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STABILITY OF LINEAR POSITIVE SYSTEMS

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We establish criteria of asymptotic stability for positive differential systems in the form of conditions of monotone invertibility of linear operators. The structure of monotone and monotonically invertible operators in the space of matrices is investigated.

1. Introduction

We consider the class of linear differential systems

$$\frac{dH(t)}{dt} + MH(t) = G(t), \quad H(0) = Y, \quad t \geq 0, \tag{1}$$

where $M: \mathscr{E} \to \mathscr{E}$ is a bounded operator. The phase space $\mathscr{E}$, where the operator M acts, is partially ordered with the use of a certain cone $\mathcal{K} \subset \mathscr{E}$. This assumption enables us to use the theory of monotone and monotonically invertible operators with respect to the cone for the investigation of system (1) [1–3].

For certain classes of operators M and cones $\mathcal{K}$, a condition for the stability of system (1) is presented in the form of conditions for the solvability of the corresponding (algebraic) equation

$$MX = Y \tag{2}$$

on $\mathcal{K}$. For example, the asymptotic stability of a matrix differential Lyapunov equation of the type (1) with the operator $MX = -AX - XA^*$ is equivalent to the existence of positive-definite matrices $X = X^* > 0$ and $Y = Y^* > 0$ that satisfy the algebraic Lyapunov equation (2) and the monotone invertibility of the operator M with respect to the cone of Hermitian nonnegative-definite matrices $\mathcal{K}$.

In the present paper, we establish criteria of asymptotic stability in a Banach space for the class of positive systems (1) whose evolution operators are monotone with respect to the normal reproducing cone $\mathcal{K}$. The solutions $H(t)$ of these systems belong to the cone $\mathcal{K}$ if $Y \in \mathcal{K}$ and $G(\tau) \in \mathcal{K}$, $0 \leq \tau \leq t$. To deduce criteria we use the integral representation of solutions of Eq. (2) and the Krein–Bonsall–Karlin theorems on the spectral radius of a monotone operator [1]. We also present certain results of investigations concerning the description of the structure of classes of monotone and monotonically invertible operators with respect to cones of nonnegative and nonnegative-definite matrices.

Note that the property of positivity of system (1) with respect to the cone of nonnegative matrices reduces to conditions of off-diagonal nonpositivity of the matrix of the operator M [1, 4].

2. Monotone and Monotonically Invertible Linear Operators

Let $M: \mathscr{E} \to \mathscr{E}$ be a linear operator acting in a certain semiordered space with a normal reproducing cone $\mathcal{K} \subset \mathscr{E}$. An operator M is called *monotone* if $X \geq Y$ yields $MX \geq MY$. Here and below, the inequality $X \geq Y$ $(X > Y)$ means that $X - Y \in \mathcal{K}$ $(X - Y \in \mathcal{K}_0)$, where $\mathcal{K}_0$ is the set of interior points of the cone $\mathcal{K}$. A monotone

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operator $\hat{M}$ is called a *majorant (minorant)* of the monotone operator M if the operator $\hat{M} - M$ $(M - \hat{M})$ is monotone. A monotone operator M is called *extremal* if it cannot be represented in the form of a sum of linearly independent minorants. All minorants of the extremal operator M have the form αM, where $0 \le \alpha \le 1$. An operator M is called *strictly monotone (strongly monotone)* if $MX > MY$ for $X > Y$ $(X \ge Y,\ X \ne Y)$. Strongly monotone (extremal) operators are interior (extreme) points of a solid cone of monotone operators [1]. An operator M is called *monotonically invertible* if it is invertible and the inverse operator M^{-1} is monotone. The monotone invertibility of an operator M is equivalent to the existence of a solution $X \ge 0$ $(X > 0)$ of Eq. (2) for any $Y \ge 0$ $(Y > 0)$.

The classes of monotone, strictly monotone, strongly monotone, and monotonically invertible operators can be described in the form of the corresponding inclusions

$$M\mathcal{K} \subset \mathcal{K}, \quad M\mathcal{K}_0 \subset \mathcal{K}_0, \quad M\mathcal{K}\backslash\{0\} \subset \mathcal{K}_0, \quad \mathcal{K} \subset M\mathcal{K}.$$

The continuity of the operator M implies that, for a closed solid cone $\mathcal{K}$, the inclusions $\mathcal{K} \subset M\mathcal{K}$ and $\mathcal{K}_0 \subset M\mathcal{K}_0$ are equivalent.

We consider the class of linear operators that can be represented in the form

$$M = L - P, \quad P\mathcal{K} \subset \mathcal{K} \subset L\mathcal{K}, \tag{3}$$

where the operators P and L are monotone and monotonically invertible, respectively.

Lemma 1. *The monotone invertibility of operator (3) is equivalent to the inequality* $\rho(T) < 1$, *where* $\rho(T)$ *is the spectral radius of the pencil of operators* $T(\lambda) = \lambda L - P$. *For a solid cone* $\mathcal{K}$, *operator (3) is monotonically invertible if and only if there exist* $X \in \mathcal{K}_0$ *and* $Y \in \mathcal{K}_0$ *satisfying Eq. (2).*

To prove Lemma 1, we use Theorems 25.1 and 25.4 from [1, pp. 199–201] and the relation

$$LM^{-1} = PM^{-1} + I = (I - S)^{-1} = \sum_{k=0}^{\infty} S^k,$$

where I is the identity operator and $S = PL^{-1}$ is a monotone operator whose spectral radius coincides with $\rho(T) < 1$.

Note that an operator M is monotone with respect to a cone $\mathcal{K}$ in a Hilbert space if and only if the adjoint operator M^* is monotone with respect to the dual cone $\mathcal{K}^* \subset \mathcal{E}^*$. The conditions of strict and strong monotonicity are also simultaneously satisfied or not satisfied for the operators M and M^*.

3. Stability of Linear Positive Systems

Let $\mathcal{K}$ be a normal reproducing cone of the space $\mathcal{E}$. The differential system (1) is called *positive* if $H(t) \ge 0$ under the conditions $Y \ge 0$ and $G(\tau) \ge 0$, $0 \le \tau \le t$. It follows from the representation of a solution of system (1) in the form

$$H(t) = W_t Y + \int_0^t W_{t-\tau} G(\tau) d\tau, \quad W_t = e^{-Mt} \equiv \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} M^k$$

that, for a positive system, the exponential operator W_t must be monotone with respect to the cone $\mathcal{K}$ for all $t \geq 0$. Conversely, if $Y \geq 0$, $G(\tau) \geq 0$, and the operator W_t is monotone, then $H(t) \geq 0$ for any $t \geq 0$. For this reason, the property of positivity of system (1) with respect to the cone $\mathcal{K}$ can be described in the form of inclusions $W_t\mathcal{K} \subset \mathcal{K}$ or $\mathcal{K} \subset W_{-t}\mathcal{K}$ for every $t \geq 0$.

Theorem 1. *A positive system (1) is asymptotically stable if and only if the operator M is monotonically invertible.*

Proof. Consider the homogeneous system

$$\frac{dZ(t)}{dt} + MZ(t) = 0, \qquad Z(0) = Y, \quad t \geq 0. \tag{4}$$

The conditions of asymptotic stability for systems (1) and (4) coincide. Integrating system (4) with regard for the condition $Z(\infty) = 0$, we obtain

$$\int_0^\infty \frac{dZ(t)}{dt}dt + M\int_0^\infty Z(t)dt = -Y + MX = 0,$$

where

$$X = \int_0^\infty Z(t)dt, \qquad Z(t) = W_t Y. \tag{5}$$

Therefore, for an asymptotically stable system (1), integral (5) is a solution of Eq. (2). In this case, it follows from the positivity condition that $X \geq 0$ for $Y \geq 0$.

By using spectral properties of monotone operators and functions of operators, we show the validity of the converse statement. It follows from the monotonicity of the operators W_t and M^{-1} that their product $V_t = W_t M^{-1}$ is monotone. The spectra of the operators W_t and V_t are formed of the corresponding numbers $e^{-\lambda t}$ and $e^{-\lambda t}/\lambda$ for $\lambda \in \sigma(M)$. By virtue of the Krein–Bonsall–Karlin theorems [1], the spectral radius of a monotone operator is a point of its spectra. Therefore, for any $\lambda \in \sigma(M)$, the following inequalities are true:

$$e^{-\mathrm{Re}\lambda t} \leq e^{-\alpha t} = \rho(W_t), \qquad \frac{e^{-\mathrm{Re}\lambda t}}{|\lambda|} \leq \frac{e^{-\beta t}}{\beta} = \rho(V_t),$$

where α and β are certain points of the spectrum of the operator M. The right-hand sides of these inequalities take real positive values for $t \geq 0$. For sufficiently small $t < 2\pi/\rho(M))$, the first inequality implies that α is a real point of the spectrum of the operator M such that $\mathrm{Re}\,\lambda \geq \alpha$ for any $\lambda \in \sigma(M)$. In order that the second inequality be valid for an arbitrarily large value of t and any $\lambda \in \sigma(M)$, it is necessary to set $\beta = \alpha > 0$. Therefore, $\mathrm{Re}\lambda \geq \alpha > 0$ for every $\lambda \in \sigma(M)$, i.e., system (1) is asymptotically stable. The theorem is proved.

Remark 1. The property of positivity of system (4) is equivalent to the monotone invertibility of the family of operators $M_c = M - cI$, $c < \alpha$, where $\alpha = \inf\{\mathrm{Re}\,\lambda : \lambda \in \sigma(M)\}$ [3]. In this case, $\alpha \in \sigma(M)$ if $\sigma(M) \neq \varnothing$. If the operators M and M_c are monotonically invertible for $c < \alpha$, then $\alpha > 0$. Otherwise, the bilateral operator estimate $M \le M_\alpha \le M_c$ is true and the operator M_α must be monotonically invertible, which contradicts the condition $\alpha \in \sigma(M)$ [1]. Therefore, $\alpha > 0$ if $M_c^{-1} \ge 0$ for any $c \le 0$. If $0 < c < \alpha$, then an arbitrary solution of system (4) satisfies the estimate

$$\|Z(t)\| \le q e^{-ct} \|Y\|, \quad t \ge 0, \quad q > 0,$$

i.e., system (4) is exponentially stable. If α is an eigenvalue of the operator M, then system (4) has a partial solution of the form $Z(t) = e^{-\alpha t} Y$, $Y \ne 0$. Therefore, for system (4), which is positive with respect to the cone $\mathcal{K}$, the relations $\alpha > 0$ and $\mathcal{K} \subset M\mathcal{K}$ are equivalent and can be regarded as criteria for its exponential stability. If $\mathcal{K} \subset M_c \mathcal{K}$ for $c \le 0$, then system (4) is positive and exponentially stable.

To use Theorem 1, it is necessary to establish the property of positivity of system (1).

Lemma 2. *If two systems of the type (1) corresponding to operators M_1 and M_2 are positive, then system (1) with the operator $M_1 + M_2$ is also positive.*

To prove this statement, we use the relations

$$e^{-(M_1 + M_2)t} = E(t) + t^3 R(t) = \lim_{k \to \infty} \left[E\left(\frac{t}{k}\right) \right]^k,$$

$$E(t) = \frac{1}{2}\left(e^{-M_1 t} e^{-M_2 t} + e^{-M_2 t} e^{-M_1 t}\right),$$

where $R(t)$ is a certain entire operator function, and the property of closedness of the cone of monotone operators [5]. For commuting operators M_1 and M_2, we have

$$e^{-(M_1 + M_2)t} = e^{-M_1 t} e^{-M_2 t}, \quad t \ge 0.$$

Consider the class of operators (3). It follows from the monotonicity of the operator P that the operator function e^{Pt}, $t \ge 0$, is monotone. By virtue of Lemmas 1 and 2 and Theorem 1, we obtain the following statement:

Theorem 2. *Suppose that M is an operator of the type (3) and, for any $t \ge 0$, the operator e^{-Lt} is monotone with respect to the normal solid cone $\mathcal{K}$. In this case, the following assertions are equivalent:*

(i) *system (1) is asymptotically stable;*

(ii) *the operator M is monotonically invertible;*

(ii) *there exist $X > 0$ and $Y > 0$ satisfying Eq. (2);*

(iv) $\rho(T) < 1$, *where* $T(\lambda) = \lambda L - P$.

By using normal solid cones of nonnegative and nonnegative-definite matrices, we now formulate corollaries of the presented results for matrix equations.

Corollary 1. *Suppose that* $\mathcal{K} \subset R^{n\times m}$ *is a cone of* $n \times m$ *matrices with nonnegative elements and the operator*

$$M = \sum_k A_k X B_k, \quad A_k \in R^{n\times n}, \quad B_k \in R^{m\times m}, \tag{6}$$

acts in the space $R^{n\times m}$. *Furthermore, all off-diagonal elements of the matrix*

$$C = \sum_k A_k \otimes B_k^T,$$

where $\otimes$ *denotes the Kronecker product, are nonpositive. Then, the following assertions are equivalent:*

(i) system (1) is asymptotically stable;

(ii) for any positive matrix $Y > 0$, *Eq. (2) has a positive solution* $X > 0$;

(iii) there exist matrices $X > 0$ *and* $Y > 0$ *satisfying Eq. (2);*

(iv) for certain $\alpha > 0$, *the operator* $P = \alpha I - M$ *is monotone and* $\rho(P) < \alpha$.

This statement is a corollary of Theorem 2 and known properties of M-matrices. Indeed, the matrix equations (1) and (2) can be represented in the form

$$\frac{dh(t)}{dt} + Ch(t) = g(t), \quad Cx = y,$$

where $h(t)$, $g(t)$, x, and y are vectors composed of the transposed rows of the corresponding matrices $H(t)$, $G(t)$, X, and Y [6]. For this reason, the property of positivity of system (1) is equivalent to the condition of off-diagonal nonpositivity of the elements of the matrix C [1, 4]. In this case, the monotone invertibility of the operator M is equivalent to the nonnegativity of the matrix C^{-1} and, according to [4], to the location of the spectrum of the matrix C in the half-plane $\operatorname{Re}\lambda > 0$, and also to the representation $C = \alpha I - S$, where $S \geq 0$ and $\rho(S) < \alpha$.

Note that, under the conditions of Corollary 1, the inequality $C^{-1} \geq 0$ is equivalent to the positivity of all principal minors of the matrix C [4, 7].

Corollary 2. *Suppose that* $\mathcal{K} \subset C^{n\times n}$ *is a cone of Hermitian nonnegative-definite matrices and the operator*

$$M = L - P, \quad LX = -A^* X - XA, \quad PX = \sum_k A_k^* X A_k, \tag{7}$$

where $A, A_k \in C^{n\times n}$, *acts in the space* $C^{n\times n}$. *Then the following assertions are equivalent:*

(i) system (1) is asymptotically stable;

(ii) for any positive-definite right-hand side $Y = Y^* > 0$, *the matrix equation (2) has a positive-definite solution* $X = X^* > 0$;

(iii) there exist matrices $X = X^* > 0$ *and* $Y = Y^* > 0$ *satisfying Eq. (2);*

(iv) the real parts of all eigenvalues of the matrix A *are negative and* $\rho(T) < 1$, *where* $T(\lambda) = \lambda L - P$.

The assertions of Corollary 2 follow from Lemma 2 and Theorem 2. Indeed, the operator functions

$$e^{-Lt}X = e^{A^*t}Xe^{At}, \qquad e^{Pt}X = \sum_{k=0}^{\infty}\frac{t^k}{k!}P^kX, \qquad t \geq 0$$

are monotone and, by virtue of Lemma 2, system (1) with operator (7) is positive. By virtue of Theorem 2, assertions (i)–(iv) are equivalent. Here, one should take into account that the monotone invertibility of the operator L is equivalent to the location of the spectrum of the matrix A to the left of the imaginary axis (the Lyapunov theorem).

Note that a solution of system (4) with operator (7) can be regarded as the matrix of second moments for the Itô stochastic system

$$dx(t) = Ax(t)dt + \sum_k A_k x(t)dw_k(t),$$

where w_k are the components of the standard Wiener process (see, e.g., [8–10]). In this case, the asymptotic stability of system (4) is equivalent to the mean-square asymptotic stability of the trivial solution of this stochastic system [8]. For $P = 0$, relations (1) and (2) with operator (7) are, respectively, the differential and the algebraic Lyapunov equations, for which all propositions of Corollary 2 are known.

4. Structure of Monotone and Monotonically Invertible Operators with Respect to the Cone of Nonnegative-Definite Matrices

In the investigation and application of matrix equations, various representations of linear operators play an important role. In particular,

$$MX = \sum_{i,j=1}^{k}\gamma_{ij}A_iXA_j^* \equiv \sum_{p,q=1}^{n}x_{pq}H_{pq} \equiv \sum_{s=1}^{r}\sigma_sD_sXD_s^*, \tag{8}$$

where

$$\Gamma = \Gamma^*, \quad H_{pq} = B_p\Gamma B_q^*, \quad B_p = \left\| a_{\xi p}^i \right\|_{\xi,i=1}^{m,k}, \quad h_{pq}^{\xi\eta} = \sum_{s=1}^{\tau}\sigma_s d_{\xi p}^{(s)}\overline{d_{\eta q}^{(s)}},$$

and $\sigma_1, \ldots, \sigma_r$ are nonzero eigenvalues of the block matrix H [11–13]. For the indices of inertia of Hermitian matrices H and Γ, we have $i_\pm(H) \leq i_\pm(\Gamma)$, where $i_\pm(\cdot)$ are the numbers of positive and negative eigenvalues of a matrix, counting multiplicity. Note that the equality takes place in the case of linear independence of the matrix coefficients $A_1, \ldots, A_k \in C^{m\times n}$. The matrices $D_1, \ldots, D_r$ in (8) are orthonormal, i.e., $(D_s, D_g) = \delta_{sg}$, where $(P, Q) = \operatorname{tr}(QP^*)$ is the scalar product.

The linear independence of the matrices $A_1, \ldots, A_k$ is equivalent to the linear independence of the family of operators $A_iXA_j^*$. Therefore, we can represent an arbitrary linear operator $M\colon C^{n\times n} \to C^{m\times m}$ in the form (8) and, furthermore, $M\mathcal{H}_n \subset \mathcal{H}_m$, where $\mathcal{H}_n$ is the space of Hermitian matrices of order n. In this case, the adjoint operator has the form

$$M^* Y = \sum_{i,j=1}^{k} \bar{\gamma}_{ij} A_i^* Y A_j .$$

Let $\mathcal{K} \subset \mathcal{H}_n$ be a cone of nonnegative-definite Hermitian matrices of order n. We investigate the classes of monotone and monotonically invertible operators of the type (8) under the assumption that $n = m$. For $n \neq m$, all constructions and conclusions are analogous.

In terms of real matrices, we describe the class of monotone operators (8) as follows:

$$\sum_{s=1}^{r} \sigma_s \tilde{D}_s \tilde{X} \tilde{D}_s^T \geq 0 \quad \forall \tilde{X} = \begin{bmatrix} S & K \\ -K & S \end{bmatrix} \geq 0, \quad \tilde{D}_s = \begin{bmatrix} R_s & G_s \\ -G_s & R_s \end{bmatrix},$$

$$X = S + iK, \quad D_s = R_s + iG_s, \quad S^T = S, \quad K^T = -K, \quad s = 1, \ldots, r.$$

This follows from the equivalence of the matrix inequalities $X \geq 0$ and $\tilde{X} \geq 0$.

If $H \geq 0$ (in particular, $\Gamma \geq 0$), then the operator M is monotone. However, the operator M can be monotone even for $i_-(H) \neq 0$ or $i_-(\Gamma) \neq 0$. The simplest example of such an operator is the transposition operator

$$X^T = \sum_{p,q=1}^{n} x_{pq} E_{pq}, \quad E = \begin{bmatrix} E_{11} & \cdots & E_{n1} \\ \vdots & \ddots & \vdots \\ E_{1n} & \cdots & E_{nn} \end{bmatrix}, \quad i_{\pm}(E) = \frac{n(n \pm 1)}{2},$$

where E_{pq} are elements of the unit basis of the space $C^{n \times n}$. If, for a certain vector $x \in C^n$ $(z \in C^n)$, the equality

$$\operatorname{rang}[A_1 x, \ldots, A_k x] = k \qquad \left(\operatorname{rang}\left[B_1^* z, \ldots, B_n^* z\right] = k\right)$$

is satisfied, then the inequality $\Gamma \geq 0$ is equivalent to the monotonicity of the operator M.

We define the property of monotonicity of an operator M as follows:

$$G_z = \left\| z^* H_{pq} z \right\|_1^n \geq 0 \qquad \forall z \in C^n. \tag{9}$$

In this case, the conditions of strict (strong) monotonicity are equivalent to the relations $G_z \geq 0$ and $G_z \neq 0$ $\forall z \neq 0$ $(G_z > 0 \;\; \forall z \neq 0)$. Determining the principal minors of matrix (9) that correspond to given collections of the numbers of rows and columns p, we obtain the following algebraic conditions for the monotonicity of the operator M:

$$\Theta_p = \left\| \Theta_{ij}^{(p)} \right\| \geq 0, \quad \Theta_{ij}^{(p)} = \sum_{\xi, \eta} \det \begin{bmatrix} h_{p_1 p_1}^{\xi_1 \eta_1} & \cdots & h_{p_1 p_\nu}^{\xi_1 \eta_\nu} \\ \vdots & \ddots & \vdots \\ h_{p_\nu p_1}^{\xi_\nu \eta_1} & \cdots & h_{p_\nu p_\nu}^{\xi_\nu \eta_\nu} \end{bmatrix},$$

$$p = \{p_1, \ldots, p_\nu\}, \quad 1 \leq p_1 < \ldots < p_\nu \leq n, \quad \nu = 1, n,$$

$$\xi = \{\xi_1, \ldots, \xi_\nu\}, \quad i = \{i_1, \ldots, i_\nu\}, \quad 1 \leq i_1 \leq \ldots \leq i_\nu \leq n,$$

$$\eta = \{\eta_1, \ldots, \eta_\nu\}, \quad j = \{j_1, \ldots, j_\nu\}, \quad 1 \leq j_1 \leq \ldots \leq j_\nu \leq n.$$

Here, the summation is carried out over all collections of indices $\xi(\eta)$ that coincide with $i(j)$ after ordering, and the rows (columns) of the matrix Θ_p that correspond to the collections of indices $i(j)$ are arranged in the lexicographic order.

By using the spectral decomposition of nonnegative-definite matrices, one can show that conditions (9) are satisfied if and only if the blocks of the matrix H can be represented in the form [11]

$$H_{pq} = U_p U_q^* + V_q V_p^*, \quad p, q = \overline{1, n}. \tag{10}$$

The general representation of a monotone operator follows from relations (8) and (10).

Lemma 3. *A linear operator M is monotone with respect to a cone of nonnegative-definite Hermitian matrices if and only if it can be represented in the form*

$$MX = \sum_i A_i X A_i^* + \sum_j B_j X^T B_j^*. \tag{11}$$

One can show that operators of the type AXA^* and BX^TB^* are extremal. Therefore, according to Lemma 4, monotone operators can be represented as a sum of their extremal minorants. The number of extremal minorants in relation (11) can be decreased if some of them can be linearly represented in terms of the other, in particular, if the matrices A_i (or B_j) are linearly dependent.

A monotone operator M is strictly monotone if and only if, for a certain matrix $X_0 \geq 0$, we have $MX_0 > 0$. Indeed, for any matrix $X > 0$, there exists $\varepsilon > 0$ such that $X \geq \varepsilon X_0$ and, hence, $MX \geq \varepsilon MX_0 > 0$.

A strictly monotone operator can be non-invertible. For example, the Schur operator $MX = A \odot X$ is strictly monotone if and only if $A \geq 0$ and $a_{ii} > 0$ for any i, whereas a criterion for its invertibility is the validity of the inequality $a_{ij} \neq 0$ for any i and j. The linear operator $MX = (\operatorname{tr} X)I$ is strictly monotone but not invertible.

We now pass to the description of the class of monotonically invertible operators. Taking the structure of the monotone operator (11) into account, we set

$$MX = M_0 X - M_1 X - \ldots - M_r X, \quad M_j X = \begin{cases} A_j X A_j^*, & j \in J_1, \\ A_j X^T A_j^*, & j \in J_2, \end{cases} \tag{12}$$

where $A_j \in C^{n \times n}$ and J_1 and J_2 are certain subsets of indices. Lemma 1 yields the following statement:

Lemma 4. *A linear operator (12) is monotonically invertible if and only if*

$$\rho(T) < 1, \quad T(\lambda) = \lambda T_0 - T_1 - \ldots - T_r, \quad \det A_0 \neq 0, \tag{13}$$

$$T_j = \begin{cases} A_j \otimes \overline{A}_j, & j \in J_1, \\ (A_j \otimes \overline{A}_j)E, & j \in J_2, \end{cases} \quad E = \sum_{t,i=1}^{n} E_{ti} \otimes E_{it}, \tag{13}$$

where $\rho(T)$ is the spectral radius of the pencil of matrices $T(\lambda)$.

Operators of the form (12) with linearly independent matrix coefficients A_j are not monotone. If an operator is simultaneously monotone and monotonically invertible, then it is an extremal operator of the type AXA^* or

$A X^T A^*$, where A is a certain matrix of full rank. Conditions (13) can be generalized to the case of rectangular matrix coefficients under the restriction $\operatorname{rang} A_0 = m < n$. It is easy to give an example of a monotonically invertible operator that cannot be represented in the form (12). The general representation of linear monotonically invertible operators has not yet been established.

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