Research project of Kosyak Alexandre

1.Unitary representations of the infinite-dimensional groups and the Ismagilov conjecture.

2. The orbit method for infinite-dimensional groups.

3. Von-Neumann algebras and infinite-dimensional groups.

4. Representations of the braid groups and of the quantum groups.

1. It is well known that a *general approach* towards **construction irreducible representations of infinite-dimensional topological groups** *does not exist*. We try to *develop* such an approach using **dynamical system** and the **ergodic theory**. Let $(X, \mathfrak{B})$ be a *measurable space* and let $\mathrm{Aut}(X)$ denote the group of all *measurable automorphisms* of the space X. With any *measurable action* $\alpha : G \to \mathrm{Aut}(X)$ of a group G on a space X and a G-*quasi-invariant measure* μ on X one can associate a *unitary representation* $\pi^{\alpha,\mu,X} : G \to U(L^2(X,\mu))$, of the group G by the formula $(\pi_t^{\alpha,\mu,X} f)(x) = (d\mu(\alpha_{t^{-1}}(x))/\ d\mu(x))^{1/2} f(\alpha_{t^{-1}}(x))$, $f \in L^2(X,\mu)$. Let us set $\alpha(G) = \{\alpha_t \in \mathrm{Aut}(X) \mid t \in G\}$. Let $\alpha(G)'$ be the *centralizer* of the subgroup $\alpha(G)$ in $\mathrm{Aut}(X):\ \alpha(G)' = \{g \in \mathrm{Aut}(X) \mid \{g, \alpha_t\} = g\alpha_t g^{-1}\alpha_t^{-1} = e\ \forall t \in G\}$. The following conjecture has been discussed in [20,22,24, CV].

Conjecture 1 *The representation* $\pi^{\alpha,\mu,X} : G \to U(L^2(X,\mu))$ *is irreducible if and only if*

1) $\mu^g \perp \mu\ \forall g \in \alpha(G)' \backslash \{e\}$, *(where* $\perp$ *stands for singular),*

2) the measure μ *is* G-**ergodic**.

This conjecture is known as the **Ismagolov conjecture**, in the case when $X = \tilde{G}$ is a complition of the group G in a suitable topology. The corresponding representation we call *regular*. Now this conjecture *is proved for a lot of infinite-dimensional groups*, for example for inductive limit $B_0^{\mathbb{N}}$ (or $B_0^{\mathbb{Z}}$) of *nilpotent* [5,7,14,CV], *solvable* [25,CV] and *simple* groups [24,31,CV], for the group of a *circle diffeomorphisms* [8,CV] etc. Whether it holds in the **general case** over the field $\mathbb{C}$ is an **open problem**. We have construct the so-called *quasiregular representations* of the group $B_0^{\mathbb{N}}(\mathbf{k})$ of infinite upper triangular matrices *with coefficient in* a field $\mathbf{k} = \mathbb{C}$ in [19,20,22,CV] (resp. in a finite field $\mathbb{F}_p$ in [30,CV]) and give the criteria of the irreducibility and equivalence of the constructed representations. The new phenomenon is discovered: *the Ismagilov conjecture in not valid* in the case of the field $\mathbb{F}_p$.

2. Unitary representations of the infinite-dimensional nilpotent group and the orbit method. Let $\hat{G}$ be the dual of the group G. Our **aim** is to describe $\hat{G}$ for $G = \varinjlim_n G_n$ where $G_n = B(n, \mathbb{R})$ is the group of all $n \times n$ upper triangular real matrices with units on the principal diagonal.

We can consider also $\tilde{G} = \varprojlim_n G_n$. The group G is the group of infinite matrices of the form $I + x$ where x is finite (resp. arbitrary) upper triangular matrix. It is shown in [30,CV] that $\hat{G} \supset \bigcup_n \hat{G}_n = \hat{\hat{G}}$, but $\hat{G} \setminus \bigcup_n \hat{G}_n \neq \emptyset$. Namely $\hat{G} \setminus \bigcup_n \hat{G}_n \ni$ "regular" and "quasiregular" representations of the group G (see definitions in [20,28,CV]).

One may use Kirillov's orbit method [5, 6] to describe $\hat{G}_n$. For infinite-dimensional nilpotent group G there are no orbit method. We would like **to develop the orbit method** in this case.

The basic result of the method of orbits, applied to nilpotent Lie groups G_n, is the description of a one-to-one correspondence between two sets:

a) The set $\hat{G}_n$ of all equivalence classes of irreducible unitary representations of a connected and simply connected nilpotent Lie group G_n.

b) The set $O(G_n)$ of all orbits of the group G_n in the space $\mathfrak{g}_n^*$ dual to the Lie algebra $\mathfrak{g}_n = \mathrm{Lie}(G_n)$ with respect to the coadjoint representation.

In [5, 6] it is proved that all irreducible representations $\hat{G}_n$ are obtained as induced representations $\mathrm{Ind}_H^{G_n} U_{f,H}$ associated with a points $f \in \mathfrak{g}_n^*$ and the corresponding *subordinate* subgroup $H \subset G_n$. The induced representation $\mathrm{Ind}_H^{G_n} U_{f,H}$ is defined canonically in the Hilbert space $L^2(H\backslash G_n, \mu)$.

A. Kirillov [6], Chapter I, §4, p.10 says: *"The method of induced representations is not directly applicable to infinite-dimensional groups (or more precisely to a pair $G \supset H$) with an infinite-dimensional factor $H\backslash G$)".*

Firstly we need to **define the notions of induction** in the case when the space $H\backslash G$ is **infinite-dimensional**. Since the corresponding homogeneous space $H\backslash G$ is infinite-dimensional, the unique G-quasi-invariant measure on $H\backslash G$ (existing in the finite-dimensional case) should be replaced by some G-quasi-invariant measure on the **completion** $\widetilde{H\backslash G}$ of the initial space $H\backslash G$ **in a certain topology**. Hence the procedure of induction will not be unique but nevertheless well-defined (if a G-quasi-invariant measure exists). So the uniquely defined induced representation $\mathrm{Ind}_H^G U_{f,H}$ in the Hilbert space $L^2(H\backslash G, \mu)$ should be replaced by the family of induced representations $\mathrm{Ind}_H^{\tilde{G},\mu} U_{f,H}$ in the Hilbert spaces $L^2(H\backslash \tilde{G}, \mu)$ corresponding to different completions $\tilde{G}$ of the group G and different G-quasi-invariant measures μ on $H\backslash\tilde{G}$. **Secondly** the remarkable fact is that it is sufficient to consider only the Hilbert completions of the initial group G and the spaces $H\backslash G$. The **Hilbert-Lie groups** appear naturally in the representation theory of the infinite-dimensional matrix group. *Every unitary representation of the group* $\mathrm{GL}_0(\mathbb{Z},\mathbb{R}) = \varinjlim_n \mathrm{GL}(2n-1,\mathbb{R})$ *can be extended by continuity to a unitary representation* $U_2 : \mathrm{GL}_2(a) \to U(H)$ *of some Hilbert-Lie group* $\mathrm{GL}_2(a)$ *depending on the representation* [4,CV] .

Let us denote by $B_2(a)$ the completion of the subgroup $B_0^{\mathbb{N}} \subset \mathrm{GL}_0(\mathbb{Z},\mathbb{R})$ in the Hilbert-Lie group $\mathrm{GL}_2(a)$. Since $B_0^{\mathbb{N}} = \bigcap_{a\in\mathfrak{A}} B_2(a)$ (see [4,CV]) we conclude that $\widehat{B_0^{\mathbb{N}}} = \bigcup_{a\in\mathfrak{A}} \widehat{B_2(a)}$. It leaves to describe $\widehat{B_2(a)}$. The problem of developing the orbit method for the Hilbert-Lie group could be more easy, since the corresponding Lie algebra $\mathfrak{b}_2(a)$ is a Hilbert-Lie algebra the pairing and the dual $\mathfrak{b}_2(a)^*$ are well defined.

3. Von-Neumann algebras and infinite-dimensional groups. The first example of *a non type* I *factors* appeared in the work of J.von Neumann as the second commutant of the *regular representation if the discrete ICC group*. In this case the von Neumann algebra is II_1 factor. It is natural to study the *von Neumann algebra, generated* by the *representations of infinite-dimensional groups*. Let μ_b be the infinite tensor product $\mu_b = \otimes_{1\leq k<n}\mu_{b_{kn}}$ of one-dimensional Gaussian measure $d\mu_{b_{kn}} = (b_{kn}/\pi)^{1/2}\exp(-b_{kn}x_{kn}^2)dx_{kn}$ on the group $\tilde{G} = B^{\mathbb{N}}$ (resp. $\tilde{G} = B^{\mathbb{Z}}$), such that $(\mu_b)^{R_t} \sim \mu_b \sim (\mu_b)^{L_t}$, $\forall t \in G$. The right T^R and the left T^L regular representations of the group $G = B_0^{\mathbb{N}}$ (resp. $G = B_0^{\mathbb{Z}}$) are naturally defined in the Hilbert space $L^2(\tilde{G},\mu_b)$, $T^R, T^L : G \to U(L^2(\tilde{G},\mu_b))$. Let us denote by $\mathfrak{A}_G^R$ and by $\mathfrak{A}_G^L$ the corresponding von-Neumann algebras, generated by the right and the left regular representations $\mathfrak{A}_G^R = \left(T_t^R \mid t \in G\right)''$, $\quad \mathfrak{A}_G^L = \left(T_s^L \mid s \in G\right)''$ (see [12,13,CV]).

Problem 1. Describe the **commutant** $\left(\mathfrak{A}_G^R\right)'$ and find the condition on the measure μ_b **when** the corresponding **von-Neumann algebras** $\mathfrak{A}_G^R$ and $\mathfrak{A}_G^L$ are **factors**.

Theorem 2 [12,CV] *We have* $\left(\mathfrak{A}_G^R\right)' = \mathfrak{A}_G^L$ *if* $\mu_b(x^{-1}) \sim \mu_b(x)$. *Operator of the canonical conjugation* J_μ *is defined by* $(J_\mu f)(x) = \Delta_\mu^{1/2}(x)\overline{f(x^{-1})}$, $J_\mu T_t^{R,\mu} J_\mu = T_t^{L,\mu} \forall t \in G$, *where* $\Delta_\mu(x) = d\mu_b(x^{-1})/d\mu_b(x)$ *is the modular operator.*

Theorem 3 [13,CV] *Representation* $T^{R,L}(t,s) = T_t^R T_s^L$, $T^{R,L} : B_0^{\mathbb{N}} \times B_0^{\mathbb{N}} \to U(L^2(B^{\mathbb{N}},\mu_b))$ *is irreducible if* $S_{kn}^{R,L}(\mu_b) = \sum_{m=n+1}^{\infty} b_{km}/S_{nm}^L(\mu_b) = \infty$, *where* $S_{kn}^L(\mu_b) = \sum_{m=n+1}^{\infty} b_{km}/b_{nm}$.

$\mathfrak{A}^R_G$ is factor if the representation $T^{R,L}: G\times G\to U(L^2(\tilde{G},d\mu_b))$ is irreducible [13,CV]. If $(\mu_b)^{R_t}\sim\mu_b\sim(\mu_b)^{L_t}$, $\forall t\in G$ then $\mathfrak{A}^R_{B_0^{\mathbb{Z}}}$ is factor if $S^{R,L}_{kn}(\mu_b)=\sum_{m=n+1}^{\infty}b^2_{km}(b_{km}+S^R_{km}(\mu_b))^{-1}(b_{nm}+S^L_{nm}(\mu_b))^{-1}=\infty$ where $S^R_{kn}(\mu_b)=\sum_{r=-\infty}^{k-1}b_{rn}/b_{rk}$ [17,CV].

Problem 2. Determine the **type of** the corresponding **factors**. We prove that the **corresponding factor is** III_1 for the case of the group $B_0^{\mathbb{N}}$ [34,CV]. For the group $B_0^{\mathbb{Z}}$ it is proved in [29,CV] (work in progress in collaboration with I.Dynov).

4. Representations of the braid groups B_n and of the quantum groups. Our **aim** is to describe the **dual** $\hat{B}_n$ of the **braid group** B_n. It is natural to compare the **representation theory** of the **symmetric group** S_n and of the braid group B_n. We know almost everything about representation theory of the symmetric group S_n. We know the description of the *dual* $\hat{S}_n$ in terms of *Young diagrams*. We know even the *Plancherel measure* on $\hat{S}_n$. The *Young graph* explains how to decompose the restriction $\pi\mid_{S_{n-1}}$ of the representation $\pi\in\hat{S}_n$, etc.

The braid groups B_n are **defined** by the generators σ_i, $1\leq i\leq n-1$ and by the relations $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$, $\quad\sigma_i\sigma_j=\sigma_i\sigma_j$ for $\mid i-j\mid\geq 2$. The *dual* $\hat{B}_n$ of the group B_n *is known* only for the *commutative case* when $n=2$. In this case $B_2\cong\mathbb{Z}$ hence $\hat{B}_2\cong S^1$. The *representation theory* for the braid groups B_n is much more *complicated* than for S_n. The *reason* is the following. In the case of the group S_n we have the essential (*quadratic) relation* $\sigma_i^2=1$, hence $Sp\,(\pi(\sigma_i))\subseteq\{-1,1\}$. In the case of the group B_n we do not have these conditions. Since $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$ we have $Sp\,(\pi(\sigma_i))=Sp\,(\pi(\sigma_{i+1}))$, but the **spectra** $Sp\,(\pi(\sigma_i))$ may be almost **arbitrary**.

The **Hecke algebra** $H_n(q)$ see f.e. [2] appears as the factor algebra of the group algebra of the group B_n subject to the following *quadratic relation* $\sigma_i^2=(q-1)\sigma_i+q$, $1\leq i\leq n-1$, hence $Sp\,(\pi(\sigma_i))\subseteq\{-1,q\}$ and $H_n(q)\cong\mathbb{C}[S_n]$. This is a reason why the representation theory of Hecke algebras is well developed.

The *next step* is to impose the *polynomial condition* $p_k(\sigma_i)=0$ on the generators σ_i where k is the order of the polynomial $p_k(x)$. For $k=3$ the corresponding algebra is called **Birman–Murakami–Wenzl type algebra** or simple BMW algebra see [9, 12] (see also [10]) and so on.

The situation becomes much more complicated if no additional conditions on the spectra are imposed. We *shall study* this **general case**.

In [11] I.Tuba and H.Wenzl gave the **complete classification** of all *simple representations* of B_3 for **dimension** ≤ 5. In [1] E.Formanek et al. gave the **complete classification** of all *simple representations of* B_n for **dimension** $\leq n$.

In the work [32,CV] with S.Albeverio we have constructed a $\left[\frac{n+1}{2}\right]+1$ parameter family of irreducible representations of the braid group B_3 **in arbitrary dimension** $n\in\mathbb{N}$, using **a** $q-$**deformation of the Pascal triangle**. This construction extends in particular results by S.P. Humphries [3], I. Tuba and H. Wenzl [11], and E. Ferrand (2000). The **irreducibility** and the **equivalence** of the constructed representations is studied. For example the representations corresponding to different q and n are *nonequivalent*.

There is a striking **connection** [33,CV] between these **representations** of B_3 and a highest weight modules of the **quantum group** $U_q(\mathfrak{sl}_2)$, a one-parameter *deformation of the universal enveloping algebra* $U(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$. The starting point for all these considerations is some homomorphism ρ_3 of the braid group B_3 into $\mathrm{SL}(2,\mathbb{Z})$:

$$\rho_3: B_3\mapsto \mathfrak{sl}_2 \overset{\exp}{\mapsto} \mathrm{SL}(2,\mathbb{Z}),\quad \sigma_1\mapsto \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\overset{\exp}{\mapsto}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),\quad \sigma_2\mapsto \left(\begin{smallmatrix}0&0\\-1&0\end{smallmatrix}\right)\overset{\exp}{\mapsto}\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right).$$

The constructed representations may be treated as the $q-$**symmetric power** of this *fundamental representation* or as an appropriate $q-$**exponential** of the highest weight modules of $U_q(\mathfrak{sl}_2)$.

We plan to generalize these connection between the representations of the braid group B_n and the highest weight modules of the $U_q(\mathfrak{sl}_{n-1})$ **for arbitrary** n using the so-called **reduced Burau representation** $b_n^{(t)}$ see c.f. [2]. We note that in particular $\rho_3=b_3^{(-1)}$.

Let $\mathfrak{g}$ be the Lie algebra defined by a Cartan matrix $\mathbf{A}$ and let $\mathbf{B}$ be the corresponding braid group. Denote by $\mathbf{U}(\mathfrak{g})$ the quantized enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{C}(v)$, and let V be the integrable $\mathbf{U}(\mathfrak{g})-$module. In [8] G. Lusztig defined a natural action of $\mathbf{B}$ on V which permutes the weight space of V according to the action of the Weyl group on the weights. This rather *general but different approach* allows us also to construct the irreducible representations of the braid group $\mathbf{B}$ (see [7]).

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