $z = x \otimes y$, with x a positive element of $\epsilon\mathbf{C}$, and y a positive element of $q\mathbf{C}$. We shall establish the following: $\forall\epsilon > 0, \exists C > 0,\ /((1 \otimes q^2)(x \otimes y))^2 \leqslant C(de^2 \otimes 1 + 1 \otimes q^2)^4 + \epsilon(de^2 \otimes 1 + 1 \otimes q^2)^2$. The result will follow from Lemma 2.1.3. For any $\epsilon > 0$, there exists a positive real number r, such that $x \otimes 1 \leqslant r(de^2 \otimes 1) + \epsilon$ and $1 \otimes y \leqslant r(1 \otimes q^2) + \epsilon$. Since $-1 \leqslant -\overline{p} \leqslant q \leqslant p$, we have $q^2 \leqslant 1$, hence $de^2 \otimes q^2 \leqslant de^2 \otimes 1 \leqslant de^2 \otimes 1 + 1 \otimes q^2$. It follows from these inequalities that $x \otimes y \leqslant (r^2 + 2\epsilon r)(de^2 \otimes 1 + 1 \otimes q^2) + \epsilon^2$, and that

$$(1 \otimes q^2)(x \otimes y) \leqslant (r^2 + 2\epsilon r)(de^2 \otimes 1 + 1 \otimes q^2)^2 + \epsilon^2(de^2 \otimes 1 + 1 \otimes q^2).$$

Raising to the square each side of this inequality we obtain

$$\begin{aligned}((1 \otimes q^2)(x \otimes y))^2 \ &\leqslant\ 2[(r^2 + 2\epsilon r)^2(de^2 \otimes 1 + 1 \otimes q^2)^4 + \\ &\quad + \epsilon^4(de^2 \otimes 1 + 1 \otimes q^2)^2].\end{aligned}$$ □

THEOREM 2.1.5. *There exist two positive central elements M and N of $\mathcal{M}(\epsilon\mathbf{C} \otimes q\mathbf{C})$ such that:*

(i) $M + N = 1$,
(ii) $M(\epsilon\mathbf{C} \otimes Q\mathbf{C}) + N(E\mathbf{C} \otimes q\mathbf{C}) \subset \epsilon\mathbf{C} \otimes q\mathbf{C}$,
(iii) $\mathrm{Id} \otimes \pi_p(M) = 1,\ \mathrm{Id} \otimes \pi_p(N) = 0$,
(iv) *M and N are invariant under the exchange of the projections p and $\overline{p}$.*

Proof. In view of Proposition 2.1.4, it suffices to show (ii). Let x be a positive element of $\epsilon\mathbf{C}$. For any positive real ϵ, there exists a positive real number r, such that

$$x \otimes p \leqslant (rde^2 \otimes 1 + \epsilon)(1 \otimes p) \leqslant r(de^2 \otimes 1 + 1 \otimes q^2) + \epsilon.$$

It follows that

$$(x \otimes p)(1 \otimes q^2) \leqslant r(de^2 \otimes 1 + 1 \otimes q^2)^2 + \epsilon(de^2 \otimes 1 + 1 \otimes q^2).$$

Proceeding as in the proof of Proposition 2.1.4, one shows that the norm limit $\lim_{\alpha\to 0^+}(de^2 \otimes 1 + 1 \otimes q^2 + \alpha)^{-1}(x \otimes p)(1 \otimes q^2)$ exists, hence, is an element of $\epsilon\mathbf{C} \otimes q\mathbf{C}$. We omit the proof for N, which is analogue. □

2.2. CANONICAL HOMOMORPHISMS AND THE C^*-ALGEBRA $\mathcal{B}$

DEFINITION 2.2.1. Let λ be the homomorphism defined on generators by

$$\begin{aligned}
\lambda\colon \epsilon\mathbf{C} &\to M_2(\epsilon\mathbf{C} \otimes q\mathbf{C}),\\
e &\to \begin{pmatrix} e \otimes p & 0 \\ 0 & e \otimes \overline{p} \end{pmatrix},\\
F &\to \begin{pmatrix} \sqrt{M}(F \otimes 1) & \sqrt{N} \\ \sqrt{N} & -\sqrt{M}(F \otimes 1) \end{pmatrix}.
\end{aligned}$$

The map $\lambda\colon E\mathbf{C} \to M_2(E\mathbf{C} \otimes Q\mathbf{C})$ restricts to $\epsilon\mathbf{C} \to M_2(\epsilon\mathbf{C} \otimes q\mathbf{C})$ thanks to Theorem 2.1.5 (ii).

The following proposition will be useful to carry over the external product in K-homology to infinite tensor powers (see 4.2.6).

PROPOSITION 2.2.2. *The composition $\pi_p\lambda\colon \epsilon\mathbf{C} \to M_2(\epsilon\mathbf{C})$ is the embedding of $\epsilon\mathbf{C}$ in the upper left corner in 2×2 matrices.*

Proof. This follows from (iii) in 2.1.5 □

DEFINITION 2.2.3. Let F and ζ be the generators of $E\tilde{S}$. Recall that $\zeta\zeta^* = \zeta^*\zeta$ is the unit of $\tilde{S}$, and that $F^2 = 1$, is the unit of $E\tilde{S}$. Consider the unitary $u = \zeta + 1 - \zeta\zeta^*$. Since $\zeta - \zeta\zeta^* \in S$, $[F, u] \in \epsilon S$, and u is an element of $\mathcal{M}(\epsilon S)$. Set $P = (1 + F)/2$, and $T = PuP$. Let μ be the homomorphism defined, on generators by:

$$\begin{aligned}
\mu\colon q\mathbf{C} &\to M_2(\epsilon S),\\
p &\to \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix},\\
\overline{p} &\to W^*\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}W,
\end{aligned}$$

with

$$W = \begin{pmatrix} T & (P - TT^*)^{1/2} \\ -(P - T^*T)^{1/2} & T^* \end{pmatrix} + \begin{pmatrix} 1 - P & 0 \\ 0 & 1 - P \end{pmatrix}.$$

DEFINITION 2.2.4. We shall denote by j the composition $j = (\mathrm{Id} \otimes \mu)\lambda \colon \epsilon\mathbf{C} \to M_4(\epsilon\mathbf{C} \otimes \epsilon S)$. Given any integer $n \in \mathbf{N}$, we define $j_{n+1} \colon M_{4^n}(M_2(\epsilon\mathbf{C}) \otimes (\epsilon S)^{\otimes n}) \to M_{4^n}(M_2(\epsilon\mathbf{C}) \otimes (\epsilon S)^{\otimes n}) \otimes M_4(\epsilon S)$ accordingly. We set $\mathcal{B}_{n+1} = M_{4^n}(M_2(\epsilon\mathbf{C}) \otimes (\epsilon S)^{\otimes n})$, and denote by $\mathcal{B}$ the inductive limit C^*-algebra of the system $(\mathcal{B}_n, j_n)_{n \geqslant 1}$.

Note that, through the identification $\epsilon\mathbf{C} \simeq M_2(S)$, explained after Definition 3.2.3, we have $\pi_\epsilon(\epsilon(M_2(\epsilon\mathbf{C}))) \simeq M_4(\epsilon S)$ and, for any integer n, $\pi_\epsilon(\epsilon\mathcal{B}_n) \simeq (M_4(\epsilon S))^{\otimes n}$. This is the reason why we have chosen to use 2×2 matrices over $\epsilon\mathbf{C}$, rather that $\epsilon\mathbf{C}$ itself in the definition of $\mathcal{B}_n$. This is not essential but this choice will make algebras and spaces of representation easier to handle, especially in Section 4.

PROPOSITION 2.2.5. (i) *For any integer* n, $K_0(\mathcal{B}_n) = 0$, $K_1(\mathcal{B}_n) = \mathbf{Z}$, *and* j_n *is a* KK*-equivalence*; (ii) $K_0(\mathcal{B}) = 0$, $K_1(\mathcal{B}) = \mathbf{Z}$.

Proof. Recall that $\epsilon\mathbf{C}$ is KK-equivalent to S, and that ϵS is KK-equivalent to $\mathbf{C}$ ([21]). In the setting of [6], the maps (j_n) correspond to products with the Bott element. Passing to inductive limits, we obtain the K-theory of $\mathcal{B}$. □

3. Anti-Wick Operators on Infinite-Dimensional Euclidean Space $\mathbf{R}^\infty$

We shall represent the C^*-algebra $\mathcal{B}$, constructed in 2.2.4, as an algebra of operators on a (filtered) Hilbert space $\mathcal{N}^\infty$, in a way analogous to symbols of pseudodifferential operators acting by pointwise multiplication by functions on the cotangent space. We also describe the associated extension of anti-Wick operators on $\mathbf{R}^\infty$. We begin by recalling the class of anti-Wick symbols on $\mathbf{C}^n$ and the anti-Wick isometry I. We give explicit formulas describing an extension of anti-Wick symbols on $\mathbf{C}^{n+1}$ and the corresponding extension of pseudodifferential operators on $\mathbf{R}^{n+1}$. These extensions rest on a representation of the C^*-algebra $\mathcal{B}_n$ as continuous functions on the cosphere S^{2n+1} and on the usual lifting (completely positive map), which extends, along the ray to the origin, continuous functions on S^{2n+1} to the closed unit ball B^{2n+2}. The C^*-algebra$\mathcal{B}_n$ has a natural representation by continuous functions on the cylinder $\mathbf{T} \times (D^2)^n$ vanishing on the boundary of each of the disks D^2. The construction we have chosen consists in mapping this cylinder to a closed subset of the cosphere, by projecting each point along the ray to the origin. In this manner, we obtain continuous functions which extend naturally to the all cosphere. The construction carries over to infinite dimensional Euclidean space using the estimates of [13] and the special choice of the operators M, and N, made in Section 2 (see also 4.2.3).

3.1. OPERATORS ON $\mathbf{R}^n$

We recall the construction of the extension of anti-Wick operators on $\mathbf{R}^{n+1}$, and describe it as a C^*-algebra morphism, from $\epsilon(S\otimes(\epsilon S)^{\otimes n})$ to $M_{2^{n+1}}(\mathcal{K}(L^2(\mathbf{C}^{n+1})))$.

DEFINITION 3.1.1. The class of anti-Wick symbols (of order 0), $\Gamma(\mathbf{C}^n)$ consists of the functions $f \in C(\mathbf{C}^n)$, such that for any $z \in \mathbf{C}^n$, $|z| = 1$, the limit $\lim_{\rho\to\infty} f(\rho z)$ exists, uniformly in z, over all the unit sphere (see also [17]).

Set, for any $X \in \mathbf{C}^n$, $\mathcal{Z}(X) = X(1+|X|^2)^{-1/2}$. The map $\mathcal{Z}$ is a homeomorphism, from $\mathbf{C}^n$ onto the open unit ball $\overset{\circ}{B}{}^{2n} = \{z \in \mathbf{C}^n/|z| < 1\}$. An anti-Wick symbol of order 0 is a continuous function f on $\mathbf{C}^n$, such that $f\circ\mathcal{Z}^{-1}$ extends to a continuous function on the closure of $\overset{\circ}{B}{}^{2n}$.

DEFINITION 3.1.2. (i) Let $I\colon L^2(\mathbf{R}) \hookrightarrow L^2(\mathbf{C})$ be the anti-Wick isometry (see [17, 13].) We shall denote by $P^I = II^*$ the projection on the subspace $I(L^2(\mathbf{R}))$. Raising I to the tensorial power n yields the anti-Wick isometry $I^{\otimes n}\colon L^2(\mathbf{R}^n) \hookrightarrow L^2(\mathbf{C}^n)$. When no confusion can arise, we shall refer to $I^{\otimes n}$ by the letter I, independently of the integer n. We use similar convention for the projection P^I. When necessary, we shall refer to tensorial factors by subscripts. For instance, $I^{\otimes n} = I_1\otimes I_2\otimes\cdots\otimes I_n$ and use similar notations for the projection P^I.

(ii) Let f be an anti-Wick symbol. We shall denote by M_f the operator of pointwise multiplication by f on $L^2(\mathbf{C}^n)$. Explicitly,

$$M_f\xi(X) = f(X)\xi(X),\ \forall\xi \in L^2(\mathbf{C}^n),\ X \in \mathbf{C}^n.$$

The operator with anti-Wick symbol f is defined by $D_f = I^*M_fI$. The notation D_f will equally refer to the compression of M_f to the subspace $I(L^2(\mathbf{R}^n))$ of $L^2(\mathbf{C}^n)$.

If f vanishes at infinity, D_f is compact, and $\forall f, g \in \Gamma(\mathbf{C}^n)$, $D_{fg} - D_fD_g$ is compact. For completeness, we recall the formula

$$(I(\xi))(X) = \pi^{-1/4}\int \xi(t)\mathrm{e}^{-(t-a)^2/2}\mathrm{e}^{-itb}\mathrm{d}t, \quad \forall\xi \in L^2(\mathbf{R}),\ X = a+ib \in \mathbf{C}$$

(see [17]). It will be also convenient to consider functions admitting continuous extensions along the rays which are contained in some specified hyperplanes in $\mathbf{C}^n$. These partial symbols are defined in 5.1.1

Before turning to the construction of anti-Wick operators, let us recall some definitions, and fix notations, which will be necessary in the sequel. An (invertible) extension of a C^*-algebra A by a C^*-algebra B is a completely positive continuous map, of norm less or equal to 1, $\sigma\colon A \to \mathcal{M}(B)$, such that, for any $a, b \in A$, $\sigma(ab) - \sigma(a)\sigma(b) \in B$. Such a map provides a C^*-algebra homomorphism $\dot{\sigma}\colon A \to \mathcal{M}(B)/B$ (the Busby invariant of the extension). There is an exact sequence $A = D/B$, where D is any C^*-algebra isomorphic to the fibered product $A \oplus_{\dot{\sigma}} \mathcal{M}(B) = \{(a, m) \in A \oplus \mathcal{M}(B)/\dot{\sigma}(a) = m + B\}$. If A is separable, the Stinespring extension theorem shows that we can enlarge σ to a C^*-algebra homomorphism, ϕ: $A \to M_2(\mathcal{M}(B))$ and find a projection $P \in M_2(\mathcal{M}(B))$ such that $[\phi(A), P] \subset M_2(B)$, and that, $\forall a \in A$, $\sigma(a) = P\phi(a)P$. The pair (ϕ, P) is alternatively described by a homomorphism, $\mathcal{E}$, from ϵA to $M_2(B)$. The data $\mathcal{E}, \sigma$, and $\dot{\sigma}$ are in one to one correspondence, up to homotopy (a homotopy is a continuous path of maps in the relevant category). For the sake of simplicity, we shall equally refer to an extension by the morphism $\mathcal{E} = (\phi, P)$, the completely positive map σ or the Busby invariant $\dot{\sigma}$. The set of such extensions $A = D/B$ will be denoted by $\mathrm{Ext}(A, B)$. We refer the reader to [4, 9, 10] for more information.

DEFINITION 3.1.3. Denote by ζ the unitary, and by P the projection generating $E\tilde{S}$ (see 2.2.3.) Let β be the homomorphism defined on generators by

$$\begin{aligned} \beta\colon \epsilon S &\to M_2(C(D^2)), \\ \zeta &\to \begin{pmatrix} z & (1-|z|^2)^{1/2} \\ -(1-|z|^2)^{1/2} & \overline{z} \end{pmatrix} = \mathcal{U}(z), \\ P &\to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Considering S as the ideal of $C(\mathbf{T})$, consisting of the continuous functions vanishing at the argument $\theta = 0$, we define, for any integer n, $\mathrm{Id}_S \otimes \beta_n$: $S \otimes (\epsilon S)^{\otimes n} \to M_{2^n}(C(\mathbf{T} \times (D^2)^n))$ accordingly.

DEFINITION 3.1.4. Let

$$S^{2n+1} = \{(z'_1, z'_2, \ldots, z'_{n+1}) \in \mathbf{C}^{n+1} / \sum |z'_i|^2 = 1\}$$

be the unit sphere in $\mathbf{C}^{n+1}$, and

$$F_{2n+1} = \{(z'_1, z'_2, \ldots, z'_n) \in S^{2n+1} / |z'_i| \leqslant |z'_1|,\ \forall i \geqslant 1\}.$$

We denote by h_n: $\mathbf{T} \times (D^2)^n \to F_{2n+1}$ the homeomorphism defined, in coordinates, by

$$h_n((z_1, z_2, \ldots, z_{n+1})) = (z_1, z_2, \ldots, z_{n+1})/\left(\sum_{i\geqslant 1} |z_i|^2\right)^{1/2},$$

and by h_{n*}: $C(\mathbf{T} \times (D^2)^{\times n}) \to C(F_{2n+1})$ the C^*-algebra morphism $h_{n*}(f) = f \circ h_n^{-1}$.

Let $(\omega_1, \omega_2, \ldots, \omega_{n+1})$ be an $n+1$-tuple of pairwise commuting normal elements, such that $\sum_{i=1}^{n+1} \omega_i^* \omega_i \leqslant 1$. The universal C^*-algebra generated by the ω_i's is the algebra of continuous functions on B^{2n+2}. Similarly, we denote the generators of $C(S^{2n+1})$ by $z'_1, z'_2, \ldots, z'_{n+1}$.

DEFINITION 3.1.5. For any integer n, we let ρ_n be the homomorphism defined on generators by

$$\begin{aligned}\rho_n\colon C(S^{2n+1}) &\to M_2(C(B^{2n+2})),\\ z'_1 &\to \begin{pmatrix} \omega_1 & (1-\sum_{i\geqslant 1}|\omega_i|^2)^{1/2} \\ -(1-\sum_{i\geqslant 1}|\omega_i|^2)^{1/2} & \overline{\omega}_1 \end{pmatrix},\\ z'_i &\to \begin{pmatrix} \omega_i & 0 \\ 0 & \omega_i \end{pmatrix}, \quad \forall i>1.\end{aligned}$$

The map β_n of Definition 3.1.3 sends $(\epsilon S)^{\otimes n}$ to functions on $(D^2)^n$ vanishing on the boundary $\partial((D^2)^n)$. Since the map h_n sends $\mathbf{T}\times\partial((D^2)^n)$ onto the boundary of F_{2n+1}, we shall consider functions in the image of $h_{n*}\beta_n$ as functions on the sphere S^{2n+1}, which vanish outside the interior of F_{2n+1}. We can now make the following definition:

DEFINITION 3.1.6. The anti-Wick extension, $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}]\colon \epsilon(S\otimes(\epsilon S)^{\otimes n}) \to M_{2^{n+1}}(\mathcal{K}(L^2(\mathbf{C}^{n+1})))$ is the C^*-algebra morphism associated to the couple $((\mathrm{Id}_{M_2}\otimes \rho_n)(\mathrm{Id}_{M_2}\otimes h_{n*})(\mathrm{Id}_S\otimes\beta_n)\,,\ \mathrm{Id}_{M_{2^{n+1}}}\otimes P^I)$ (see 3.1.2 and the following discussion).

PROPOSITION 3.1.7. *The morphism $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}]$ is expressed, using complex coordinates $(X_1, X_2, \ldots, X_n)\in\mathbf{C}^{n+1}$ on generators by*

$$\begin{aligned}\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}]\colon \epsilon(S\otimes(\epsilon S)^{\otimes n}) &\to M_{2^{n+1}}(\mathcal{K}(L^2(\mathbf{C}^{n+1})))\\ \forall 1\leqslant j\leqslant n+1,\ \zeta_j &\to (\mathrm{Id}_{M_2})^{\otimes j-1}\otimes\mathcal{U}(X_j(1+|X_1|^2)^{-1/2})\\ &\qquad \otimes(\mathrm{Id}_{M_2})^{\otimes n+1-j}\\ \forall 1< j\leqslant n+1,\ P_j &\to (\mathrm{Id}_{M_2})^{\otimes j-1}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes(\mathrm{Id}_{M_2})^{\otimes n+1-j}\\ P_1 &\to \begin{pmatrix}P_1^I&0\\0&0\end{pmatrix}\otimes\left[\bigotimes_{j=2}^{n+1}\begin{pmatrix}P_j^I&0\\0&P_j^I\end{pmatrix}\right]=Q\end{aligned}$$

Here ζ_1 is the generator of $C(\mathbf{T})$ which corresponds to the first tensorial factor S. Similarly, $\zeta_j,\ j>1$ refers to the $(j-1)th$ tensorial factor ϵS. Similar subscript conventions are used for referring to projections.

Proof. These formulas are established by a straightforward computation. We remark that for each integer j, the unitary $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}](\zeta_j)$ is a continuous function, defined on the closed set

$$F'_{2n+1}=\{(X_1,X_2,\ldots,X_{n+1})\in\mathbf{C}^{n+1}/\ |X_j|^2\leqslant 1+|X_1|^2\},$$

and that $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}](\zeta_j)$ is a diagonal matrix on the boundary of F'_{2n+1}. Note that, for $j > 1$, the unitary $\mathcal{U}(X_j(1+|X_1|^2)^{-1/2}$ is not defined outside F'_{2n+1}. But the commutator

$$\left[\mathcal{U}(X_j(1+|X_1|^2)^{-1/2}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

vanishes on the boundary of F'_{2n+1}. In consequence, we can, without changing this morphism, extend the image of ζ_j by setting $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}](\zeta_j) = (\mathrm{Id}_{M_2})^{\otimes j-1} \otimes \mathcal{U}(X_j|X_j|^{-1}) \otimes (\mathrm{Id}_{M_2})^{\otimes n+1-j}$ on the complement of F'_{2n+1} (or use any diagonal unitary matrix). Although the extended unitary is not a continuous function, our morphism $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}]$ is well defined since the commutator vanishes on the complement of the interior of F'_{2n+1}. □

The completely positive map $\sigma_{\mathrm{AW}}[n+1]$, associated to $D_{\mathrm{AW}}[\mathbf{C}^{n+1}]$ is defined by $\sigma_{\mathrm{AW}}[n+1](x) = QD_{\mathrm{AW}}[\mathbf{C}^{n+1}](x)Q$, $\forall x \in S \otimes (\epsilon S)^{\otimes n}$. Note that $\sigma_{\mathrm{AW}}[n+1](x)$ factors through the anti-Wick isometry $I\colon L^2(\mathbf{R}^{n+1}) \hookrightarrow L^2(\mathbf{C}^{n+1})$, and can be considered as an operator on $L^2(\mathbf{R}^{n+1}) \otimes \mathbf{C}^{n+1}$. In fact, the algebra $\sigma_{\mathrm{AW}}[n+1]$ $(S \otimes (\epsilon S)^{\otimes n}) + M_{2^{n+1}}(\mathcal{K}(L^2(\mathbf{R}^{n+1})))$ is generated by anti-Wick pseudodifferential operators on $\mathbf{R}^{n+1}$.

3.2. OPERATORS ON $\mathbf{R}^\infty$ AND ANTI-WICK SYMBOLS ON $\mathbf{C}^\infty$

We shall extend the construction of anti-Wick operators to $\mathbf{R}^\infty$. For completeness, we first recall some definitions on filtered Hilbert space and compact operators relative to filtrations (see [13]; remark that our notations are slightly different).

DEFINITION 3.2.1. Let $(\mathcal{H}^n, v_n)$ be a sequence of Hilbert spaces $\mathcal{H}^n$, and isometries $v_n\colon \mathcal{H}^n \hookrightarrow \mathcal{H}^{n+1}$. We shall denote by $\mathcal{H}^\infty$ the corresponding inductive limit Hilbert space and by Φ the filtration of $\mathcal{H}^\infty$ by the subspaces $\mathcal{H}^n$. For any integer n, we denote by $\omega_n\colon \mathcal{H}^n \hookrightarrow \mathcal{H}^\infty$ the canonical isometry and by P^n be the range projection of ω_n. For $m > n$, we let $v_n^m\colon \mathcal{H}^n \hookrightarrow \mathcal{H}^m$, be the composition of the $v_k's$, $n \leqslant k < m$. A sequence of operators on $(\mathcal{H}^\infty, \Phi)$ is a sequence of bounded linear operators $(T_n \in \mathcal{L}(\mathcal{H}^n))_{n\in\mathbf{N}}$. The sequence (T_n) is locally uniformly convergent to an operator $T \in \mathcal{L}(\mathcal{H}^\infty)$ if, for any integer n_0, the sequence $(\omega_n T_n v_{n_0}^n \omega_{n_0}^*)_{n \geqslant n_0}$ is norm convergent to $T\omega_{n_0}\omega_{n_0}^*$.

We recall from [13] that $L_{\mathrm{diag}}(\mathcal{H}^\infty)$ is the subalgebra of $\mathcal{L}(\mathcal{H}^\infty)$, consisting of the operators T such that $[T, P^m] = 0$, for all integers m, greater or equal to some integer N. We denote by $\mathcal{L}_{\mathrm{diag}}(\mathcal{H}^\infty)$ the Banach subalgebra of operators $T \in \mathcal{L}(\mathcal{H}^\infty)$, such that the sum $\sum_n \|[P^n, T]\|$ is finite, and by $\mathcal{L}_{\mathrm{loc}}(\mathcal{H}^\infty)$ the norm closure of $\mathcal{L}_{\mathrm{diag}}(\mathcal{H})$. Elements of $\mathcal{L}_{\mathrm{loc}}(\mathcal{H}^\infty)$ are said to be local (relative to Φ). We let $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$ be the ideal of $\mathcal{L}_{\mathrm{loc}}(\mathcal{H}^\infty)$, of operators whose compressions to the subspaces $\mathcal{H}^n$ are compact operators on $\mathcal{H}^n$. The following inclusions of closed two sided ideals hold: $\mathcal{K}(\mathcal{H}^\infty) \subset \mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty) \subset \mathcal{L}_{\mathrm{loc}}(\mathcal{H}^\infty)$. An operator $T \in \mathcal{L}_{\mathrm{loc}}(\mathcal{H}^\infty)$ is Fredholm relative to Φ if $T + \mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$ is an invertible element of $\mathcal{O}_{\mathrm{loc}}(\mathcal{H}^\infty) = \mathcal{L}_{\mathrm{loc}}(\mathcal{H}^\infty)/\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$. The set of Fredholm operators relative to Φ will be denoted by $\mathcal{F}_{\mathrm{loc}}(\mathcal{H}^\infty)$. Fredholm operators relative to Φ are classified by the map $\Lambda -$ ind. Given an operator $T \in \mathcal{F}_{\mathrm{loc}}(\mathcal{H}^\infty)$, there exists an integer N, such that, for any $m \geqslant N$, $P^m T P^m$ is a Fredholm operator on the space $\mathcal{H}^m$. We define $\Lambda - \mathrm{ind}(T)$ to be the class of the sequence $(\mathrm{ind}(P^m T P^m))_{m \geqslant N}$, in the group $\Pi_{i=1}^\infty \mathbf{Z}/\bigoplus_{i=1}^\infty \mathbf{Z}$. There is a corresponding dimension map, $\Lambda - \dim$, which is an isomorphism from $K_0(\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty))$ to $\Pi_{i=1}^\infty \mathbf{Z}/\bigoplus_{i=1}^\infty \mathbf{Z}$. Note that local uniform convergence implies strong convergence. The following criterion will be useful to establish local uniform convergence and the localness of the limit of sequences of operators on $(\mathcal{H}^\infty, \Phi)$.

PROPOSITION 3.2.2. *Let (T_n) be a sequence of operators on the filtered Hilbert space $(\mathcal{H}^\infty, \Phi)$. We assume that $\mathcal{H}^{n+1} = \mathcal{H}^n \otimes \mathcal{H}^1$. If $\sum_n \|(T_n \otimes \mathrm{Id} - T_{n+1})v_n\| < \infty$, the sequence (T_n) converges locally uniformly to an operator $T \in \mathcal{L}(\mathcal{H}^\infty)$, and*

$$\begin{aligned} \forall n \in \mathbf{N}, \quad \|[P^n, T]\| \leqslant & \sum_{j \geqslant n} \|(T_j \otimes \mathrm{Id} - T_{j+1})v_j\| + \\ & + \sum_{j \geqslant n} \|v_j^*(T_j \otimes \mathrm{Id} - T_{j+1})\| \end{aligned}$$

Proof. See [13]. □

We shall now proceed to define an extension of anti-Wick operators on $\mathbf{R}^\infty$.

DEFINITION 3.2.3. Let $\beta_{\mathrm{AW}}[\mathbf{R} \times \mathbf{C}^n]$ be the homomorphism defined on generators, by

$$\begin{aligned} \beta_{\mathrm{AW}}[\mathbf{R} \times \mathbf{C}^n] \colon S \otimes (\epsilon S)^{\otimes n} &\to M_{2^n}(S \otimes C_0(\mathbf{C}^n)), \\ \forall y \in S,\ y &\to y\,, \\ \forall 2 \leqslant j \leqslant n+1,\ \zeta_j &\to \mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}](\zeta_j)\,, \\ \forall 2 \leqslant j \leqslant n+1,\ P_j &\to \mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}](P_j). \end{aligned}$$

In other words, we have $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}] = (\beta_{\mathrm{AW}}[\mathbf{R} \times \mathbf{C}^n], Q)$ (the notation Q was introduced in Proposition 3.1.7).

The C^*-algebra $\epsilon\mathbf{C}$ is isomorphic to the algebra $M_2(S)$. For instance, the map sending

$$e \quad \text{to} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$F \quad \text{to} \quad \begin{pmatrix} 1/\sqrt{1+x^2} & x/\sqrt{1+x^2} \\ x/\sqrt{1+x^2} & -1/\sqrt{1+x^2} \end{pmatrix}$$

is an isomorphism, under which the canonical grading γ, of $\epsilon\mathbf{C}$ ($\gamma(e) = e$, $\gamma(F) = -F$) corresponds to the map sending the matrix

$$\begin{pmatrix} f_{1,1}(x) & f_{1,2}(x) \\ f_{2,1}(x) & f_{2,2}(x) \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} f_{1,1}(-x) & -f_{1,2}(-x) \\ -f_{2,1}(-x) & f_{2,2}(-x) \end{pmatrix}.$$

The morphism $\beta_{\mathrm{AW}}[\mathbf{R} \times \mathbf{C}^n]$, once enlarged to 2×2 matrices, defines a homomorphism from $\epsilon\mathbf{C} \otimes (\epsilon S)^{\otimes n}$ to $M_{2^n}(\epsilon\mathbf{C} \otimes C_0(\mathbf{C}^n))$ that we shall denote in the same way. The notation γ will refer equally to the grading of $\epsilon\mathbf{C}$ and to the grading of $M_2(S)$.

We are interested in the study of an analog of the operators of pointwise multiplication for an infinite tensor power of the space $L^2(\mathbf{R})$ and of corresponding anti-Wick pseudodifferential operators. We construct the filtered Hilbert space $\mathcal{H}^\infty$, by letting

$$\mathcal{H} = L^2(\mathbf{R}), \quad \mathcal{H}^1 = (\mathcal{H} \oplus \mathcal{H}) \otimes \mathbf{C}^4, \quad \text{and} \quad \mathcal{H}^{n+1} = \mathcal{H}^n \otimes \mathcal{H}^1.$$

We choose a sequence of unit vectors $e_n \in \mathcal{H}$, and define the isometries $v_n(\xi) = \xi \otimes (e_{n+1} \oplus 0) \otimes (1, 0, 0, 0)$. This amounts to considering, in the setting of [1], an infinite tensor product of Hilbert spaces, with stabilisation $\bigotimes_{n=1}^\infty e_n$. The associated inductive limit space is our analog of L^2 functions on $\mathbf{R}^\infty$. The cotangent space is dealt with in a similar way: set $\mathcal{N} = L^2(\mathbf{C})$, $\mathcal{N}^1 = (\mathcal{N} \oplus \mathcal{N}) \otimes \mathbf{C}^4$. Let $I\colon \mathcal{H} \hookrightarrow \mathcal{N}$ be the anti-Wick isometry (see [17, 13]). We set $\epsilon_n = I(e_n)$, and construct the space $\mathcal{N}^\infty$ with stabilisation $\bigotimes_{n=1}^\infty \epsilon_n$. There is an isometry $I^\infty\colon \mathcal{H}^\infty \hookrightarrow \mathcal{N}^\infty$ which extends the $I^{\otimes n}$'s in a natural way. Anti-Wick operators on $\mathbf{R}^\infty$ will be the compression to $I^\infty(\mathcal{H}^\infty)$ of suitably defined symbols acting on $\mathcal{N}^\infty$.

Let x be element of $\mathcal{B}_{n+1}$. Then x acts, through $\beta_{\mathrm{AW}}[\mathbf{R} \times \mathbf{C}^n]$, on the space $((\mathcal{N} \oplus \mathcal{N}) \otimes \mathbf{C}^4)^{\otimes(n+1)}$. This action is described as follows: Consider $\mathcal{B}_1 = M_2(\epsilon\mathbf{C}) \simeq M_4(S)$. The algebra S acts on $\mathcal{N} \oplus \mathcal{N}$, by the formula given in Proposition 3.1.7, and $\mathcal{B}_1$ acts on $(\mathcal{N} \oplus \mathcal{N}) \otimes \mathbf{C}^4 = \mathcal{N}^1$. Now, assume that the action of $\mathcal{B}_n$ on $\mathcal{N}^n$ is defined. We have $\mathcal{B}_{n+1} \simeq \mathcal{B}_n \otimes M_4(\epsilon S)$. Again, the action of ϵS is given by the formula in 3.1.7, so that $\mathcal{B}_{n+1}$ acts on $\mathcal{N}^n \otimes (\mathcal{N} \oplus \mathcal{N}) \otimes \mathbf{C}^4 \simeq \mathcal{N}^{n+1}$. This action consists of pointwise multiplication by matrix-valued functions. Denote by eval: $C_0(\mathbf{C}^n \times \mathbf{C}) \to C_0(\mathbf{C}^n)$ the evaluation at the origin; explicitly $[\mathrm{eval}(f)](X) = f(X, 0)$, $\forall X \in \mathbf{C}^n$. We shall now explain the construction of a C^*-morphism from $\mathcal{B}$ to $\mathcal{L}_{\mathrm{loc}}(\mathcal{N}^\infty)$. From Definition 3.2.3, we obtain (considering ϵS as a subalgebra of the multipliers of $S \otimes \epsilon S$), a C^*-morphism from ϵS to $M_2(C_0(\mathbf{C}))$ that we shall denote by $\tilde{\beta}$ (hence, $\beta_{\mathrm{AW}}[\mathbf{R} \times \mathbf{C}] = \mathrm{Id}_S \otimes \tilde{\beta}$). Composing $\tilde{\beta}$ with the evaluation map yields a C^*-morphism from ϵS to $M_2(\mathbf{C})$. This composition sends the generator

$$\zeta \quad \text{to} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the projection

$$P \quad \text{to} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using Definition 2.2.3, we compute the composition

$$(\mathrm{Id}_{M_4} \otimes \mathrm{eval})(\mathrm{Id}_{M_2} \otimes \tilde{\beta})\mu(x) = \mathrm{Diag}(\pi_p(x), 0, \pi_{\overline{p}}(x), 0) \in M_4(\mathbf{C}), \forall x \in q\mathbf{C}.$$

Now, we come to the composition $\mathrm{eval}\tilde{\beta}\mu\lambda\colon \epsilon\mathbf{C} \to M_8(\epsilon\mathbf{C})$. (In several places,we shall shorten notations, omitting to denote identity maps on matrices.) Recall that the definition of λ is given in 2.2.1. It follows from Proposition 2.2.2, and from the invariance of M and N under the exchange of p and $\overline{p}$ (see 2.1.5), that for any x element of $\epsilon\mathbf{C}$, $\mathrm{eval}\tilde{\beta}\mu\lambda(x) = \mathrm{Diag}(x, 0, 0, 0, 0, \gamma(x), 0, 0)$. (Here we have chosen the idenfication $M_4 \otimes M_2 \simeq M_8$, which consists in considering an element of $M_4 \otimes M_2$ as a 4×4 matrix, whose entries are 2×2 blocks.) We can, in the same manner, compute the image of an element $x \in \epsilon\mathbf{C} \otimes (\epsilon S)^{\otimes n}$. Such an element is sent by the map

$$\mathrm{Id}_{M_2(\epsilon\mathbf{C})} \otimes (\mu \otimes \mathrm{Id}_{(\epsilon S)^{\otimes n}})(\lambda \otimes \mathrm{Id}_{(\epsilon S)^{\otimes n}}),$$

to

$$M_4((\epsilon\mathbf{C} \otimes \epsilon S) \otimes (\epsilon S)^{\otimes n}) \simeq \epsilon\mathbf{C} \otimes (\epsilon S)^{\otimes n} \otimes M_4(\epsilon S).$$

Applying $\mathrm{eval}\tilde{\beta}$ on the last tensorial factor, we obtain

$$\mathrm{eval}\tilde{\beta}\mu\lambda(x) = \mathrm{Diag}(x, 0, 0, 0, 0, \gamma(x), 0, 0).$$

The C^*-algebra $\mathcal{B}$ is the inductive limit of the system $(\mathcal{B}_n, j_n)$, $\mathcal{B}_n$ is a matrix algebra over $\epsilon\mathbf{C} \otimes (\epsilon S)^{\otimes n-1}$, and j_n is the composition $(\mathrm{Id}_{\epsilon\mathbf{C}} \otimes \mu)\lambda$, enlarged to a map on $\mathcal{B}_n$ by tensoring with identity on matrices (see 2.2.4). It follows that, for any

$$x = m \otimes x_1 \otimes x_2 \in \mathcal{B}_n, \quad m \in M_{4^{(n+1)/2}}(\mathbf{C}), \quad x_1 \in \epsilon\mathbf{C}, \quad x_2 \in (\epsilon S)^{\otimes n-1},$$

we have $\mathrm{eval}\tilde{\beta} j_n(x) = \mathrm{Diag}(m \otimes x_1 \otimes x_2, 0, 0, 0, 0, m \otimes \gamma(x_1) \otimes x_2, 0, 0)$. Proceeding as in [13], we shall obtain a C^*-morphism from $\mathcal{B}$ to $\mathcal{L}_{\mathrm{loc}}(\mathcal{N}^\infty)$, in the following way: The space $\mathcal{N}^\infty$ is the inductive limit of the spaces $\mathcal{N}^n, \mathcal{N}^{n+1} = \mathcal{N}^n \otimes \mathcal{N}^1$ and isometries sending the vector $\xi \in \mathcal{N}^n$ to the vector whose first component is $\xi \otimes \epsilon_{n+1}$, and whose (seven) other components are equal to 0. For each integer k, choose a real number $0 < r_k$, and a continuous increasing function $\phi_k\colon \mathbf{R}_+ \to [0, 1]$, such that $\phi_k(r) = 0, \forall r \leqslant r_k$, and $\phi_k(r) \to 1$ when $r \to \infty$. Replacing, in the definition of $\beta_{\mathrm{AW}}[\mathbf{R} \times \mathbf{C}^n]$ each generator X_k by $X_k\phi_k(|X_k|)$, we obtain a new C^*-algebra morphism, homotopic to the original one, that we shall denote by $\beta_{\mathrm{AW}}[\mathbf{R} \times \mathbf{C}^n; (\phi_k)_{k \leqslant n}]$. Since ϵ_k is a function of rapid decrease, we can choose r_k in such a way that $\int_{|x+iy|>r_k} |\epsilon_k(x+iy)|^2 \mathrm{d}x\mathrm{d}y < 2^{-2(k+1)}$. The results of the above computation at the origin remain now valid for evaluation at any point of the closed disk $|z| \leqslant r_k$. It follows that, for any $x \in \mathcal{B}_n$ and for any $\xi \in \mathcal{N}^n$, with $\|x\| = \|\xi\| = 1$, we have (denoting by $v_n^{n+1}\colon \mathcal{N}^n \to \mathcal{N}^{n+1}$ the isometries defining the inductive system):

$$\begin{aligned}&\|(\beta_{\mathrm{AW}}[\mathbf{R} \times \mathbf{C}^{n-1}, (\phi_k)](x) \otimes \mathrm{Id}_{\mathcal{N}^1})(v_n^{n+1}(\xi)) - (\beta_{\mathrm{AW}}[\mathbf{R} \times \mathbf{C}^n, (\phi_k)] \times \\ &\quad \times (j_{n+1}(x)))(v_n^{n+1}(\xi))\| \leqslant 2^{-n}\end{aligned}$$

In view of Proposition 3.2.2, we conclude that the sequence of morphisms $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}; (\phi_k)_{k\leqslant n}]$ defines a C^*-morphism from $\epsilon\mathcal{B}$ to $\mathcal{K}_{\mathrm{loc}}(\mathcal{N}^\infty)$. This morphism is the extension of the anti-Wick operators on $\mathbf{R}^\infty$. Replacing, in the definition of $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}; (\phi_k)_{k\leqslant n}]$, the projection Q by the projection $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (\mathrm{Id}_{M_2})^{\otimes n}$, yields an element of $\mathrm{Ext}(S \otimes (\epsilon S)^{\otimes n}, C_0(\mathbf{C}^{n+1}))$. The inductive limit homomorphism is the extension of anti-Wick symbols on $\mathbf{C}^\infty$. (Note that there is no simple manner to express these anti-Wick symbols on $\mathbf{C}^\infty$ as functions on a space, but, they arise as operators on the filtered Hilbert space $\mathcal{N}^\infty$ in a natural way.)

DEFINITION 3.2.4. We shall denote by $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^\infty; (\phi_n)]\colon \epsilon\mathcal{B} \to \mathcal{K}_{\mathrm{loc}}(\mathcal{N}^\infty)$ the extension of anti-Wick operators on $\mathbf{R}^\infty$, described above.

4. Infinite Tensor Power of Some *KK*-Elements

We shall study another method for constructing extensions of the C^*-algebra $\mathcal{B}$ by $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$, which is based on external product in KK-theory. We describe extensions of the C^*-algebra S by $\mathcal{K}(L^2(\mathbf{R}))$ by Dirac pairs. Such a pair consists of an operator T, and of a projection Q, such that TT^* and T^*T are equal to the unit on the range of Q, modulo compact operators. A homomorphism from ϵS to $\mathcal{K}(L^2(\mathbf{R}) \oplus L^2(\mathbf{R}))$ is canonically associated to the pair (T, Q). Consistent triples are defined, such that a filtered Hilbert space $\mathcal{N}^\infty$, and a representation of the strict inductive limit $\bigcup_n \mathcal{B}_n$ into $L_{\mathrm{diag}}(\mathcal{N}^\infty)$ can be constructed. This representation is the infinite tensor product of the representations attached to the Dirac pairs. The construction is carried over in details for infinite tensor power of the abstract Toeplitz extension, and of the anti-Wick extension on $\mathbf{R}$. Note that these products are not extensions of pseudodifferential operators on $\mathbf{R}^n$. But the advantage of these infinite tensor power constructions is that we shall obtain, from the Toeplitz extension, an extension of $\mathcal{B}$ by the usual compact operators on $\mathcal{H}^\infty$, to which we shall compare the infinite tensor power of the anti-Wick extension. The link with the extension of anti-Wick operators on $\mathbf{R}^\infty$ will be established in Section 5.

4.1. ASYMPTOTIC HOMOTOPY

Recall that an asymptotic morphism from a C^*-algebra A to a C^*-algebra B is a family of maps $(\psi_t)_{t\in[0,\infty)}$, from A to B, such that,

$$\forall\, a \in A,\ b \in A,\ \|\psi_t(ab) - \psi_t(a)\psi_t(b)\| \to 0, \|\psi_t(\lambda a + \mu b) - (\lambda\psi_t(a) + \\ + \mu\psi_t(b))\| \to 0, \forall\lambda, \mu \in \mathbf{C}, \quad \text{and} \quad \|\psi_t(a^*) - \psi_t(a)^*\| \to 0,$$

when t goes to infinity (the real t will be called the asymptotic parameter). The map $t \to \psi_t(a)$ is required to be norm continuous, for any $a \in A$. Asymptotic morphisms were defined by A. Connes and N. Higson (see [5, 7]), in relation with exact functors on C^*-algebras. We shall be interested in a special class of asymptotic morphisms, those which can be realized as homomorphisms, from A, to a C^*-algebra containing B.

DEFINITION 4.1.1. Let $(\mathcal{H}^\infty, \Phi)$ be a filtered Hilbert space, and denote by P^n the projections on the subspaces of the filtration (see 3.2.1). Let $(\psi_t)_{t\in[0,\infty)}$ be an asymptotic morphism, from a C^*-algebra A to $\mathcal{K}(\mathcal{H}^\infty)$. The asymptotic morphism (ψ_t) is said to be realizable if, for any $a \in A$, the net $(\psi_t(a))_{t\in[0,\infty)}$ is strongly convergent to an operator $\psi_\infty(a) \in \mathcal{K}_{\text{loc}}(\mathcal{H}^\infty)$, and if, for any integer n_0, the net $(P^{n_0}\psi_t(a)P^{n_0})_{t\in[0,\infty)}$ is norm convergent in $\mathcal{K}(P^{n_0}\mathcal{H}^\infty)$. We define, for such an asymptotic morphism, $\psi_\infty(a) = \lim_{t\to\infty}\psi_t(a)$, $\forall a \in A$.

PROPOSITION 4.1.2. *Let (ψ_t): $A \to \mathcal{K}(\mathcal{H}^\infty)$ be a realizable asymptotic morphism. Denote by ψ_∞: $A \to \mathcal{K}_{\text{loc}}(\mathcal{H}^\infty)$ the associated morphism of C^*-*algebras. *The induced map $(\psi_\infty)_*$: $K_*(A) \to K_*(\mathcal{K}_{\text{loc}}(\mathcal{H}^\infty))$ is the composition of the map $(\psi_t)_*$: $K_*(A) \to K_*(\mathcal{K}(\mathcal{H}^\infty))$ induced by the asymptotic morphism (ψ_t), and of the canonical embedding $K_*(\mathcal{K}(\mathcal{H}^\infty)) \hookrightarrow K_*(\mathcal{K}_{\text{loc}}(\mathcal{H}^\infty))$.*

Proof. This is essentially a reformulation of the results of [13]. The map Λ-dim: $K_0(\mathcal{K}_{\text{loc}}(\mathcal{H}^\infty)) \simeq \Pi\mathbf{Z}/\bigoplus\mathbf{Z}$ extends the usual dimension map on $K_0(\mathcal{K}(\mathcal{H}^\infty))$. □

DEFINITION 4.1.3. Let A be a C^*-algebra and $\psi^{(0)}$, $\psi^{(1)}$ be two C^*-morphisms from A to $\mathcal{K}_{\text{loc}}(\mathcal{H}^\infty)$. The morphisms $\psi^{(0)}$ and $\psi^{(1)}$ are homotopic in r.asympt $(A, \mathcal{K}_{\text{loc}}(\mathcal{H}^\infty))$ (or asymptotically homotopic) if there exists a homotopy of asymptotic morphisms, with homotopy parameter s, $\Psi^{(s)}$: $A \to \mathcal{K}(\mathcal{H}^\infty)[0, 1]$, such that, for every $s \in [0, 1]$, the asymptotic morphism $\Psi^{(s)}$ is realizable, $\Psi^{(0)}_\infty = \psi^{(0)}$, and $\Psi^{(1)}_\infty = \psi^{(1)}$.

4.2. DIRAC PAIRS AND CONSISTENT TRIPLES

We now come to the construction of infinite tensor powers in K-homology. An efficient way of handling Fredholm modules over ϵS will be to consider Dirac pairs, as defined below. Consistent triples will furnish a filtered Hilbert space, on which our infinite products will act in a natural way.

DEFINITION 4.2.1. (i) Let $\mathcal{H}$ be a Hilbert space, let T be a linear operator on $\mathcal{H}$, and Q be a projection on $\mathcal{H}$. The pair (T, Q) is called a Dirac pair (on $\mathcal{H}$), if $\|T\| \leqslant 1$, $[Q, T] \in \mathcal{K}(\mathcal{H})$, $Q(1 - T^*T) \in \mathcal{K}(\mathcal{H})$, $Q(1 - TT^*) \in \mathcal{K}(\mathcal{H})$.

(ii) We shall denote by $\mathcal{D}(T, Q)$: $\epsilon S \to \mathcal{K}(\mathcal{H} \oplus \mathcal{H})$, the C^*-algebra morphism, defined, on generators, by

$$[\mathcal{D}(T, Q)](\zeta) = \mathcal{U}(T), \quad [\mathcal{D}(T, Q)](P) = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}.$$

DEFINITION 4.2.2. Let L be a linear operator on $\mathcal{H} \oplus \mathcal{H}$. Consider the partition $\{1, 2, 3, 4\} = \{i, j\} \cup \{k, l\}$, with $k < l$. Let $U^{(i,j)}$ be the unitary on $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$

defined by $U^{(i,j)}(\xi_i, \xi_j, \xi_k, \xi_l) = (\xi_1, \xi_2, \xi_3, \xi_4)$.

We define

$$L^{(i,j)}(\xi_1, \xi_2, \xi_3, \xi_4) = U^{(i,j)}((L(\xi_i, \xi_j), 0, 0)).$$

In particular, if L is the identity on $\mathcal{H} \oplus \mathcal{H}$, then $L^{(1,3)}(\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_1, 0, \xi_3, 0)$.

PROPOSITION 4.2.3. *Let μ: $q\mathbf{C} \to M_2(\epsilon S)$ be the morphism defined in* 2.2.3. *Let $\mathcal{H}$ be a Hilbert space and (T, Q) be a Dirac pair on $\mathcal{H}$. The composition $(Id_{M_2} \otimes \mathcal{D}(T, Q)) \circ \mu$ is given on generators by*

$$\begin{aligned} (\mathrm{Id}_{M_2} \otimes \mathcal{D}(T, Q)) \circ \mu\colon q\mathbf{C} &\to \mathcal{K}(Q(\mathcal{H}) \oplus Q(\mathcal{H}) \oplus Q(\mathcal{H}) \oplus Q(\mathcal{H})), \\ p &\to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{(1,3)}, \\ \overline{p} &\to [W^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W]^{(1,3)}, \quad W = \mathcal{U}(QTQ), \end{aligned}$$

(*The canonical inclusion*

$$\mathcal{K}(Q(\mathcal{H}) \oplus Q(\mathcal{H}) \oplus Q(\mathcal{H}) \oplus Q(\mathcal{H})) \hookrightarrow \mathcal{K}(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus e)$$

is understood; *the superscript* $(1, 3)$ *refers to Definition* 4.2.2.)

Proof. Let u be the unitary introduced in Definition 2.2.3. Remark that $\mathcal{D}(T, Q)(\zeta) = \mathcal{D}(T, Q)(u)$, hence $\mathcal{D}(T, Q)(e) = 1$. Set

$$\Omega = \mathcal{U}(QTQ + 1 - Q), \qquad \tilde{\Omega} = \mathrm{Diag}\left(\begin{pmatrix} \Omega & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

An explicit computation yields

$$\begin{aligned} (\mathrm{Id}_{M_2} \otimes \mathcal{D}(T, Q)) \circ \mu(p) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{(1,3)}, \\ (\mathrm{Id}_{M_2} \otimes \mathcal{D}(T, Q)) \circ \mu(\overline{p}) &= \tilde{\Omega}^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{(1,3)} \tilde{\Omega}. \end{aligned}$$

In restriction to $q\mathbf{C}$, this homomorphism remains unchanged if the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{(1,3)}$ is replaced everywhere by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)^{(1,3)}$. The decomposition $1 = Q + (1 - Q)$ gives the result. □

DEFINITION 4.2.4. Let $(\mathcal{H}^n, v_n)$ be a sequence of Hilbert spaces and isometries. Assume that there is a Hilbert space $\mathcal{H}^1$, such that, for every integer n, $\mathcal{H}^{n+1} = \mathcal{H}^n \otimes \mathcal{H}^1$. Let A be a C^*-algebra, suppose we are given a sequence of homomorphisms $j_n\colon A^{\otimes n} \to A^{\otimes n+1}$, and a sequence of representations $\lambda_n\colon A \to \mathcal{L}(\mathcal{H}^1)$, such that

$$\forall n \in \mathbf{N},\ \forall \xi \in \mathcal{H}^n,\ \forall a_1, a_2, \dots, a_n \in A,\ [(\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_{n+1}) j_n \times \\ \times (a_1 \otimes a_2 \otimes \cdots \otimes a_n)](v_n(\xi)) = v_n([(\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_n) \times \\ \times (a_1 \otimes a_2 \otimes \cdots \otimes a_n)](\xi)).$$

Then, $\forall\, a_1, a_2, \dots, a_n \in A$, the sequence

$$((\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_{n+p}) j_{n+p-1} \dots j_{n+1} j_n \\ (a_1 \otimes a_2 \otimes \cdots \otimes a_n) \in \mathcal{L}(\mathcal{H}^{n+p}))_{p\in\mathbf{N}},$$

defines an operator on $\mathcal{H}^\infty = \varinjlim(\mathcal{H}^n, v_n)$. We obtain a representation of $A^\infty = \varinjlim(A^{\otimes n}, j_n)$. We shall refer to this representation as the infinite tensor product of the sequence of representations (λ_n) and denote it by $\bigotimes_{n=1}^\infty \lambda_n$.

This infinite tensor product representation can be described explicitly as follows: for any integer m, denote by ω_m the canonical embedding of $\mathcal{H}^m$ into $\mathcal{H}^\infty$. Let $p,\ q$ be any pair of integers and choose any integer $m \geqslant \max(p, q)$. Let ξ be element of $\mathcal{H}^q$ and let x be element of $A_1 \otimes A_2 \otimes \cdots \otimes A_p$. Then

$$\left[\left(\bigotimes_{n=1}^{\infty} \lambda_n\right)(x)\right](\omega_q(\xi)) = \omega_m\left(\left[\left(\bigotimes_{n=1}^{m} \lambda_n\right)(j_{m-1} \dots j_p(x))\right](v_q^m(\xi))\right).$$

DEFINITION 4.2.5. A consistent triple on a Hilbert space $\mathcal{H}$ is a triple (T, Q, ξ), where (T, Q) is a Dirac pair on $\mathcal{H}$, and $\xi \in \mathcal{H}$ is a unit vector such that $QTQ(\xi) = 0$.

THEOREM 4.2.6. *Let $\mathcal{H}$ be a Hilbert space. Assume that for each integer n, a consistent triple (T_n, Q_n, e_n) on $\mathcal{H}$ is given. Set $\mathcal{H}^1 = (\mathcal{H} \oplus \mathcal{H}) \otimes \mathbf{C}^4$. We define a sequence of Hilbert spaces and isometries $(\mathcal{H}^n, v_n)$, by $\forall n \in \mathbf{N},\ \mathcal{H}^{n+1} = \mathcal{H}^n \otimes \mathcal{H}^1,\ v_n(\xi) = \xi \otimes (e_{n+1} \oplus 0) \otimes (1, 0, 0, 0),\ \forall \xi \in \mathcal{H}^n$ and set $\mathcal{H}^\infty = \varinjlim(\mathcal{H}^n, v_n)$. Then, $(\bigotimes_{n=1}^\infty \mathrm{Id}_{M_4} \otimes \mathcal{D}(T_n, Q_n)) \circ \pi_\epsilon$ is a well defined homomorphism from $\epsilon\mathcal{B}$ to $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$.*

Proof. For each integer $n > 0$, $\pi_\epsilon(\epsilon\mathcal{B}_n) = (M_4(\epsilon S))^{\otimes n}$. It follows from 4.2.3 that, for any x, element of $q\mathbf{C}$ and for any integer n,

$$[(\mathrm{Id}_{M_2} \otimes \mathcal{D}(T_n, Q_n)) \circ \mu(x)]((e_n \oplus 0) \otimes (1, 0)) = \pi_p(x).((e_n \oplus 0) \otimes (1, 0)).$$

We conclude from 2.2.2, that Definition 4.2.4, can be applied, with $A_p = M_4(\epsilon S)$ for every integer p. □

DEFINITION 4.2.7. We shall refer to the infinite tensor product representation of theorem 4.2.6 by using the notation $\bigotimes_{n=1}^\infty(\mathcal{D}(T_n, Q_n), e_n)$, or the notation: $\bigotimes_{n=1}^\infty \mathcal{D}(T_n, Q_n)$, if no confusion can arise.

PROPOSITION 4.2.8. *Let* $((T_n, Q_n, e_n))_{n\in\mathbf{N}}$ *and* $((T'_n, Q_n, e_n))_{n\in\mathbf{N}}$ *be two sequences of consistent triples on a Hilbert space* $\mathcal{H}$, *such that* $\forall n \in \mathbf{N}$, $Q_n(T_n - T'_n) \in \mathcal{K}(\mathcal{H})$. *Then* $(\bigotimes_{n=1}^{\infty} \mathcal{D}(T_n, Q_n)) \circ \pi_\epsilon$ *is homotopic to* $(\bigotimes_{n=1}^{\infty} \mathcal{D}(T'_n, Q_n)) \circ \pi_\epsilon$, *in* $r.\mathrm{asympt}(\epsilon\mathcal{B}, \mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty))$.

Proof. Let, for $s \in [0, 1]$, $n \in \mathbf{N}$, $T_n^{(s)} = (1-s)\,T_n + s\,T'_n$. Then $(T_n^{(s)}, Q_n, e_n)$ is a consistent triple. Let p be any positive integer, and let $t \in [0, 1]$. We define the asymptotic morphism with asymptotic parameter $p+t$ and homotopy parameter s by

$$\begin{aligned}\Phi_{p+t}^{(s)} \;=\; &\otimes\left(\bigotimes_{k=1}^{p} \mathcal{D}((T_k^{(s)}, Q_k), e_k)) \otimes \mathcal{D}((T_{p+1}^{(ts)}, Q_{p+1}), e_{p+1}\right) \otimes \\ &\otimes\left(\bigotimes_{q=p+2}^{\infty} \mathcal{D}((T_q, Q_q), e_q)\right)\pi_\epsilon.\end{aligned}$$

It is easily checked that $\Phi_{p+t}^{(s)}$ is a homotopy of realizable asymptotic morphisms with endpoints

$$\Phi_\infty^{(0)} = \bigotimes_{n=1}^{\infty}(\mathcal{D}(T_n, Q_n), e_n) \quad \text{and} \quad \Phi_\infty^{(1)} = \bigotimes_{n=1}^{\infty}(\mathcal{D}(T'_n, Q_n), e_n). \qquad \square$$

COROLLARY 4.2.9. *Let* $((T_n, Q_n, e_n))_{n\in\mathbf{N}}$ *be a sequence of consistent triples on a Hilbert space* $\mathcal{H}$, *such that* $\forall n \in \mathbf{N}$, $Q_n(e_n) = e_n$. *The following hold*:

(i) $\forall n \in \mathbf{N}$, $(Q_n T_n Q_n, 1, e_n)$ *is a consistent triple on* $Q_n(\mathcal{H})$.

(ii) *The operator* $\bigotimes_{n=1}^{\infty}(Q_n \otimes \mathrm{Id}_{M_4(\mathbf{C})})$ *is a well defined projection on* $\mathcal{H}^\infty$. *We shall denote it by* Q^∞.

(iii) $(\bigotimes_{n=1}^{\infty} \mathcal{D}(T_n, Q_n)) \circ \pi_\epsilon$ *is homotopic*, *in* $r.\mathrm{asympt}(\epsilon\mathcal{B}, \mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty))$, *to the composition of* $(\bigotimes_{n=1}^{\infty} \mathcal{D}(Q_n T Q_n, 1)) \circ \pi_\epsilon$, *with the canonical embedding of* $\mathcal{K}_{\mathrm{loc}}(Q^\infty(\mathcal{H}^\infty))$ into $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$.

Proof. (i), and (ii) are obvious. Assertion (iii) follows from Proposition 4.2.8, and from the equality

$$T_n + \mathcal{K}(\mathcal{H}) = (Q_n T_n Q_n \oplus (1 - Q_n) T_n (1 - Q_n)) + \mathcal{K}(\mathcal{H}),$$

which holds for every integer n. It is easy to check that

$$\mathcal{D}((Q_n T_n Q_n \oplus (1 - Q_n) T_n (1 - Q_n)), Q_n)$$

is the composition of $\mathcal{D}(Q_n T_n Q_n, 1)$, with the embedding of $\mathcal{K}(Q_n(\mathcal{H}) \oplus Q_n(\mathcal{H}))$ into $\mathcal{K}(\mathcal{H} \oplus \mathcal{H})$. $\square$

4.3. INFINITE TENSOR POWER OF THE TOEPLITZ EXTENSION

In this subsection, we let $\mathcal{H}$ be the space $l^2(\mathbf{N})$ and denote the canonical basis of $\mathcal{H}$ by (δ_n).

DEFINITION 4.3.1. Let V be the unilateral shift (coisometry of index 1: $V(\delta_n) = \delta_{n-1}, \forall n > 0,\ V(\delta_0) = 0$). We let $\mathcal{D}_\tau = \mathcal{D}(V, 1)\colon \epsilon S \to \mathcal{K}(l^2(\mathbf{N}) \oplus l^2(\mathbf{N}))$. Note that $(V, 1, \delta_0)$ is a consistent triple on $l^2(\mathbf{N})$.

PROPOSITION 4.3.2. *The composition* $\mathrm{Id}_{M_2} \otimes \mathcal{D}_\tau \circ \mu$ *is expressed on generators by*

$$\begin{aligned} \mathrm{Id}_{M_2} \otimes \mathcal{D}_\tau \circ \mu\colon q\mathbf{C} &\to M_4(\mathcal{K}(l^2(\mathbf{N})), \\ p &\to \begin{pmatrix} [\delta_0] & 0 \\ 0 & 0 \end{pmatrix}^{(1,3)}, \\ \overline{p} &\to 0. \end{aligned}$$

Proof. This follows from 4.2.3. □

DEFINITION 4.3.3. We shall denote by $\mathcal{D}_\tau^{\otimes\infty}$, the inductive limit representation $\bigotimes_{n=1}^{\infty}(\mathcal{D}(V, 1), \delta_0)$. The morphism $\mathcal{D}_\tau^{\otimes\infty} \circ \pi_\epsilon\colon \epsilon\mathcal{B} \to \mathcal{K}(\mathcal{H}^\infty) \subset \mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$ will be referred to as the infinite tensor power of the Toeplitz extension.

Remark 4.3.4. Note that, although the infinite tensor product construction 4.2.7 yields, in general, a map to $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$, the infinite tensor power of the Toeplitz extension, $\mathcal{D}_\tau^{\otimes\infty} \circ \pi_\epsilon$ is a homomorphism from $\epsilon\mathcal{B}$ to $\mathcal{K}(\mathcal{H}^\infty)$, since $[\delta_0]$ is an operator of rank 1. This shows that the associated connecting map, from $K_1(\mathcal{B})$ to $K_0(\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty))$ factors through the embedding of $K_0(\mathcal{K}(\mathcal{H}^\infty)) = \mathbf{Z}$ into $K_0(\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)) = \Pi\mathbf{Z}/\bigoplus\mathbf{Z}$, as the subgroup which consists of the classes of constant sequences of integers.

4.4. INFINITE TENSOR POWER OF ANTI-WICK EXTENSIONS

In what follows, we denote by M_X the unbounded operator on $L^2(\mathbf{C})$, defined by $(M_X(\xi))(X) = X\xi(X)$. We refer the reader to ([16], chapter 5) for details concerning the harmonic oscillator, and the basis of Hermite functions.

PROPOSITION 4.4.1. *Consider the operator* I^*M_XI, *with domain the space of smooth functions of rapid decrease. Let* D_X *be its closure and write* $D_X = \mathrm{phase}(D_X)|D_X|$ *the polar decomposition of* D_X.

(i) *There exists a unitary* $U\colon l^2(\mathbf{N}) \to L^2(\mathbf{R})$, *such that* $U^*\mathrm{phase}(D_X)U = V$.
(ii) *Let* $\psi\colon \mathbf{R}_+ \to [0, 1]$ *be a continuous function, such that* $\lim_{r\to\infty}\psi(r) = 1$. *Then*, $\mathrm{phase}(D_X) - \mathrm{phase}(D_X)\psi(|D_X|)$ *is a compact operator on* $L^2(\mathbf{R})$.
(iii) $\mathrm{phase}(D_X)\psi(|D_X|) - D_{X|X|^{-1}\psi(|X|)}$ *is a compact operator on* $L^2(\mathbf{R})$.

Proof. Set

$$\rho_0(t) = \pi^{-1/4}\exp(-t^2/2), \quad \forall t \in \mathbf{R}, \quad A = 2^{-1/2}(t + \partial_t),$$
$$A^* = 2^{-1/2}(t - \partial_t).$$

Let $(\rho_n)_{n\in\mathbf{N}}$ be the basis of Hermite functions. Recall that $\rho_n = (n!)^{-1/2}(A^{*n})\rho_0$, $\forall n > 0$. We have

$$D_X(\rho_n) = (2n)^{1/2}\rho_{n-1},\ \forall n > 0,\ D_X(\rho_0) = 0$$

and

$$D_X^*(\rho_n) = (2(n+1))^{1/2}\rho_{n+1},\ \forall n \geqslant 0.$$

The unitary U sends the vector δ_n to the vector ρ_n; (ii): Observe that $\mathrm{phase}(D_X) - \mathrm{phase}(D_X)\psi(|D_X|)$ maps ρ_n to $(1 - \psi((2n)^{1/2}))\rho_{n-1}$. For showing (iii), consider the extension of anti-Wick operators on $\mathbf{R}$. It follows from (ii), that the image of $\mathrm{phase}(D_X)\psi(|D_X|)$, and of $D_{X|X|^{-1}\psi(|X|)}$ modulo the compact operators are both equal to the canonical unitary generating $C(\mathbf{T})$. (Indeed, by (i), $\mathrm{phase}(D_X)$ is an essentially normal Fredholm operator, of index -1, whose essential spectrum is the whole unit circle of the complex plane.) □

DEFINITION 4.4.2. Let $\psi\colon \mathbf{R}_+ \to [0, 1]$ be a continuous function, such that $\lim_{r\to\infty}\psi(r) = 1$. Let $I\colon L^2(\mathbf{R}) \hookrightarrow L^2(\mathbf{C})$ be the anti-Wick isometry, and denote by P^I the projection on the subspace $I(L^2(\mathbf{R}))$. We define

$$\mathcal{D}[\mathbf{C};\psi] = \mathcal{D}(M_{X|X|^{-1}\psi(|X|)}, P^I)\colon \epsilon S \to \mathcal{K}(L^2(\mathbf{C}) \oplus L^2(\mathbf{C}))$$

and

$$\mathcal{D}[\mathbf{R};\psi] = \mathcal{D}(D_{X|X|^{-1}\psi(|X|)}, 1)\colon \epsilon S \to \mathcal{K}(L^2(\mathbf{R}) \oplus L^2(\mathbf{R})).$$

Remark that the anti-Wick extension $\mathcal{D}_{\mathrm{AW}}[\mathbf{C};\phi]$ is recovered when $\psi(r) = r\phi(r)(1+r^2\phi(r)^2)^{-1/2}$. Since continuity of ϕ is necessary to define $\mathcal{D}_{\mathrm{AW}}[\mathbf{C};\phi]$, we shall assume that our functions ψ are such that $r^{-1}\psi(r)$ is itself continuous. This will give a simple correspondence with anti-Wick extensions (See Definition 4.4.4 (ii), below.)

PROPOSITION 4.4.3. *Let $\psi\colon \mathbf{R}_+ \to [0, 1]$ be a continuous function, such that $r \to r^{-1}\psi(r)$ is continuous, and, $\lim_{r\to\infty}\psi(r) = 1$. Let ρ_0 be the unit vector in $L^2(\mathbf{R})$, generating the kernel of* $\mathrm{phase}(D_X)$ *(see the proof of* 4.4.1*). Then* $(\mathrm{phase}(D_X), 1, \rho_0)$ *and* $(D_{X|X|^{-1}\psi(|X|)}, 1, \rho_0)$ *are consistent triples on* $L^2(\mathbf{R})$; $D(M_{X|X|^{-1}\psi(|X|)}, P^I, I(\rho_0))$ *is a consistent triple on* $L^2(\mathbf{C})$).

Proof. We need only to show that $(D_{X|X|^{-1}\psi(|X|)}, 1, \rho_0)$ is a consistent triple on $L^2(\mathbf{R})$. It is clear that $(D_{X|X|^{-1}\psi(|X|)}, 1)$ is a Dirac pair. Let us show that $D_{X|X|^{-1}\psi(|X|)}(\rho_0) = 0$. Let (ρ_m) be the basis of Hermite functions. The equalities

$$I^*M_{\bar{X}}(\eta) = (t + \partial_t)I^*\eta \quad \text{and} \quad M_X I(\xi) = I(t - \partial_t)\xi,$$

hold for any smooth functions of rapid decrease η, on $\mathbf{C}$, and ξ, on $\mathbf{R}$. Set $r^2 = X\bar{X}$. It follows from [16] that ρ_0 is an eigenfunction of $D_{\mathcal{P}}$, for any polynomial $\mathcal{P}(r^2)$.

We can check, for any integer m, the convergence of the power series expansion $\sum_n I^*(-1)^n(r/2)^{2n}/n!I(\rho_m)$ in $L^2(\mathbf{R})$, and conclude that $D_{\exp(-(r/2)^2)}$ is diagonal in the basis of Hermite functions, with eigenvalues smaller or equal to 1. Using the equality $D_{r^{2n}\exp(-(r/2)^2)} = (t+\partial_t)^n D_{\exp(-(r/2)^2)}(t-\partial_t)^n$, we can, in the same manner, show that for any integer k, the operator $D_{\exp(-kr^2)}$ is diagonal in the basis of Hermite functions, and extend the result to any anti-Wick operator whose symbol is a function of r, vanishing at infinity. Since ρ_0 generates the kernel of $t+\partial_t$, the proof is complete (see also the proof of 4.4.1). □

DEFINITION 4.4.4. Let (ψ_n) be a sequence of continuous functions, from $\mathbf{R}_+$ to $[0, 1]$, such that $\forall n \in \mathbf{N},\ r \to r^{-1}\psi_n(r)$ is continuous and $\lim_{r\to\infty}\psi_n(r) = 1$.

(i) We define

$$\mathcal{D}^{\otimes\infty}[\mathbf{C}; (\psi_n)] = \bigotimes_{n=1}^{\infty}(\mathcal{D}[\mathbf{C}; \psi_n], I(\rho_0))$$

and

$$\mathcal{D}^{\otimes\infty}[\mathbf{R}; (\psi_n)] = \bigotimes_{n=1}^{\infty}(\mathcal{D}[\mathbf{R}; \psi_n], \rho_0).$$

(ii) Given a sequence of continuous functions (ϕ_n), such that $\psi_n(r) = r\phi_n(r)(1+r^2\phi_n(r)^2)^{-1/2}$, $\forall n \in \mathbf{N}$, we shall denote the infinite tensor power representation $\mathcal{D}^{\otimes\infty}[\mathbf{C}; (\psi_n)]$ by $\mathcal{D}^{\otimes\infty}_{\mathrm{AW}}[\mathbf{C}; (\phi_n)]$ (see the remark following 4.4.2).

(iii) We shall denote the morphism $\bigotimes_{n=1}^{\infty}(\mathcal{D}(\mathrm{phase}(D_X), 1), \rho_0)$ by $\mathcal{D}^{\otimes\infty}_{\tau}$.

Remark 4.4.5. In the above definition, the morphism $\bigotimes_{n=1}^{\infty}(\mathcal{D}(\mathrm{phase}(D_X), 1); \rho_0)\pi_\epsilon$ is identified to the infinite tensor power of the Toeplitz extension. This is justified by [4.4.1 (i)].

THEOREM 4.4.6. *Let (ψ_n) be a sequence of continuous functions, from $\mathbf{R}_+$ to $[0, 1]$, such that, $\forall n \in \mathbf{N},\ r \to r^{-1}\psi_n(r)$ is continuous and $\lim_{r\to\infty}\psi_n(r) = 1$. The morphisms $\mathcal{D}^{\otimes\infty}[\mathbf{C}; (\psi_n)]\pi_\epsilon$ and $\mathcal{D}^{\otimes\infty}_{\tau}\pi_\epsilon$ (see 4.4.4) are homotopic, in r.asympt$(\epsilon\mathcal{B}, \mathcal{K}_{\mathrm{loc}}(\mathcal{N}^\infty))$.*

Proof. Corollary 4.2.9 shows that $\mathcal{D}^{\otimes\infty}[\mathbf{C}; (\psi_n)]\pi_\epsilon$ is homotopic, in r.asympt$(\epsilon\mathcal{B}, \mathcal{K}_{\mathrm{loc}}(\mathcal{N}^\infty))$, to $\mathcal{D}^{\otimes\infty}[\mathbf{R}; (\psi_n)]\pi_\epsilon$, composed with the embedding of $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$ into $\mathcal{K}_{\mathrm{loc}}(\mathcal{N}^\infty)$. It follows from proposition (4.4.1 (ii), (iii)) that, for each integer n, $(D_{X|X|^{-1}\psi(|X|)}, 1)$ is a compact perturbation of $(\mathrm{phase}(D_X), 1)$. The result follows from (4.4.1 (i)), and from Proposition 4.2.8. □

5. Anti-Wick Extension on $\mathbf{R}^\infty$ Versus Infinite Tensor Power of the Toeplitz Extension

In this section, we study the relations between the constructions of Sections 3 and 4. We remark that a homomorphism from a C^*-algebra A to $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$ defines an asymptotic representation of A into the compact operators on $\mathcal{H}^\infty$. We use a notion of local homotopy for such C^*-morphisms based on homotopy of the associated asymptotic representation. This one is weaker than the usual homotopy for which equicontinuity of the family of completely positive maps obtained by compressions to the subspaces of the filtration is required. However, the Λ-ind map remains invariant under local homotopy. This will provide a link between the extension of anti-Wick operators on $\mathbf{R}^\infty$ and the infinite tensor power of the Toeplitz extension. Then we are enabled to compute the K-theory connecting map of these extensions, and to show that any elliptic anti-Wick operator on $\mathbf{R}^\infty$ is homotopic in $\mathcal{F}_{\mathrm{loc}}(\mathcal{H}^\infty)$ to an ordinary Fredholm operator on $\mathcal{H}^\infty$. This refines previous results obtained in [13].

Note: Simplified notations will be used in the proofs of this section: we shall denote by $\mathcal{D}_{\mathrm{AW}}$ the morphism $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}]$. Similar notation will be used for $\mathcal{D}^I_{\mathrm{AW}}[\mathbf{C}^{n+1}]$, $\sigma_{\mathrm{AW}}[n+1]$, etc. The Hilbert space $\mathcal{H}$ will be identified to the subspace $I(\mathcal{H})$, of $\mathcal{N}$. Similar notations hold for $\mathcal{H}^n$, $\mathcal{H}^\infty$. Consequently, the anti-Wick operator D_f will be considered as the compression to $\mathcal{H}$ of the operator of pointwise multiplication M_f.

5.1. HOMOTOPIES IN FINITE-DIMENSIONAL SPACE

In order to describe our homotopies, it will be convenient to introduce the following partial symbols:

DEFINITION 5.1.1. Let $(X_1, X_2, \ldots X_n)$ denote the coordinates in $\mathbf{C}^n$, with $X_j \in \mathbf{C}$, $\forall 1 \leqslant j \leqslant n$. Let

$$\{1, 2, \ldots, n\} = \{k_1, k_2, \ldots, k_j\} \cup \{l_1, l_1, \ldots, l_{n-j}\}$$

be a partition of the set of indices, and $\mathbf{C}^n = \mathbf{C}^j \times \mathbf{C}^{n-j}$ be the corresponding decomposition of $\mathbf{C}^n$. We shall denote by $\Gamma_{(k_1,k_2,\ldots,k_j)}$ the algebra of continuous functions of the variables $X_{l_1}, X_{l_2}, \ldots, X_{l_{n-j}}$, with values in $\Gamma(\mathbf{C}^j)$.

The extension of pseudodifferential operators $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}]$ corresponds to the algebra of full symbols $\Gamma(\mathbf{C}^{n+1})$. The extension $\bigotimes_{k=1}^{n+1} \mathcal{D}_{\mathrm{AW}}[\mathbf{C}]$ is associated to tensor products of elements of $\Gamma(\mathbf{C})$. These partial symbols will arise as intermediate steps between these two.

DEFINITION 5.1.2. Define $\mathcal{D}^I_{\mathrm{AW}}[\mathbf{C}^{n+1}] : \epsilon(S \otimes (\epsilon S)^{\otimes n}) \to M_{2^{n+1}}(\mathcal{K}(L^2(\mathbf{C}^{n+1})))$ by

$$\begin{aligned} \mathcal{D}^I_{\mathrm{AW}}[\mathbf{C}^{n+1}] : \epsilon(S \otimes (\epsilon S)^{\otimes n}) &\to M_{2^{n+1}}(\mathcal{K}(L^2(\mathbf{C}^{n+1}))), \\ \forall 1 \leqslant j \leqslant n+1,\ \zeta_j &\to (\mathrm{Id}_{M_2})^{\otimes j-1} \otimes \mathcal{U}(X_j(1+|X_1|^2)^{-1/2}) \otimes \\ &\quad \otimes (\mathrm{Id}_{M_2})^{\otimes n+1-j}, \end{aligned}$$

$$\forall 1 < j \leqslant n+1,\ P_j \ \to\ (\mathrm{Id}_{M_2})^{\otimes j-1} \otimes \begin{pmatrix} P_j^I & 0 \\ 0 & 0 \end{pmatrix} \otimes (\mathrm{Id}_{M_2})^{\otimes n+1-j},$$

$$P_1 \ \to\ \begin{pmatrix} P_1^I & 0 \\ 0 & 0 \end{pmatrix} \otimes \left[\bigotimes_{j=2}^{n+1} \begin{pmatrix} P_j^I & 0 \\ 0 & P_j^I \end{pmatrix} \right] = Q.$$

As in the proof of Proposition 3.1.7, the unitaries $\mathcal{U}(X_j(1+|X_1|^2)^{-1/2})$ are extended by $\mathcal{U}(X_j|X_j|^{-1})$ outside the domain $|X_j| \leqslant (1+|X_1|^2)^{1/2}$

PROPOSITION 5.1.3. *Let us denote by* $\sigma_{\mathrm{AW}}[n+1]$: $S \otimes (\epsilon S)^{\otimes n} \to \mathcal{L}(\mathcal{H}^{n+1})$ *the completely positive map associated to the morphism* $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^{n+1}]$ (*see Proposition* 3.1.7). *Let* $\sigma^I_{\mathrm{AW}}[n+1]$ *be the completely positive map associated to* $\mathcal{D}^I_{\mathrm{AW}}[\mathbf{C}^{n+1}]$. *Then, for any x element of* $S \otimes (\epsilon S)^{\otimes n}$, $\sigma_{\mathrm{AW}}[n+1](x) - \sigma^I_{\mathrm{AW}}[n+1](x)$ *is a compact operator.*

Proof. (See also 3.1.7.)

(a) Set $\mathcal{D}_{\mathrm{AW}}(P_j) = \mathcal{D}^I_{\mathrm{AW}}(P_j) + \mathcal{D}^{I\perp}_{\mathrm{AW}}(P_j)$, and denote by $\mathcal{D}^{I\perp}_{\mathrm{AW}}$ the morphism corresponding to the projections $\mathcal{D}^{I\perp}_{\mathrm{AW}}(P_j)$. Recall that $Q = \mathcal{D}_{\mathrm{AW}}(P_1) = \mathcal{D}^I_{\mathrm{AW}}(P_1)$, and, for any $x \in S \otimes (\epsilon S)^{\otimes n}$: $\sigma_{\mathrm{AW}}(x) = Q\mathcal{D}_{\mathrm{AW}}(x)Q$. We have to check that $\sigma_{\mathrm{AW}}(x) - \sigma^I_{\mathrm{AW}}(x) \in \mathcal{K}(\mathcal{H}^{n+1})$. The difference $\sigma_{\mathrm{AW}}(x) - \sigma^I_{\mathrm{AW}}(x)$ is a sum of terms of the following type: $T = QT_1T_2T_3T_4Q$, where

$T_1 = \mathcal{D}_{\mathrm{AW}}(x_1 \cdots \otimes x_k),\ x_1,\ x_2, \ldots x_k \in S,$
T_2 is a monomial in $\mathcal{D}_{\mathrm{AW}}(\zeta_{k+1}), \mathcal{D}_{\mathrm{AW}}(\zeta_{k+2}), \ldots \mathcal{D}_{\mathrm{AW}}(\zeta_{k+j})$,
$T_3 = \mathcal{D}^{I\perp}_{\mathrm{AW}}(P_{k+1})\mathcal{D}^{I\perp}_{\mathrm{AW}}(P_{k+2}) \ldots \mathcal{D}^{I\perp}_{\mathrm{AW}}(P_{k+j})$, and
T_4 is a monomial in $\mathcal{D}_{\mathrm{AW}}(\zeta_{k+j+1}) \ldots \mathcal{D}_{\mathrm{AW}}(\zeta_{n+1})$.

(b) Let $\psi \in \Gamma(\mathbf{C}^{n+1})$ be any anti-Wick symbol vanishing on a neighborhood of the hyperplane $X_1 = 0$. Terms of type ψT_2, and ψT_4 are elements of $\Gamma_{(1,k+1,\ldots,k+j)}$ and of $\Gamma_{(1,k+j+1,\ldots,n+1)}$, respectively.

(c) Let $a(X_1, X_l)$ be a continuous function of the complex variables X_1 and X_l. Assume that $a \in \Gamma_{(l)}$. Set $f(X_1) = P_l^I a(X_1, X_l) P_l^{I\perp}$; f is a continuous function of X_1, with values in $\mathcal{K}(L^2(\mathbf{R}_l^2))$. On each X_1-compact, f is the norm limit of functions $f_n(X_1) = P_l^I a_n(X_1, X_l) P_l^{I\perp}$, with $a_n(X_1, X_l)$, continuous with compact $X_1 \times X_l$ support. (Since $f(X_1)$ is compact, and is the strong limit of the compact operators $f_n(X_1)$.)

(d) Write $T_2 = T_2^{(1)} + T_2^{(2)}$, where $T_2^{(1)}$ (resp $T_2^{(2)}$) is a continuous function with X_1-compact support (resp. vanishing on an open set containing the hyperplane $X_1 = 0$). It follows from (c), that $T_1T_2^{(1)}$ can be approximated by functions with compact support. It follows from (b) that $T_1T_2^{(2)}$ is an element of $\Gamma_{(1,2,\ldots,k+j)}$.

(e) One can check that $QT_1T_2T_3T_4Q = [Q, T_1T_2]T_3[T_4, Q]$ It follows from (d), that

$$[Q, T_1T_2] \in \mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_{k+j}) s\mathcal{L}(\mathcal{H}_{k+j+1} \otimes \otimes \mathcal{H}_{k+j+2} \cdots \otimes \mathcal{H}_{n+1}).$$

It follows from (b) that

$$[T_4, Q] \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_{k+j}) \otimes \mathcal{K}(\mathcal{H}_{k+j+1} \otimes \mathcal{H}_{k+j+2} \cdots \otimes \mathcal{H}_{n+1}),$$

hence the result. □

PROPOSITION 5.1.4. *Let*

$$Q' = \begin{pmatrix} P^I & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{Id}_{M_2} \otimes \mathrm{Id}_{M_2} \cdots \otimes \mathrm{Id}_{M_2}.$$

Define

$$\begin{aligned} \mathcal{D}'^I_{\mathrm{AW}}[\mathbf{C}^{n+1}](\zeta_j) &= \mathcal{D}^I_{\mathrm{AW}}[\mathbf{C}^{n+1}](\zeta_j),\ \forall j \leqslant 1,\ \mathcal{D}'^I_{\mathrm{AW}}[\mathbf{C}^{n+1}](P_j) \\ &= \mathcal{D}^I_{\mathrm{AW}}[\mathbf{C}^{n+1}](P_j),\ \forall j > 1,\ \mathcal{D}'^I_{\mathrm{AW}}[\mathbf{C}^{n+1}](P_1) = Q'. \end{aligned}$$

Denote by $\sigma'_{\mathrm{AW}}[n+1]$ the completely positive map associated to $\mathcal{D}'^I_{\mathrm{AW}}[\mathbf{C}^{n+1}]$. Then, $\sigma_{\mathrm{AW}}[n+1](x)$ is homotopic to $\sigma'_{\mathrm{AW}}[n+1](x)$ in $\mathrm{Ext}(S \otimes (\epsilon S)^{\otimes n}, \mathcal{K}(\mathcal{H}^{n+1}))$ *(see Section* 3*).*

Proof. Remark that, for any $x \in S \otimes (\epsilon S)^{\otimes n}$, there is a direct sum decomposition $Q'\mathcal{D}^I_{\mathrm{AW}}Q'(x) = \sum^{\oplus}_{i \geqslant 0} P(i)\mathcal{D}^I_{\mathrm{AW}}(x)P(i)$, modulo compact operators. Here $P(0) = Q$, and each projection $P(i), i \geqslant 1$ contains exactly one tensorial factor $(P^I)^{\perp}$. Explicitly,

$$\begin{aligned} P(i) &= \begin{pmatrix} P^I & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{\otimes(i-1)} \otimes \\ &\otimes \begin{pmatrix} (P^I)^{\perp} & 0 \\ 0 & (P^I)^{\perp} \end{pmatrix} \otimes \begin{pmatrix} P^I & 0 \\ 0 & P^I \end{pmatrix}^{\otimes(n-i)}. \end{aligned}$$

This implies that the corresponding direct summand extends to a map from $S \otimes (\epsilon S)^{\otimes i-1} \otimes E_1 S \otimes (\epsilon S)^{\otimes n+1-i}$ to $\mathcal{L}(\mathcal{H}^{n+1})$. The extended map remains completely positive and multiplicative modulo the compact operators. Since $E_1 S$ is a contractible C^*-algebra, we can conclude that every direct summand corresponding to a block with index $i \geqslant 1$ is homotopic to 0. □

PROPOSITION 5.1.5. *The morphisms $\mathcal{D}'^I_{\mathrm{AW}}[\mathbf{C}^{n+1}]$* (see 5.1.4) *and $\bigotimes_{k=1}^n \mathcal{D}(X_k(1+|X_k|^2)^{-1/2}, P^I_k)$ are homotopic in* $\mathrm{Hom}(\epsilon\mathcal{B}_n, \mathcal{K}(\mathcal{N}^n))$.

Proof. The morphism $\mathcal{D}'^I_{\mathrm{AW}}$ is determined by the couple $(\beta'^I_{\mathrm{AW}}, Q')$, with

$$\begin{aligned} \beta'^I_{\mathrm{AW}} : S \otimes (\epsilon S)^{\otimes n} &\to S \otimes M_{2^n}(\mathcal{K}(L^2(\mathbf{C}^n))), \\ y &\to y\,, \forall y \in S, \\ \zeta_j &\to \mathcal{D}'^I_{\mathrm{AW}}(\zeta_j)\,, \forall 2 \leqslant j \leqslant n+1, \\ P_j &\to \mathcal{D}'^I_{\mathrm{AW}}(P_j)\,, \forall 2 \leqslant j \leqslant n+1. \end{aligned}$$

(See also 3.2.3, and the description of the identification $M_2(S) \sim \epsilon\mathbf{C}$.) In the tensor product decomposition

$$(L^2(\mathbf{C}) \oplus L^2(\mathbf{C}))^{\otimes(n+1)} = (L^2(\mathbf{C}) \oplus L^2(\mathbf{C})) \otimes (L^2(\mathbf{C}) \oplus L^2(\mathbf{C}))^{\otimes n},$$

the algebra S acts by pointwise multiplication on $(L^2(\mathbf{C})\oplus L^2(\mathbf{C}))$, and β'^{I}_{AW} factors through $C_0(\mathbf{R}, \mathcal{L}((L^2(\mathbf{C}) \oplus L^2(\mathbf{C}))^{\otimes n}))$. Hence, for any $x \in S \otimes \epsilon S^{\otimes n}$, we shall equally consider $\beta'^{I}_{\mathrm{AW}}(x)$ as a continuous function of the variable X_1, with values in $\mathcal{K}((L^2(\mathbf{C}) \oplus L^2(\mathbf{C}))^{\otimes n})$. We define a homotopy in a way similar to 4.2.8, by

$$\Phi^s = \mathrm{Id}_S \bigotimes_{k=2}^{n} [\mathcal{D}(T_k^{(s)}, P_k^I)]\pi_\epsilon, \text{ where } T_k^{(s)} = s\, X_k(1+|X_1|^2)^{-1/2} + \\ + (1-s)\, X_k(1+|X_k|^2)^{-1/2},$$

on the domain $|X_k|(1+|X_1|^2)^{-1/2} \leqslant 1$, and

$$T_k^{(s)} = s\, X_k|X_k|^{-1} + (1-s)\, X_k(1+|X_k|^2)^{-1/2},$$

on the complement.

5.2. K-THEORY MAPS

DEFINITION 5.2.1. Let A be a C^*-algebra, let $\mathcal{H}$ be a Hilbert space. An asymptotic representation of A into $\mathcal{K}(\mathcal{H})$ is a sequence of linear, completely positive maps of norm less or equal to one, $(\lambda_n\colon A \to \mathcal{K}(\mathcal{H}))$, such that, for any pair (a, b), of elements of A, $\lim_{n\to\infty} \|\lambda_n(ab) - \lambda_n(a)\lambda_n(b)\| = 0$. We shall denote by $[\lambda_n]$ such an asymptotic representation.

PROPOSITION 5.2.2. *Let* $\mathcal{H}^\infty = \varinjlim(\mathcal{H}^n, v_n)$ *be a filtered Hilbert space, and, for each integer* n, *denote by* P^n *the projection on the subspace* $\mathcal{H}^n \subset \mathcal{H}$. *Set, for any* $k \in \mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$, $\kappa_n(k) = P^n k P^n$. *The sequence* $[\kappa_n]$ *is an asymptotic representation of* $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$, *into* $\mathcal{K}(\mathcal{H}^\infty)$. *Any* C^*-algebra *morphism* $\psi\colon A \to \mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$ *yields an asymptotic representation of* A *into* $\mathcal{K}(\mathcal{H}^\infty)$, *that we shall denote by* $[\psi_n]$.

Proof. This is a consequence of the density of $L_{\mathrm{diag}}(\mathcal{H}^\infty)$ in $\mathcal{L}_{\mathrm{loc}}(\mathcal{H}^\infty)$. □

DEFINITION 5.2.3. (i) Let $[\psi_n^{(0)}]$, $[\psi_n^{(1)}]$ be two asymptotic representations of a C^*-algebra A into $\mathcal{K}(\mathcal{H})$. Then $[\psi_n^{(0)}]$, and $[\psi_n^{(1)}]$ are locally homotopic, if there exists a family $[\psi_n^{(t)}]_{t\in[0,1]}$ of asymptotic representations of A into $\mathcal{K}(\mathcal{H})$, such that, for every integer n and for each $a \in A$, the map $t \to \psi_n^{(t)}(a)$ is norm continuous.

(ii) Let $\psi^{(0)}$ and $\psi^{(1)}$ be two C^*-algebra morphisms from A to $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$. Then $\psi^{(0)}$ and $\psi^{(1)}$ are locally homotopic if there exists a family of homomorphisms $\psi^{(t)}\colon A \to \mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$, $t \in [0, 1]$, such that the corresponding family of asymptotic representations is a local homotopy with endpoints $[\psi_n^{(0)}]$ and $[\psi_n^{(1)}]$.

Note that if $\psi^{(0)}$ and $\psi^{(1)}$ are homotopic in r.asympt$(\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty))$, then, $\psi^{(0)}$ and $\psi^{(1)}$ are locally homotopic (see 4.1.3 and 5.2.2).

PROPOSITION 5.2.4. *Let* $\psi^{(0)}$ *and* $\psi^{(1)}$ *be locally homotopic. Then*: $\psi_*^{(0)} = \psi_*^{(1)}\colon K_*(A) \to K_*(\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty))$.

Proof. It is shown, in [13], that the isomorphism from $K_*(\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty))$ to $\Pi\mathbf{Z}/\bigoplus\mathbf{Z}$ is provided by the dimension map Λ-dim. Let p be a projection in $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$. There exists an integer N, such that the spectrum of $P^n p P^n$ is contained in $[0, 1/4] \cup [3/4, 1]$, for any $n \geqslant N$. Denote by $\mathcal{X}$ the characteristic function of $[3/4, 1]$, and by dim the usual dimension map on $\mathcal{K}(\mathcal{H}^\infty)$. Then, $\Lambda\text{-}\dim(p)$ is the class of the sequence $(\dim(\mathcal{X}(P^n p P^n)))_{n\geqslant N}$. This map is invariant under local homotopies. □

Note. In what follows, $\mathcal{H}$ denotes the Hilbert space $L^2(\mathbf{R})$, and $\mathcal{H}^\infty$ is the infinite tensor power of the corresponding space $\mathcal{H}^1 = (\mathcal{H}\oplus\mathcal{H})\otimes\mathbf{C}^4$, with stabilisation $\bigotimes_{n=1}^\infty \pi^{-1/4}\exp(-t^2/2)$. In consequence, $\mathcal{N}^\infty$ is the infinite tensor power of $\mathcal{N}^1$, with stabilisation $\bigotimes_{n=1}^\infty I(\pi^{-1/4}\exp(-t^2/2))$ (The definition of $\mathcal{H}^1$, $\mathcal{N}^1$, and the construction of the infinite tensor power spaces are described in Subsection 3.2).

PROPOSITION 5.2.5. *The morphism* $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^\infty; (\phi_n)]\colon \epsilon\mathcal{B} \to \mathcal{K}_{\mathrm{loc}}(\mathcal{N}^\infty)$ *is locally homotopic to* $\mathcal{D}_{\mathrm{AW}}^{\otimes\infty}[\mathbf{C}]\pi_\epsilon$.

Proof. Recall that $\mathcal{D}_{\mathrm{AW}}^{\otimes\infty}[\mathbf{C}] = \bigotimes_{k=1}^\infty \mathcal{D}(X_k(1+|X_k|^2)^{-1/2}, P_k^I)$ (see also 4.4.2, and 4.4.4). It follows from Proposition 5.1.3, and from proposition 5.1.5, that, for each integer n, the anti-Wick extension on $\mathbf{R}^n$ is homotopic to the nth tensor power of the anti-Wick extension on $\mathbf{R}$. The anti-Wick extension on $\mathbf{R}^\infty$, is described in Subsection 3.2. For each integer n, $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^n; (\phi_k)_{k\leqslant n}]$ does not depend, up to homotopy, on the choice of the functions $(\phi_1, \phi_2, \dots, \phi_n)$, provided that $\lim_{r\to\infty}\phi_k(r) = 1$, $\forall 1\leqslant k\leqslant n$. The homotopy described in 5.1.3, through 5.1.5, extends to a path of maps from $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^\infty; (\phi_n)]$ to $\mathcal{D}_{\mathrm{AW}}^{\otimes\infty}[\mathbf{C}; (\phi_n)]\pi_\epsilon$; this path is not continuous, but is a local homotopy. We conclude from Theorem 4.4.6 that, up to local homotopy, $\mathcal{D}_{\mathrm{AW}}^{\otimes\infty}[\mathbf{C}; (\phi_n)]\pi_\epsilon$ does not depend on the choice of the functions (ϕ_n), provided that $\lim_{r\to\infty}\phi_n(r) = 1$, for any $n \in \mathbf{N}$. The family of functions $\phi_n(r) = 1$, $\forall r\geqslant 0$, $\forall n \in \mathbf{N}$, corresponds to the infinite tensor power representation in the statement of the proposition. □

We have seen, in Subsection 3.1, that an invertible extension of a C^*-algebra A by a C^*-algebra B is a completely positive map, σ, from A to the multipliers algebra of B, which fulfills suitable conditions. The map σ corresponds to a pair (ϕ, P), that is, to a C^*-algebra morphism, from ϵA to $M_2(B)$, in such a way that, for any a element of A, $\sigma(a) = P\phi(a)P$. Let D be the C^*-algebra generated by the set $\{\sigma(a)/a \in A\}\cup B$. We have an exact sequence $A = D/B$, and a connecting map, $\partial : K_{*+1}(A) \to K_*(B)$. This map coincides with the map induced by the pair (ϕ, P) in K-theory, composed with the isomorphism $K_*(\epsilon A) \simeq K_{*+1}(A)$. In the following we shall consider ∂ as a map from $K_*(\epsilon A)$ to $K_*(B)$. In our case, A will be the inductive limit C^*-algebra $\mathcal{B}$ and B will be the C^*-algebra $\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$. The map ϕ sends $\mathcal{B}$ to $\mathcal{L}_{\mathrm{loc}}(\mathcal{N}^\infty)$ and P corresponds to the projection on the subspace $\mathcal{H}^\infty$ of $\mathcal{N}^\infty$.

THEOREM 5.2.6. (i) *The connecting map*, ∂_{AW}, *from* $K_*(\epsilon\mathcal{B})$ *to* $K_*(\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty))$ *of the extension of anti-Wick operators on* $\mathbf{R}^\infty$ *is the composition of the map sending the generator of* $K_*(\epsilon\mathcal{B}) = \mathbf{Z}$ *to the generator of* $K_*(\mathcal{K}(\mathcal{H}^\infty)) = \mathbf{Z}$, *with the canonical embedding* $K_*(\mathcal{K}(\mathcal{H}^\infty)) \hookrightarrow K_*(\mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty))$.

(ii) *Denote by* D *the anti-Wick extension* C^*-algebra *on* $\mathbf{R}^\infty$ (*that is*, $D = \sigma_{\mathrm{AW}}(\mathcal{B}) + \mathcal{K}_{\mathrm{loc}}(\mathcal{H}^\infty)$). *Let* $x \in \tilde{D}$ *whose image in* $\tilde{\mathcal{B}}$ *is invertible. Then*, x, *is homotopic*, *in* $\mathcal{F}_{\mathrm{loc}}(\mathcal{H}^\infty)$, *to a Fredholm operator on the Hilbert space* $\mathcal{H}^\infty$.

Proof. (i) The morphism $\mathcal{D}_{\mathrm{AW}}[\mathbf{C}^\infty; (\phi_n)]$ is locally homotopic to $\bigotimes_{k=1}^\infty \mathcal{D}(X_k(1 + |X_k|^2)^{-1/2}, P_k^I)$. Let ∂_τ be the K-theory connecting map of the extension $\mathcal{D}_\tau^{\otimes\infty} \circ \pi_\epsilon$. Theorem 4.4.6 and Proposition 4.1.2 then show that ∂_{AW} is the composition of ∂_τ, and of the embedding of $\mathbf{Z}$ in $\Pi\mathbf{Z}/\bigoplus\mathbf{Z}$, as the subgroup of classes of constant sequences. Since $\mathcal{D}_\tau$ is the usual Toeplitz extension, it maps the generators of $K_*(\epsilon S)$ to the generator of $K_*(\mathbf{C}) = \mathbf{Z}$. An analogous assertion holds for finite tensor powers of $\mathcal{D}_\tau$. Since K-theory is compatible with inductive limits, we conclude that ∂_τ sends the generator of $K_*(\epsilon\mathcal{B})$ to the generator of $\mathbf{Z}$, and the proof of (i) is complete (see also the proof of 4.4.6).

(ii) It follows from (i), that for any x, as in (ii), Λ-ind(x), is an element of the image of $\mathbf{Z}$ in $\Pi\mathbf{Z}/\oplus\mathbf{Z}$. Since $L_{\mathrm{diag}}(\mathcal{H}^\infty)$ is dense in $\mathcal{L}_{\mathrm{loc}}(\mathcal{H}^\infty)$ (and the linear group of $\mathcal{O}_{\mathrm{loc}}(\mathcal{H}^\infty)$ is open), x is homotopic, in $\mathcal{F}_{\mathrm{loc}}(\mathcal{H}^\infty)$, to a diagonal operator $\bigoplus_{n\geqslant 0} T_n$. Here $T_0 \in \mathcal{L}(\mathcal{H}^{m_0})$, $T_n \in \mathcal{L}(\mathcal{H}^{m_0+n} \ominus \mathcal{H}^{m_0+n-1})$, $\forall n > 0$, and m_0 is a fixed integer. By homotopy invariance of the map Λ-ind, there exists an integer N, such that $\mathrm{ind}(T_n) = 0$, $\forall n \geqslant N$. This implies that $\bigoplus_{n\geqslant N} T_n$ is homotopic to an invertible element of $\mathcal{L}_{\mathrm{loc}}(\bigoplus_{n\geqslant N}(\mathcal{H}^n \ominus \mathcal{H}^{n-1}))$ (see [13], proposition 8, p. 34). □

Remark 5.2.7. Let $D_0 \subset D$ be the smallest C^*-algebra containing $\sigma_{\mathrm{AW}}(\mathcal{B})$. The previous result makes it reasonable to define an elliptic anti-Wick operator on $\mathbf{R}^\infty$ as an element of $\tilde{D}_0$ whose image in $\tilde{\mathcal{B}}$ is invertible.

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