Ukrainian Mathematical Journal, Vol. 57, No. 5, 2005

OPERATORS OF GENERALIZED TRANSLATION AND HYPERGROUPS CONSTRUCTED FROM SELF-ADJOINT DIFFERENTIAL OPERATORS

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We construct new examples of operators of generalized translation and convolutions in eigenfunctions of certain self-adjoint differential operators.

In 1950–1953, Berezans'kyi (partially together with S. Krein) developed a detailed general theory of hypercomplex systems with countable and continuous bases (see [1] and the bibliography therein). This theory made it possible to construct deep generalizations of harmonic analysis and the theory of almost periodic functions. In foreign countries, the investigation of objects related to hypercomplex systems (hypergroups) was begun only in 1973. One of important problems in the theory of hypercomplex systems and hypergroups is the construction of various specific examples of such objects. A broad class of examples is based on operators of generalized translation associated with specific differential and difference operators [1, 2]. In the present paper, we show that self-adjoint differential operators of the Udnov–Schechter type that act simultaneously in a domain and on the boundary can also be used for this purpose.

1. Algebraic Structures Associated with First-Order Operators

1.1. Self-Adjoint Operator. In the Hilbert space $H = L_2\left(-\frac{1}{2}, \frac{1}{2}\right) \oplus \mathbf{E}^1$, we consider the operator A defined by the equality

$$A\begin{pmatrix} \varphi(x) \\ \frac{1}{2}\left[\varphi\left(-\frac{1}{2}\right)+\varphi\left(\frac{1}{2}\right)\right] \end{pmatrix} = \begin{pmatrix} -i\varphi'(x) \\ i\left[\varphi\left(\frac{1}{2}\right)-\varphi\left(-\frac{1}{2}\right)\right] \end{pmatrix} \tag{1}$$

on functions $\varphi \in W_2^1\left(-\frac{1}{2}, \frac{1}{2}\right)$ from the Sobolev space of all absolutely continuous functions on the segment $\left[-\frac{1}{2}, \frac{1}{2}\right]$ with square integrable derivative φ'.

Theorem 1. *The operator A in the Hilbert space H is a self-adjoint operator with purely discrete spectrum. Its eigenvalues λ_n, $n \in Z$, enumerated in ascending order and satisfying the condition $\lambda_0 = 0$, $\lambda_{-n} = -\lambda_n$, are roots of the characteristic equation*

$$e^{i\lambda} = \frac{2 - i\lambda}{2 + i\lambda}. \tag{2}$$

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Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 57, No. 5, pp. 659–668, May, 2005. Original article submitted February 3, 2005.

 0041–5995/05/5705–0782

For large $n \gg 1$, *the eigenvalues have the following asymptotics:*

$$\lambda_n = \pi(2n-1) + \varepsilon_n + O\left(\frac{1}{n^3}\right), \quad \varepsilon_n = \frac{4}{\pi(2n-1)}. \tag{3}$$

The functions $\varphi_n = e^{i\lambda_n x}$, $n \in Z$, *form a system of functions complete in the space* $L_2\left(-\frac{1}{2}, \frac{1}{2}\right)$ *and orthogonal with respect to the scalar product*

$$\langle \varphi, \psi \rangle = (\varphi, \psi)_{L_2} + \varphi_r \cdot \overline{\psi}_r, \quad \varphi_r = \frac{1}{2}\left[\varphi\left(\frac{1}{2}\right) + \varphi\left(-\frac{1}{2}\right)\right], \quad \varphi, \psi \in C\left(-\frac{1}{2}, \frac{1}{2}\right).$$

Proof. Let $U = \operatorname{col}(u(x), u_r)$ denote a vector from H constructed from a function $u \in W_2^1\left(-\frac{1}{2}, \frac{1}{2}\right)$. The vector u belongs to the domain of definition of the operator A and $Au = \operatorname{col}(-iu'(x), iu_s) \in H$, where $u_s = u\left(\frac{1}{2}\right) - u\left(-\frac{1}{2}\right)$. We show that the domain of definition of the operator A is everywhere dense in H. If this is not true, then there exists a vector $\Psi = \operatorname{col}(\psi(x), a) \neq 0$, $\psi \in L_2$, $a \in C^1$, such that $(U, \Psi)_H = 0$ $\forall u \in W_2^1$. For functions $u \in C_0^\infty\left(-\frac{1}{2}, \frac{1}{2}\right)$, the equality $(U, \Psi)_H = 0$ reduces to the equality $(u, \psi)_{L_2} = 0$ $\forall u \in C_0^\infty$. Taking into account that the space C_0^∞ is dense in the space L_2, we conclude that $\psi \equiv 0$. The equality $(U, \Psi)_H = 0$ turns into $u_r \bar{a} = 0$ $\forall u \in W_2^1$. Setting $u \equiv 1$, we get $a = 0$, which contradicts the condition $\Psi = \operatorname{col}(\psi(x), a) \neq 0$. Therefore, the domain of definition of the operator A is everywhere dense in the space H. The operator A is symmetric in the Hilbert space H. Indeed, for the vectors $U = \operatorname{col}(u(x), u_r)$ and $V = \operatorname{col}(v(x), v_r)$, where $u, v \in W_2^1$, by virtue of the definition of the operator A and the Green formula we have

$$(AU, V)_H - (U, AV)_H = (-iu', v)_{L_2} + (iu_s, v_r)_{E^1} - (u, -iv')_{L_2} - (u_r, iv_s)_{E^1}$$

$$= -i\left[u\left(\frac{1}{2}\right)\overline{v}\left(\frac{1}{2}\right) - u\left(-\frac{1}{2}\right)\overline{v}\left(-\frac{1}{2}\right)\right] + i[u_s \overline{v}_r + u_r \overline{v}_s] \equiv 0.$$

Let us show that the operator A is self-adjoint in H. To this end, it suffices to show that the range of values of the operator $A \pm iI$ is the entire space H, i.e., the equation $(A \pm iI)U = \Psi$ is uniquely solvable for any vector $\Psi = \operatorname{col}(\psi(x), a) \in H$. This equation reduces to the differential equation $-iu'(x) \pm iu(x) = \psi(x)$ with the boundary condition $iu_s \pm iu_r = a$. This problem is uniquely solvable for any $\psi \in L_2$ and $a \in C$, and its solution $u \in W_2^1$ can easily be represented in explicit form.

To find the eigenvectors $\Phi_\lambda = \operatorname{col}(\varphi_\lambda, \psi_\lambda)$ and eigenvalues λ, it is necessary to find a nontrivial solution of the equation $A\Phi_\lambda = \lambda\Phi_\lambda$. This problem reduces to the differential equation $-i\varphi'_\lambda = \lambda\varphi_\lambda$ with the boundary condition $-i\varphi_{\lambda,s} = \lambda\varphi_{\lambda,r}$. The latter problem has the nontrivial solutions $\varphi_\lambda(x) = e^{i\lambda x}$ only if λ is a solution of the characteristic equation (2). Equation (2) has simple real solutions λ_n, $n \in Z$; one can enumerate them in ascending order, setting $\lambda_0 = 0$. Then $\lambda_{-n} = -\lambda_n$, and, for large n, solutions of the characteristic equation (2) have asymptotics (3). Since the eigenfunctions $\Phi_{\lambda_n} = \operatorname{col}(\varphi_{\lambda_n}, \varphi_{\lambda,r})$ of the self-adjoint operator A form a system of functions complete and orthogonal in H, we have $(\Phi_{\lambda_n}, \Phi_{\lambda_m})_H = 0$ for $n \neq m$. This equality is equivalent to the condition of the orthogonality of the functions $\{e^{i\lambda_n x}\}_{n\in Z}$ with respect to the scalar product $\langle \cdot, \cdot \rangle$ defined by (4). Moreover,

$$(\Phi_{\lambda_n}, \Phi_{\lambda_m})_H = \langle e^{i\lambda_n x}, e^{i\lambda_m x} \rangle = \delta_{n,m} N_n^2, \tag{5}$$

where $N_n^2 = 1 + [\lambda_n^2/4 + 1]^{-1}$ and $\delta_{n,m}$ is the Kronecker symbol.

The theorem is proved.

Theorem 2. *In order that the system of functions* $\{e^{i\lambda_n x}\}_{n\in Z}$, $\lambda_0 = 0$, *form a system of functions complete in* $L_2\left(-\frac{1}{2}, \frac{1}{2}\right)$ *and orthogonal with respect to the scalar product (4), it is necessary and sufficient that the numbers* λ_n *be all solutions of Eq. (2).*

Proof. Using the condition $\langle e^{i\lambda_n x}, 1 \rangle = 0$, we obtain the characteristic equation (2) for the numbers λ_n. If the numbers $\lambda_n \neq \lambda_m$ are solutions of Eq. (2), then $\langle e^{i\lambda_n x}, e^{i\lambda_m x} \rangle = 0$, i.e., they are orthogonal with respect to the scalar product (2). The completeness of the system $\{e^{i\lambda_n x}\}_{n\in Z}$ (2) follows from Theorem 1 because the vectors $\Phi_{\lambda_n} = \operatorname{col}(e^{i\lambda_n x}, \cos(\lambda_n/2))$ form a complete orthogonal system in the space H.

The theorem is proved.

Remark 1. Theorems 1 and 2 remain true in the case where the scalar product (2) is replaced by a more general one, namely

$$\langle \varphi, \psi \rangle = (\varphi, \psi)_{L_2\left(-\frac{1}{2}, \frac{1}{2}\right)} + \alpha^2 \varphi_r \overline{\psi}_r,$$

the operator A is defined by the equality $A \operatorname{col}(u(x), \alpha u_r) = \operatorname{col}(-iu'(x), i\alpha^{-1} u_s)$, and the characteristic equation (2) is replaced by $e^{i\lambda} = \dfrac{2 - i\lambda\alpha^2}{2 + i\lambda\alpha^2}$.

1.2. Evolution Equation and Operators of Generalized Translation. Consider the Cauchy problem for an evolution equation in the space H:

$$\frac{dU}{dt} = iAU, \qquad U\big|_{t=0} = F. \tag{6}$$

Since the operator A is self-adjoint in the space H, problem (6) is uniquely solvable on the entire axis $-\infty < t < +\infty$ for an arbitrary initial condition $F \in H$. Let $F = \operatorname{col}(f(x), f_r)$, where $f \in C\left(-\frac{1}{2}, \frac{1}{2}\right)$. Then problem (6) has the solution

$$U(t) = \operatorname{col}\left(u(t,x), \frac{1}{2}\left[u\left(t, \frac{1}{2}\right) + u\left(t, -\frac{1}{2}\right)\right]\right),$$

where the function $u(t,x)$ is a solution of the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \qquad \frac{\partial}{\partial t}\left[u\left(t,\frac{1}{2}\right)+u\left(t,-\frac{1}{2}\right)\right] = 2\left[u\left(t,-\frac{1}{2}\right)-u\left(t,\frac{1}{2}\right)\right], \\ &u(0,x) = f(x), \qquad u\left(0,\frac{1}{2}\right)+u\left(0,-\frac{1}{2}\right) = f\left(\frac{1}{2}\right)+f\left(-\frac{1}{2}\right). \end{aligned} \tag{7}$$

Theorem 3. *A solution of problem (7) can be represented in the form* $u(t,x) = \hat{f}(x+t)$, *where the function* $\hat{f}(x)$ *is an extension of the function* f *defined on the segment* $\left[-\frac{1}{2},\frac{1}{2}\right]$ *to the entire axis. Moreover,*

$$\hat{f}(x) = \begin{cases} -f(x+1)+2e^{2(x+1/2)}f_r+4\int\limits_{x+1}^{1/2} e^{2(x+1-\tau)}f(\tau)\,d\tau, & -\frac{3}{2}\le x\le -\frac{1}{2}, \\ f(x), & -\frac{1}{2}\le x\le\frac{1}{2}, \\ -f(x-1)+2e^{-2(x-1/2)}f_r+4\int\limits_{-1/2}^{x-1} e^{-2(x-1-\tau)}f(\tau)\,d\tau, & \frac{1}{2}\le x\le\frac{3}{2}. \end{cases} \tag{8}$$

If $f(x) = e^{i\lambda_n x}$, *where* λ_n *are eigenvalues of the operator* A, *then* $\hat{f}(x) = e^{i\lambda_n x}$ *for all* x. *If*

$$f(x) = \sum_n f_n e^{i\lambda_n x}$$

is the Fourier series of a function $f \in C\left(-\frac{1}{2},\frac{1}{2}\right)$, *then its extension* $\hat{f}$ *can be represented by the same series:*

$$\hat{f} = \sum_n f_n e^{i\lambda_n x}, \qquad x\in R^1.$$

Proof. It follows from the equation $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$ that $u(t,x) = \hat{f}(x+t)$. Using the initial condition $u(0,x) = f(x)$, we establish that $\hat{f}(x) = f(x)$ for $-\frac{1}{2}\le x\le\frac{1}{2}$. Using the remaining relation in (7), we get

$$\hat{f}'\left(t+\frac{1}{2}\right)+2\hat{f}\left(t+\frac{1}{2}\right) = -\hat{f}'\left(t-\frac{1}{2}\right)+2\hat{f}\left(t-\frac{1}{2}\right). \tag{9}$$

Integrating equality (9) for $\frac{1}{2}\le t\le\frac{3}{2}$ and $-\frac{3}{2}\le t\le -\frac{1}{2}$ and taking into account that $\hat{f}(x) = f(x)$, we obtain relation (8).

If the vector of initial conditions $F = \mathrm{col}(e^{i\lambda_n x}, \cos(\lambda_n/2)) \equiv \Phi_{\lambda_n}$ in the Cauchy problem (6) is an eigenvector of the operator A with eigenvalue λ_n, then the solution has the form $U(t) = e^{i\lambda_n t}\Phi_{\lambda_n}$. Therefore, $u(t,x) = e^{i\lambda_n t} e^{i\lambda_n x} = \hat{f}(x+t)$ and, hence, $\hat{f}(x) = u(x,0) = e^{i\lambda_n x}$. Thus, $e^{i\hat{\lambda}_n x} = e^{i\lambda_n x}$ for all x. Hence,

$$\hat{f}(x) = \sum_n \hat{f}_n e^{i\lambda_n x} = \sum_n f_n e^{i\lambda_n x}.$$

The theorem is proved.

Definition 1. *The operator* T^t *of generalized translation that corresponds to problem (7) is defined by the equality*

$$T^t f(x) = u(t,x) = \hat{f}(x+t), \tag{10}$$

where $\hat{f}$ *is the extension of a function* f *from the segment* $\left[-\frac{1}{2}, \frac{1}{2}\right]$ *to the entire axis according to Theorem 3.*

Theorem 4. *If* λ_n *are eigenvalues of the operator* A, *i.e., solutions of the characteristic equation (2), then*

$$T^t e^{i\lambda_n x} = e^{i\lambda_n t} e^{i\lambda_n x}. \tag{11}$$

If

$$f(x) = \sum_n f_n e^{i\lambda_n x}$$

is the Fourier series of a function f *in the system of functions* $\{e^{i\lambda_n x}\}_{n\in Z}$ *orthogonal with respect to the scalar product (4), i.e.,*

$$f_n = \left\langle f(x), e^{i\lambda_n x} \right\rangle \frac{1}{N_n^2}, \tag{12}$$

then

$$T^t f(x) = \sum_n f_n e^{i\lambda_n t} e^{i\lambda_n x}. \tag{13}$$

The family of operators T^t *forms a unitary one-parameter group of operators with respect to the scalar product (4):*

$$T^{t_1} T^{t_2} = T^{t_1+t_2}, \quad T^0 = I, \quad \langle T^t \varphi, T^t \psi \rangle = \langle \varphi, \psi \rangle. \tag{14}$$

Proof. By virtue of Theorem 3, the extension of the function $e^{i\lambda_n x}$ coincides with this function. Therefore, by virtue of (10), we obtain relation (11). Applying the linear operator T^t to the Fourier series of the function $f(x)$ in the system $\{e^{i\lambda_n x}\}_{n\in Z}$ and taking (11) into account, we get relation (13). The last equality in (14) follows from the Parseval equality for Fourier series in a complete orthogonal system of functions, i.e.,

$$\langle f, g\rangle = \sum_n f_n \bar{g}_n \frac{1}{N_n^2}, \tag{15}$$

and the explicit form of the Fourier coefficients of the function $T^t f$, namely, $(T^t f)_n = f_n e^{i\lambda_n t}$, which follows from (13).

The theorem is proved.

1.3. Convolution

Definition 2. *The convolution of two continuous functions* $f, g \in C\left(-\frac{1}{2}, \frac{1}{2}\right)$ *is defined by the scalar product* $\langle\cdot,\cdot\rangle$ *(4) and the operator of generalized translation* T^t *as follows:*

$$(f * g)(t) = \langle T^t f, g^*\rangle, \tag{16}$$

where $g^*(x) = \bar{g}(-x)$ *is an involution.*

Theorem 5. *The convolution defined in Definition 2 is an associative commutative multiplication.*

Proof. Let us determine the convolution of two characters $e^{i\lambda_n x}$ and $e^{i\lambda_m x}$. Taking into account that $T^t e^{i\lambda_n x} = e^{i\lambda_n t} e^{i\lambda_m x}$ and $[e^{i\lambda_m x}]^* = e^{i\lambda_m x}$ and using equality (16), we get

$$\left(e^{i\lambda_n x} * e^{i\lambda_m x}\right)(t) = \left\langle e^{i\lambda_n t} e^{i\lambda_n x}, e^{i\lambda_m x}\right\rangle = e^{i\lambda_n t}\delta_{nm} N_n^2. \tag{17}$$

Let functions $f, g \in C\left(-\frac{1}{2}, \frac{1}{2}\right)$ be given. Representing these functions by the Fourier series

$$f(x) = \sum_n f_n e^{i\lambda_n x} \quad \text{and} \quad g(x) = \sum_n g_n e^{i\lambda_n x}$$

and taking into account inequality (17), we obtain

$$(f * g)(x) = \sum_n f_n g_n N_n^{-2} e^{i\lambda_n x}.$$

It follows from representation (18) that the Fourier coefficients of the convolution are expressed in terms of the Fourier coefficients of the convolution multipliers, i.e., convolution (16) is commutative.

Representing the Fourier coefficients of the repeated convolutions $(f * g) * h$ and $f * (g * h)$ in terms of the product of Fourier coefficients of convolution multipliers, we get

$$[(f*g)*h]_n = f_n g_n h_n N_n^{-4} = [f*(g*h)]_n.$$

Thus, convolution (16) is associative.

The theorem is proved.

Using Definition 2, the explicit form of the scalar product (4), the explicit form of the translation operator $T^t f(x) = \hat{f}(x+t)$, and the explicit form (8) of the extension $\hat{f}$ of the function f, we easily obtain the following relation for the convolution of two functions:

$$(f*g)(x) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} f(\xi)g(\eta)c(\xi,\eta,x)\,d\xi\,d\eta,$$

where $c(\xi,\eta,x) = (\chi_{[-1/2,\xi]} * \chi_{[-1/2,\eta]})(x)$ and $\chi_{[-1/2,\xi]}$ is the characteristic function of the segment $\left[-\frac{1}{2},\xi\right]$.

This convolution can also be represented in the following explicit form:

$$(f*g)(x) = \int_{x-1/2}^{1/2} f(x-s)g(s)\,ds - \int_{-1/2}^{x-1/2} f(x-1-s)g(s)\,ds + 4\int\int_{D_x} e^{-2(x-1+\xi+\eta)} f(\xi)g(\eta)\,d\xi\,d\eta$$

$$+ e^{-2x}\left[f_r \cdot g_r + 2f_r \int_{-1/2}^{x-1/2} e^{1+\eta} g(\eta)\,d\eta + 2g_r \int_{-1/2}^{x-1/2} e^{1+\xi} f(\xi)\,d\xi\right],$$

where $x > 0$ and $D_x = \{(\xi,\eta)\colon -1/2 \le \xi, \eta \le 1/2,\ \xi + \eta \le x - 1\}$.

1.4. Structure Constants. The collection of characters $\{e^{i\lambda_n x}\}_{n\in Z}$ forms an orthogonal system with respect to the convolution multiplication (17). The ordinary product of two characters can be expressed as their linear combination

$$e^{i\lambda_n x}\, e^{i\lambda_m x} = \sum_k c_{n,m,k}\, e^{i\lambda_k x},$$

where $c_{n,m,k}$ are structure constants. Since this representation is the Fourier series of the function $e^{i\lambda_n x}\, e^{i\lambda_m x}$, we have

$$c_{n,m,k} = \frac{1}{N_k^2}\left\langle e^{i\lambda_n x} e^{i\lambda_m x}, e^{i\lambda_k x}\right\rangle.$$

For $\lambda_n + \lambda_m - \lambda_k \ne 0$, we get

$$c_{n,m,k} = (-1)^{n+m+k+1}\left[\left(4+\lambda_n^2\right)\left(4+\lambda_m^2\right)\left(4+\lambda_k^2\right)\right]^{-1/2} \frac{2\lambda_n\lambda_m(\lambda_n+\lambda_m)}{N_k^2(\lambda_n+\lambda_m-\lambda_k)}.$$

Furthermore, $c_{n,0,k} = c_{0,n,k} = \delta_{n,k}$ and $c_{n,-n,0} = 1$.

2. Algebraic Structures Associated with Second-Order Differential Operators

2.1. Self-Adjoint Operator. In the space $H = L_2(0, 1) \oplus \mathbf{E}^1$, we consider the operator B defined by the equality

$$B \operatorname{col}\big(u(x), u(1)\big) = \operatorname{col}\big(-u''(x), u'(1)\big) \tag{19}$$

on functions $u(x)$ from the Sobolev space $W_2^2(0,1)$ that satisfy the boundary condition $u'(0) = 0.$

Theorem 6. *The operator B is a self-adjoint operator in the Hilbert space H with purely discrete spectrum. Its eigenvalues λ_n^2, enumerated in ascending order, are roots of the characteristic equation*

$$\tan \lambda_n = -\lambda_n, \qquad \lambda_0 = 0, \quad \lambda_n \geq 0. \tag{20}$$

The functions $\varphi_n = \cos \lambda_n x$, $n = 0, 1, \ldots$, form a system of functions complete in the space H and orthogonal with respect to the scalar product

$$\langle \varphi, \psi \rangle = (\varphi, \psi)_{L_2} + \varphi(1) \cdot \overline{\psi}(1), \qquad \varphi, \psi \in C(0, 1). \tag{21}$$

Moreover,

$$\langle \cos \lambda_n x, \cos \lambda_m x \rangle = \delta_{n,m} N_n^2, \tag{22}$$

where

$$N_n^2 = \frac{1}{2}\big(1 + \cos^2 \lambda_n\big) = \frac{1}{2}\left(1 + \frac{1}{1+\lambda_n^2}\right), \quad n > 0, \qquad \text{and} \qquad N_0^2 = 2.$$

The proof of Theorem 6 is analogous to the proof of Theorem 1.

2.2. Generalized Translation

Definition 3. *The linear operator of generalized translation T^t on the basis $\cos \lambda_n x$, where λ_n are roots of the characteristic equation (20), is defined by the equality*

$$T^t \cos \lambda_n x = \cos \lambda_n t \, \cos \lambda_n x. \tag{23}$$

Definition 4. *The extension $\hat{f}$ of a continuous function $f \in C(0, 1)$ to the entire axis is defined by the Fourier series of the function f in the complete system of functions $\{\cos \lambda_n x\}_{n \geq 0}$ orthogonal with respect to the scalar product (21):*

$$\hat{f}(x) = \sum_{n=0}^{\infty} f_n \cos \lambda_n x, \tag{24}$$

where $f_n = \langle f(x), \cos \lambda_n x \rangle \dfrac{1}{N_n^2}$ are the Fourier coefficients of the function f.

Theorem 7. *The operator of generalized translation* T^t *acts on an arbitrary continuous function* $f \in C(0,1)$ *as follows:*

$$T^t f(x) = \frac{1}{2}\left[\hat{f}(x+t) + \hat{f}(x-t)\right], \tag{25}$$

where $\hat{f}$ *is the extension of the function* f *to the entire axis according to Definition 4.*

Proof. Let

$$f(x) = \sum_{n=0}^{\infty} f_n \cos\lambda_n x$$

be the Fourier series of a function $f \in C(0,1)$. Acting by the operator T^t on this series and taking equality (23) into account, we get

$$T^t f(x) = \sum_{n=0}^{\infty} f_n \cos\lambda_n t \cos\lambda_n x = \frac{1}{2}\sum_{n=0}^{\infty} f_n\left[\cos\lambda_n(x+t) + \cos\lambda_n(x-t)\right] = \frac{1}{2}\left[\hat{f}(x+t) + \hat{f}(x-t)\right].$$

Theorem 8. *Let* $U(t) = \mathrm{col}\left(u(t,x), u(t,1)\right) \in H$ *be a solution of the Cauchy problem*

$$\frac{d^2U}{dt^2} + BU = 0, \qquad U(0) = \mathrm{col}\left(f(x), f(1)\right), \qquad \left.\frac{dU}{dt}\right|_{t=0} = 0. \tag{26}$$

Then $u(t,x) = T^t f(x)$, *where* T^t *is the operator of generalized translation.*

Proof. If $f(x) = \cos\lambda_n x$, where λ_n^2 are eigenvalues of the self-adjoint operator B, then $u(t,x) = \cos\lambda_n t \cos\lambda_n x$, i.e., equality (23) is true. The solution of problem (26) obtained by the method of eigenfunctions can be represented in the form

$$u(t,x) = \frac{1}{2}\left[\hat{f}(x+t) + \hat{f}(x-t)\right], \tag{27}$$

which leads to representation (24) for the operators T^t. On the other hand, substituting (27) into (26), we obtain the following differential equation for the function $\hat{f}$:

$$\hat{f}'(1+t) + \hat{f}(1+t) = \hat{f}'(1-t) + \hat{f}(1-t).$$

Integrating this equation, we get

$$\hat{f}(x+1) = -f(1-x) + 2e^{-x} f(1) + 2\int_0^x e^{-(x-\tau)} f(1-\tau)\,d\tau, \qquad 0 \le x \le 1.$$

The theorem is proved.

2.3. Convolution

Definition 5. *The convolution of two functions* $f, g \in C(0,1)$ *is defined by the equality*

$$(f * g)(x) = \left\langle f(y), T^{-y}\overline{g}(x)\right\rangle. \tag{28}$$

Theorem 9. *Convolution (28) is associative and commutative.*

Proof. Let us determine the convolution of characters. We have

$$(\cos\lambda_n(\cdot) * \cos\lambda_m(\cdot))(x) = \int_0^1 \cos\lambda_n y \cos\lambda_m y \cos\lambda_m x\, dy + \cos\lambda_n \cos\lambda_m \cos\lambda_m x = \delta_{n,m}\cos\lambda_n x \cdot N_n^2.$$

The subsequent proof repeats the proof of Theorem 5.

Using the explicit form of the operators of translation T^t, we determine the explicit form of the convolution:

$$\begin{aligned}(f * g)(x) = {} & \frac{1}{2}\int_0^x f(y) g(x-y)\, dy + \frac{1}{2}\int_0^{1-x} [f(x+y)g(y) + f(y)g(x+y)]\, dy \\ & - \frac{1}{2}\int_{-x/2}^{x/2} f\left(1 - \frac{1}{2}x + s\right) g\left(1 - \frac{1}{2}x - s\right) ds \\ & + \exp(2-x)\iint_{D_x} f(y) g(s) \exp(-x-s)\, dy\, ds + f(1) g(1) \exp(-x) \\ & + f(1)\int_{1-x}^1 g(y)\exp(1-x-y)\, dy + g(1)\int_{1-x}^1 f(y)\exp(1-x-y)\, dy,\end{aligned}$$

where

$$D_x = \{(s,y) \in \mathbf{R}^2 \mid s \le 1,\ y \le 1,\ s + y \ge 2 - x\}.$$

2.4. Structure Constants. The elements of the system $\{\cos\lambda_n x\}_{n=0}^{\infty}$ are characters for the generalized translations (23). Using structure constants $c_{n,m,k}$, we express the ordinary product of two characters via their linear combination:

$$\cos\lambda_n x \cos\lambda_m x = \sum_k c_{n,m,k} \cos\lambda_k x.$$

Since this representation is a Fourier series, we have

$$c_{n,m,k} = \frac{1}{N_k^2}\langle \cos\lambda_n x \cos\lambda_m x, \cos\lambda_k x\rangle.$$

Using the formula

$$\cos a \cos b \cos c = \frac{1}{4}(\cos(a+b+c) + \cos(a+b-c) + \cos(a-b+c) + \cos(-a+b+c)),$$

we get

$$\begin{aligned}(\cos\lambda_n x \cos\lambda_m x, \cos\lambda_k x)_H &= \int_0^1 \cos\lambda_n x \cos\lambda_m x \cos\lambda_k x\, dx + \cos\lambda_n \cos\lambda_m \cos\lambda_k \\ &= \frac{1}{4}\left[\frac{\sin(\lambda_n+\lambda_m+\lambda_k)}{\lambda_n+\lambda_m+\lambda_k} + \frac{\sin(\lambda_n+\lambda_m-\lambda_k)}{\lambda_n+\lambda_m-\lambda_k}\right. \\ &\qquad \left. + \frac{\sin(\lambda_n-\lambda_m+\lambda_k)}{\lambda_n-\lambda_m+\lambda_k} + \frac{\sin(-\lambda_n+\lambda_m+\lambda_k)}{-\lambda_n+\lambda_m+\lambda_k}\right] + \cos\lambda_n \cos\lambda_m \cos\lambda_k .\end{aligned}$$

Using the characteristic equation (20), we obtain the following relation for $n, m, k \neq 0$:

$$c_{n,m,k} = \frac{2(-1)^{n+m+k}\lambda_n^2\lambda_m^2\lambda_k^2\left[\left(1+\lambda_n^2\right)\left(1+\lambda_m^2\right)\left(1+\lambda_k^2\right)\right]^{-1/2}}{N_k^2\left[\lambda_n^4+\lambda_m^4+\lambda_k^4-2\left(\lambda_n^2\lambda_m^2+\lambda_n^2\lambda_k^2+\lambda_m^2\lambda_k^2\right)\right]}.$$

Furthermore,

$$c_{n,m,0} = \frac{1}{2}\delta_{n,m} N_n^2 \qquad \text{and} \qquad c_{0,m,k} = c_{m,0,k} = \delta_{m,k}.$$

3. Conclusions

In the present paper, we have given two examples of the construction of operators of generalized translation based on eigenfunctions of self-adjoint differential operators. From the abstract point of view, this construction consists of the restriction of the basis on which the operators of translation are defined. Let Ω be a topological space, let $C(\Omega)$ be the space of functions continuous on Ω, and let operators of generalized translation T^t be defined on $C(\Omega)$ [2]. Assume that functions $\chi(x,\lambda)$, $x \in \Omega$, $\lambda \in \Lambda$, form a family of characters, i.e., $T^t\chi(x,\lambda) = \chi(t,\lambda)\chi(x,\lambda)$. Consider a subspace $\Omega_0 \subset \Omega$. Let a scalar product $\langle\cdot,\cdot\rangle$ be given on the space $C(\Omega_0)$. Furthermore, assume that there exists a family of characters $\{\chi(x,\lambda_n)\}_{n=1}^{\infty}$ that forms a complete orthogonal system with respect to this scalar product. Then every function $f(x) \in C(\Omega_0)$ can be represented by the Fourier series

$$f(x) = \sum_n f_n \chi(x,\lambda_n), \qquad x \in \Omega_0.$$

Since the characters are also defined for $x \in \Omega$, the Fourier series is defined for all $x \in \Omega$ and is a natural extension $\hat{f}$ of the function f defined on Ω_0. Therefore, the operators of generalized translation T^t defined on $C(\Omega)$ generate operators of generalized translation T_0^t on $C(\Omega_0)$ according to the formula

$$T_0^t f = (T^t \hat{f}) \restriction_{C(\Omega_0)},$$

i.e., the function f is naturally extended from Ω_0 to Ω, then the operator T^t is used, and the obtained function $T^t \hat{f}$ is considered as a function on Ω_0. Apparently, this procedure allows one to obtain a series of other informative examples of operators of generalized translation.

This work was partially supported by the DFG (project No. 436 UKR113/72).

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