POISSON SUMMATION ON ADELES

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Abstract. These are notes of my talk on the seminar on J.Tate's thesis held at MPI in Bonn in Spring 2006.

1. General Poisson summation formula

Let V is locally compact abelian group, V^* it's group of characters, μ a Haar measure on V. Suppose there exist a bihomomorphism $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{R}/\mathbb{Z}$ s.t. the map

$$y \in V \mapsto e^{2\pi i\langle y,\cdot\rangle} \in V^* \tag{1}$$

provides an isomorphism of V and V^*.

Let $\Gamma \subset V$ be a discrete countable subgroup with compact quotient V/Γ, and $D \subset V$ be a relatively compact fundamental domain for Γ. Then

Theorem 1. *If continuous function $f \in L_1(V,\mu)$ satisfies*
(1) $\sum_{\xi\in\Gamma} f(x+\xi)$ is uniformly (absolutely) convergent for $x \in D$
(2) $\sum_{\eta\in\Gamma^\perp} |\hat{f}(\eta)|$ is convergent
then

$$\sum_{\xi\in\Gamma} f(\xi) = \frac{1}{\mu(D)} \sum_{\eta\in\Gamma^\perp} \hat{f}(\eta).$$

Proof. Note that periodic w.r.t. Γ functions can be considered as functions on V/Γ and vise versa. Then, the functional $C(V/\Gamma) \to \mathbb{C}$

$$\phi \mapsto \int_D \phi(x)\mu(dx)$$

is linear, continious and translation invariant. To prove the translation invarianse we use countability of Γ:

$$\int_D \phi(y+x)\mu(dx) = \int_{y+D} \phi(x)\mu(dx),$$

decomposing $y + D = \cup_{\xi\in\Gamma}\xi + D_\xi$, $D_\xi \subset D$, we note that $D = \cup D_\xi$, so

$$= \sum_\xi \int_{\xi+D_\xi} \phi(x)\mu(dx) = \sum_\xi \int_{D_\xi} \phi(x-\xi)\mu(dx)$$

$$= \sum_\xi \int_{D_\xi} \phi(x)\mu(dx) = \int_D \phi(x)\mu(dx).$$

So, this funcional satisfies the characteristic properties of Haar integral. Then there exist a Haar measure ν on V/Γ such that

$$\int_D \phi(x)\mu(dx) = \int_{V/\Gamma} \phi(x)\nu(dx).$$

Obviously, $\nu(V/\Gamma) = \mu(D)$.

Consider $\phi(x) = \sum_{\xi\in\Gamma} f(x+\xi)$. This is a periodic continious function. Since $\Gamma^\perp$ is indetified with V/Γ^* under (1), by inversion formula for Fourier transform we get

$$\phi(x) = \frac{1}{\nu(V/\Gamma)} \sum_{\eta\in\Gamma^\perp} \hat{\phi}(\eta) e^{2\pi i\langle x,\eta\rangle}.$$

Now we calculate the Fourier transform for $\eta \in \Gamma^\perp$:

$$\hat{\phi}(\eta) = \int_D \phi(x) e^{-2\pi i\langle \eta,x\rangle}\mu(dx) = \int_D \sum_{\xi\in\Gamma} f(x+\xi) e^{-2\pi i\langle \eta,x\rangle}\mu(dx)$$

(by uniform convergence)

$$= \sum_{\xi\in\Gamma}\int_D f(x+\xi)e^{-2\pi i\langle\eta,(x+\xi)\rangle}\mu(dx) = \sum_{\xi\in\Gamma}\int_{\xi+D} f(x)e^{-2\pi i\langle\eta,x\rangle}\mu(dx)$$

$$= \int_V f(x)e^{-2\pi i\langle\eta,x\rangle}\mu(dx) = \hat{f}(\eta).$$

So,

$$\sum_\xi f(\xi+x) = \phi(x) = \frac{1}{\mu(D)} \sum_{\eta\in\Gamma^\perp} \hat{f}(\eta)e^{2\pi i\langle x,\eta\rangle}.$$

Put $x = 0$ and get the result. □

Example (functional equation for theta functions) Let $V = \mathbb{R}^n$, μ — Lebesque measure on V, $\langle\cdot,\cdot\rangle$ — positively definite quadratic form. Define $\Theta_\Gamma(t) = \sum_{x\in\Gamma} e^{-\pi t\langle x,x\rangle}$ for $t > 0$. Then

$$\Theta_\Gamma(t) = \frac{1}{\mu(V/\Gamma)}\frac{1}{t^{\frac{n}{2}}}\Theta_{\Gamma^\perp}(\frac{1}{t}).$$

If $\Gamma^\perp = \Gamma$ and $\langle x,x\rangle \in 2\mathbb{Z}$ for all $x \in \Gamma$, then one can define holomorphic version

$$\theta_\Gamma(z) = \sum_{x\in\Gamma} e^{\pi i z\langle x,x\rangle}.$$

Since $\theta_\Gamma(it) = \Theta_\Gamma(t)$, θ_Γ is a modular form of weight $\frac{n}{2}$.

2. Poisson summation on adeles

Let k – number field, A – it's adeles. We have an embedding $\phi : k \to A$

$$x \mapsto (x, x, x, \dots)$$

as a discrete subgroup with compact quotient (from the talk of Nils Frohbegr). So, we want to specify
(I) a pairing $\langle\cdot,\cdot\rangle$: $A \times A \to \mathbb{R}/\mathbb{Z}$
(II) a fundamental domain D for $k \subset A$
(III) a dual group $k^\perp$
to use Poisson summation formula (Theorem 1)

(I) Recall that for each place β of k we have identified the additive group of the local field k_β with it's character group k_β^* via pairing

$$\langle x,y\rangle_\beta = \Lambda_\beta(xy) = \lambda_p(Tr_{k_\beta|\mathbb{Q}_p}(xy))$$

where $\beta|p$, p – the place of $\mathbb{Q}$. And the dual group is a restricted product of duals $k_\beta^* \cong k_\beta$ with respect to inverse differents δ_β^{-1} (defined for finite places) with componentwise action (from the talk of Daniel Rohleder). So, since $\delta_\beta = o_\beta$ for all unramified $k_\beta : \mathbb{Q}_p$ and all but finite nuber of places are unramified, we get $A \cong A^*$ via the pairing

$$\langle(x_\beta),(y_\beta)\rangle = \sum_\beta \Lambda_\beta(x_\beta y_\beta) \tag{2}$$

(only finite number of terms are non-zero).

(II) Let $A = A^0 \times A^\infty$ where A^0 are finite adeles and $A^\infty = \prod_{\beta|\infty} k_\beta$ is a Minkovsky space isomorphic to $\mathbb{R}^n$, $n = [k : \mathbb{Q}]$. Let π^0, π^∞ be correspondent projections. Then $\pi^\infty \circ \phi : k \to A^\infty$ is a classical Minkovsky map (see Appendix below). Under this map any ideal in $a \subset o_k$ is mapped into discrete lattice with covolume equal to $\sqrt{|d(a)|}$. Consider a $\mathbb{Z}$-basis of o_k, $o_k = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_n$. Let $x_i^\infty \in A^\infty$ be the image of x_i. Then the parallelotope $D^\infty = \{\sum \alpha_i x_i^\infty | 0 \le \alpha_i < 1\}$is a fundamental domain for the image of o_k, and it's volume $\mu^\infty(D^\infty) = \sqrt{|d|}$. Let $O = \prod_\beta o_\beta \subset A^0$. Then obviously

$$D = O \times D^\infty$$

is a (relatively compact) fundamental domain for k in A. Indeed, since $A = k + A_{S_\infty}$ and $k \cap A_{S_\infty} = o_k$(Theorem 1.1 in the talk of Nils Frohberg) for every $x \in A$ we can find $y \in k$ s.t. $x - y \in A_{S_\infty}$ and this y is defined up to the element of o_k. Now we choose $z \in o_k$ s.t. $(x-y)^\infty - z \in D^\infty$, and the choise is unique. So, $y + z$ is uniquely defined s.t. $x - (y+z) \in D$.

Proposition 1. *For our measure μ (fixed in Tate's thesis) on A*

$$\mu(D) = 1.$$

Proof. $\mu(D) = \mu^0(O) \times \mu^\infty(D^\infty) = \mu^0(O) \times \sqrt{|d|}$. $\mu^0(O) = \prod \mu^\beta(o_\beta) = \prod (N\delta_\beta)^{-1/2}$. Since $\delta = \prod_\beta (\delta_\beta \cap o_k)$ (J. Neukirch, "Algebraic Number Theory", chapter III, Corollary 2.3) we have

$$|d| = [o_k : \delta] = \prod_\beta [o_k : \delta_\beta \cap o_k] = \prod [o_\beta : \delta_\beta],$$

since $o_k \cap \delta_\beta$ is a power of the ideal $\beta \subset o_k$. This implies $\mu(D) = 1$. □

(III) We are going to prove that with the pairing (2) $k^{perp} = k$. Let us define the function from A to $\mathbb{R}/\mathbb{Z}$

$$\Lambda(x) = \sum_\beta \Lambda_\beta(x_\beta),$$

so that $\langle x, y \rangle = \Lambda(xy)$.

Proposition 2. *For $x \in k$ we have $\Lambda(x) = 0$.*

Proof. Recall that λ_p for places p of $\mathbb{Q}$ are additive, so

$$\Lambda(x) = \sum_p \lambda_p \left(\sum_{\beta|p} Tr_{k_\beta|\mathbb{Q}_p} x \right)$$

(since $Tr_{k|\mathbb{Q}}(x) = \sum_{\beta|p} Tr_{k_\beta|\mathbb{Q}_p} x$, for finite places see Appendix below)

$$= \sum_p \lambda_p(Tr_{k|\mathbb{Q}} x).$$

So we need to prove our statement only for $k = \mathbb{Q}$. Let $x = \frac{m}{n}$ with m, n coprime. Let $n = \prod p^{r_p}$. Then one can find integers s_p such that $m = \sum s_p \frac{n}{p^{r_p}}$ since the numbers $\frac{n}{p^{r_p}}$ have no divizor in common. Then for finite places $\lambda_p(\frac{m}{n}) = \frac{s_p}{p^{r_p}}$ and $\lambda_\infty(\frac{m}{n}) = -\frac{m}{n}$ modulo $\mathbb{Z}$, so that

$$\sum \frac{s_p}{p^{r_p}} - \frac{m}{n} = \frac{1}{n} \left(\sum_p s_p \frac{n}{p^{r_p}} - m \right) = 0.$$

□

Theorem 2. $k^{\perp} = k$

Proof. Note that just by definition $k^{\perp}$ is a k-vector space, and by Proposition 2 $k \subset k^{\perp}$. Since $k^{\perp}$ is $(A/k)^*$, it is discrete. Then $k^{\perp}/k \subset A/k$ is a discrete subset of a compact set, whence it is finite. But for k-vector space $k^{\perp}$ the index $[k^{\perp} : k]$ can be finite only if it is 1, since k is not a finite field. □

Finally, due to (I),(II) and (III) we can formulate Theorem 1 for adeles:

Theorem 3. *If continuous function $f \in L_1(A, \mu)$ satisfies*
(1) $\sum_{\xi \in k} f(x + \xi)$ is uniformly (absolutely) convergent for $x \in D$
(2) $\sum_{\xi \in k} |\hat{f}(\xi)|$ is convergent
then

$$\sum_{\xi \in k} f(\xi) = \sum_{\xi \in k} \hat{f}(\xi).$$

Appendix: Minkovsky space

(J. Neukirch, "Algebraic Number Theory", chapter I)

Let $A^{\infty}_{\mathbb{C}} = \mathbb{C}^n$ and $\sigma_1, \ldots, \sigma_n$ are embegdings of k into $\mathbb{C}$. Then the image of

$$x \mapsto (\sigma_1(x), \ldots, \sigma_n(x)) \in A^{\infty}_{\mathbb{C}}$$

lies in the $\mathbb{R}$-subspace $A^{\infty} \subset A^{\infty}_{\mathbb{C}}$ of vectors stable under conjugation $(x_i)' = (y_i = \bar{x}_{i'})$ (where $\sigma_{i'}$ is complex conjugate to σ_i). $A^{\infty} \cong \mathbb{R}^n$ is called a Minkovsky space. In $A^{\infty}_{\mathbb{C}}$ the scalar product is given by $\langle x, y \rangle = \sum_i x_i \bar{y}_i$. This scalar product restricted to A^{∞} takes values in $\mathbb{R}$, and is called a Minkovsky metric.

Proposition 3. *For any ideal $a \subset o_k$ it's image in Minkovsky spase is a cocompact lattice with covolume $\sqrt{|d(a)|}$.*

Proof. Consider a basis $a = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n$. Put $M = (M_{ij} = \sigma_j(a_i))$. Then $d(a) = detM \neq 0$, what exactly means that images $\vec{a}_i$ of basis elements in Minkovsky space are linearly independent. Then the square of volume $vol(A^{\infty}/a)^2$ is the determinant of the Gramm matrix $(\langle \vec{a}_i, \vec{a}_j \rangle)$. But $(\langle \vec{a}_i, \vec{a}_j \rangle) = A\bar{A}^t$, so $det(\langle \vec{a}_i, \vec{a}_j \rangle) = |detA|^2 = d^2$. □

Minkovsky space is a product of pieces correspondent to real embeddings (each piece is simply $\mathbb{R}$) and pairs of conjugate complex embeddings (each piece is $\mathbb{C}$, or $\mathbb{R}^2$). The Minkovsky metric on real pieces is just usual distance in $\mathbb{R}$, while on $\mathbb{C}$ it is $\sqrt{2}$ times usual distance. (This means that on $\mathbb{C}$-pieces the measure is 2 times Lebesque measure, exactly as we have chosen in Tate's thesis.) Indeed, consider a piece of A^{∞} correspondent to a pair of complex embeddings. These are vectors $(x, \bar{x}); x \in \mathbb{C}$ of $A^{\infty}_{\mathbb{C}}$, and their scalar product is

$$\langle (x, \bar{x}), (y, \bar{y}) \rangle = x\bar{y} + \bar{x}y = 2 * Re(x\bar{y}).$$

In particular $||(x, \bar{x})||^2 = 2|x|^2$. We identify this peace with $\mathbb{C}$ by the first coordinate, whence the statement.

Appendix: places above the finite prime p

(J. Neukirch, "Algebraic Number Theory", II.8)

Proposition 4. $k \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{\beta | p} k_{\beta}$ *as a $\mathbb{Q}_p$-algebras.*

Proof. Let $k = \mathbb{Q}(\alpha)$, and $f \in \mathbb{Q}[X]$ be a minimal polynomial for alpha. Consider decomposition $f(X) = \prod_{i=1}^{r} f_i(X)^{m_i}$ in $\mathbb{Q}_p[X]$. Since f is separable (has all different roots in $\bar{\mathbb{Q}}$), all $m_i = 1$. Then k can be embedded into $\bar{\mathbb{Q}_p}$ in exactly r different ways (up to conjugation), and so we get r different extensions of p-valuation on

$\mathbb{Q}$ to k. By Extension Theorem (8.1 in Neukirch) there are no other extensions of valuation. Since valuations correspond to primes above p, we can renumber factors f_i so that $k_\beta = \mathbb{Q}_p[X]/(f_\beta)$. Since $k \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{Q}_p[X]/(f)$, it is isomorphic to $\prod_\beta \mathbb{Q}_p[X]/(f_\beta)$ by Chinese Reminder Theorem. □

Corollary 5. *(I)* $[k : \mathbb{Q}] = \sum_{\beta|p}[k_\beta : \mathbb{Q}_p]$
(II) For $x \in k$ $Tr_{k|\mathbb{Q}}(x) = \sum_{\beta|p} Tr_{k_\beta|\mathbb{Q}_p}(x)$.
(II) For $x \in k$ $N_{k|\mathbb{Q}}(x) = \sum_{\beta|p} N_{k_\beta|\mathbb{Q}_p}(x)$.

Proof. For (II),(III) one considers a characteristic polynomial of a multiplication by x in both isomorphic algebras from the Theorem. □