Symmetry, Integrability and Geometry: Methods and Applications SIGMA **20** (2024), 082, 40 pages

First Cohomology with Trivial Coefficients of All Unitary Easy Quantum Group Duals

Alexander MANG

Hamburg University, Bundesstraße 55, 20146 Hamburg, Germany
E-mail: *alex@alexandermang.net*
URL: https://alexandermang.net

Received September 24, 2023, in final form August 22, 2024; Published online September 12, 2024
https://doi.org/10.3842/SIGMA.2024.082

Abstract. The first quantum group cohomology with trivial coefficients of the discrete dual of any unitary easy quantum group is computed. That includes those potential quantum groups whose associated categories of two-colored partitions have not yet been found.

Key words: discrete quantum group; quantum group cohomology; trivial coefficients; easy quantum group; category of partitions

2020 Mathematics Subject Classification: 20G42; 05A18

1 Introduction

1.1 Background and context

In [4], Banica and Speicher provided a way of constructing compact quantum groups (in the sense of [41, 43, 44]) by solving infinite combinatorics puzzles: They introduced three operations on the collection of all equivalence relations of disjoint unions of finite sets and showed that each subset which is closed under these operations gives rise to a compact quantum group. An uncountable number of such sets and of the resulting so-called "easy" quantum groups and, in fact, all there can be, have since been found in [3, 4, 30, 31, 32, 39]. In [34], Tarrago and Weber extended Banica and Speicher's operations to the collection of all "two-colored" partitions, thus providing even more quantum groups to find. The classification program they initiated to determine all sets closed under the operations is still ongoing (see [15, 23, 25, 26, 27, 28, 29, 34]). The construction has since been further extended to two-colored partitions with arbitrarily many "colors" by Freslon in [13], to "three-dimensional" sets by Cébron and Weber in [8] and to equivalence relations on graphs instead of sets by Mančinska and Roberson in [24].

An issue that all these constructions share is that it is difficult to tell which of the resulting compact quantum groups are new and which are isomorphic to already known ones. In particular, each solution to the combinatorics puzzle does not only provide one quantum group but an entire countably infinite series, one for each dimension of its fundamental representation. And already Banica and Speicher themselves observed in [4, Proposition 2.4 (4)] that, at least in some cases, the quantum groups of one solution are isomorphic to those of another, just shifted by one dimension. That underlines the importance of studying quantum group invariants with the potential of telling easy quantum groups apart. Of course, these are often very difficult to compute like, e.g., the L^2-cohomology of [21] of discrete quantum groups. But perhaps at least the cohomology with trivial coefficients is a reasonable goal to strive for.

The present article computes the first order of the quantum group cohomology with trivial coefficients of the discrete duals of all of Tarrago and Weber's so-called unitary easy quantum groups. That includes even the potential ones whose combinatorics puzzles have not been solved yet. Said cohomology can be realized as the first Hochschild cohomology of the trivial bimodule of an augmented algebra presented in terms of generators and relations. As with any augmented algebra the space of 1-coboundaries is then trivial and the task thus boils down to solving the generally infinite system of linear equations in the finitely many generators determining the 1-cocycles.

The results of the present article might be useful for the computation of the second order begun by Bichon, Das, Franz, Gerhold, Kula and Skalski in [7, 9] as well as Wendel in [40]. The former six investigated the cohomology of certain easy quantum groups out of a different motivation. In particular, they were interested in the Calabi–Yau property of [14], a generalization of Poincaré duality, and the classification of Schürmann triples. Namely, a quantum group whose second cohomology vanishes has the AC property, defined in [12], which is important in the study of quantum Lévy processes because it guarantees the existence of an associated Schürmann triple.

In [7, 9], Bichon, Das, Franz, Gerhold, Kula and Skalski had already laid out a potential strategy for computing the second cohomology of any easy quantum group (later refined in [10] to address universal unitary quantum groups). This strategy is based on two key insights and goes as follows. They interpreted quantum group cohomology as Hochschild cohomology and chose the Hochschild complex as their resolution. Thus, they were faced with having to compute the quotient of the 2-cocycles by the 2-coboundaries. By a very clever use of the universal property of the quantum groups in question, they managed to solve the linear system of equations determining the space of 2-coboundaries. This use of the universal property is the first key tool (see [7, Lemma 5.4] and [9, Lemma 4.1]).

Understanding the 2-cocycles then allowed them to define a "defect map", an injective linear map from 2-cohomology to a certain finite-dimensional vector space of matrices. Thus, at this point they only needed to determine the image of this defect map in order to compute the second cohomology. This is where their second key insight comes into play. Namely, although being interested only in the second-order cohomology, they incidentally also computed the first. That is because they wanted to make use of the multiplicative structure of the cohomology ring. They showed that, at least for the specific quantum groups they investigated, each 2-cocycle was cohomologous to a linear combination of cup products of 1-cocycles. Thus, rather than having to probe the potentially infinite-dimensional vector space of all 2-cocycles as the domain of the defect map they could confine themselves to determining the image of the restriction to cup products, a finite-dimensional space.

In short, when trying to compute the second cohomology of any easy quantum group it might be helpful, perhaps even necessary, to know the first cohomology. Hence, the main result of the present article might also constitute an intermediate step in computing the higher cohomologies of all easy quantum groups.

1.2 Main result

Let $M_n(\mathbb{C})$ be the $\mathbb{C}$-vector space of $(n \times n)$-matrices with complex entries and I the identity $(n \times n)$-matrix. Moreover, call a matrix "small" if each of its rows and each of its columns sums to 0.

Then, the below theorem extends the results of [7, 9] as well as [40].

Theorem. *Let $n \in \mathbb{N}$, let G be any unitary easy compact $(n \times n)$-matrix quantum group, let u be its fundamental representation and let $\mathcal{C}$ be the category of two-colored partitions associated with G. Say that $\mathcal{C}$ has property*

(1) *if and only if each block of each two-colored partition of $\mathcal{C}$ has at most two points,*
(2) *if and only if each block of each two-colored partition of $\mathcal{C}$ has at least two points,*
(3) *if and only if each block of each two-colored partition of $\mathcal{C}$ with at least two points contains as many white lower and black upper points as it does black lower and white upper points,*

(4) *if and only if each two-colored partition of $\mathcal{C}$ has as many white lower and black upper points as it has black lower and white upper points.*

An isomorphism of complex vector spaces from the first quantum group cohomology of $\widehat{G}$ with trivial coefficients to the subspace

$$\{v \in M_n(\mathbb{C}) \wedge A(\mathcal{C}, v)\} \cong \mathbb{C}^{\oplus \beta_1(\widehat{G})}$$

of $M_n(\mathbb{C})$ is defined by the rule which assigns to (the one-elemental cohomology class of) any 1-cocycle η the matrix $(\eta(u_{j,i}))_{(j,i)\in\{1,\ldots,n\}^{\times 2}}$, where $\beta_1(\widehat{G})$ and for any $v \in M_n(\mathbb{C})$ the predicate $A(\mathcal{C}, v)$ are as follows:

If $\mathcal{C}$ is ...,	then $A(\mathcal{C}, v)$ is "..."	and $\beta_1(\widehat{G})$ is ...
$1 \wedge 2 \wedge 3$	$\top$	n^2
$1 \wedge \neg 2 \wedge 3 \wedge 4$	$\exists \lambda \in \mathbb{C}\colon v - \lambda I$ is small	$(n-1)^2 + 1$
$1 \wedge \neg 2 \wedge 3 \wedge \neg 4$	v is small	$(n-1)^2$
$1 \wedge 2 \wedge \neg 3 \wedge 4$	$\exists \lambda \in \mathbb{C}\colon v - \lambda I$ is skew-symmetric	$\frac{1}{2}n(n-1) + 1$
$1 \wedge 2 \wedge \neg 3 \wedge \neg 4$	v is skew-symmetric	$\frac{1}{2}n(n-1)$
$1 \wedge \neg 2 \wedge \neg 3 \wedge 4$	$\exists \lambda \in \mathbb{C}\colon v - \lambda I$ is skew-symmetric and small	$\frac{1}{2}(n-1)(n-2) + 1$
$1 \wedge \neg 2 \wedge \neg 3 \wedge \neg 4$	v is skew-symmetric and small	$\frac{1}{2}(n-1)(n-2)$
$\neg 1 \wedge 2 \wedge 3 \wedge 4$	v is diagonal	n
$\neg 1 \wedge \neg 3 \wedge 4$	$\exists \lambda \in \mathbb{C}\colon v - \lambda I = 0$	1
$\neg 1 \wedge \neg 3 \wedge \neg 4$	$v = 0$	0

And these are all the cases that can occur.

1.3 Structure of the article

Excluding the introduction, the article is divided into five sections.

- ☐ Section 2 recalls the definitions of compact quantum groups and the quantum group cohomology with trivial coefficients of their discrete duals.
- ☐ Following that, Section 3 provides particular examples of compact quantum groups by presenting the definitions of categories of two-colored partitions and unitary easy quantum groups.
- ☐ For the convenience of the reader, the definition of the first Hochschild cohomology and important results about it are recalled in Section 4.
- ☐ Section 5 defines the vector spaces of matrices appearing in the main result and computes their dimensions.
- ☐ The proof of the main theorem is contained in Section 6. Starting from a characterization of the first cohomology of a universal algebra recalled in Section 4 the first quantum group cohomology as defined in Section 2 is computed of the discrete duals of the quantum groups defined in Section 3.

1.4 Notation

In the following, $0 \notin \mathbb{N}$. Rather, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $[\![k]\!] := \{i \in \mathbb{N} \wedge i \leq k\}$ for any $k \in \mathbb{N}_0$, in particular, $[\![0]\!] = \varnothing$. The symbol $\times$ will denote the Cartesian product of sets, with the convention $S^{\times 0} := \{\varnothing\}$ for any set S. Throughout, all algebras are meant to be associative and unital. The symbols $\triangleright$ and $\triangleleft$ are used to denote the left respectively right actions of any algebra on any bimodule. Moreover, given any vector spaces V and W over any field the symbol

$[V, W]$ will stand for the vector space of linear maps from V to W. Furthermore, for any vector space X and any (possibly infinite) set E the notation $X^{\times E}$ will be used for the E-fold direct product vector space of X (not to be confused with the direct sum $X^{\oplus E}$). For any field $\mathbb{K}$ and any set E, the free $\mathbb{K}$-algebra over E will be denoted by $\mathbb{K}\langle E\rangle$. For any $R \subseteq \mathbb{K}\langle E\rangle$, we will write $\mathbb{K}\langle E \mid R\rangle$ for the universal $\mathbb{K}$-algebra with generators E and relations R.

2 Quantum groups and their cohomology

The most general kind of "quantum group" in the sense considered here are the locally compact quantum groups introduced by Kustermans and Vaes in [17, 18, 19, 20]. Two subcategories of these are Woronowicz's compact quantum groups defined in [41, 43, 44] and Van Daele's discrete quantum groups studied in [35, 36].

While of those two each is equivalent to the dual category of the other via Pontryagin duality, it is customary to ascribe the cohomology discussed in the present article to the discrete quantum group rather than its compact dual in order to preserve the analogy with the group case. At the same time, the particular quantum groups treated in this article are usually considered to be compact rather than discrete.

And it is in fact most convenient for the purpose of the present article to adopt the latter perspective and work with compact quantum groups. The fact that the quantum group cohomology is actually that of discrete quantum groups will be glossed over by only giving the definition of the composition of the cohomology functor with the Pontryagin transformation. However, the custom will be respected when it comes to notation.

2.1 Compact quantum groups

Quantum groups can be defined both on an analytic, namely von-Neumann- or C^*-algebraic level, and on a purely algebraic level. For the purposes of discussing quantum group cohomology, it fully suffices to consider the latter definition, given in [11]. In that sense, an (*algebraic*) *compact quantum group* G is the formal dual of a Hopf $*$-algebra $\mathbb{C}[\widehat{G}]$ which admits a faithful positive integral. A big class of examples is provided in Section 3. For any G, in the present article $\mathbb{C}[\widehat{G}]$ is generated as a $*$-algebra by the matrix coefficients of a single finite-dimensional unitary comodule M. The coefficient matrix u of a choice of such an M is often called a *fundamental representation*. The axioms imply in particular that, if $u^\bullet$ is the matrix of the conjugate comodule of M and if $u^\circ := u$, then $\mathbb{C}[\widehat{G}]$ is generated as an algebra (as opposed to a $*$-algebra) by the union of the entries of u° and $u^\bullet$. If A is the underlying algebra and ϵ the counit of the Hopf $*$-algebra $\mathbb{C}[\widehat{G}]$, it is entirely sufficient to think of G as the augmented algebra (A, ϵ) and keep in mind that for the examples in this article a generating set of A can be given consisting of the entries of two matrices u° and $u^\bullet$ of the same size.

2.2 Quantum group cohomology

One of many equivalent ways of introducing quantum group cohomology is via Hochschild cohomology. For any compact quantum group G and any $p \in \mathbb{N}_0$, if A is the underlying algebra and ϵ the counit of $\mathbb{C}[\widehat{G}]$ and if ${}_\epsilon\mathbb{C}_\epsilon$ denotes the A-bimodule given by the $\mathbb{C}$-vector space $\mathbb{C}$ equipped with the left and right A-actions defined by $a \otimes \lambda \mapsto \epsilon(a)\lambda$ respectively $\lambda \otimes a \mapsto \lambda\epsilon(a)$ for any $a \in A$ and $\lambda \in \mathbb{C}$, then the *p-th quantum group cohomology with trivial coefficients* of the discrete dual $\widehat{G}$ of G is defined as

$$H^p(\widehat{G}) := H^p_{\mathrm{HS}}(A, {}_\epsilon\mathbb{C}_\epsilon),$$

the p-th Hochschild cohomology of A with coefficients in ${}_\epsilon\mathbb{C}_\epsilon$.

3 Categories of two-colored partitions and unitary easy quantum groups

The quantum groups whose quantum group cohomology is investigated in the present article are the discrete duals of so-called easy quantum groups. They can be defined via Tannaka–Krein duality (see [42]) using the combinatorics of so-called two-colored partitions (which will be explained in Definition 3.6). Throughout the article, it will be important to distinguish the notions of a two-colored partition and a set-theoretical partition in the following sense.

Notation 3.1. Let X be any set.

(a) A *set-theoretical partition* of X is the quotient set of any equivalence relation on X or, equivalently, any set of non-empty pairwise disjoint subsets of X whose union is X.

(b) Given any two set-theoretical partitions p and q of X, write $p \leq q$ if p is *finer* than q, i.e., if for any $\mathsf{B} \in p$ there exists $\mathsf{C} \in q$ with $\mathsf{B} \subseteq \mathsf{C}$.

(c) For any two set-theoretical partitions p and q of X, let $\zeta(p, q) := 1$ if $p \leq q$ and let $\zeta(p, q) := 0$ otherwise.

(d) Furthermore, for any set-theoretical partitions p_1 and p_2 of X the *join* of p_1 and p_2 is the unique set-theoretical partition s of X which satisfies $p_1 \leq s$ and $p_2 \leq s$ and which is minimal with that property with respect to the partial order $\leq$.

(e) For any mapping $f\colon X \to Y$ from X to any set Y and for any subset $B \subseteq Y$, let $f^{\leftarrow}(B) := \{x \in X \wedge f(x) \in B\}$ denote the *pre-image of B under f*. Moreover, let $\operatorname{ran}(f) := \{f(x) \mid x \in X\}$ and $\ker(f) := \{f^{\leftarrow}(\{y\}) \mid y \in \operatorname{ran}(f)\}$ be the *image* and *kernel* of f, respectively.

(f) For any set-theoretical partition p of X, write π_p for the associated *projection*, the mapping $X \to p$ which maps any $\mathsf{x} \in X$ to the unique $\mathsf{B} \in p$ with $\mathsf{x} \in \mathsf{B}$. And for any second set Y and any mapping $f\colon X \to Y$ with $p \leq \ker(f)$, let f/p denote the *quotient mapping*, the unique mapping $p \to Y$ with $(f/p) \circ \pi_p = f$.

Example 3.2. If $X = \{1, 2, 3, 4, 5, 6\}$, then $p = \{\{1\}, \{2, 4\}, \{3, 5, 6\}\}$ is a set-theoretical partition of X and the projection π_p is the mapping $X \to p$, which sends 1 to $\{1\}$, sends both 2 and 4 to $\{2, 4\}$ and sends each of 3, 5 and 6 to $\{3, 5, 6\}$.

Moreover, if $Y = \{a, b, c, d\}$ and $|Y| = 4$ and if $f\colon X \to Y$ maps each of 1, 2 and 4 to a and each of 3, 5 and 6 to c, then the kernel of f is $\ker(f) = \{\{1, 2, 4\}, \{3, 5, 6\}\}$. Since then $p \leq \ker(f)$ the quotient mapping f/p exists and maps both $\{1\}$ and $\{2, 4\}$ to a and $\{3, 5, 6\}$ to c.

In contrast, if $f(4)$ was not given by a but by b instead, then $\ker(f)$ would equal $\{\{1, 2\}, \{4\}, \{3, 5, 6\}\}$, in which case p would not be finer than $\ker(f)$ since there would be no $\mathsf{C} \in \ker(f)$ with $\{2, 4\} \subseteq \mathsf{C}$. There would be no f/p with $(f/p) \circ \pi_p = f$.

3.1 Two-colored partitions and their categories

Two-colored partitions can be defined as follows. For further details see [34], where they were first introduced, generalizing the (uncolored) "partitions" considered in [4].

Assumptions 3.3.

(a) Let $^{\blacksquare}(\cdot)$ and $_{\blacksquare}(\cdot)$ be any two injections with common domain $\mathbb{N}$ and with disjoint ranges.

(b) Let $\circ$ and $\bullet$ be arbitrary with $\circ \neq \bullet$.

Definition 3.4.

(a) For any $\{k, \ell\} \subseteq \mathbb{N}_0$, we call $\Pi_\ell^k := \{^{\blacksquare}a, {}_{\blacksquare}b \mid a \in [\![k]\!] \wedge b \in [\![\ell]\!]\}$ the set of k upper and ℓ lower *points*.

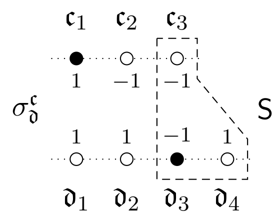
(b) Given any $\{k, \ell\} \subseteq \mathbb{N}_0$, any set X and any mappings $g\colon [\![k]\!] \to X$ and $j\colon [\![\ell]\!] \to X$ denote by $g \,^{\blacksquare}{}_{\blacksquare}\, j$ the mapping $\Pi_\ell^k \to X$ with $^{\blacksquare}a \mapsto g(a)$ for any $a \in [\![k]\!]$ and $_{\blacksquare}b \mapsto j(b)$ for any $b \in [\![\ell]\!]$.

(*c*) $\circ$ and $\bullet$ are called the two *colors* and are said to be *dual* to each other, in symbols, $\overline{\circ} := \bullet$ and $\overline{\bullet} := \circ$. They moreover have the *color values* $\sigma(\circ) := 1$ and $\sigma(\bullet) := -1$.

(*d*) For any $\{k,\ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$ and any $\mathfrak{d}\colon [\![\ell]\!] \to \{\circ,\bullet\}$ the *color sum* of $(\mathfrak{c},\mathfrak{d})$ is the $\mathbb{Z}$-valued measure $\sigma^{\mathfrak{c}}_{\mathfrak{d}}$ on Π^k_ℓ with density $-\sigma(\mathfrak{c}_a)$ on ${}^{\blacksquare}a$ for any $a \in [\![k]\!]$ and density $\sigma(\mathfrak{d}_b)$ on ${}_{\blacksquare}b$ for any $b \in [\![\ell]\!]$. Moreover, $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} := \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\Pi^k_\ell)$ is called the *total color sum* of $(\mathfrak{c},\mathfrak{d})$.

(*e*) For brevity, let $|\mathfrak{c}| := k$ for any $k \in \mathbb{N}_0$ and any $\mathfrak{c}\colon [\![k]\!] \to \{\circ,\bullet\}$.

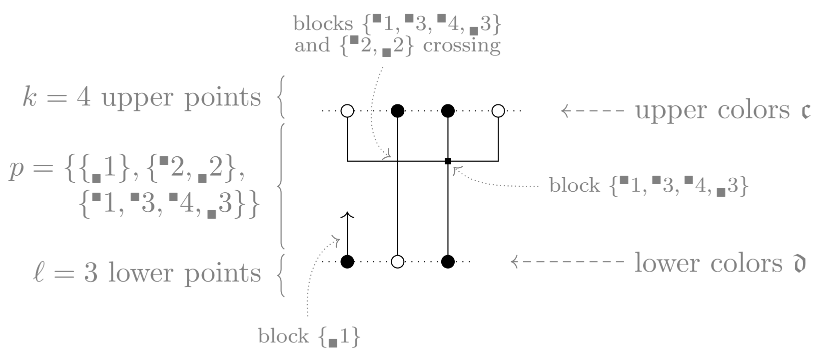
Example 3.5. Consider $\mathfrak{c}\colon [\![3]\!] \to \{\circ,\bullet\}$ and $\mathfrak{d}\colon [\![4]\!] \to \{\circ,\bullet\}$ with $\mathfrak{c}_2 = \mathfrak{c}_3 = \circ$ and $\mathfrak{c}_1 = \bullet$ and $\mathfrak{d}_1 = \mathfrak{d}_2 = \mathfrak{d}_4 = \circ$ and $\mathfrak{d}_3 = \bullet$.



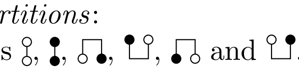
The color sum $\sigma^{\mathfrak{c}}_{\mathfrak{d}}$ has density 1 at each of ${}^{\blacksquare}1$, ${}_{\blacksquare}1$, ${}_{\blacksquare}2$ and ${}_{\blacksquare}4$ and density -1 at each of ${}^{\blacksquare}2$, ${}^{\blacksquare}3$ and ${}_{\blacksquare}3$. Consequently, the subset $\mathsf{S} = \{{}^{\blacksquare}3, {}_{\blacksquare}3, {}_{\blacksquare}4\}$ of Π^3_4 has color sum $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{S}) = \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}^{\blacksquare}3\}) + \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}_{\blacksquare}3\}) + \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{{}_{\blacksquare}4\}) = -1 - 1 + 1 = -1$. The total color sum is $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} = 1$.

Definition 3.6.

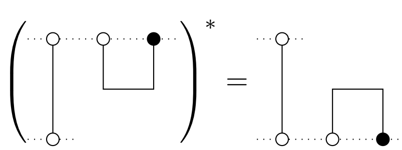
(*a*) A *two-colored partition* is any triple $(\mathfrak{c},\mathfrak{d},p)$ for which there exist $\{k,\ell\} \subseteq \mathbb{N}_0$ such that $\mathfrak{c}$ and $\mathfrak{d}$ are mappings from $[\![k]\!]$ respectively $[\![\ell]\!]$ to $\{\circ,\bullet\}$, the upper and lower *colorings*, and such that p, the collection of *blocks*, is a set-theoretical partition of the set Π^k_ℓ of points.



(*b*) Any set $\mathcal{C}$ of two-colored partitions meeting the following conditions is called a *category of two-colored partitions*:

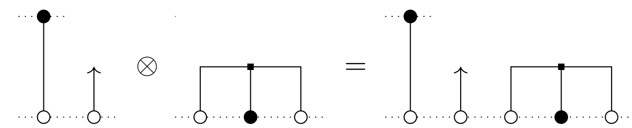
(i) $\mathcal{C}$ contains .

(ii) $\mathcal{C}$ is closed under forming adjoints, that is, horizontal reflection. More precisely, $(\mathfrak{d},\mathfrak{c},p^*) \in \mathcal{C}$ for any $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$, where, if $\{k,\ell\} \subseteq \mathbb{N}_0$ are such that p is a set-theoretical partition of Π^k_ℓ, then $p^* := \{\{{}^{\blacksquare}b \mid b \in [\![\ell]\!] \wedge {}_{\blacksquare}b \in \mathsf{B}\} \cup \{{}_{\blacksquare}a \mid a \in [\![k]\!] \wedge {}^{\blacksquare}a \in \mathsf{B}\}\}_{\mathsf{B}\in p}$ is the *adjoint* of p.

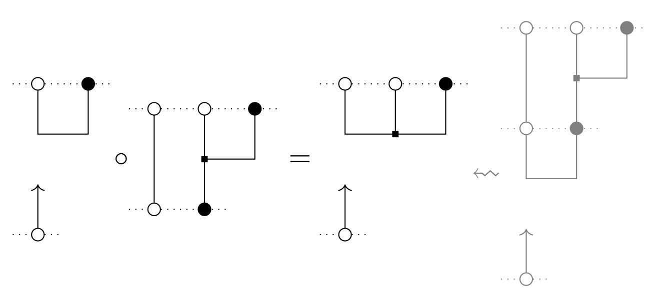


(iii) $\mathcal{C}$ is closed under tensor products, i.e., horizontal concatenation. Formally, $(\mathfrak{c}_1 \otimes \mathfrak{c}_2, \mathfrak{d}_1 \otimes \mathfrak{d}_2, p_1 \otimes p_2) \in \mathcal{C}$ for any $(\mathfrak{c}_1,\mathfrak{d}_1,p_1) \in \mathcal{C}$ and $(\mathfrak{c}_2,\mathfrak{d}_2,p_2) \in \mathcal{C}$, where, if k_t and ℓ_t are such that p_t is a set-theoretical partition of $\Pi^{k_t}_{\ell_t}$ for each $t \in [\![2]\!]$, then $\mathfrak{c}_1 \otimes \mathfrak{c}_2 \in \{\circ,\bullet\}^{\times(k_1+k_2)}$ is defined by $a \mapsto \mathfrak{c}_1(a)$ if $a \leq k_1$ and $a \mapsto \mathfrak{c}_2(a-k_1)$ if $k_1 < a$ and, analogously, $\mathfrak{d}_1 \otimes \mathfrak{d}_2 \in \{\circ,\bullet\}^{\times(\ell_1+\ell_2)}$ is defined by $b \mapsto \mathfrak{d}_1(b)$ if $b \leq \ell_1$ and $b \mapsto \mathfrak{d}_2(b-\ell_1)$

if $\ell_1 < b$, and where $p_1 \otimes p_2 := p_1 \cup \{\{{}^{\blacksquare}(k_1+a) \mid a \in [\![k_2]\!] \wedge {}^{\blacksquare}a \in \mathsf{B}\} \cup \{{}_{\blacksquare}(\ell_1+b) \mid b \in [\![\ell_2]\!] \wedge {}_{\blacksquare}b \in \mathsf{B}\}\}_{\mathsf{B}\in p_2}$ is the *tensor product* of (p_1, p_2).

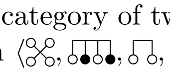


(iv) $\mathcal{C}$ is closed under composition, i.e., vertical concatenation in the following sense. If for two set-theoretical partitions the lower coloring of the first agrees with the upper coloring of the second, then the composition has the same upper coloring as the first and the same lower coloring as the second. Any blocks of the first which only include upper points are inherited by the composition, as are any blocks of the second which only include lower points. The remaining blocks of the composition are formed by the following procedure. The collection of all non-empty intersections of blocks of the first two-colored partition with the set of lower points is a set-theoretical partition of the latter. Likewise, a set-theoretical partition of the set of upper points of the second two-colored partition is given by the collection of all non-empty intersections of blocks of the second two-colored partition with it. If the lower points of the first two-colored partition and the upper points of the second are identified according to the numbering, the two set-theoretical partitions just described have a join. For each element of the join, consider the union of the following two sets. The first is the (possibly empty) set of upper points of the first two-colored partition which are contained in a block of the first two-colored partition which intersects the element of the join if the latter is interpreted as a set of lower points of the first two-colored partition. Similarly, the second is the (possibly empty) set of lower points of the second two-colored partition which are contained in a block of the second two-colored partition which intersects the element of the join if the latter is interpreted as a set of upper points of the second two-colored partition. Provided that the union of these two sets is not empty, it constitutes a block of the composition of the two-colored partitions. And all blocks of the composition arise in one of the three aforementioned ways. In formulas: $(\mathfrak{c}, \mathfrak{e}, qp) \in \mathcal{C}$ for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $(\mathfrak{d}, \mathfrak{e}, q) \in \mathcal{C}$, where if $\{k, \ell, m\} \subseteq \mathbb{N}_0$ are such that p is a set-theoretical partition of Π_ℓ^k and q one of Π_m^ℓ, and if s is the join of the two set-theoretical partitions $\{\{j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{A}\}\}_{\mathsf{A}\in p} \backslash \{\varnothing\}$ and $\{\{i \in [\![\ell]\!] \wedge {}^{\blacksquare}i \in \mathsf{C}\}\}_{\mathsf{C}\in q} \backslash \{\varnothing\}$ of $[\![\ell]\!]$, then $qp := \{\mathsf{A} \in p \wedge \mathsf{A} \subseteq \Pi_0^k\} \cup \{\mathsf{C} \in q \wedge \mathsf{C} \subseteq \Pi_m^0\} \cup \{\bigcup\{\mathsf{A} \cap \Pi_0^k \mid \mathsf{A} \in p \wedge \exists j \in B\colon {}_{\blacksquare}j \in \mathsf{A}\} \cup \bigcup\{\mathsf{C} \cap \Pi_m^0 \mid \mathsf{C} \in q \wedge \exists i \in B\colon {}^{\blacksquare}i \in \mathsf{C}\}\}_{B\in s} \backslash \{\varnothing\}$ is the *composition* of (q, p).



(*c*) For any set $\mathcal{G}$ of two-colored partition, we write $\langle \mathcal{G} \rangle$ for the intersection of all categories of two-colored partitions containing $\mathcal{G}$ and we say that $\mathcal{G}$ *generates* $\langle \mathcal{G} \rangle$.

Example 3.7.

(*a*) Of course, the set of all two-colored partitions forms the maximal category of two-colored partitions. It follows from [34, Theorem 8.3] that it coincides with .

(*b*) Another category of two-colored partitions is given by $\langle \times^{\circ\bullet}_{\circ\bullet} \rangle$, the category of two-colored pair partitions with neutral blocks, i.e., all $(\mathfrak{c}, \mathfrak{d}, p)$ with $|\mathsf{B}| = 2$ and $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ for any $\mathsf{B} \in p$, (see [26, Proposition 5.3]).

(*c*) By [34, Theorem 7.2], the minimal category of two-colored partitions $\langle \varnothing \rangle$ is the subset of all elements of $\langle \times^{\circ\bullet}_{\circ\bullet} \rangle$ which are *non-crossing*. The precise definition of being non-crossing is unimportant here; informally, it means that blocks can be "drawn without intersections".

Not much familiarity with two-colored partitions and their categories is required in order to prove the main result. In particular, the full classification of all categories of two-colored partitions can remain open. However, we will need to divide the landscape of all possible categories as follows.

Definition 3.8. We say that any category $\mathcal{C}$ of two-colored partitions is

(*a*) *case* $\mathcal{O}$ if $\uparrow_{\circ}\uparrow_{\bullet} \notin \mathcal{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \notin \mathcal{C}$,
(*b*) *case* $\mathcal{B}$ if $\uparrow_{\circ}\uparrow_{\bullet} \in \mathcal{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \notin \mathcal{C}$,
(*c*) *case* $\mathcal{H}$ if $\uparrow_{\circ}\uparrow_{\bullet} \notin \mathcal{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$,
(*d*) *case* $\mathcal{S}$ if $\uparrow_{\circ}\uparrow_{\bullet} \in \mathcal{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$,
(*e*) *class* NNSB if $\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{B}) = 0$ for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and any $\mathsf{B} \in p$ with $2 \leq |\mathsf{B}|$,
(*f*) *class* NP if $\Sigma^{\mathfrak{c}}_{\mathfrak{d}} = 0$ for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$.

The names NNSB and NP reflect the defining conditions of having only <u>n</u>eutral <u>n</u>on-<u>s</u>ingleton <u>b</u>locks respectively only <u>n</u>eutral two-colored <u>p</u>artitions, where "neutral" means vanishing color sum. For the motivation behind the names $\mathcal{O}$, $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{S}$ see Remark 3.22 below.

Remark 3.9. For any of the known categories of two-colored partitions, it is easy to determine whether it has a given property in Definition 3.8 or not. Any known category which is not case $\mathcal{H}$ is covered by [28] (see Section 7 there for the correspondence to the results of [15, 26, 27, 34]) and any known case-$\mathcal{H}$ category by [15, 23, 34] or [25, Chapter 1].

Cases $\mathcal{O}$*,* $\mathcal{B}$*,* $\mathcal{S}$*.* Any category $\mathcal{R}_{f,v,s,l,k,x}$ in the main theorem of [28] is case $\mathcal{O}$ if and only if $f = \{2\}$, case $\mathcal{B}$ if and only if $f = \{1, 2\}$ and case $\mathcal{S}$ if and only if $f = \mathbb{N}$ (and never case $\mathcal{H}$). It is class NNSB if and only if $v = \{0\}$ or $v = \pm\{0, 1\}$ and class NP if and only if $s = \{0\}$.

Case $\mathcal{H}$*.* In [34], neither of the categories $\mathcal{H}_{\text{glob}}(k)$ of Theorem 7.1 and $\mathcal{H}_{\text{grp,glob}}(k)$ of Theorem 8.3 is class NNSB. And, each is class NP if and only if $k = 0$. The category $\mathcal{H}'_{\text{loc}}$ in Theorem 7.2 is both class NNSB and class NP. Each of $\mathcal{H}_{\text{loc}}(k, d)$ from Theorem 7.2 and $\mathcal{H}_{\text{grp,loc}}(k, d)$ from Theorem 8.3 is class NNSB if and only if $k = d = 0$ and is class NP if and only if $k = 0$.

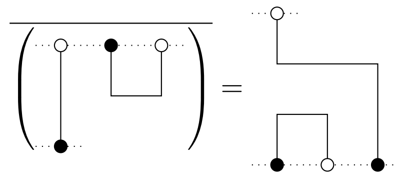
The case-$\mathcal{H}$ categories in [15, Table 1] which are not already covered by [34] are $\mathcal{H}_{\text{hl,glob}}(k, 0)$, $\mathcal{H}_{\text{hl,glob}}(k, s)$, $\mathcal{H}_{\pi}(k, s)$, $\mathcal{H}_{\pi}(k, \infty)$ and $\mathcal{H}_{A}(k)$ (where the categories $\mathcal{H}_{\text{hl,glob}}(k, 0)$ and $\mathcal{H}_{\text{hl,glob}}(k, s)$ can each also be written as $\mathcal{H}_{A}(k)$ for certain A). None one of these are class NNSB. And any one is class NP if and only if $k = 0$.

In [23], no group-theoretical category $\mathcal{R}$ of two-colored partitions in the sense of Definition 4.1.5 is class NNSB. And, any such category is class NP if and only if $F_{\infty}(\mathcal{R})$ in the sense of Definition 4.3.21 contains no word with different numbers of generators and inverses of generators.

Lastly, the category $\mathcal{W}_{\mathcal{R}}$ in the sense of the main result of [25, Chapter 1] is both class NNSB and class NP for any parameter $\mathcal{R}$.

Beyond those case distinctions, we will also need to know the following elementary facts about categories of two-colored partitions.

Definition 3.10. Dual two-colored partitions are obtained by simultaneous horizontal reflection, vertical reflection and color inversion. More precisely, given any $\{k, \ell\} \subseteq \mathbb{N}_0$, any $\mathfrak{c} \in \{\circ, \bullet\}^{\times k}$, any $\mathfrak{d} \in \{\circ,\bullet\}^{\times \ell}$ and any set-theoretical partition p of Π_ℓ^k the *dual* of $(\mathfrak{c},\mathfrak{d},p)$ is the triple $(\overline{\mathfrak{d}},\overline{\mathfrak{c}},\overline{p})$, where $\overline{\mathfrak{d}} \in \{\circ,\bullet\}^{\times \ell}$ is defined by $j \mapsto \overline{\mathfrak{d}_{\ell-j+1}}$, where $\overline{\mathfrak{c}} \in \{\circ,\bullet\}^{\times k}$ is defined by $i \mapsto \overline{\mathfrak{c}_{k-i+1}}$, and where $\overline{p} := \{\{{}^{\blacksquare}(\ell-j+1) \mid j \in [\![\ell]\!] \wedge {}_{\blacksquare}j \in \mathsf{B}\} \cup \{{}_{\blacksquare}(k-i+1) \mid i \in [\![k]\!] \wedge {}^{\blacksquare}i \in \mathsf{B}\}\}_{\mathsf{B}\in p}$ is the *dual* of p.



Lemma 3.11. *Let $\mathcal{C}$ be any category of two-colored partitions.*

(a) $(\overline{\mathfrak{d}},\overline{\mathfrak{c}},\overline{p}) \in \mathcal{C}$ *for any* $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$.
(b) $\overset{\uparrow}{\circ}\overset{\uparrow}{\bullet} \in \mathcal{C}$ *if and only if there exist* $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ *and* $\mathsf{B} \in p$ *such that* $|\mathsf{B}| < 2$.
(c) $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$ *if and only if there exist* $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ *and* $\mathsf{B} \in p$ *such that* $|\mathsf{B}| > 2$.
(d) *If* $\overset{\uparrow}{\circ}\overset{\uparrow}{\bullet} \in \mathcal{C}$ *and* $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$, *then* $\lceil\uparrow\rceil\uparrow_{\circ\bullet\circ\bullet} \in \mathcal{C}$.
(e) *If* $\overset{\uparrow}{\circ}\overset{\uparrow}{\bullet} \in \mathcal{C}$, *then* $\{\overset{\uparrow}{\circ}{}^{\otimes|\Sigma_{\mathfrak{d}}^{\mathfrak{c}}|}, \overset{\uparrow}{\bullet}{}^{\otimes|\Sigma_{\mathfrak{d}}^{\mathfrak{c}}|}\} \subseteq \mathcal{C}$ *for any* $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ *with* $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$.

Proof. Part *(a)* is implied by [34, Lemma 1.1 (a)]. Parts *(b)* and *(c)* are [34, Lemmas 1.3 (b) and 2.1 (a)] and [34, Lemmas 1.3 (d) and 2.1 (b)], respectively. Part *(d)* follows immediately from [34, Lemma 1.3 (b)].

In order to see part *(e)*, use [34, Lemma 1.1 (a)] to first "rotate" any potential upper point of $(\mathfrak{c},\mathfrak{d},p)$ down (in an arbitrary direction). "Disconnect" then each and every point from its block with the help of [34, Lemma 1.3 (b)]. Following that, keep "erasing" neighboring points of different colors, as [34, Lemma 1.1 (b)] permits, until no such points remain. None of these transformations have affected the total color sum. The resulting two-colored partition is either $\overset{\uparrow}{\circ}{}^{\otimes|\Sigma_{\mathfrak{d}}^{\mathfrak{c}}|}$ or $\overset{\uparrow}{\bullet}{}^{\otimes|\Sigma_{\mathfrak{d}}^{\mathfrak{c}}|}$. Passing to the adjoint of the dual as allowed by *(a)* hence shows the claim. ■

Lemma 3.12.

(a) *Any case-$\mathcal{O}$ or case-$\mathcal{H}$ category of two-colored partitions that is class* NNSB *is class* NP.
(b) *No case-$\mathcal{S}$ category of two-colored partitions is class* NNSB.

Proof. (a) Let $\mathcal{C}$ be case-$\mathcal{O}$ or case-$\mathcal{H}$ and class NNSB. Then, for any $(\mathfrak{c},\mathfrak{d},p) \in \mathcal{C}$ and any $\mathsf{B} \in p$ on the one hand $2 \leq |\mathsf{B}|$ by the first assumption and thus on the other hand $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ by the second assumption. Since that demands $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = \sum_{\mathsf{B}\in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) = 0$ the category $\mathcal{C}$ is necessarily class NP.

(b) Since any case-$\mathcal{S}$ category contains both $\overset{\uparrow}{\circ}\overset{\uparrow}{\bullet}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}$, it must also contain $\lceil\uparrow\rceil\uparrow_{\circ\bullet\circ\bullet}$ by Lemma 3.11 *(d)*. The fact that $\{{}_{\blacksquare}1, {}_{\blacksquare}3\} \in \lceil\uparrow\rceil\uparrow$ and $\sigma^{\varnothing}_{\circ\bullet\circ\bullet}(\{{}_{\blacksquare}1, {}_{\blacksquare}3\}) = 2 \neq 0$ hence shows that such a category is not class NNSB. ■

3.2 Unitary easy quantum groups

"Easy" quantum groups are now defined by transforming the elements of a given category of two-colored partitions into relations for the generators of a universal algebra that can be given the structure of a compact quantum group. To be more precise, an entire series of compact quantum groups indexed by $\mathbb{N}$ arises in this way.

Assumptions 3.13. In the following, fix any $n \in \mathbb{N}$ and any $2n^2$-elemental set $E = \{u_{j,i}^{\circ}, u_{j,i}^{\bullet}\}_{i,j=1}^{n}$ and define the two families $u^{\circ} := (u_{j,i}^{\circ})_{(j,i)\in[\![n]\!]^{\times 2}}$ and $u^{\bullet} := (u_{j,i}^{\bullet})_{(j,i)\in[\![n]\!]^{\times 2}}$.

The transformation of two-colored partitions into relations is accomplished by the following formula, where ζ was defined in Notation 3.1 *(c)*.

Notation 3.14. For any $\{k,\ell\}\subseteq\mathbb{N}_0$, any $\mathfrak{c}\in\{\circ,\bullet\}^{\times k}$ and $\mathfrak{d}\in\{\circ,\bullet\}^{\times\ell}$, any set-theoretical partition p of Π_ℓ^k and any $g\in[\![n]\!]^{\times k}$ and $j\in[\![n]\!]^{\times\ell}$, let in $\mathbb{C}\langle E\rangle$

$$\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}:=\sum_{i\in[\![n]\!]^{\times\ell}}\zeta(p,\ker(g\,{}^{\blacksquare}{}_{\blacksquare}\,i))\overrightarrow{\prod_{b=1}^{\ell}}u_{j_b,i_b}^{\mathfrak{d}_b}-\sum_{h\in[\![n]\!]^{\times k}}\zeta(p,\ker(h\,{}^{\blacksquare}{}_{\blacksquare}\,j))\overrightarrow{\prod_{a=1}^{k}}u_{h_a,g_a}^{\mathfrak{c}_a}.$$

For example, the two-colored partitions $\,{}^{\circ}_{\circ}\!\vert$ and $\,{}^{\bullet}_{\bullet}\!\vert$ induce the trivial relation 0. The relations induced by $\sqcap_{\circ\bullet}$, $\sqcup^{\bullet\circ}$, $\sqcap_{\bullet\circ}$ and $\sqcup^{\circ\bullet}$ will be of the utmost importance.

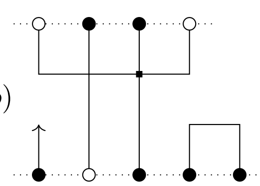
Lemma 3.15. *For any $g\in[\![n]\!]^{\times 2}$ and $j\in[\![n]\!]^{\times 2}$, the following hold:*

$$\mathrm{r}_{\circ\bullet}^{\varnothing}(\sqcap)_{j,\varnothing}=\sum_{i=1}^{n}u_{j_1,i}^{\circ}u_{j_2,i}^{\bullet}-\delta_{j_1,j_2}1,\qquad \mathrm{r}_{\varnothing}^{\bullet\circ}(\sqcup)_{\varnothing,g}=\delta_{g_1,g_2}1-\sum_{h=1}^{n}u_{h,g_1}^{\bullet}u_{h,g_2}^{\circ},$$
$$\mathrm{r}_{\bullet\circ}^{\varnothing}(\sqcap)_{j,\varnothing}=\sum_{i=1}^{n}u_{j_1,i}^{\bullet}u_{j_2,i}^{\circ}-\delta_{j_1,j_2}1,\qquad \mathrm{r}_{\varnothing}^{\circ\bullet}(\sqcup)_{\varnothing,g}=\delta_{g_1,g_2}1-\sum_{h=1}^{n}u_{h,g_1}^{\circ}u_{h,g_2}^{\bullet}.$$

Proof. Only the proof for $\mathrm{r}_{\circ\bullet}^{\varnothing}(\sqcap)_{j,\varnothing}$ is given. With the names of Notation 3.14 then $k=0$ and $\ell=2$ and $\mathfrak{c}=\varnothing$ and $\mathfrak{d}=\circ\bullet$ and $p=\{\{{}_{\blacksquare}1,{}_{\blacksquare}2\}\}$ and $g=\varnothing$. On the one hand, for any $i\in[\![n]\!]^{\times 2}$ the set-theoretical partition $\ker(g\,{}^{\blacksquare}{}_{\blacksquare}\,i)$ can only take two values, namely $\{\{{}_{\blacksquare}1,{}_{\blacksquare}2\}\}$ if $i_1=i_2$ and $\{\{{}_{\blacksquare}1\},\{{}_{\blacksquare}2\}\}$ if $i_1\neq i_2$. Whereas $\ker(g\,{}^{\blacksquare}{}_{\blacksquare}\,i)$ even agrees with p in the former case, p is not finer than $\ker(g\,{}^{\blacksquare}{}_{\blacksquare}\,i)$ in the latter case. Hence, only if $i_1=i_2$ does $\zeta(p,\ker(g\,{}^{\blacksquare}{}_{\blacksquare}\,i))$ evaluate to 1. Consequently, the first of the two sums in the definition of $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}$ effectively runs only over the pairs (i,i) for $i\in[\![n]\!]$. That explains the term $\sum_{i=1}^{n}u_{j_1,i}^{\circ}u_{j_2,i}^{\bullet}$ in the claim. On the other hand, $k=0$ by convention implies $[\![n]\!]^{\times k}=\{\varnothing\}$. Thus, $\varnothing$ is the only h over which the second sum in the definition of $\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}$ runs. Just like before, $\ker(h\,{}^{\blacksquare}{}_{\blacksquare}\,j)$ is then either $\{\{{}_{\blacksquare}1,{}_{\blacksquare}2\}\}$ or $\{\{{}_{\blacksquare}1\},\{{}_{\blacksquare}2\}\}$, depending on whether if $j_1=j_2$ or not. In other words, $\zeta(p,\ker(h\,{}^{\blacksquare}{}_{\blacksquare}\,j))=\delta_{j_1,j_2}$. By $k=0$ the set of indices a over which the product $\overrightarrow{\prod}_{a=1}^{k}u_{h_a,g_a}^{\mathfrak{c}_a}$ runs is the empty set $\varnothing$ (whereas the set of indices h before was not $\varnothing$ but $\{\varnothing\}$). By common convention, a product with empty index set is 1. That is how the second term $-\delta_{j_1,j_2}1$ in the claim comes about. ■

In general, the relations can become quite complicated.

Example 3.16. If $k=4$ and $\ell=5$ and $\mathfrak{c}=\circ\bullet\bullet\circ$ and $\mathfrak{d}=\bullet\circ\bullet\bullet\bullet$ and $p=\{\{{}_{\blacksquare}1\},\{{}^{\blacksquare}2,{}_{\blacksquare}2\},\{{}_{\blacksquare}4,{}_{\blacksquare}5\},\{{}^{\blacksquare}1,{}^{\blacksquare}3,{}^{\blacksquare}4,{}_{\blacksquare}3\}\}$,

$(\mathfrak{c},\mathfrak{d},p)$ 

then for any $g\in[\![n]\!]^{\times k}$ and $j\in[\![n]\!]^{\times\ell}$,

$$\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}=\delta_{g_1,g_3}\delta_{g_1,g_4}\left(\sum_{i=1}^{n}u_{j_1,i}^{\bullet}\right)u_{j_2,g_2}^{\circ}u_{j_3,g_1}^{\bullet}\left(\sum_{i=1}^{n}u_{j_4,i}^{\bullet}u_{j_5,i}^{\bullet}\right)-\delta_{j_4,j_5}\,u_{j_3,g_1}^{\circ}u_{j_2,g_2}^{\bullet}u_{j_3,g_3}^{\bullet}u_{j_3,g_4}^{\circ}.$$

The following definition of "easy" quantum groups is the algebraic version of [33, Definition 5.1]. Recall that $n\in\mathbb{N}$ is fixed per Assumptions 3.13.

Notation 3.17. For any set $\mathcal{P}$ of two-colored partitions, let

$$R_{\mathcal{P}}:=\left\{\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}\,\middle|\,(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}\wedge g\in[\![n]\!]^{\times|\mathfrak{c}|}\wedge j\in[\![n]\!]^{\times|\mathfrak{d}|}\right\}$$

and let $J_{\mathcal{P}}$ be the two-sided ideal of $\mathbb{C}\langle E\rangle$ generated by $R_{\mathcal{P}}$.

Definition 3.18. For any category $\mathcal{C}$ of two-colored partitions, the *unitary easy compact quantum group* of $(\mathcal{C}, n)$ is given by $(\mathbb{C}\langle E \mid R_{\mathcal{C}}\rangle, *, \Delta)$, where $*$ and Δ are respectively the unique anti-multiplicative anti-linear self-map of $\mathbb{C}\langle E \mid R_{\mathcal{C}}\rangle$ and the unique multiplicative linear map from $\mathbb{C}\langle E \mid R_{\mathcal{C}}\rangle$ to the tensor product algebra of $\mathbb{C}\langle E \mid R_{\mathcal{C}}\rangle$ with itself which satisfy respectively

$$(u^{\mathfrak{c}}_{j,i} + J_{\mathcal{C}})^* = u^{\bar{\mathfrak{c}}}_{j,i} + J_{\mathcal{C}} \qquad \text{and} \qquad \Delta(u^{\mathfrak{c}}_{j,i} + J_{\mathcal{C}}) = \sum_{s=1}^{n} (u^{\mathfrak{c}}_{j,s} + J_{\mathcal{C}}) \otimes (u^{\mathfrak{c}}_{s,i} + J_{\mathcal{C}})$$

for any $\{i, j\} \subseteq [\![n]\!]$ and $\mathfrak{c} \in \{\circ, \bullet\}$.

Remark 3.19. The definition of unitary easy quantum groups is usually given in terms of universal $*$-algebras, not universal algebras, cf. [33, Definition 5.1]. The variant given above is equivalent, as explained hereafter. For any $m \in \mathbb{N}_0$ and any $e \in [\![n]\!]^{\times m}$ let $\overline{e} \in [\![n]\!]^{\times m}$ be defined by $i \mapsto e_{m-i+1}$ for any $i \in [\![m]\!]$. Let $\{k, \ell\} \subseteq \mathbb{N}_0$, let $\mathfrak{c} \in \{\circ, \bullet\}^{\times k}$, let $\mathfrak{d} \in \{\circ, \bullet\}^{\times \ell}$, let $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$, let $g \in [\![n]\!]^{\times k}$ and let $j \in [\![n]\!]^{\times \ell}$. Then, with respect to the $*$-map in Definition 3.18,

$$\begin{aligned}
(\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g})^* &= \sum_{i \in [\![n]\!]^{\times \ell}} \zeta(p, \ker(g \,{}^{\blacksquare}{}_{\blacksquare}\, i)) \overleftarrow{\prod_{b=1}^{\ell}} (u^{\mathfrak{d}_b}_{j_b, i_b})^* - \sum_{h \in [\![n]\!]^{\times k}} \zeta(p, \ker(h \,{}^{\blacksquare}{}_{\blacksquare}\, j)) \overleftarrow{\prod_{a=1}^{k}} (u^{\mathfrak{c}_a}_{h_a, g_a})^* \\
&= \sum_{i \in [\![n]\!]^{\times \ell}} \zeta((\overline{p})^*, \ker(\overline{g} \,{}^{\blacksquare}{}_{\blacksquare}\, \overline{i})) \overrightarrow{\prod_{b=1}^{\ell}} u^{\bar{\mathfrak{d}}_b}_{\bar{j}_b, \bar{i}_b} - \sum_{h \in [\![n]\!]^{\times k}} \zeta((\overline{p})^*, \ker(\overline{h} \,{}^{\blacksquare}{}_{\blacksquare}\, \overline{j})) \overrightarrow{\prod_{a=1}^{k}} u^{\bar{\mathfrak{c}}_a}_{\bar{h}_a, \bar{g}_a} \\
&= \mathrm{r}^{\bar{\mathfrak{c}}}_{\bar{\mathfrak{d}}}((\overline{p})^*)_{\bar{j}, \bar{g}}.
\end{aligned}$$

Since $\mathcal{C}$ also contains the two-colored partition $(\bar{\mathfrak{c}}, \bar{\mathfrak{d}}, (\overline{p})^*)$ by Lemma 3.11 *(a)*, the switch from the universal $*$-algebra to the universal algebra makes no difference.

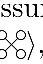
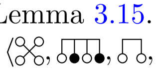
For the idea of the proof of the following, see [33, Remark 5.2].

Proposition 3.20. *For any category $\mathcal{C}$ of two-colored partitions, the unitary easy compact quantum group of $(\mathcal{C}, n)$ is a compact quantum group whose co-unit is given by the unique multiplicative linear functional ϵ with*

$$\epsilon(u^{\mathfrak{c}}_{j,i} + J_{\mathcal{C}}) = \delta_{j,i}$$

for any $\{i, j\} \subseteq [\![n]\!]$ and $\mathfrak{c} \in \{\circ, \bullet\}$. It can be seen as a compact $(n \times n)$-matrix quantum group with fundamental representation induced by u°.

Example 3.21. Let $\mathcal{C}$ be a category of two-colored partitions and let G be the unitary quantum group of $(\mathcal{C}, n)$.

- *(a)* If $\mathcal{C}$ is the minimal category $\langle \varnothing \rangle$, then G is the *free unitary quantum group* U_n^+ introduced by Wang in [38]. Its algebra can be presented as the universal algebra generated by E (fixed in Assumptions 3.13) subject to only the relations of Lemma 3.15.
- *(b)* For $\mathcal{C} = \langle$$\rangle$, we recover the classical unitary group U_n, the universal commutative(!) algebra subject to the relations of Lemma 3.15.
- *(c)* Should $\mathcal{C}$ be the maximal category $\langle$$\rangle$ of all two-colored partitions, then G is the symmetric group S_n.

Currently, there is no complete list of all unitary easy quantum groups because the classification of all categories of two-colored partitions is not yet finished.

Remark 3.22. The names $\mathcal{O}$, $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{S}$ of the four cases from Definition 3.8 were introduced by Tarrago and Weber in [34, Definition 2.2] and refer respectively to the orthogonal group O_n, bistochastic group B_n, hyperoctahedral group H_n and symmetric group S_n. (A "bistochastic" matrix is understood to be an orthogonal matrix each of whose rows and columns sums to 1.) Tarrago and Weber showed that for each $\mathcal{X} \in \{\mathcal{O}, \mathcal{B}, \mathcal{H}, \mathcal{S}\}$ there exists a category of two-colored partitions which is case $\mathcal{X}$ and maximally so. And, in the sense of Definition 3.18, the maximal case-$\mathcal{O}$ category is the one associated with O_n, the maximal case-$\mathcal{B}$ category the one associated with B_n, the maximal case-$\mathcal{H}$ category the one associated with H_n and the maximal case-$\mathcal{S}$ category the one associated with S_n.

4 First Hochschild cohomology of universal algebras

For the convenience of the reader, Section 4 recalls the definition of and some elementary results about the first Hochschild cohomology. Throughout, let $\mathbb{K}$ be any field.

4.1 First Hochschild cohomology

In Section 4.1, our algebra shall remain abstract. Section 4.2 will then recall which conclusions can be drawn if a presentation of the algebra in terms of generators and relations is given.

Assumptions 4.1. Let A be any $\mathbb{K}$-algebra and X any A-bimodule.

That means in particular that X is a $\mathbb{K}$-vector space implicitly equipped with $\mathbb{K}$-linear maps $\triangleright\colon A\otimes X \to X$ and $\triangleleft\colon X\otimes A \to X$, the left and right *actions* of A, such that $a_1\triangleright(a_2\triangleright x) = (a_1a_2)\triangleright x$ and $(x \triangleleft a_2) \triangleleft a_1 = x \triangleleft (a_2a_1)$ and $(a_1 \triangleright x) \triangleleft a_2 = a_1 \triangleright (x \triangleleft a_2)$ for any $x \in X$ and $\{a_1, a_2\} \subseteq A$.

Example 4.2. For any augmentation ϵ of A, i.e., any $\mathbb{K}$-algebra morphism from A to $\mathbb{K}$, the $\mathbb{K}$-vector space $\mathbb{K}$ becomes an A-bimodule X if equipped with the actions defined by $a \triangleright \lambda = \epsilon(a)\lambda$ respectively $\lambda \triangleleft a = \lambda\epsilon(a)$ for any $\lambda \in \mathbb{K}$ and $a \in A$. It is often called the *trivial bimodule* of (A, ϵ).

4.1.1 The fundamental definitions

The following definitions were first given by Hochschild in [16].

Definition 4.3.

(*a*) The *X-valued Hochschild 1-cocycles of A* are the $\mathbb{K}$-vector subspace $Z^1_{\mathrm{HS}}(A, X)$ of $[A, X]$ formed by all elements η such that for any $\{a_1, a_2\} \subseteq A$,

$$\big(\partial^1\eta\big)(a_1 \otimes a_2) := a_1 \triangleright \eta(a_2) - \eta(a_1a_2) + \eta(a_1) \triangleleft a_2 = 0.$$

(*b*) The *X-valued Hochschild 1-coboundaries of A* are the $\mathbb{K}$-vector subspace $B^1_{\mathrm{HS}}(A, X)$ of $[A, X]$ formed by all elements η such that there exists $x \in X$ with for any $a \in A$,

$$\eta(a) = \big(\partial^0 x\big)(a) := a \triangleright x - x \triangleleft a.$$

It can be seen that $\eta(1) = 0$ for any $\eta \in Z^1_{\mathrm{HS}}(A, X)$ and that $B^1_{\mathrm{HS}}(A, X)$ is a $\mathbb{K}$-vector subspace of $Z^1_{\mathrm{HS}}(A, X)$.

Definition 4.4. We call the quotient $\mathbb{K}$-vector space $H^1_{\mathrm{HS}}(A, X)$ of $Z^1_{\mathrm{HS}}(A, X)$ with respect to $B^1_{\mathrm{HS}}(A, X)$ the *first Hochschild cohomology of A with X-coefficients.*

Example 4.5. In the case of Example 4.2, i.e., for trivial coefficients, the only 1-coboundary of A is the zero map because $(\partial^0\lambda)(a) = \epsilon(a)\lambda - \lambda\epsilon(a) = 0$ for any $\lambda \in \mathbb{K}$ and $a \in A$. Hence, $Z^1_{\mathrm{HS}}(A, X) \cong H^1_{\mathrm{HS}}(A, X)$ in that instance.

4.1.2 Algebra hom characterization of 1-coycles

1-cocycles can be characterized as certain algebra homomorphisms by means of a folk theorem recorded as [22, Lemma 1.9]. The latter uses the following construction.

Definition 4.6. Let $A \oplus_0^1 X$ denote the $\mathbb{K}$-vector space $A \oplus X$ equipped with the $\mathbb{K}$-linear map $\mathbb{K} \to A \oplus X$ with $1 \mapsto (1, 0)$ and the $\mathbb{K}$-linear map $(A \oplus X)^{\otimes 2} \to A \oplus X$ defined by for any $\{a_1, a_2\} \subseteq A$ and $\{x_1, x_2\} \subseteq X$,

$$(a_1, x_1) \otimes (a_2, x_2) \mapsto (a_1a_2, a_1 \triangleright x_2 + x_1 \triangleleft a_2).$$

Lemma 4.7. *$A \oplus_0^1 X$ is a $\mathbb{K}$-algebra.*

Proof. $(1, 0)$ is a unit because for any $a \in A$ and any $x \in X$,

$$(a, x) \cdot (1, 0) = (a1, a \triangleright 0 + x \triangleleft 1) = (a, x) = (1a, 1 \triangleright x + 0 \triangleleft a) = (1, 0) \cdot (a, x).$$

Moreover, for any $\{a_1, a_2, a_3\} \subseteq A$ and any $\{x_1, x_2, x_3\} \subseteq X$,

$$\begin{aligned}
((a_1, x_1) \cdot (a_2, x_2)) \cdot (a_3, x_3) &= (a_1a_2, a_1 \triangleright x_2 + x_1 \triangleleft a_2) \cdot (a_3, x_3) \\
&= (a_1a_2a_3, (a_1a_2) \triangleright x_3 + (a_1 \triangleright x_2 + x_1 \triangleleft a_2) \triangleleft a_3) \\
&= (a_1a_2a_3, a_1a_2 \triangleright x_3 + a_1 \triangleright x_2 \triangleleft a_3 + x_1 \triangleleft a_2a_3) \\
&= (a_1a_2a_3, a_1 \triangleright (a_2 \triangleright x_3 + x_2 \triangleleft a_3) + x_1 \triangleleft (a_2a_3)) \\
&= (a_1, x_1) \cdot (a_2a_3, a_2 \triangleright x_3 + x_2 \triangleleft a_3) \\
&= (a_1, x_1) \cdot ((a_2, x_2) \cdot (a_3, x_3)),
\end{aligned}$$

which shows that the multiplication is associative. ■

The following is then the folk theorem mentioned in [22, Lemma 1.9].

Lemma 4.8. *For any $\psi \in [A, X]$, the map $A \to A \oplus X$ with $a \mapsto (a, \psi(a))$ for any $a \in A$ defines a $\mathbb{K}$-algebra homomorphism $A \to A \oplus_0^1 X$ if and only if $\psi \in Z^1_{\mathrm{HS}}(A, X)$.*

Proof. If the map in the claim is denoted by f_ψ, then $f_\psi(1) = (1, \psi(1))$ and for any $\{a_1, a_2\} \subseteq A$, obviously, $f_\psi(a_1a_2) = (a_1a_2, \psi(a_1a_2))$ and

$$f_\psi(a_1) \cdot f_\psi(a_2) = (a_1, \psi(a_1)) \cdot (a_2, \psi(a_2)) = (a_1a_2, a_1 \triangleright \psi(a_2) + \psi(a_1) \triangleleft a_2).$$

The two values coincide if and only if $\psi(a_1a_2) = a_1 \triangleright \psi(a_2) + \psi(a_1) \triangleleft a_2$, which is to say if and only if $\psi \in Z^1_{\mathrm{HS}}(A, X)$. As then $\psi(1) = 0$ the claim is true. ■

The next result will be required later in the proof of, ultimately, Proposition 4.19.

Lemma 4.9. *For any $m \in \mathbb{N}$, any $\{a_i\}_{i=1}^m \subseteq A$ and any $\{x_i\}_{i=1}^m \subseteq X$, in $A \oplus_0^1 X$,*

$$\overrightarrow{\prod}_{i=1}^m (a_i, x_i) = \left(\overrightarrow{\prod}_{i=1}^m a_i, \sum_{i=1}^m \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \triangleright x_i \triangleleft \left(\overrightarrow{\prod}_{j=i+1}^m a_j\right)\right).$$

Proof. The cases $m \in \{1, 2, 3\}$ are, respectively, trivial, the definition of the multiplication of $A \oplus_0^1 X$ and an intermediate result in the proof of Lemma 4.7. Generally,

$$\begin{aligned}
&\left(\overrightarrow{\prod}_{i=1}^{m-1} a_i, \sum_{i=1}^{m-1} \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \triangleright x_i \triangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m-1} a_j\right)\right) \cdot (a_m, x_m) \\
&\quad = \left(\left(\overrightarrow{\prod}_{i=1}^{m-1} a_i\right) a_m, \left(\overrightarrow{\prod}_{i=1}^{m-1} a_i\right) \triangleright x_m + \sum_{i=1}^{m-1} \left(\overrightarrow{\prod}_{j=1}^{i-1} a_j\right) \triangleright x_i \triangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m-1} a_j\right) \triangleleft a_m\right).
\end{aligned}$$

Hence, the claim is true. ■

4.2 Conclusions for universal algebras

Using Lemma 4.8, it is possible to give a canonical equational characterization of the 1-cocycles if a presentation of the algebra in terms of generators and relations is given.

Assumptions 4.10. Let E be any set, $R \subseteq \mathbb{K}\langle E\rangle$ arbitrary, J the two-sided ideal of $\mathbb{K}\langle E\rangle$ generated by R, and X any $\mathbb{K}\langle E \mid R\rangle$-bimodule.

Definition 4.11. Let

$$F^1_{E,R,X}\colon \mathbb{K}\langle E\rangle \to [X^{\times E}, X],\, p \mapsto F^{1,p}_{E,R,X}$$

be the unique $\mathbb{K}$-linear map with for any $m \in \mathbb{N}$ and any $\{e_i\}_{i=1}^m \subseteq E$, if $p = \overrightarrow{\prod}_{i=1}^m e_i$, then for any $x \in X^{\times E}$,

$$F^{1,p}_{E,R,X}(x) = \sum_{i=1}^{m} \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright x_{e_i} \triangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\right),$$

and with $F^{1,1}_{E,R,X} = 0$.

Example 4.12. For any augmentation ϵ of $A = \mathbb{K}\langle E \mid R\rangle$, if X is the trivial bimodule of (A, ϵ) in the sense of Example 4.2, then for any $m \in \mathbb{N}$ and any $\{e_i\}_{i=1}^m \subseteq E$, if $p = e_1 \cdots e_m$, then $F^{1,p}_{E,R,X}$ is a linear map which assigns to any family $x = (x_e)_{e \in E}$ of elements of $\mathbb{K}$ the number

$$F^{1,p}_{E,R,X}(x) = \sum_{i=1}^{m} \left(\prod_{j \in [\![m]\!] \backslash \{i\}} \epsilon\,(e_j + J) \right) x_{e_i}.$$

Definition 4.13.

(a) Let $Z^1_{E,R,X}$ denote the $\mathbb{K}$-vector subspace of $X^{\times E}$ of all elements x with $F^{1,r}_{E,R,X}(x) = 0$ for any $r \in R$.

(b) Write $B^1_{E,R,X}$ for the $\mathbb{K}$-vector subspace of $X^{\times E}$ formed by all elements x for which there exists $z \in X$ with $x_e = (e + J) \triangleright z - z \triangleleft (e + J)$ for any $e \in E$.

Lemma 4.14. *For any $p \in \mathbb{K}\langle E\rangle$ and any $z \in X$, if $x \in X^{\times E}$ is such that $x_e = (e + J) \triangleright z - z \triangleleft (e + J)$ for any $e \in E$, then*

$$F^{1,p}_{E,R,X}(x) = (p + J) \triangleright z - z \triangleleft (p + J).$$

In particular, $B^1_{E,R,X}$ is a $\mathbb{K}$-vector subspace of $Z^1_{E,R,X}$.

Proof. The claimed identity is clear if $p = 1$. If there are $m \in \mathbb{N}$ and $\{e_i\}_{i=1}^m \subseteq E$ with $p = e_1 \cdots e_m$, then by definition,

$$\begin{aligned}
F^{1,p}_{E,R,X}(x) &= \sum_{i=1}^{m} \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright x_{e_i} \triangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\right) \\
&= \sum_{i=1}^{m} \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright ((e_i + J) \triangleright z - z \triangleleft (e_i + J)) \triangleleft \left(\overrightarrow{\prod}_{j=i+1}^{m} e_j + J\right) \\
&= \left(\sum_{i=2}^{m+1} \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright z \triangleleft \left(\overrightarrow{\prod}_{j=i}^{m} e_j + J\right)\right) \\
&\quad - \left(\sum_{i=1}^{m} \left(\overrightarrow{\prod}_{j=1}^{i-1} e_j + J\right) \triangleright z \triangleleft \left(\overrightarrow{\prod}_{j=i}^{m} e_j + J\right)\right)
\end{aligned}$$

$$= \left(\overrightarrow{\prod}_{i=1}^{m} e_i + J\right) \triangleright z - z \triangleleft \left(\overrightarrow{\prod}_{i=1}^{m} e_i + J\right).$$

Thus, the identity holds for arbitrary $p \in \mathbb{K}\langle E\rangle$ by $\mathbb{K}$-linearity. It follows in particular that, if $p \in R$, then $F^{1,p}_{E,R,X}(x) = 0$ since $p + J$ is the zero vector of $\mathbb{K}\langle E \mid R\rangle$ in this case. That proves the claim about $B^1_{E,R,X}$. ■

In particular, the preceding lemma enables us to consider the following space.

Definition 4.15. Let $H^1_{E,R,X}$ be the $\mathbb{K}$-vector quotient space of $Z^1_{E,R,X}$ with respect to $B^1_{E,R,X}$.

The following notation allows referencing easily a multitude of algebra morphisms whose existence is implied by the universal property of $\mathbb{K}\langle E\rangle$.

Notation 4.16. Let B be any $\mathbb{K}$-algebra and let $(b_e)_{e\in E} \in B^{\times E}$ be arbitrary. The *evaluation* of p at $(b_e)_{e\in E}$ in B is given by $p((b_e)_{e\in E}) := g(p)$, where g is the unique $\mathbb{K}$-algebra morphism $\mathbb{K}\langle E\rangle \to B$ with $e \mapsto b_e$ for any $e \in E$.

Lemma 4.17. *For any $p \in \mathbb{K}\langle E\rangle$ and any $x \in X^{\times E}$, evaluating p at $(e+J, x_e)_{e\in E}$ in the algebra $\mathbb{K}\langle E \mid R\rangle \oplus^1_0 X$ yields*

$$p((e + J, x_e)_{e\in E}) = (p + J, F^{1,p}_{E,R,X}(x)).$$

Proof. Because $F^{1,1}_{E,R,X}(x) = 0$ by definition, the claim is true if $p = 1$. If there exist $m \in \mathbb{N}$ and $\{e_i\}_{i=1}^m \subseteq E$ with $p = e_1 \cdots e_m$, then the claim follows immediately from Lemma 4.9 and Definition 4.11. For arbitrary p, the assertion therefore holds by $\mathbb{K}$-linearity. ■

Remark 4.18. Given any $\mathbb{K}$-algebra B, any $\mathbb{K}$-algebra morphism $f\colon \mathbb{K}\langle E \mid R\rangle \to B$ and any $p \in \mathbb{K}\langle E\rangle$,

$$f(p + J) = p((f(e + J))_{e\in E}),$$

where the right-hand side is an evaluation of p in B.

Indeed, if g is the unique $\mathbb{K}$-algebra morphism $\mathbb{K}\langle E\rangle \to B$ with $e \mapsto b_e := f(e + J)$ for any $e \in E$, then $f(p + J) = g(p) = p((b_e)_{e\in E}) = p((f(e + J))_{e\in E})$, where the first identity holds by the uniqueness of g and where the second is nothing but an application of Notation 4.16.

Now we can give a useful characterization of the spaces of 1-cocycles of universal algebras.

Proposition 4.19.

(a) *A commutative diagram of $\mathbb{K}$-linear maps is given by*

$$\begin{array}{ccc} Z^1_{\mathrm{HS}}(\mathbb{K}\langle E \mid R\rangle, X) & \hookrightarrow\!\!\!\!\rightarrow & Z^1_{E,R,X} \\ \subseteq\uparrow & & \uparrow\subseteq \\ B^1_{\mathrm{HS}}(\mathbb{K}\langle E \mid R\rangle, X) & \hookrightarrow\!\!\!\!\rightarrow & B^1_{E,R,X}, \end{array}$$

where the horizontal arrows both assign to any element η of their respective domains the tuple

$$(\eta(e + J))_{e\in E}.$$

Moreover, the horizontal arrows are both $\mathbb{K}$-linear isomorphisms. Their respective inverses both assign to any element x of their respective domains the mapping $\mathbb{K}\langle E \mid R\rangle \to X$ with

$$p + J \mapsto F^{1,p}_{E,R,X}(x)$$

for any $p \in \mathbb{K}\langle E\rangle$.

(b) *There exists an isomorphism of* $\mathbb{K}$*-vector spaces*

$$H^1_{\mathrm{HS}}(\mathbb{K}\langle E \mid R\rangle, X) \lhook\joinrel\twoheadrightarrow H^1_{E,R,X}$$

such that the class of any $\eta \in Z^1_{\mathrm{HS}}(\mathbb{K}\langle E \mid R\rangle, X)$ *is sent to the class of the* $x \in Z^1_{E,R,X}$ *with* $x_e = \eta(e+J)$ *for any* $e \in E$. *The inverse isomorphism sends the class of any* $x \in Z^1_{E,R,X}$ *to the class of the* $\eta \in Z^1_{\mathrm{HS}}(\mathbb{K}\langle E \mid R\rangle, X)$ *with* $\eta(p+J) = F^{1,p}_{E,R,X}(x)$ *for any* $p \in \mathbb{K}\langle E\rangle$.

Proof. (a) Abbreviate $A := \mathbb{K}\langle E \mid R\rangle$ and $B := A \oplus^1_0 X$.

Step 1. Upper horizontal arrow is well defined. (*And a bit more.*) First, we prove that for any $\eta \in Z^1_{\mathrm{HS}}(A, X)$, if $x_e := \eta(e+J)$ for any $e \in E$, then $\eta(p+J) = F^{1,p}_{E,R,X}(x)$ for any $p \in \mathbb{K}\langle E\rangle$. That then in particular shows that $F^{1,r}_{E,R,X}(x) = 0$ for any $r \in R$.

By $\eta \in Z^1_{\mathrm{HS}}(A, X)$, according to Lemma 4.8, the rule that $a \mapsto (a, \eta(a))$ for any $a \in A$ defines a $\mathbb{K}$-algebra homomorphism $f\colon A \to B$. Hence, for any $p \in \mathbb{K}\langle E\rangle$ it must hold that $(p+J, \eta(p+J)) = f(p+J) = p((f(e+J))_{e\in E}) = p((e+J, \eta(e+J))_{e\in E}) = p((e+J, x_e)_{e\in E}) = (p+J, F^{1,p}_{E,R,X}(x))$ in B, where the second and last identities are implied by Remark 4.18 and Lemma 4.17, respectively. Hence, $F^{1,p}_{E,R,X}(x) = \eta(p+J)$ for any $p \in \mathbb{K}\langle E\rangle$, as claimed.

Step 2. Alleged inverse upper horizontal arrow well defined. Next, we show that for any $x \in X^{\times E}$ with $F^{1,r}_{E,R,X}(x) = 0$ for any $r \in R$ there exists $\eta \in Z^1_{\mathrm{HS}}(A, X)$ with $\eta(p+J) = F^{1,p}_{E,R,X}(x)$ for any $p \in \mathbb{K}\langle E\rangle$. One consequence of this is then that the alleged inverse upper horizontal arrow is well defined.

For any $r \in R$ because $F^{1,r}_{E,R,X}(x) = 0$ and $r \in J$ we can infer by Lemma 4.17 that $r((e+J, x_e)_{e\in E}) = (J, 0)$ in B. The universal property of A therefore guarantees the existence of a unique $\mathbb{K}$-algebra homomorphism $f\colon A \to B$ with $f(e+J) = (e+J, x_e)$ for any $e \in E$. More generally, for any $p \in \mathbb{K}\langle E\rangle$ it must hold that $f(p+J) = p((f(e+J))_{e\in E}) = (p+J, F^{1,p}_{E,R,X}(x))$, where the two identities are again due to Remark 4.18 and Lemma 4.17. In other words, if $\eta(p+J) := F^{1,p}_{E,R,X}(x)$ for any $p \in \mathbb{K}\langle E\rangle$, then the rule that $a \mapsto (a, \eta(a))$ for any $a \in A$ defines a $\mathbb{K}$-algebra homomorphism $A \to B$, namely f. According to Lemma 4.8, that demands $\eta \in Z^1_{\mathrm{HS}}(A, X)$. Hence, the initial claim is true.

Step 3. Upper horizontal arrow has alleged inverse. It suffices to prove that for any $\eta \in Z^1_{\mathrm{HS}}(A, X)$ and any $x \in X^{\times E}$ with $F^{1,r}_{E,R,X}(x) = 0$ for any $r \in R$ the statements that $x_e = \eta(e+J)$ for any $e \in E$ and that $\eta(p+J) = F^{1,p}_{E,R,X}(x)$ for any $p \in \mathbb{K}\langle E\rangle$ are equivalent. Clearly, the second implies the first by the fact that $F^{1,e}_{E,R,X}(x) = x_e$ for any $e \in E$ by definition. And that the other implication holds was shown in Step 1.

Step 4. Vertical arrows well defined. That the left vertical arrow is well defined is clear. That the same is true for the right vertical arrow was shown in Lemma 4.14.

Step 5. Lower horizontal arrow and its inverse. From the definition of the lower horizontal arrow and that of $B^1_{E,R,X}$, it is clear that the lower horizontal arrow is well defined. Conversely, the inverse of the upper horizontal arrow restricts to the inverse of the lower horizontal arrow. That is because for any $x \in X^{\times E}$ with $F^{1,r}_{E,R,X}(x) = 0$ for any $r \in R$, for the unique $\eta \in Z^1_{\mathrm{HS}}(A, X)$ with $\eta(p+J) = F^{1,p}_{E,R,X}(x)$ for any $p \in \mathbb{K}\langle E\rangle$ and for any $z \in X$, if $x_e = (e+J) \triangleright z - z \triangleleft (e+J)$ for any $e \in E$, then $\eta(p+J) = F^{1,p}_{E,R,X}(x) = (p+J) \triangleright z - z \triangleleft (p+J) = (\partial^0 z)(p+J)$ by Lemma 4.14.

Step 6. Commutativity of the diagram. Because the two horizontal arrows are defined by the same rule and since the vertical arrows are set inclusions the diagram commutes.

(b) Follows directly from *(a)* and is only stated for emphasis. ■

Example 4.20. Let X be the trivial bimodule with respect to an augmentation ϵ of $\mathbb{K}\langle E \mid R\rangle$ as in Example 4.2. Because then $B^1_{\mathrm{HS}}(\mathbb{K}\langle E \mid R\rangle, X) = \{0\}$ by Example 4.5 what Proposition 4.19 implies is that also $B^1_{E,R,X} = \{0\}$ and that therefore $H^1_{\mathrm{HS}}(\mathbb{K}\langle E \mid R\rangle, X) \cong Z^1_{E,R,X}$.

Remark 4.21. If V is the $\mathbb{K}$-vector space underlying the $\mathbb{K}\langle E \mid R\rangle$-bimodule X, then the rules $p \blacktriangleright v := (p+J) \triangleright v$ and $v \blacktriangleleft p := v \triangleleft (p+J)$ for any $p \in \mathbb{K}\langle E\rangle$ and $v \in V$ define left respectively right $\mathbb{K}\langle E\rangle$-actions $\blacktriangleright$ and $\blacktriangleleft$ on V which turn it into a $\mathbb{K}\langle E\rangle$-bimodule Y, the *restriction of scalars* of X along the canonical projection $\mathbb{K}\langle E\rangle \to \mathbb{K}\langle E \mid R\rangle$. With this definition there is no difference between the linear maps $F^1_{E,R,X}$ and $F^1_{E,\varnothing,Y}$. (But, of course, there is in general still a difference between $Z^1_{E,\varnothing,Y} = V^{\times E}$ and $Z^1_{E,R,X} = \{x \in V^{\times E} \wedge \forall r \in R\colon F^{1,r}_{E,\varnothing,Y}(x) = 0\}$ and likewise between $B^1_{E,\varnothing,Y}$ and $B^1_{E,R,X}$.) The advantage of the notation $F^1_{E,R,X}$ is that one can work immediately with the given bimodule X and does not have to introduce Y first. Then again, talking about $F^1_{E,\varnothing,Y}$ can be advantageous too, e.g., in the instance of considering simultaneously multiple different R and thus multiple different X for which though the restrictions of scalars Y all happen to be the same.

5 Certain spaces of scalar matrices and their dimensions

The vector spaces of matrices appearing in the main result are characterized and their dimensions are computed. Recall that for any $n \in \mathbb{N}$ any $v \in M_n(\mathbb{C})$ is called *skew-symmetric* if $v = -v^{\mathrm{t}}$.

Definition 5.1. We call any $v \in M_n(\mathbb{C})$ *small* if $\sum_{i=1}^n v_{j,i} = 0$ for any $j \in [\![n]\!]$ and $\sum_{j=1}^n v_{j,i} = 0$ for any $i \in [\![n]\!]$, i.e., if each row and each column sums to zero.

Lemma 5.2. *For any $n \in \mathbb{N}$ and $v \in M_n(\mathbb{C})$ the following equivalences hold.*

(a) *There is $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small if and only if $\sum_{s=1}^n v_{j,s} - \sum_{s=1}^n v_{s,i} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$. Moreover, then $\lambda = \sum_{s=1}^n v_{j,s} = \sum_{s=1}^n v_{s,i}$ for any $\{i,j\} \subseteq [\![n]\!]$.*

(b) *There is $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is skew-symmetric if and only if for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$ both $v_{j,i} + v_{i,j} = 0$ and $v_{j,j} - v_{i,i} = 0$. Moreover, then $\lambda = v_{i,i}$ for any $i \in [\![n]\!]$.*

(c) *There are $\{\lambda_1, \lambda_2\} \subseteq \mathbb{C}$ such that $v - \lambda_1 I$ is skew-symmetric and $v - \lambda_2 I$ small if and only if there is $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is both skew-symmetric and small. Moreover, then $\lambda = \lambda_1 = \lambda_2$.*

Proof. For $n = 1$, all claims hold trivially. Hence, suppose $2 \leq n$ in the following.

(a) If $\lambda \in \mathbb{C}$ is such that $w := v - \lambda I$ is small, then for any $\{i,j\} \subseteq [\![n]\!]$ it follows $0 = \sum_{s=1}^n w_{j,s} = \sum_{s=1}^n (v_{j,s} - \lambda\delta_{j,s}) = \sum_{s=1}^n v_{j,s} - \lambda$ and $0 = \sum_{s=1}^n w_{s,i} = \sum_{s=1}^n (v_{s,i} - \lambda\delta_{s,i}) = \sum_{s=1}^n v_{s,i} - \lambda$, which proves $\sum_{s=1}^n v_{j,s} = \lambda = \sum_{s=1}^n v_{s,i}$. Of course, then $\sum_{s=1}^n v_{j,s} - \sum_{s=1}^n v_{s,i} = \lambda - \lambda = 0$ for any $\{i,j\} \subseteq [\![n]\!]$.

Conversely, if $\sum_{s=1}^n v_{j,s} - \sum_{s=1}^n v_{s,i} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$ and if we let $\lambda := \sum_{s=1}^n v_{1,s}$ and $w := v - \lambda I$, then for any $\{i,j\} \subseteq [\![n]\!]$, first, $\lambda = \sum_{s=1}^n v_{j,s} = \sum_{s=1}^n v_{s,i}$ and thus, second, $\sum_{s=1}^n w_{j,s} = \sum_{s=1}^n (v_{j,s} - \lambda\delta_{j,s}) = \sum_{s=1}^n v_{j,s} - \lambda = 0$ and, likewise, $\sum_{s=1}^n w_{s,i} = \sum_{s=1}^n (v_{s,i} - \lambda\delta_{s,i}) = \sum_{s=1}^n v_{s,i} - \lambda = 0$. Hence, w is small then.

(b) If for $\lambda \in \mathbb{C}$ the matrix $w := v - \lambda I$ is skew-symmetric, then $0 = w_{j,i} + w_{i,j} = (v_{j,i} - \lambda\delta_{j,i}) + (v_{i,j} - \lambda\delta_{i,j}) = v_{j,i} + v_{i,j} - 2\lambda\delta_{j,i}$ for any $\{i,j\} \subseteq [\![n]\!]$. Consequently, if $i \neq j$, this means $0 = v_{j,i} + v_{i,j}$ and, if $i = j$, we find $0 = 2v_{i,i} - 2\lambda$, i.e., $\lambda = v_{i,i}$. And that implies in particular $v_{j,j} - v_{i,i} = \lambda - \lambda = 0$ for any $\{i,j\} \subseteq [\![n]\!]$.

If, conversely, $v_{j,i} + v_{i,j} = 0$ and $v_{j,j} - v_{i,i} = 0$ for any $\{i,j\} \subseteq [\![n]\!]$ with $i \neq j$ and if we let $\lambda := v_{1,1}$ and $w := v - \lambda I$, then, on the one hand, $\lambda = v_{i,i}$ for any $i \in [\![n]\!]$ and, on the other hand, for any $\{i,j\} \subseteq [\![n]\!]$, generally, $w_{j,i} + w_{i,j} = (v_{j,i} - \lambda\delta_{j,i}) + (v_{i,j} - \lambda\delta_{i,j}) = v_{j,i} + v_{i,j} - 2\lambda\delta_{j,i}$, which in case $i \neq j$ simply means $w_{j,i} + w_{i,j} = v_{j,i} + v_{i,j} = 0$ and which for $i = j$ amounts to $w_{j,i} + w_{i,j} = 2v_{i,i} - 2\lambda = 2\lambda - 2\lambda = 0$. In conclusion, w is skew-symmetric then.

(c) One implication is clear. If, conversely, $\{\lambda_1, \lambda_2\} \subseteq \mathbb{C}$ are such that $v - \lambda_1 I$ is skew-symmetric and $v - \lambda_2 I$ is small, then $\lambda_1 = v_{1,1}$ by (b) and $\lambda_2 = \sum_{j=1}^n v_{j,1} = \sum_{i=1}^n v_{1,j}$ by (a). Subtracting the two identities $\sum_{j=1}^n v_{j,1} = \lambda_1 + \sum_{j=2}^n v_{j,1}$ and $\sum_{i=1}^n v_{1,i} = \lambda_1 + \sum_{i=2}^n v_{1,i}$ from each other therefore yields $0 = \sum_{j=2}^n v_{j,1} - \sum_{i=2}^n v_{1,i}$. Since also $v_{i,1} = -v_{1,i}$ for each $i \in [\![n]\!]$

with $1 < i$ by (b), that is the same as saying $0 = 2\sum_{j=2}^{n} v_{j,1}$. And $\sum_{j=2}^{n} v_{j,1} = 0$ then implies $\lambda_2 = \lambda_1 + \sum_{j=2}^{n} v_{j,1} = \lambda_1$, which is all we needed to see. ■

Lemma 5.3. *For any $n \in \mathbb{N}$ and each statement A below, the set $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is a complex vector subspace of $M_n(\mathbb{C})$ and has the listed dimension.*

	$A(v)$	$\dim_{\mathbb{C}}\{v \in M_n(\mathbb{C}) \wedge A(v)\}$
(a)	$\top$	n^2
(b)	$\exists\lambda \in \mathbb{C}\colon v - \lambda I$ is small	$(n-1)^2 + 1$
(c)	v is small	$(n-1)^2$
(d)	$\exists\lambda \in \mathbb{C}\colon v - \lambda I$ is skew-symmetric	$\frac{1}{2}n(n-1) + 1$
(e)	v is skew-symmetric	$\frac{1}{2}n(n-1)$
(f)	$\exists\lambda \in \mathbb{C}\colon v - \lambda I$ is skew-symmetric and small	$\frac{1}{2}(n-1)(n-2) + 1$
(g)	v is skew-symmetric and small	$\frac{1}{2}(n-1)(n-2)$
(h)	v is diagonal	n
(i)	$\exists\lambda \in \mathbb{C}\colon v - \lambda I = 0$	1
(j)	$v = 0$	0

Proof. (a) It is well known that, if for any $\{k, \ell\} \subseteq [\![n]\!]$ the matrix $E^n_{\ell,k} \in M_n(\mathbb{C})$ has $\delta_{\ell,j}\delta_{k,i}$ as its (j,i)-entry for any $\{i,j\} \subseteq [\![n]\!]$, then the family $(E^n_{\ell,k})_{(\ell,k)\in[\![n]\!]^{\times 2}}$ is a $\mathbb{C}$-linear basis of $\{v \in M_n(\mathbb{C}) \wedge A(v)\} = M_n(\mathbb{C})$.

(b) Since A can be expressed by a homogenous system of linear equations by Lemma 5.2 (a) the set $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is indeed a vector space. Hence, it suffices to show that the mapping $\varphi_n\colon M_{n-1}(\mathbb{C}) \oplus \mathbb{C} \to \{v \in M_n(\mathbb{C}) \wedge A(v)\}$ defined by the rule that $(u, \lambda) \mapsto v$, where for any $\{i, j\} \subseteq [\![n]\!]$,

$$v_{j,i} = \begin{cases} u_{j,i} + \lambda\delta_{j,i}, & j < n \wedge i < n, \\ -\sum\limits_{\ell=1}^{n-1} u_{\ell,i}, & j = n \wedge i < n, \\ -\sum\limits_{k=1}^{n-1} u_{j,k}, & j < n \wedge i = n, \\ \sum\limits_{k,\ell=1}^{n-1} u_{\ell,k} + \lambda, & j = n \wedge i = n, \end{cases}$$

for any $u \in M_{n-1}(\mathbb{C})$ and $\lambda \in \mathbb{C}$, is a $\mathbb{C}$-linear isomorphism. We begin by proving that φ_n is well defined. For any $u \in M_{n-1}(\mathbb{C})$ and $\lambda \in \mathbb{C}$, if $\varphi_n(u, \lambda) = v$ and if $w = v - \lambda I$, then, on the one hand, for any $i \in [\![n]\!]$ with $i < n$, by definition,

$$\sum_{j=1}^{n} w_{j,i} = \sum_{j=1}^{n-1}(v_{j,i} - \lambda\delta_{j,i}) + v_{n,i} = \sum_{j=1}^{n-1} u_{j,i} + \left(-\sum_{\ell=1}^{n-1} u_{\ell,i}\right) = 0.$$

and also,

$$\sum_{j=1}^{n} w_{j,n} = \sum_{j=1}^{n-1}(v_{j,n} - \lambda\delta_{j,n}) + (v_{n,n} - \lambda) = \sum_{j=1}^{n-1}\left(-\sum_{k=1}^{n-1} u_{j,k}\right) + \sum_{k,\ell=1}^{n-1} u_{\ell,k} = 0.$$

On the other hand, for any $j \in [\![n]\!]$ with $j < n$,

$$\sum_{i=1}^{n} w_{j,i} = \sum_{i=1}^{n-1}(v_{j,i} - \lambda\delta_{j,i}) + v_{j,n} = \sum_{i=1}^{n-1} u_{j,i} + \left(-\sum_{k=1}^{n-1} u_{j,k}\right) = 0$$

and also,

$$\sum_{i=1}^{n} w_{n,i} = \sum_{i=1}^{n-1}(v_{n,i} - \lambda\delta_{n,i}) + (v_{n,n} - \lambda) = \sum_{i=1}^{n-1}\left(-\sum_{\ell=1}^{n-1} u_{\ell,i}\right) + \sum_{k,\ell=1}^{n-1} u_{\ell,k} = 0.$$

Together these four conclusions prove that w is small, i.e., that $A(v)$ holds.

Conversely, by Lemma 5.2 (a) a well-defined $\mathbb{C}$-linear map $\psi_n\colon \{v \in M_n(\mathbb{C}) \wedge A(v)\} \to M_{n-1}(\mathbb{C}) \oplus \mathbb{C}$ is obtained as follows: For any $v \in M_n(\mathbb{C})$ with $A(v)$, if $\lambda \in \mathbb{C}$ is such that $v - \lambda I$ is small, then $v \mapsto (u, \lambda)$, where for any $\{k, \ell\} \subseteq [\![n-1]\!]$,

$$u_{\ell,k} = v_{\ell,k} - \lambda\delta_{\ell,k}.$$

It remains to show $\psi_n \circ \varphi_n = \mathrm{id}$ and $\varphi_n \circ \psi_n = \mathrm{id}$. And, indeed, for any $u \in M_{n-1}(\mathbb{C})$ and $\lambda \in \mathbb{C}$, if $v = \varphi_n(u, \lambda)$, then we have already seen that $w = v - \lambda I$ is small. For any $\{k, \ell\} \subseteq [\![n]\!]$, by definition, $w_{\ell,k} = v_{\ell,k} - \lambda\delta_{\ell,k} = (u_{\ell,k} + \lambda\delta_{\ell,k}) - \lambda\delta_{\ell,k} = u_{\ell,k}$, which proves $\varphi_n(v) = (u, \lambda)$ and thus $\psi_n \circ \varphi_n = \mathrm{id}$.

Conversely, for any $v \in M_n(\mathbb{C})$ such that $A(v)$ is satisfied, if $(u, \lambda) = \psi_n(v)$, then we already know $\lambda = \sum_{\ell=1}^{n} v_{\ell,i} = \sum_{k=1}^{n} v_{j,k}$ for any $\{k, \ell\} \subseteq [\![n]\!]$ by Lemma 5.2 (a). If $v' = \varphi_n(u, \lambda)$, then for any $\{i, j\} \subseteq [\![n]\!]$ with $i < n$ and $j < n$ it hence follows by definition $v'_{j,i} = u_{j,i} + \lambda\delta_{j,i} = (v_{j,i} - \lambda\delta_{j,i}) + \lambda\delta_{j,i} = v_{j,i}$ as well as by $\lambda = \sum_{\ell=1}^{n} v_{\ell,i}$,

$$v'_{n,i} = -\sum_{\ell=1}^{n-1} u_{\ell,i} = -\sum_{\ell=1}^{n-1}(v_{\ell,i} - \lambda\delta_{\ell,i}) = \lambda - \sum_{\ell=1}^{n-1} v_{\ell,i} = v_{n,i}$$

and by $\lambda = \sum_{k=1}^{n} v_{k,j}$,

$$v'_{j,n} = -\sum_{k=1}^{n-1} u_{j,k} = -\sum_{k=1}^{n-1}(v_{j,k} - \lambda\delta_{j,k}) = \lambda - \sum_{k=1}^{n-1} v_{j,k} = v_{j,n}$$

and, lastly,

$$\begin{aligned} v'_{n,n} &= \sum_{k,\ell=1}^{n-1} u_{\ell,k} + \lambda = \sum_{k,\ell=1}^{n-1} (v_{\ell,k} - \lambda\delta_{\ell,k}) + \lambda = \sum_{\ell=1}^{n-1}\left(\sum_{k=1}^{n-1} v_{\ell,k} - \lambda\right) + \lambda \\ &= \sum_{\ell=1}^{n-1}(-v_{\ell,n}) + \lambda = v_{n,n}, \end{aligned}$$

where we have used $\lambda = \sum_{k=1}^{n} v_{\ell,k}$ for any $\ell \in [\![n]\!]$ in the next-to-last step and $\lambda = \sum_{\ell=1}^{n} v_{\ell,n}$ in the last. Thus, we have shown $v' = v$ and thus $\varphi_n \circ \psi_n = \mathrm{id}$, which concludes the proof in this case.

(c) By Lemma 5.2 (a), the space $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is exactly the image of $M_{n-1}(\mathbb{C}) \oplus \{0\}$ under φ_n.

(d) Lemma 5.2 (b) showed that A can be equivalently expressed as a system of homogenous linear equations, thus proving $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ to be a vector space. Let $\Gamma_n = \{(j,i) \mid \{i,j\} \subseteq [\![n]\!] \wedge j < i\} \cup \{\varnothing\}$ as well as $B^n_{(j,i)} = T^n_{j,i} = E^n_{j,i} - E^n_{i,j}$ for any $\{i,j\} \subseteq [\![n]\!]$ with $j < i$

and $B^n_\varnothing = I$. Then, the claim will be verified once we show that $(B^n_\gamma)_{\gamma\in\Gamma_n}$ is a $\mathbb{C}$-linear basis of $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$.

The family $(B^n_\gamma)_{\gamma\in\Gamma_n}$ is $\mathbb{C}$-linearly independent. Indeed, if $(a_\gamma)_{\gamma\in\Gamma_n} \in \mathbb{C}^{\oplus\Gamma_n}$ is such that $\sum_{\gamma\in\Gamma_n} a_\gamma B^n_\gamma = 0$, then by $I = \sum_{i=1}^n E^n_{i,i}$,

$$0 = \sum_{\substack{(j,i)\in[\![n]\!]^{\times 2}\\ \wedge j<i}} a_{(j,i)}(E^n_{j,i} - E^n_{i,j}) + a_\varnothing \sum_{i=1}^n E^n_{i,i} = \sum_{(j,i)\in[\![n]\!]^{\times 2}} \begin{Bmatrix} a_{(j,i)}, & j<i \\ -a_{(i,j)}, & i<j \\ a_\varnothing, & j=i \end{Bmatrix} E^n_{j,i},$$

which demands $(a_\gamma)_{\gamma\in\Gamma_n} = 0$ since $(E^n_{\ell,k})_{(\ell,k)\in[\![n]\!]^{\times 2}}$ is $\mathbb{C}$-linearly independent.

It remains to prove that $\{B^n_\gamma \mid \gamma \in \Gamma_n\}$ spans $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$. If $v \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$ are such that $w = v - \lambda I$ is skew-symmetric, then $v_{j,i} = -v_{i,j}$ and $\lambda = v_{j,j} = v_{i,i}$ for any $\{i,j\} \subseteq [\![n]\!]$ with $j \neq i$ by Lemma 5.2 (*b*). Hence, if we let $a_\varnothing = \lambda$ and $a_{(j,i)} = w_{j,i} = v_{j,i}$ for any $\{i,j\} \subseteq [\![n]\!]$ with $j < i$, then

$$\sum_{\gamma\in\Gamma_n} a_\gamma B^n_\gamma = \sum_{(j,i)\in[\![n]\!]^{\times 2}} \begin{Bmatrix} a_{(j,i)}, & j<i \\ -a_{(i,j)}, & i<j \\ a_\varnothing, & j=i \end{Bmatrix} E^n_{j,i} = \sum_{(j,i)\in[\![n]\!]^{\times 2}} \begin{Bmatrix} v_{j,i}, & j<i \\ -v_{i,j}, & i<j \\ \lambda, & j=i \end{Bmatrix} E^n_{j,i} = v.$$

Thus, $(B^n_\gamma)_{\gamma\in\Gamma_n}$ is a $\mathbb{C}$-linear basis.

(e) The proof of the previous claim shows that any $v \in M_n(\mathbb{C})$ is skew-symmetric if and only if it is in the span of $\{B^n_\gamma \mid \gamma \in \Gamma_n\}$ and has coefficient 0 with respect to $B^n_\varnothing$. Hence, $\{T^n_{j,i} \mid \{i,j\} \subseteq [\![n]\!] \wedge j < i\}$ is a $\mathbb{C}$-linear basis of $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$.

(f) All three parts (*a*)–(*c*) of Lemma 5.2 combined imply that $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is the solution set to a homogenous system of linear equations and thus a vector space. Hence, it suffices to prove that φ_n restricts to a mapping $\{u \in M_{n-1}(\mathbb{C}) \wedge u = -u^{\mathrm{t}}\} \to \{v \in M_n(\mathbb{C}) \wedge A(v)\}$ and ψ_n to one in the reverse direction.

For any skew-symmetric $u \in M_{n-1}(\mathbb{C})$ and any $\lambda \in \mathbb{C}$, if $v = \varphi_n(u,\lambda)$ and $w = v - \lambda I$, then for any $\{i,j\} \subseteq [\![n]\!]$ with $i < n$ and $j < n$ we have already seen that $w_{j,i} = u_{j,i}$, implying $w_{j,i} + w_{i,j} = u_{j,i} + u_{i,j} = 0$ by $u = -u^{\mathrm{t}}$. Moreover, for the same reason,

$$\begin{aligned} w_{n,i} + w_{i,n} &= (v_{n,i} - \lambda\delta_{n,i}) + (v_{i,n} - \lambda\delta_{i,n}) = v_{n,i} + v_{i,n} \\ &= \left(-\sum_{\ell=1}^{n-1} u_{\ell,i}\right) + \left(-\sum_{k=1}^{n-1} u_{i,k}\right) = -\sum_{\ell=1}^{n-1}(u_{\ell,i} + u_{i,\ell}) = 0 \end{aligned}$$

and

$$\begin{aligned} w_{j,n} + w_{n,j} &= (v_{j,n} - \lambda\delta_{j,n}) + (v_{n,j} - \lambda\delta_{n,j}) = v_{j,n} + v_{n,j} \\ &= \left(-\sum_{k=1}^{n-1} u_{j,k}\right) + \left(-\sum_{\ell=1}^{n-1} u_{\ell,j}\right) = -\sum_{k=1}^{n-1}(u_{j,k} + u_{k,j}) = 0 \end{aligned}$$

as well as

$$w_{n,n} + w_{n,n} = 2(v_{n,n} - \lambda) = 2\sum_{k,\ell=1}^{n-1} u_{\ell,k} = \sum_{k,\ell=1}^{n-1} (u_{\ell,k} + u_{k,\ell}) = 0,$$

which completes the proof that φ_n restricts to a map into $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$.

Conversely, if $v \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$ are such that $w = v - \lambda I$ is skew-symmetric and small, then $\lambda = v_{i,i}$ for any $i \in [\![n]\!]$ by Lemma 5.2 (*b*). For $(u,\lambda) = \psi_n(v)$ and any $\{k,\ell\} \subseteq [\![n-1]\!]$, by definition, $u_{\ell,k} = w_{\ell,k}$ and thus $u_{\ell,k} + u_{k,\ell} = w_{\ell,k} + w_{k,\ell} = 0$ by $w = -w^{\mathrm{t}}$. Hence, ψ_n maps $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ into $\{u \in M_{n-1}(\mathbb{C}) \wedge u = -u^{\mathrm{t}}\} \oplus \mathbb{C}$.

(g) As we have just shown, any $v \in M_{\mathbb{C}}(n)$ is skew-symmetric and small if and only if it lies in the image of $\{u \in M_{n-1}(\mathbb{C}) \wedge u = -u^{\mathrm{t}}\} \oplus \{0\}$ under φ_n. And φ_n is a $\mathbb{C}$-linear isomorphism from this space to $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$.

(h) It is well known that $(E^n_{i,i})_{i \in [\![n]\!]}$ is a $\mathbb{C}$-linear basis of $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$.

(i) In this case, $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is the $\mathbb{C}$-linear span of I in $M_n(\mathbb{C})$.

(j) Here, $\{v \in M_n(\mathbb{C}) \wedge A(v)\}$ is the zero $\mathbb{C}$-linear space. ■

6 First cohomology of unitary easy quantum group duals

This section computes the first quantum group cohomology with trivial coefficients (see Section 2) of the discrete dual of any unitary easy compact quantum group (see Section 3). That is achieved by applying the characterization of the first Hochschild cohomology recalled in Section 4 while using the results of Section 5 as auxiliaries.

6.1 Equations derived from the presentation

Resume the Assumptions 3.13 and the abbreviations from Notations 3.14 and 3.17. In particular, n and E are then defined. Remark 4.21 motivates moreover the following shorthand.

Notation 6.1.

(*a*) Let Y be the $\mathbb{C}\langle E\rangle$-bimodule $\mathbb{C}$ with left and right actions given by $u^{\mathfrak{c}}_{j,i} \triangleright x := \delta_{j,i} x$ respectively $x \triangleleft u^{\mathfrak{c}}_{j,i} := \delta_{j,i} x$ for any $\{i,j\} \subseteq [\![n]\!]$, any $\mathfrak{c} \in \{\circ, \bullet\}$ and any $x \in \mathbb{C}$.

(*b*) Let $F_p := F^{1,p}_{E,\varnothing,Y}$ for any $p \in \mathbb{C}\langle E\rangle$.

Then, by Section 4 for any category $\mathcal{C}$ of two-colored partitions the first cohomology with trivial coefficients of the discrete dual of any easy quantum group associated with $(\mathcal{C}, n)$ can be realized as a solution space to a system of linear equations involving maps of the form F_r for certain $r \in \mathbb{C}\langle E\rangle$ induced by $\mathcal{C}$ and n.

Proposition 6.2. *For any category $\mathcal{C}$ of two-colored partitions, if G is the unitary easy compact quantum group of $(\mathcal{C}, n)$, then there exists an isomorphism of $\mathbb{C}$-vector spaces*

$$H^1(\widehat{G}) \longrightarrow \{x \in \mathbb{C}^{\times E} \wedge \forall r \in R_{\mathcal{C}} \colon F_r(x) = 0\},$$

which maps (the one-elemental cohomology class of) any 1-cycle η to the tuple x with $x_e = \eta(e + J_{\mathcal{C}})$ for any $e \in E$.

Proof. By Definition 3.18, the algebra underlying the Hopf $*$-algebra $\mathbb{C}[\widehat{G}]$ is the universal algebra $\mathbb{C}\langle E \mid R_{\mathcal{C}}\rangle$. According to Section 2.2, the vector space $H^1(\widehat{G})$ is defined as $H^1_{\mathrm{HS}}(\mathbb{C}\langle E \mid R_{\mathcal{C}}\rangle, X)$ where $X = {}_{\epsilon}\mathbb{C}_{\epsilon}$ is trivial bimodule of $\mathbb{C}\langle E \mid R_{\mathcal{C}}\rangle$ with respect to the counit ϵ of $\mathbb{C}[\widehat{G}]$. By Proposition 3.20, this counit is such that its restriction of scalars along the canonical projection $\mathbb{C}\langle E\rangle \to \mathbb{C}\langle E \mid R_{\mathcal{C}}\rangle$ is precisely Y. Hence, the claim follows by Example 4.20 and Remark 4.21. ■

The task laid out by Proposition 6.2 is clear. We need to solve the set of linear equations in $\mathbb{C}^{\times E}$ on the right-hand side of the isomorphism there – for each category of two-colored partitions. Eventually, in Section 6.6 namely, solving these equations will require case distinctions for different kinds of categories of two-colored partitions. However, there are a great number of simplifications we can make to the equation system before it needs to come to that. Moreover, this reduces the number of cases we eventually have to consider immensely.

6.2 Simplifying each individual equation

As a first step towards solving the equations of Proposition 6.2 we consider each equation in isolation and simplify its definition. In other words, we seek a better formula for the values of the functional F_r for r of the form $\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}$ for arbitrary two-colored partitions $(\mathfrak{c},\mathfrak{d},p)$ and $g\in[\![n]\!]^{\times|\mathfrak{c}|}$ and $j\in[\![n]\!]^{\times|\mathfrak{d}|}$.

It will be convenient to have a shorthand for mappings constructed by prescribing a specified value to a specified point and otherwise inheriting the graph of a given mapping with the same domain.

Notation 6.3. For any $\{k,\ell\}\subseteq\mathbb{N}_0$, any mapping $f\colon\Pi^k_\ell\to[\![n]\!]$, any $\mathtt{z}\in\Pi^k_\ell$ and any $s\in[\![n]\!]$ write $f\downarrow_{\mathtt{z}} s$ for the mapping $\Pi^k_\ell\to[\![n]\!]$ with $\mathtt{z}\mapsto s$ and with $\mathtt{y}\mapsto f(\mathtt{y})$ for any $\mathtt{y}\in\Pi^k_\ell\backslash\{\mathtt{z}\}$.

Then, combining Notation 3.14 and Definition 4.11 yields the following description of the functionals we are investigating.

Lemma 6.4. *For any $\{k,\ell\}\subseteq\mathbb{N}_0$, any $\mathfrak{c}\in\{\circ,\bullet\}^{\times k}$, any $\mathfrak{d}\in\{\circ,\bullet\}^{\times\ell}$, any set-theoretical partition p of Π^k_ℓ, any $g\in[\![n]\!]^{\times k}$, any $j\in[\![n]\!]^{\times\ell}$ and any $x\in\mathbb{C}^{\times E}$, if $r=\mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}$ and $f=g\,{}^{\blacksquare}{}_{\blacksquare}\,j$ and $\mathfrak{w}=\mathfrak{c}\,{}^{\blacksquare}{}_{\blacksquare}\,\mathfrak{d}$, then*

$$F_r(x)=\sum_{\mathtt{z}\in\Pi^k_\ell}\sum_{s=1}^{n}\zeta(p,\ker(f\downarrow_{\mathtt{z}} s))\begin{Bmatrix}-x_{u^{\mathfrak{w}(\mathtt{z})}_{s,f(\mathtt{z})}} & \text{if } \mathtt{z}\in\Pi^k_0\\ x_{u^{\mathfrak{w}(\mathtt{z})}_{f(\mathtt{z}),s}} & \text{if } \mathtt{z}\in\Pi^0_\ell\end{Bmatrix}.$$

Proof. For any $x\in\mathbb{C}^{\times E}$, by Example 4.12,

$$\begin{aligned}F_r(x)=&\sum_{i\in[\![n]\!]^{\times\ell}}\zeta(p,\ker(g\,{}^{\blacksquare}{}_{\blacksquare}\,i))\sum_{b=1}^{\ell}\Bigg(\prod_{\substack{q\in[\![\ell]\!]\\ \wedge\, q\neq b}}\delta_{j_b,i_b}\Bigg)\,x_{u^{\mathfrak{d}_b}_{j_b,i_b}}\\ &-\sum_{h\in[\![n]\!]^{\times k}}\zeta(p,\ker(h\,{}^{\blacksquare}{}_{\blacksquare}\,j))\sum_{a=1}^{k}\Bigg(\prod_{\substack{q\in[\![k]\!]\\ \wedge\, q\neq a}}\delta_{h_a,g_a}\Bigg)\,x_{u^{\mathfrak{c}_a}_{h_a,g_a}}.\end{aligned}$$

After commuting the sums and evaluating the sums over i respectively h (as far as possible), this is identical to

$$\begin{aligned}&\sum_{b=1}^{\ell}\sum_{i_b=1}^{n}\zeta(p,\ker(g\,{}^{\blacksquare}{}_{\blacksquare}\,(j_1,\ldots,j_{b-1},i_b,j_{b+1},\ldots,j_\ell)))\,x_{u^{\mathfrak{d}_b}_{j_b,i_b}}\\ &\quad-\sum_{a=1}^{k}\sum_{h_a=1}^{n}\zeta(p,\ker((g_1,\ldots,g_{a-1},h_a,g_{a+1},\ldots,g_k)\,{}^{\blacksquare}{}_{\blacksquare}\,j))\,x_{u^{\mathfrak{c}_a}_{h_a,g_a}}.\end{aligned}$$

That agrees with the right-hand side of the claimed identity. ■

While Lemma 6.4 has given a more concise form to the equations under investigation, it can be improved upon significantly. Firstly, one can give a simpler criterion for when in the sum on the right-hand side of the identity in Lemma 6.4 a factor $\zeta(p,\ker(f\downarrow_{\mathtt{z}} s))$ is non-zero.

Lemma 6.5. *For any $\{k,\ell\}\subseteq\mathbb{N}_0$, any set-theoretical partition p of Π^k_ℓ, any mapping $f:\Pi^k_\ell\to[\![n]\!]$, any $\mathtt{z}\in\Pi^k_\ell$ and any $s\in[\![n]\!]$, the statements $p\leq\ker(f\downarrow_{\mathtt{z}} s)$ and*

$$p\backslash\{\pi_p(\mathtt{z})\}\cup\{\pi_p(\mathtt{z})\backslash\{\mathtt{z}\},\{\mathtt{z}\}\}\backslash\{\varnothing\}\leq\ker(f)\quad\wedge\quad\pi_p(\mathtt{z})\backslash\{\mathtt{z}\}\subseteq f^{\leftarrow}(\{s\})$$

are equivalent.

Proof. We show each implication separately. Below, we will use many times the fact that for any $t \in [\![n]\!]$,

$$\begin{aligned}(f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{t\})\backslash\{\mathsf{z}\} &= \{\mathsf{a} \in \Pi_\ell^k \wedge (f \downarrow_{\mathsf{z}} s)(\mathsf{a}) = t \wedge \mathsf{a} \neq \mathsf{z}\}\\ &= \{\mathsf{a} \in \Pi_\ell^k \wedge f(\mathsf{a}) = t \wedge \mathsf{a} \neq \mathsf{z}\}\\ &= f^{\leftarrow}(\{t\})\backslash\{\mathsf{z}\}.\end{aligned}$$

Step 1. First, suppose $p \leq \ker(f \downarrow_{\mathsf{z}} s)$. Then, there exists $t \in \operatorname{ran}(f \downarrow_{\mathsf{z}} s)$ such that $\pi_p(\mathsf{z}) \subseteq (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{t\})$. Because $\mathsf{z} \in \pi_p(\mathsf{z})$ this requires $\mathsf{z} \in (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{t\})$ and thus $t = s$ by $(f \downarrow_{\mathsf{z}} s)(\mathsf{z}) = s$. It follows $\pi_p(\mathsf{z}) \subseteq (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{s\})$ and thus in particular $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{s\})\backslash\{\mathsf{z}\} = f^{\leftarrow}(\{s\})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})$, which is one half of what we had to show.

It is trivially true that $\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{f(\mathsf{z})\}) \in \ker(f)$. We have already seen that $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\}) \in \ker(f)$. For any $\mathsf{B} \in p$ with $\mathsf{B} \neq \pi_p(\mathsf{z})$, i.e., $\mathsf{z} \notin \mathsf{B}$, there exists by assumption $t' \in \operatorname{ran}(f \downarrow_{\mathsf{z}} s)$ with $\mathsf{B} \subseteq (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{t'\})$. We conclude $\mathsf{B} = \mathsf{B}\backslash\{\mathsf{z}\} \subseteq (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{t'\})\backslash\{\mathsf{z}\} = f^{\leftarrow}(\{t'\})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{t'\}) \in \ker(f)$. Thus, the other half of the claim, $p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$, holds as well. That proves one implication.

Step 2. In order to show the converse implication we assume that both $p\backslash\{\pi_p(\mathsf{z})\}\cup\{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$ and $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})$ and then we distinguish two cases.

Case 2.1. If $\{\mathsf{z}\} \in p$ and thus $\pi_p(\mathsf{z}) = \{\mathsf{z}\}$ and $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} = \varnothing$, then the assumption is simply equivalent to the statement $p \leq \ker(f)$. Naturally, $\{\mathsf{z}\} \subseteq (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{s\}) \in \ker(f)$ by $(f \downarrow_{\mathsf{z}} s)(\mathsf{z}) = s$. For any $\mathsf{B} \in p$ with $\mathsf{B} \neq \{\mathsf{z}\}$ there exists by our premise a value $t \in \operatorname{ran}(f)$ with $\mathsf{B} \subseteq f^{\leftarrow}(\{t\})$. Thus, also $\mathsf{B} = \mathsf{B}\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{t\})\backslash\{\mathsf{z}\} = (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{t\})\backslash\{\mathsf{z}\} \in \ker(f \downarrow_{\mathsf{z}} s)$. In conclusion, $p \leq \ker(f \downarrow_{\mathsf{z}} s)$.

Case 2.2. In the instance that $\{\mathsf{z}\} \notin p$ the initial assumption simplifies to the statement $p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\} \leq \ker(f)$ and $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})$. The latter condition implies $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})\backslash\{\mathsf{z}\} = (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{s\})\backslash\{\mathsf{z}\} \subseteq (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{s\})$ and thus by $(f \downarrow_{\mathsf{z}} s)(\mathsf{z}) = s$ also $\pi_p(\mathsf{z}) = \pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \cup \{\mathsf{z}\} \subseteq (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{s\}) \cup \{\mathsf{z}\} \subseteq (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{s\}) \in \ker(f \downarrow_{\mathsf{z}} s)$. On the other hand, for any $\mathsf{B} \in p$ with $\mathsf{B} \neq \pi_p(\mathsf{z})$, which is to say $\mathsf{z} \notin \mathsf{B}$, there exists by assumption $t \in \operatorname{ran}(f)$ with $\mathsf{B} \subseteq f^{\leftarrow}(\{t\})$. It follows $\mathsf{B} = \mathsf{B}\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{t\})\backslash\{\mathsf{z}\} = (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{t\})\backslash\{\mathsf{z}\} \subseteq (f \downarrow_{\mathsf{z}} s)^{\leftarrow}(\{t\}) \in \ker(f \downarrow_{\mathsf{z}} s)$. Hence, altogether, $p \leq \ker(f \downarrow_{\mathsf{z}} s)$, which concludes the proof. ■

Lemma 6.5 can now be used to give a necessary criterion for the right-hand side of the identity in Lemma 6.4 to be non-zero as a whole. Namely, p and f must meet one of three conditions:

(i) The labeling f maps any points belonging to the same element of p to the same value.

(ii) There is an element of size two of p whose elements f maps to different values. Besides that f is as in (i).

(iii) There is an element of p of size three or larger, all but one of whose elements are assigned the same value by f and whose remaining element f sends to a different value. Apart from that, f is as in (i).

Definition 6.6. Let $\{k, \ell\} \subseteq \mathbb{N}_0$, let p be any set-theoretical partition of Π_ℓ^k and let $f\colon \Pi_\ell^k \to [\![n]\!]$. Then, we say that (p, f) is

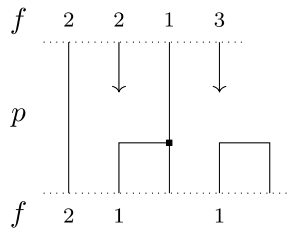
(*a*) *case R1* if $p \neq \varnothing$ and $p \leq \ker(f)$,

(*b*) *case R2* if there exists $\{\mathsf{z}_1, \mathsf{z}_2\} \in p$ such that $f(\mathsf{z}_1) \neq f(\mathsf{z}_2)$ and such that for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \{\mathsf{z}_1, \mathsf{z}_2\}$ there is $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$, in which case the set $\{\mathsf{z}_1, \mathsf{z}_2\}$ is called *critical data* of (p, f),

(*c*) *case R3* if there exist $\mathsf{Z} \in p$ and $\mathsf{z} \in \mathsf{Z}$ and $s \in [\![n]\!]$ such that $3 \leq |\mathsf{Z}|$, such that $f(\mathsf{z}) \neq s$, such that $f(\mathsf{y}) = s$ for any $\mathsf{y} \in \mathsf{Z}$ with $\mathsf{y} \neq \mathsf{z}$ and such that for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \mathsf{Z}$ there is $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$, in which case $(\mathsf{Z}, \mathsf{z}, s)$ are called *critical data* of (p, f),

(*d*) *case R4* otherwise.

Example 6.7. For $3 \leq n$, consider $k := 4$ and $\ell := 5$, the set-theoretical partition $p := \{\{{}_{\blacksquare}1, {}^{\blacksquare}1\}, \{{}^{\blacksquare}2\}, \{{}_{\blacksquare}2, {}_{\blacksquare}3, {}^{\blacksquare}3\}, \{{}^{\blacksquare}4\}, \{{}_{\blacksquare}4, {}_{\blacksquare}5\}\}$ of Π^k_ℓ and various different mappings $f\colon \Pi^k_\ell \to [\![n]\!]$ which all have in common that each of ${}_{\blacksquare}2$, ${}_{\blacksquare}4$ and ${}^{\blacksquare}3$ is mapped to 1, that each of ${}_{\blacksquare}1$, ${}^{\blacksquare}1$ and ${}^{\blacksquare}2$ is mapped to 2 and that ${}^{\blacksquare}4$ is mapped to 3. Thus, at most the values of ${}_{\blacksquare}3$ and ${}_{\blacksquare}5$ differ between different f:



(*a*) If $f({}_{\blacksquare}3) = f({}_{\blacksquare}5) = 1$, then (p, f) is case R1.
(*b*) If $f({}_{\blacksquare}3) = 1$ and $f({}_{\blacksquare}5) = 2$, if $\mathsf{z}_1 := {}_{\blacksquare}4$ and $\mathsf{z}_2 := {}_{\blacksquare}5$, then (p, f) is case R2 with critical data $\{\mathsf{z}_1, \mathsf{z}_2\} = \{{}_{\blacksquare}4, {}_{\blacksquare}5\}$.
(*c*) If $f({}_{\blacksquare}3) = 2$ and $f({}_{\blacksquare}5) = 1$, if $\mathsf{Z} := \{{}_{\blacksquare}2, {}_{\blacksquare}3, {}^{\blacksquare}3\}$, if $\mathsf{z} := {}_{\blacksquare}3$ and $s := 2$, then (p, f) is case R3 with critical data $(\mathsf{Z}, \mathsf{z}, s) = (\{{}_{\blacksquare}2, {}_{\blacksquare}3, {}^{\blacksquare}3\}, \{{}_{\blacksquare}3\}, 2)$.
(*d*) If $f({}_{\blacksquare}3) = f({}_{\blacksquare}5) = 2$, then (p, f) is case R4.

Lemma 6.8.
(*a*) *In each of the cases R2 and R3 critical data are unique.*
(*b*) *The cases R1–R4 are mutually exclusive and exhaustive.*

Proof. Let $\{k, \ell\} \subseteq \mathbb{N}_0$, let p be any set-theoretical partition of Π^k_ℓ and let $f\colon \Pi^k_\ell \to [\![n]\!]$ be arbitrary.

(a) *Case R2.* Suppose that (p, f) is case R2 and that both $\{\mathsf{z}_1, \mathsf{z}_2\}$ and $\{\mathsf{z}'_1, \mathsf{z}'_2\}$ are critical data of (p, f). If $\{\mathsf{z}_1, \mathsf{z}_2\} \neq \{\mathsf{z}'_1, \mathsf{z}'_2\}$ were true, then by the assumption on $\{\mathsf{z}_1, \mathsf{z}_2\}$ there would exist $\mathsf{B} \in \ker(f)$ with $\{\mathsf{z}'_1, \mathsf{z}'_2\} \subseteq \mathsf{B}$, meaning $f(\mathsf{z}'_1) = f(\mathsf{z}'_2)$, contrarily to our assumption. Hence, $\{\mathsf{z}_1, \mathsf{z}_2\} = \{\mathsf{z}'_1, \mathsf{z}'_2\}$ must be true instead.

Case R3. Now, let (p, f) be case R3 and let both $(\mathsf{Z}, \mathsf{z}, s)$ and $(\mathsf{Z}', \mathsf{z}', s')$ be critical data of (p, f). If $\mathsf{Z} \neq \mathsf{Z}'$ held, the assumption on Z would imply the existence of $\mathsf{B} \in \ker(f)$ with $\mathsf{Z}' \subseteq \mathsf{B}$. In particular, it would follow $f(\mathsf{y}') = f(\mathsf{z}')$ for any $\mathsf{y}' \in \mathsf{Z}'$ with $\mathsf{y}' \neq \mathsf{z}'$, of which there exists at least one by $3 \leq |\mathsf{Z}'|$. Because that would contradict the assumption, we must have $\mathsf{Z} = \mathsf{Z}'$ instead.

Furthermore, supposing $\mathsf{z} \neq \mathsf{z}'$ demands of any $\mathsf{y} \in \mathsf{Z}\backslash\{\mathsf{z}, \mathsf{z}'\}$ both $f(\mathsf{y}) = s$ by the assumption on z and s and $f(\mathsf{y}) = s'$ by the one on z' and s'. Hence, as $\mathsf{Z}\backslash\{\mathsf{z}, \mathsf{z}'\} \neq \varnothing$ by $3 \leq |\mathsf{Z}'|$, if $\mathsf{z}' \neq \mathsf{z}$, then $s = s'$. That would be a contradiction because the property of z' also requires $s \neq f(\mathsf{z}) = s'$ in that case. Hence, only $\mathsf{z} = \mathsf{z}'$ can be true.

Lastly, because the assumptions on s and s' imply $f(\mathsf{y}) = s$ respectively $f(\mathsf{y}) = s'$ for any $\mathsf{y} \in \mathsf{Z}$ with $\mathsf{y} \neq \mathsf{z} = \mathsf{z}'$ and because $\mathsf{Z}\backslash\{\mathsf{z}\} \neq \varnothing$, we must have $s = s'$ as well.

(b) It is enough to prove that cases R1–R3 are mutually exclusive. If (p, f) is case R2, then it cannot be case R1 because $f(\mathsf{z}_1) \neq f(\mathsf{z}_2)$ excludes the existence of $\mathsf{B} \in \ker(f)$ with $\{\mathsf{z}_1, \mathsf{z}_2\} \subseteq \mathsf{B}$, which would be necessary for $p \leq \ker(f)$ to hold. Similarly, (p, f) being case R3 forbids it being case R1 as well because the existence of $\mathsf{y} \in \mathsf{Z}\backslash\{\mathsf{z}\} \neq \varnothing$ with $f(\mathsf{z}) \neq s = f(\mathsf{y})$ does not allow any $\mathsf{B} \in \ker(f)$ with $\mathsf{Z} \subseteq \mathsf{B}$ to exist, which $p \leq \ker(f)$ would require. Lastly, if (p, f) were simultaneously case R2 and case R3, then $\{\mathsf{z}_1, \mathsf{z}_2\} \neq \mathsf{Z}$ would follow from $3 \leq |\mathsf{Z}|$, thus demanding by the property of Z the existence of $\mathsf{B} \in \ker(f)$ with $\{\mathsf{z}_1, \mathsf{z}_2\} \subseteq \mathsf{B}$, in contradiction to $f(\mathsf{z}_1) \neq f(\mathsf{z}_2)$. ■

Lemma 6.9. *For any $\{k, \ell\} \subseteq \mathbb{N}_0$, any set-theoretical partition p of Π^k_ℓ and any mapping $f\colon \Pi^k_\ell \to [\![n]\!]$ there exist $\mathsf{z} \in \Pi^k_\ell$ and $s \in [\![n]\!]$ such that $p \leq \ker(f \downarrow_{\mathsf{z}} s)$ if and only if (p, f) is not case R4.*

Proof. Each implication is shown individually.

Step 1. First, we suppose that (p, f) is not case R4 and deduce the existence of $\mathsf{z} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ with $p \leq \ker(f \downarrow_{\mathsf{z}} s)$. By Lemma 6.5, that is the same as finding $\mathsf{z} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ such that $p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$ and $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})$. By Lemma 6.8 (*b*), the pair (p, f) is case R1, case R2 or case R3. These three cases are treated individually.

Case 1.1. If (p, f) is case R1, then by $p \neq \varnothing$ we can find and fix some $\mathsf{z} \in \Pi_\ell^k$ and put $s := f(\mathsf{z})$. From $p \leq \ker(f)$, it then follows $\pi_p(\mathsf{z}) \subseteq f^{\leftarrow}(\{s\})$ and thus in particular $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})$, which is one part of what we have to show. The other part, $p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$ is a consequence of the fact $p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} \leq p$ and the assumption $p \leq \ker(f)$.

Case 1.2. Next, let (p, f) be case R2 and let $\{\mathsf{z}_1, \mathsf{z}_2\}$ be its critical data. If we define $\mathsf{z} := \mathsf{z}_1$ and $s := f(\mathsf{z}_2)$, then $\pi_p(\mathsf{z}) = \{\mathsf{z}_1, \mathsf{z}_2\}$ and thus $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} = \{\mathsf{z}_2\} \subseteq f^{\leftarrow}(\{s\})$. On the other hand, $p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} = p\backslash\{\{\mathsf{z}_1, \mathsf{z}_2\}\} \cup \{\{\mathsf{z}_1\}, \{\mathsf{z}_2\}\} \leq \ker(f)$ because, by assumption, for each $\mathsf{A} \in p\backslash\{\{\mathsf{z}_1, \mathsf{z}_2\}\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B} \in \ker(f)$ and, of course, $\{\mathsf{z}_1\} \subseteq f^{\leftarrow}(\{f(\mathsf{z}_1)\}) \in \ker(f)$ and $\{\mathsf{z}_2\} \subseteq f^{\leftarrow}(\{s\})$.

Case 1.3. Finally, let (p, f) be case R3 and let $(\mathsf{Z}, \mathsf{z}, s)$ be its critical data. Then, obviously, $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} = \mathsf{Z}\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})$ by assumption. And, $p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} = p\backslash\{\mathsf{Z}\} \cup \{\mathsf{Z}\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\} \leq \ker(f)$ because, by assumption, for any $\mathsf{A} \in p\backslash\{\mathsf{Z}\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$ and because $\mathsf{Z}\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\}) \in \ker(f)$ and $\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{f(\mathsf{z})\}) \in \ker(f)$. That proves one implication.

Step 2. In order to show the converse implication, we assume that there exist $\mathsf{z} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ such that $p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$ and $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})$ (which we can by Lemma 6.5) and derive that (p, f) is case R1, case R2 or case R3, thus proving that (p, f) is not case R4 by Lemma 6.8 (*b*). Note that the existence of z requires $p \neq \varnothing$. Again, a case distinction is in order.

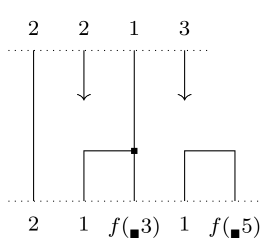
Case 2.1. First, let $f(\mathsf{z}) = s$. Then $\pi_p(\mathsf{z}) = \pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \cup \{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\}) \in \ker(f)$ by $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})$. Thus, $p \leq \ker(f)$ by $p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$. In other words, we have shown (p, f) to be case R1.

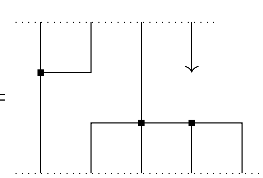
Case 2.2. Similarly, if $\pi_p(\mathsf{z}) = \{\mathsf{z}\}$, then $p = p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$. Thus, (p, f) is case R1.

Case 2.3. If $f(\mathsf{z}) \neq s$ and $|\pi_p(\mathsf{z})| = 2$, then we put $\mathsf{z}_1 := \mathsf{z}$ and we let z_2 be the unique element of $\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}$. It follows $\{\mathsf{z}_2\} = \pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})$ and thus $f(\mathsf{z}_2) = s \neq f(\mathsf{z}) = f(\mathsf{z}_1)$ by our assumptions. And, the premise $p\backslash\{\{\mathsf{z}_1, \mathsf{z}_2\}\} \cup \{\{\mathsf{z}_1\}, \{\mathsf{z}_2\}\} = p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$ means that for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \{\mathsf{z}_1, \mathsf{z}_2\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$. Hence, (p, f) is case R2 with critical data $\{\mathsf{z}_1, \mathsf{z}_2\}$.

Case 2.4. The last remaining possibility is that $f(\mathsf{z}) \neq s$ and $3 \leq |\pi_p(\mathsf{z})|$. Putting $\mathsf{Z} := \pi_p(\mathsf{z})$ implies $\mathsf{Z}\backslash\{\mathsf{z}\} = \pi_p(\mathsf{z})\backslash\{\mathsf{z}\} \subseteq f^{\leftarrow}(\{s\})$ by assumption, which is to say $f(\mathsf{y}) = s \neq f(\mathsf{z})$ for any $\mathsf{y} \in \mathsf{Z}$ with $\mathsf{y} \neq \mathsf{z}$. On the other hand, since $p\backslash\{\mathsf{Z}\} \cup \{\mathsf{Z}\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\} = p\backslash\{\pi_p(\mathsf{z})\} \cup \{\pi_p(\mathsf{z})\backslash\{\mathsf{z}\}, \{\mathsf{z}\}\}\backslash\{\varnothing\} \leq \ker(f)$, for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \mathsf{Z}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$. In other words, (p, f) is case R3 with critical data $(\mathsf{Z}, \mathsf{z}, s)$. Thus, both implications are true. ■

Example 6.10. For each of the first three f of Example 6.7 (but not the fourth) one can give at least one $(\mathsf{z}, s) \in \Pi_\ell^k \times [\![n]\!]$ such that $p \leq \ker(f \downarrow_{\mathsf{z}} s)$ (and, in fact, $\ker(f \downarrow_{\mathsf{z}} s) = \{\{{}^{\blacksquare}1, {}^{\blacksquare}2, {}_{\blacksquare}1\}, \{{}^{\blacksquare}3, {}_{\blacksquare}2, {}_{\blacksquare}3, {}_{\blacksquare}4, {}_{\blacksquare}5\}, \{{}^{\blacksquare}4\}\}$).

$(p, f) =$ 

$\ker(f \downarrow_{\mathsf{z}} s) =$ 

(a) If $f({}_{\blacksquare}3) = f({}_{\blacksquare}5) = 1$, then let $\mathsf{z} := {}_{\blacksquare}3$ and $s := 1$.
(b) If $f({}_{\blacksquare}3) = 1$ and $f({}_{\blacksquare}5) = 2$, then let $\mathsf{z} := {}_{\blacksquare}5$ and $s := 1$.
(c) If $f({}_{\blacksquare}3) = 2$ and $f({}_{\blacksquare}5) = 1$, then let $\mathsf{z} := {}_{\blacksquare}3$ and $s := 1$.

Finally, in each of the three cases R1–R3 the next lemma explains for which (z, s) the corresponding summand on the right-hand side of the identity in Lemma 6.4 has a non-zero factor $\zeta(p, \ker(f \downarrow_{\mathsf{z}} s))$.

Lemma 6.11. *Let $\{k, \ell\} \subseteq \mathbb{N}_0$, let p be any set-theoretical partition of Π_ℓ^k, let $f\colon \Pi_\ell^k \to [\![n]\!]$ be any mapping and let $\mathsf{z}' \in \Pi_\ell^k$ and $s' \in [\![n]\!]$ be arbitrary.*

(a) *If (p, f) is case R1, then $p \leq \ker(f \downarrow_{\mathsf{z}'} s')$ if and only if either $|\pi_p(\mathsf{z}')| = 1$ or both $2 \leq |\pi_p(\mathsf{z}')|$ and $s' = f(\mathsf{z}')$.*
(b) *If (p, f) is case R2 with critical data $\{\mathsf{z}_1, \mathsf{z}_2\}$, then $p \leq \ker(f \downarrow_{\mathsf{z}'} s')$ if and only if either both $\mathsf{z}' = \mathsf{z}_1$ and $s' = f(\mathsf{z}_2)$ or both $\mathsf{z}' = \mathsf{z}_2$ and $s' = f(\mathsf{z}_1)$.*
(c) *If (p, f) is case R3 with critical data $(\mathsf{Z}, \mathsf{z}, s)$, then $p \leq \ker(f \downarrow_{\mathsf{z}'} s')$ if and only if $\mathsf{z}' = \mathsf{z}$ and $s' = s$.*

Proof. By Lemma 6.5, the statement $p \leq \ker(f \downarrow_{\mathsf{z}'} s')$ is equivalent to the conjunction of $p \backslash \{\pi_p(\mathsf{z}')\} \cup \{\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\}, \{\mathsf{z}'\}\} \backslash \{\varnothing\} \leq \ker(f)$ and $\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\} \subseteq f^{\leftarrow}(\{s'\})$.

(a) Because $p \backslash \{\pi_p(\mathsf{z}')\} \cup \{\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\}, \{\mathsf{z}'\}\} \backslash \{\varnothing\} \leq p$, in the situation of (a), where $p \leq \ker(f)$, we only need to determine when $\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\} \subseteq f^{\leftarrow}(\{s'\})$. If $|\pi_p(\mathsf{z}')| = 1$, that is, $\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\} = \varnothing$, this condition is trivially satisfied. And if $2 \leq |\pi_p(\mathsf{z}')|$, then $\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\} \subseteq f^{\leftarrow}(\{s'\})$ holds if and only if $s' = f(\mathsf{z}')$ because $\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\} \subseteq \pi_p(\mathsf{z}') \subseteq f^{\leftarrow}(\{f(\mathsf{z}')\})$ by assumption. That proves (a).

(b) In case (b), if $\mathsf{z}' \notin \{\mathsf{z}_1, \mathsf{z}_2\}$, then $\{\mathsf{z}_1, \mathsf{z}_2\} \in p \backslash \{\pi_p(\mathsf{z}')\} \cup \{\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\}, \{\mathsf{z}'\}\} \backslash \{\varnothing\}$. However, because $f(\mathsf{z}_1) \neq f(\mathsf{z}_2)$ there cannot exist any $\mathsf{B} \in \ker(f)$ with $\{\mathsf{z}_1, \mathsf{z}_2\} \subseteq \mathsf{B}$. Hence, $\mathsf{z}' \notin \{\mathsf{z}_1, \mathsf{z}_2\}$ excludes $p \backslash \{\pi_p(\mathsf{z}')\} \cup \{\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\}, \{\mathsf{z}'\}\} \backslash \{\varnothing\} \leq \ker(f)$ and thus $p \leq \ker(f \downarrow_{\mathsf{z}'} s')$.

Hence, $p \leq \ker(f \downarrow_{\mathsf{z}'} s')$ requires the existence of $i \in [\![2]\!]$ with $\mathsf{z}' = \mathsf{z}_i$. If so, then $p \backslash \{\pi_p(\mathsf{z}')\} \cup \{\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\}, \{\mathsf{z}'\}\} \backslash \{\varnothing\} = p \backslash \{\{\mathsf{z}_1, \mathsf{z}_2\}\} \cup \{\{\mathsf{z}_1\}, \{\mathsf{z}_2\}\} \leq \ker(f)$ since by assumption for any $\mathsf{A} \in p$ with $\mathsf{A} \notin \{\mathsf{z}_1, \mathsf{z}_2\}$ there exists $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$. Thus, in this case, $p \leq \ker(f \downarrow_{\mathsf{z}'} s')$ is equivalent to $\{\mathsf{z}_{3-i}\} = \{\mathsf{z}_1, \mathsf{z}_2\} \backslash \{\mathsf{z}_i\} = \pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\} \subseteq f^{\leftarrow}(\{s'\})$, i.e., to $f(\mathsf{z}_{3-i}) = s'$, just as (b) claimed.

(c) Finally, under the assumptions of (c), whenever $\mathsf{z}' \notin \mathsf{Z}$, then $\mathsf{Z} \in p \backslash \{\pi_p(\mathsf{z}')\} \cup \{\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\}, \{\mathsf{z}'\}\} \backslash \{\varnothing\} \not\leq \ker(f)$ by the existence of $\mathsf{y} \in \mathsf{Z} \backslash \{\mathsf{z}\} \neq \varnothing$ with $f(\mathsf{z}) \neq s = f(\mathsf{y})$. Consequently, $p \not\leq \ker(f \downarrow_{\mathsf{z}'} s')$ if $\mathsf{z}' \notin \mathsf{Z}$.

For $\mathsf{z}' \in \mathsf{Z}$, because by assumption there is for any $\mathsf{A} \in p$ with $\mathsf{A} \neq \mathsf{Z} = \pi_p(\mathsf{z}')$ a $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$ the condition $p \backslash \{\pi_p(\mathsf{z}')\} \cup \{\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\}, \{\mathsf{z}'\}\} \backslash \{\varnothing\} \leq \ker(f)$ simplifies to the existence of $\mathsf{B} \in \ker(f)$ with $\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\} \subseteq \mathsf{B}$, which is subsumed by the second condition. In other words, if $\mathsf{z}' \in \mathsf{Z}$, then $p \leq \ker(f \downarrow_{\mathsf{z}'} s')$ if and only if $\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\} \subseteq f^{\leftarrow}(\{s'\})$.

If $\mathsf{z}' \neq \mathsf{z}$, then $\pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\} \not\subseteq f^{\leftarrow}(\{s'\})$ because, by $3 \leq |\mathsf{Z}|$, there exist $\mathsf{y} \in \mathsf{Z} \backslash \{\mathsf{z}, \mathsf{z}'\}$ with $f(\mathsf{y}) = s \neq f(\mathsf{z})$ by assumption. Hence, $p \leq \ker(f \downarrow_{\mathsf{z}'} s')$ requires $\mathsf{z}' = \mathsf{z}$. And in that case it is equivalent to $\mathsf{Z} \backslash \{\mathsf{z}\} = \pi_p(\mathsf{z}') \backslash \{\mathsf{z}'\} \subseteq f^{\leftarrow}(\{s'\})$, which is satisfied if and only if $s' = s$ because $f(\mathsf{y}) = s$ for any $\mathsf{y} \in \mathsf{Z} \backslash \{\mathsf{z}\} \neq \varnothing$. Thus, the assertion of (c) is true as well and, thus, so is the claim overall. ∎

In regard of Lemmas 6.9 and 6.11, we can now improve upon Lemma 6.4 as follows.

Lemma 6.12. *Let $\{k, \ell\} \subseteq \mathbb{N}_0$, let $\mathfrak{c} \in \{\circ, \bullet\}^{\times k}$, let $\mathfrak{d} \in \{\circ, \bullet\}^{\times \ell}$, let p be any set-theoretical partition of Π_ℓ^k, let $g \in [\![n]\!]^{\times k}$, let $j \in [\![n]\!]^{\times \ell}$, let $r := \mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}$, let $f := g \,{}^{\blacksquare}{}_{\blacksquare}\, j$, let $\mathfrak{w} := \mathfrak{c} \,{}^{\blacksquare}{}_{\blacksquare}\, \mathfrak{d}$, and let $x \in \mathbb{C}^{\times E}$.*

(i) If (p, f) is case R1, then

$$F_r(x) = \sum_{\substack{\mathsf{z}\in\Pi_\ell^k \\ \wedge\,|\pi_p(\mathsf{z})|=1}} \sum_{s=1}^{n} \begin{Bmatrix} -x_{u^{\mathfrak{w}(\mathsf{z})}_{s,f(\mathsf{z})}} & \text{if } \mathsf{z}\in\Pi_0^k \\ x_{u^{\mathfrak{w}(\mathsf{z})}_{f(\mathsf{z}),s}} & \text{if } \mathsf{z}\in\Pi_\ell^0 \end{Bmatrix} + \sum_{\substack{\mathsf{z}\in\Pi_\ell^k \\ \wedge\,2\le|\pi_p(\mathsf{z})|}} \begin{Bmatrix} -1 & \text{if } \mathsf{z}\in\Pi_0^k \\ 1 & \text{if } \mathsf{z}\in\Pi_\ell^0 \end{Bmatrix} x_{u^{\mathfrak{w}(\mathsf{z})}_{f(\mathsf{z}),f(\mathsf{z})}}.$$

(ii) If (p, f) is case R2 with critical data $\{\mathsf{z}_1, \mathsf{z}_2\}$, then

$$F_r(x) = \begin{Bmatrix} -x_{u^{\mathfrak{w}(\mathsf{z}_1)}_{f(\mathsf{z}_2),f(\mathsf{z}_1)}} & \text{if } \mathsf{z}_1\in\Pi_0^k \\ x_{u^{\mathfrak{w}(\mathsf{z}_1)}_{f(\mathsf{z}_1),f(\mathsf{z}_2)}} & \text{if } \mathsf{z}_1\in\Pi_\ell^0 \end{Bmatrix} + \begin{Bmatrix} -x_{u^{\mathfrak{w}(\mathsf{z}_2)}_{f(\mathsf{z}_1),f(\mathsf{z}_2)}} & \text{if } \mathsf{z}_2\in\Pi_0^k \\ x_{u^{\mathfrak{w}(\mathsf{z}_2)}_{f(\mathsf{z}_2),f(\mathsf{z}_1)}} & \text{if } \mathsf{z}_2\in\Pi_\ell^0 \end{Bmatrix}.$$

(iii) If (p, f) is case R3 with critical data $(\mathsf{Z}, \mathsf{z}, s)$, then

$$F_r(x) = \begin{Bmatrix} -x_{u^{\mathfrak{w}(\mathsf{z})}_{s,f(\mathsf{z})}} & \text{if } \mathsf{z}\in\Pi_0^k \\ x_{u^{\mathfrak{w}(\mathsf{z})}_{f(\mathsf{z}),s}} & \text{if } \mathsf{z}\in\Pi_\ell^0 \end{Bmatrix}.$$

(iv) If (p, f) is case R4, then $F_r(x) = 0$.

Proof. By Lemma 6.4,

$$F_r(x) = \sum_{\mathsf{z}'\in\Pi_\ell^k} \sum_{s'=1}^{n} \zeta(p, \ker(f \downarrow_{\mathsf{z}'} s')) \begin{Bmatrix} -x_{u^{\mathfrak{w}(\mathsf{z}')}_{s',f(\mathsf{z}')}} & \text{if } \mathsf{z}'\in\Pi_0^k \\ x_{u^{\mathfrak{w}(\mathsf{z}')}_{f(\mathsf{z}'),s'}} & \text{if } \mathsf{z}'\in\Pi_\ell^0 \end{Bmatrix}.$$

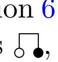
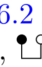
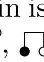
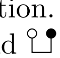
From this identity, we see immediately that $F_r(x) \neq 0$ requires the existence of $\mathsf{z} \in \Pi_\ell^k$ and $s \in [\![n]\!]$ with $p \le \ker(f \downarrow_{\mathsf{z}} s)$. Thus, Lemma 6.9 verifies *(iv)*. It remains to treat the cases *(i)*–*(iii)*.

(i) In the situation of *(i)*, for any $\mathsf{z}' \in \Pi_\ell^k$ and $s' \in [\![n]\!]$ we know from Lemma 6.11 *(a)* that $p \le \ker(f \downarrow_{\mathsf{z}'} s')$ if and only if either $|\pi_p(\mathsf{z}')| = 1$ or both $2 \le |\pi_p(\mathsf{z}')|$ and $s' = f(\mathsf{z}')$. Thus, the above formula for $F_r(x)$ simplifies to the one in *(i)*.

(ii) Under the assumptions of *(ii)*, Lemma 6.11 *(b)* tells us for any $\mathsf{z}' \in \Pi_\ell^k$ and $s' \in [\![n]\!]$ that $p \le \ker(f \downarrow_{\mathsf{z}'} s')$ if and only if either both $\mathsf{z}' = \mathsf{z}_1$ and $s' = f(\mathsf{z}_2)$ or both $\mathsf{z}' = \mathsf{z}_2$ and $s' = f(\mathsf{z}_1)$. That proves the formula for $F_r(x)$ in *(ii)*.

(iii) Finally, if the premises of *(iii)* are satisfied, then for any $\mathsf{z}' \in \Pi_\ell^k$ and $s' \in [\![n]\!]$ Lemma 6.11 *(c)* lets us infer that $p \le \ker(f \downarrow_{\mathsf{z}'} s')$ if and only if $\mathsf{z}' = \mathsf{z}$ and $s' = s$. In particular, at most one summand is non-zero. It follows that $F_r(x)$ is given by the expression in *(iii)*. ■

6.3 Halving the number of variables

Until now we have only considered each equation in the systems from Proposition 6.2 in isolation. The next simplification will take into account that the two-colored partitions , ,  and  are present in any category of two-colored partitions. That fact can be used to eliminate half the variables (as, e.g., in [22, Lemma 1.7]). This is the only explicit elimination of variables that will be made in the entire proof of the main theorem.

Lemma 6.13. *For any $g \in [\![n]\!]^{\times 2}$ and $j \in [\![n]\!]^{\times 2}$ and any $x \in \mathbb{C}^{\times E}$, if r is given by*

(a) $\mathrm{r}^{\varnothing}_{\circ\bullet}(\sqcap)_{j,\varnothing}$, *then* $F_r(x) = x_{u^{\circ}_{j_1,j_2}} + x_{u^{\bullet}_{j_2,j_1}}$.

(b) $\mathrm{r}^{\varnothing}_{\bullet\circ}(\sqcap)_{j,\varnothing}$, *then* $F_r(x) = x_{u^{\bullet}_{j_1,j_2}} + x_{u^{\circ}_{j_2,j_1}}$.

(c) $\mathrm{r}^{\bullet\circ}_{\varnothing}(\sqcup)_{\varnothing,g}$, *then* $F_r(x) = -x_{u^{\bullet}_{g_2,g_1}} - x_{u^{\circ}_{g_1,g_2}}$.

(d) $\mathrm{r}^{\circ\bullet}_{\varnothing}(\sqcup)_{\varnothing,g}$, *then* $F_r(x) = -x_{u^{\circ}_{g_2,g_1}} - x_{u^{\bullet}_{g_1,g_2}}$.

Proof. Only the proof of *(a)* is given. The others are similar. Using Definition 4.11, the result of Lemma 3.15 that $r = \mathrm{r}^{\varnothing}_{\circ\bullet}(\sqcap)_{j,\varnothing} = \sum_{i=1}^{n} u^{\circ}_{j_1,i} u^{\bullet}_{j_2,i} - \delta_{j_1,j_2} 1$ implies

$$F_r(x) = \sum_{i=1}^{n} (x_{u^{\circ}_{j_1,i}} \triangleleft u^{\bullet}_{j_2,i} + u^{\circ}_{j_1,i} \triangleright x_{u^{\bullet}_{j_2,i}}) = \sum_{i=1}^{n} (\delta_{j_2,i}\, x_{u^{\circ}_{j_1,i}} + \delta_{j_1,i}\, x_{u^{\bullet}_{j_2,i}}) = x_{u^{\circ}_{j_1,j_2}} + x_{u^{\bullet}_{j_2,j_1}}$$

because $F_1 = 0$. ■

Notation 6.14. Let $v \in M_n(\mathbb{C})$ be arbitrary.

(*a*) Let $x^v \in \mathbb{C}^{\times E}$ be such that for any $\{i,j\} \subseteq [\![n]\!]$,

$$x^v_{u^{\circ}_{j,i}} := v_{j,i} \qquad \text{and} \qquad x^v_{u^{\bullet}_{j,i}} := -v_{i,j}.$$

(*b*) For any set $\mathcal{P}$ of two-colored partitions let $A(\mathcal{P}, v)$ denote the statement that $F_r(x^v) = 0$ for any $r \in R_{\mathcal{P}}$.

Ultimately, it will be shown that in the case of categories of two-colored partitions the predicates A defined in Notation 6.14 are equivalent to those used in the formulation of the main theorem.

Proposition 6.15. *For any category $\mathcal{C}$ of two-colored partitions, if G is the unitary easy compact quantum group of $(\mathcal{C}, n)$, then there exists an isomorphism of $\mathbb{C}$-vector spaces*

$$H^1(\widehat{G}) \hookrightarrow\!\!\!\!\rightarrow \{v \in M_n(\mathbb{C}) \wedge A(\mathcal{C}, v)\},$$

which maps (the one-elemental cohomology class of) any 1-cocycle η to the matrix v with $v_{j,i} = \eta(u^{\circ}_{j,i} + J_{\mathcal{C}})$ for any $\{i,j\} \subseteq [\![n]\!]$.

Proof. By Proposition 6.2, it suffices to show that the rule $x \mapsto (x_{u^{\circ}_{j,i}})_{(j,i)\in[\![n]\!]^{\times 2}}$ gives a $\mathbb{C}$-linear isomorphism

$$\{x \in \mathbb{C}^{\times E} \wedge \forall r \in R_{\mathcal{C}} \colon F_r(x) = 0\} \hookrightarrow\!\!\!\!\rightarrow \{v \in M_n(\mathbb{C}) \wedge \forall r \in R_{\mathcal{C}} \colon F_r(x^v) = 0\}.$$

The claimed isomorphism is well defined: Let $x \in \mathbb{C}^{\times E}$ be such that $F_r(x) = 0$ for any $r \in R_{\mathcal{C}}$. Then, for any $j \in [\![n]\!]^{\times 2}$ because $\mathrm{r}^{\varnothing}_{\circ\bullet}(\sqcap)_{j,\varnothing} \in R_{\mathcal{C}}$ in particular

$$x_{u^{\circ}_{j_1,j_2}} + x_{u^{\bullet}_{j_2,j_1}} = 0$$

by Lemma 6.13, i.e., $x_{u^{\bullet}_{j_2,j_1}} = -x_{u^{\circ}_{j_1,j_2}}$. Hence, if we let $v := (x_{u^{\circ}_{j,i}})_{(j,i)\in[\![n]\!]^{\times 2}}$, then for any $\{i,j\} \subseteq [\![n]\!]$ by definition not only $x^v_{u^{\circ}_{j,i}} = v_{j,i} = x_{u^{\circ}_{j,i}}$ but also

$$x^v_{u^{\bullet}_{j,i}} = -v_{i,j} = -x_{u^{\circ}_{i,j}} = x_{u^{\bullet}_{j,i}},$$

which is to say $x^v = x$. Thus, per assumption, in particular $F_r(x^v) = F_r(x) = 0$ for any $r \in R_{\mathcal{C}}$. That proves that the map is well defined.

It is clear that the mapping is $\mathbb{C}$-linear. Moreover, it is injective because, if again $x \in \mathbb{C}^{\times E}$ is such that $F_r(x) = 0$ for any $r \in R_{\mathcal{C}}$ and if again $v := (x_{u^{\circ}_{j,i}})_{(j,i)\in[\![n]\!]^{\times 2}}$, then $v = 0$ necessitates $x^v = 0$ by definition of x^v and thus $x = 0$ by the identity $x^v = x$ established in the preceding paragraph.

To show surjectivity, we let $v \in M_n(\mathbb{C})$ be arbitrary with $F_r(x^v) = 0$ for any $r \in R$ and abbreviate $x := x^v$. Then, of course, $F_r(x) = 0$ for any $r \in R_{\mathcal{C}}$. Because moreover $x_{u^{\circ}_{j,i}} = x^v_{u^{\circ}_{j,i}} = v_{j,i}$ for any $\{i, j\} \subseteq [\![n]\!]$ the tuple x is a preimage of v. Thus, the claim is true. ■

The next lemma correspondingly eliminates the variables $(x_{u^{\bullet}_{j,i}})_{(j,i)\in[\![n]\!]^{\times 2}}$ from the formula obtained in Lemma 6.12 for the individual equations in the systems from Proposition 6.2. Recall from Notation 3.1 that f/p denotes the quotient mapping of any mapping f with respect to any set-theoretical partition p of its domain and recall the definition of the color sum $\sigma^{\mathfrak{c}}_{\mathfrak{d}}$ of two color tuples $\mathfrak{c}$ and $\mathfrak{d}$ from Definition 3.4.

Lemma 6.16. *Let $(\mathfrak{c}, \mathfrak{d}, p)$ be any two-colored partition, let $g \in [\![n]\!]^{\times|\mathfrak{c}|}$, let $j \in [\![n]\!]^{\times|\mathfrak{d}|}$, let $r := \mathrm{r}^{\mathfrak{c}}_{\mathfrak{d}}(p)_{j,g}$, let $f := g \blacksquare j$, and let $v \in M_n(\mathbb{C})$.*

(i) If (p, f) is case R1, then

$$F_r(x^v) = \sum_{\substack{\mathsf{A}\in p\\ \wedge\,|\mathsf{A}|=1}} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{A}) \sum_{s=1}^{n} \begin{Bmatrix} v_{(f/p)(\mathsf{A}),s} & \text{if } \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{A}) = 1 \\ v_{s,(f/p)(\mathsf{A})} & \text{if } \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{A}) = -1 \end{Bmatrix} + \sum_{\substack{\mathsf{A}\in p\\ \wedge\,2\le|\mathsf{A}|}} \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\mathsf{A}) v_{(f/p)(\mathsf{A}),(f/p)(\mathsf{A})}.$$

(ii) If (p, f) is case R2 with critical data $\{\mathsf{z}_1, \mathsf{z}_2\}$, then

$$F_r(x^v) = \frac{1}{2}\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathsf{z}_1, \mathsf{z}_2\})(v_{f(\mathsf{z}_1),f(\mathsf{z}_2)} + v_{f(\mathsf{z}_2),f(\mathsf{z}_1)}).$$

(iii) If (p, f) is case R3 with critical data $(\mathsf{Z}, \mathsf{z}, s)$, then

$$F_r(x^v) = \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathsf{z}\}) \begin{Bmatrix} v_{f(\mathsf{z}),s} & \text{if } \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathsf{z}\}) = 1 \\ v_{s,f(\mathsf{z})} & \text{if } \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathsf{z}\}) = -1 \end{Bmatrix}.$$

(iv) If (p, f) is case R4, then $F_r(x^v) = 0$.

Proof. We only have to show that in each of the first three cases the right-hand sides of the identities for $F_r(x^v)$ in the claim agree with the corresponding ones of Lemma 6.12. For the purposes of this proof, let $v^{\mathrm{b}(1)} := v$ and $v^{\mathrm{b}(-1)} := v^{\mathrm{t}}$ and recall $\sigma(\circ) = 1$ and $\sigma(\bullet) = -1$. Then, for any $\mathfrak{e} \in \{\circ, \bullet\}$ and $\{i, j\} \subseteq [\![n]\!]$ the definitions imply

$$x^v_{u^{\mathfrak{e}}_{j,i}} = \begin{Bmatrix} v_{j,i} & \text{if } \mathfrak{e} = \circ \\ -v_{i,j} & \text{if } \mathfrak{e} = \bullet \end{Bmatrix} = \sigma(\mathfrak{e})(v^{\mathrm{b}(\sigma(\mathfrak{e}))})_{j,i}.$$

If $k := |\mathfrak{c}|$ and $\ell := |\mathfrak{d}|$ and $\mathfrak{w} := \mathfrak{c} \blacksquare \mathfrak{d}$, then it follows for any $\mathsf{z} \in \Pi^k_\ell$ and any $s \in [\![n]\!]$ that

$$\begin{aligned}\begin{Bmatrix} -x^v_{u^{\mathfrak{w}(\mathsf{z})}_{s,f(\mathsf{z})}} & \text{if } \mathsf{z} \in \Pi^k_0 \\ x^v_{u^{\mathfrak{w}(\mathsf{z})}_{f(\mathsf{z}),s}} & \text{if } \mathsf{z} \in \Pi^0_\ell \end{Bmatrix} &= \begin{Bmatrix} -\sigma(\mathfrak{w}(\mathsf{z}))(v^{\mathrm{b}(\sigma(\mathfrak{w}(\mathsf{z})))})_{s,f(\mathsf{z})} & \text{if } \mathsf{z} \in \Pi^k_0 \\ \sigma(\mathfrak{w}(\mathsf{z}))(v^{\mathrm{b}(\sigma(\mathfrak{w}(\mathsf{z})))})_{f(\mathsf{z}),s} & \text{if } \mathsf{z} \in \Pi^0_\ell \end{Bmatrix} \\ &= \begin{Bmatrix} \sigma(\overline{\mathfrak{w}(\mathsf{z})})(v^{\mathrm{b}(\sigma(\overline{\mathfrak{w}(\mathsf{z})}))})_{f(\mathsf{z}),s} & \text{if } \mathsf{z} \in \Pi^k_0 \\ \sigma(\mathfrak{w}(\mathsf{z}))(v^{\mathrm{b}(\sigma(\mathfrak{w}(\mathsf{z})))})_{f(\mathsf{z}),s} & \text{if } \mathsf{z} \in \Pi^0_\ell \end{Bmatrix} \\ &= \sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathsf{z}\})(v^{\mathrm{b}(\sigma^{\mathfrak{c}}_{\mathfrak{d}}(\{\mathsf{z}\}))})_{f(\mathsf{z}),s}\end{aligned}$$

by the definition of the color sum and, analogously,

$$\begin{Bmatrix} -x^v_{u^{\mathfrak{w}(\mathsf{z})}_{f(\mathsf{z}),s}} & \text{if } \mathsf{z}\in\Pi_0^k \\ x^v_{u^{\mathfrak{w}(\mathsf{z})}_{s,f(\mathsf{z})}} & \text{if } \mathsf{z}\in\Pi_\ell^0 \end{Bmatrix} = \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\})(v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\}))})_{s,f(\mathsf{z})}.$$

We now distinguish the three relevant cases.

(i) In the situation of *(i)*, by Lemma 6.12 the number $F_r(x^v)$ is given by

$$\sum_{\substack{\mathsf{z}\in\Pi_\ell^k\\ \wedge\,|\pi_p(\mathsf{z})|=1}} \sum_{s=1}^{n} \begin{Bmatrix} -x^v_{u^{\mathfrak{w}(\mathsf{z})}_{s,f(\mathsf{z})}} & \text{if } \mathsf{z}\in\Pi_0^k \\ x^v_{u^{\mathfrak{w}(\mathsf{z})}_{f(\mathsf{z}),s}} & \text{if } \mathsf{z}\in\Pi_\ell^0 \end{Bmatrix} + \sum_{\substack{\mathsf{z}\in\Pi_\ell^k\\ \wedge\,2\leq|\pi_p(\mathsf{z})|}} \begin{Bmatrix} -x^v_{u^{\mathfrak{w}(\mathsf{z})}_{f(\mathsf{z}),f(\mathsf{z})}} & \text{if } \mathsf{z}\in\Pi_0^k \\ x^v_{u^{\mathfrak{w}(\mathsf{z})}_{f(\mathsf{z}),f(\mathsf{z})}} & \text{if } \mathsf{z}\in\Pi_\ell^0 \end{Bmatrix}.$$

By what was shown initially, this can be rewritten identically as

$$\sum_{\substack{\mathsf{z}\in\Pi_\ell^k\\ \wedge\,|\pi_p(\mathsf{z})|=1}} \sum_{s=1}^{n} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\})(v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\}))})_{f(\mathsf{z}),s} + \sum_{\substack{\mathsf{z}\in\Pi_\ell^k\\ \wedge\,2\leq|\pi_p(\mathsf{z})|}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\})(v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\}))})_{f(\mathsf{z}),f(\mathsf{z})}.$$

And that is exactly what was claimed because $\ker(f)\leq p$ and $\sum_{\mathsf{z}\in\mathsf{A}}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\})=\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})$ for any $\mathsf{A}\in p$.

(ii) Under the assumptions of *(ii)*, Lemma 6.12 tells us that $F_r(x^v)$ can be computed as

$$\begin{Bmatrix} -x^v_{u^{\mathfrak{w}(\mathsf{z}_1)}_{f(\mathsf{z}_2),f(\mathsf{z}_1)}} & \text{if } \mathsf{z}_1\in\Pi_0^k \\ x^v_{u^{\mathfrak{w}(\mathsf{z}_1)}_{f(\mathsf{z}_1),f(\mathsf{z}_2)}} & \text{if } \mathsf{z}_1\in\Pi_\ell^0 \end{Bmatrix} + \begin{Bmatrix} -x^v_{u^{\mathfrak{w}(\mathsf{z}_2)}_{f(\mathsf{z}_1),f(\mathsf{z}_2)}} & \text{if } \mathsf{z}_2\in\Pi_0^k \\ x^v_{u^{\mathfrak{w}(\mathsf{z}_2)}_{f(\mathsf{z}_2),f(\mathsf{z}_1)}} & \text{if } \mathsf{z}_2\in\Pi_\ell^0 \end{Bmatrix},$$

which, by our initial observations, is identical to

$$\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\})(v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\}))})_{f(\mathsf{z}_2),f(\mathsf{z}_1)} + \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_2\})(v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_2\}))})_{f(\mathsf{z}_1),f(\mathsf{z}_2)}.$$

Since $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_i\})\in\{-1,1\}$ for each $i\in[\![2]\!]$, either $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\})=\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_2\})$, in which case we infer

$$\begin{aligned} F_r(x^v) &= \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\})((v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\}))})_{f(\mathsf{z}_2),f(\mathsf{z}_1)} + (v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\}))})_{f(\mathsf{z}_1),f(\mathsf{z}_2)}), \\ &= \frac{1}{2}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1,\mathsf{z}_2\})(v_{f(\mathsf{z}_1),f(\mathsf{z}_2)} + v_{f(\mathsf{z}_2),f(\mathsf{z}_1)}), \end{aligned}$$

or $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\})=-\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_2\})$, implying

$$F_r(x^v) = \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\})(v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\}))})_{f(\mathsf{z}_2),f(\mathsf{z}_1)} - \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\})(v^{\mathrm{b}(-\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1\}))})_{f(\mathsf{z}_1),f(\mathsf{z}_2)} = 0.$$

And that is precisely what we needed to show in this case.

(iii) Finally, if the premises of *(iii)* are satisfied, according to Lemma 6.12 and by our initial findings,

$$F_r(x^v) = \begin{Bmatrix} -x^v_{u^{\mathfrak{w}(\mathsf{z})}_{s,f(\mathsf{z})}} & \text{if } \mathsf{z}\in\Pi_0^k, \\ x^v_{u^{\mathfrak{w}(\mathsf{z})}_{f(\mathsf{z}),s}} & \text{if } \mathsf{z}\in\Pi_\ell^0 \end{Bmatrix} = \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\})(v^{\mathrm{b}(\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\}))})_{f(\mathsf{z}),s}.$$

Since this is just what we claimed, that concludes the proof. ■

6.4 All equations of any single two-colored partition

As an intermediate step to solving the systems of linear equations of Proposition 6.15, we now study the system of equations induced not by an entire category of two-colored partitions but only any single two-colored partition.

Definition 6.17. With respect to any two-colored partition $(\mathfrak{c},\mathfrak{d},p)$ we say that any $v\in M_n(\mathbb{C})$ meets

(*a*) *condition P1* if for any $h\colon p\to [\![n]\!]$,

$$\sum_{\substack{\mathsf{A}\in p\\ \wedge\,|\mathsf{A}|=1}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) \sum_{s=1}^{n} \begin{cases} v_{h(\mathsf{A}),s} & \text{if } \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})=1\\ v_{s,h(\mathsf{A})} & \text{if } \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})=-1 \end{cases} + \sum_{\substack{\mathsf{A}\in p\\ \wedge\,2\leq|\mathsf{A}|}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, v_{h(\mathsf{A}),h(\mathsf{A})} = 0.$$

(*b*) *condition P2* if there is no $\mathsf{Y}\in p$ with $|\mathsf{Y}|=2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})\neq 0$ or if $v_{j,i}+v_{i,j}=0$ for any $\{i,j\}\subseteq[\![n]\!]$ with $i\neq j$.

(*c*) *condition P3* if there is no $\mathsf{Z}\in p$ with $3\leq|\mathsf{Z}|$ or if $v_{j,i}=0$ for any $\{i,j\}\subseteq[\![n]\!]$ with $i\neq j$.

Lemma 6.18. *For any two-colored partition $(\mathfrak{c},\mathfrak{d},p)$ and any $v\in M_n(\mathbb{C})$, the statement $A(\{(\mathfrak{c},\mathfrak{d},p)\},v)$ is equivalent to v meeting simultaneously all the three conditions P1–P3 with respect to $(\mathfrak{c},\mathfrak{d},p)$.*

Proof. Both implications are proved separately.

Step 1. First implication. First, suppose that conditions P1–P3 are satisfied, let $g\in[\![n]\!]^{\times|\mathfrak{c}|}$ and $j\in[\![n]\!]^{\times|\mathfrak{d}|}$ be arbitrary and let $r:=\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}$. We show that $F_r(x^v)=0$. If $f:=g\,{}^{\blacksquare}_{\blacksquare}\,j$, then (p,f) falls into one of the four cases R1–R4 by Lemma 6.8 (*b*).

Case 1.1. If (p,f) is case R1, then we can define $h:=f/p$. And, then by case (*i*) of Lemma 6.16 condition P1 says precisely that $F_r(x^v)=0$.

Case 1.2. Next, suppose that (p,f) is case R2 with critical data $\{\mathsf{z}_1,\mathsf{z}_2\}$. Then $F_r(x^v)=\frac{1}{2}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1,\mathsf{z}_2\})(v_{f(\mathsf{z}_1),f(\mathsf{z}_2)}+v_{f(\mathsf{z}_2),f(\mathsf{z}_1)})$ by case (*ii*) of Lemma 6.16. Hence, if $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1,\mathsf{z}_2\})=0$ we have nothing to prove. Otherwise, condition P2 guarantees that $v_{b,a}+v_{a,b}=0$ for any $\{a,b\}\subseteq[\![n]\!]$, thus showing $F_r(x^v)=0$ since $f(\mathsf{z}_2)\neq f(\mathsf{z}_1)$.

Case 1.3. Now, let (p,f) be case R3 with critical data $(\mathsf{Z},\mathsf{z},s)$. Then condition P3 implies that v is diagonal. Since by case (*iii*) of Lemma 6.16 the number $F_r(x^v)$ is given by $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\})\,v_{f(\mathsf{z}),s}$ or $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\})\,v_{s,f(\mathsf{z})}$ that proves $F_r(x^v)=0$ in this case because $f(\mathsf{z})\neq s$.

Case 1.4. Lastly, if (p,f) is case R4, then $F_r(x^v)=0$ by case (*iv*) of Lemma 6.16. Hence, there is nothing to show.

Step 2. Second implication. To show the converse we assume $F_r(x^v)=0$ for any $r\in R_{\{(\mathfrak{c},\mathfrak{d},p)\}}$ and prove that then conditions P1–P3 are met. Let $k:=|\mathfrak{c}|$ and $\ell:=|\mathfrak{d}|$.

Step 2.1. If $p=\varnothing$, condition P1 is trivially satisfied. Otherwise, for any $h\colon p\to[\![n]\!]$ let $f:=h\circ\pi_p$, let $g\in[\![n]\!]^{\times k}$ and $j\in[\![n]\!]^{\times\ell}$ be such that $g\,{}^{\blacksquare}_{\blacksquare}\,j:=f$ and let $r:=\mathrm{r}_{\mathfrak{d}}^{\mathfrak{c}}(p)_{j,g}$. Then (p,f) is case R1. Moreover, $F_r(x^v)$ is exactly the left-hand side of the equation in condition P1 by Lemma 6.16 (*i*). This proves condition P1 to be satisfied because $F_r(x^v)=0$ by assumption.

Step 2.2. Now, let $\mathsf{Y}\in p$ be such that $|\mathsf{Y}|=2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})\neq 0$ and let $\{a,b\}\subseteq[\![n]\!]$ be arbitrary with $a\neq b$. We find $\mathsf{z}_1\in\Pi_\ell^k$ and $\mathsf{z}_2\in\Pi_\ell^k$ such that $\mathsf{z}_1\neq\mathsf{z}_2$ and $\{\mathsf{z}_1,\mathsf{z}_2\}=\mathsf{Y}$. If we let $f\colon\Pi_\ell^k\to[\![n]\!]$ be such that $\mathsf{z}_1\mapsto a$ and $\mathsf{y}\mapsto b$ for any $\mathsf{y}\in\Pi_\ell^k\backslash\{\mathsf{z}_1\}$, then $f(\mathsf{z}_1)\neq f(\mathsf{z}_2)$ by $a\neq b$ and for any $\mathsf{A}\in p$ with $\mathsf{A}\neq\{\mathsf{z}_1,\mathsf{z}_2\}$ there is $\mathsf{B}\in\ker(f)$ with $\mathsf{A}\subseteq\mathsf{B}$, namely $\mathsf{B}=\Pi_\ell^k\backslash\{\mathsf{z}_1\}$. In other words, (p,f) is case R2. Hence, $F_r(x^v)=\frac{1}{2}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1,\mathsf{z}_2\})(v_{b,a}+v_{a,b})$ by Lemma 6.16 (*ii*). Since $F_r(x^v)=0$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}_1,\mathsf{z}_2\})\neq 0$ by assumption, condition P2 is thus met as well.

Step 2.3. Lastly, suppose $\mathsf{Z}\in p$ and $3\leq|\mathsf{Z}|$ and let $\{a,b\}\subseteq[\![n]\!]$ be arbitrary with $a\neq b$. Fix any $\mathsf{z}\in\mathsf{Z}$, let $s:=b$ and define $f\colon\Pi_\ell^k\to[\![n]\!]$ by demanding $\mathsf{z}\mapsto a$ and $\mathsf{y}\mapsto b$ for any

$\mathsf{y} \in \Pi_\ell^k \backslash \{\mathsf{z}\}$. Then, $f(\mathsf{z}) \neq s$ by $a \neq b$ and $f(\mathsf{y}) = s$ for any $\mathsf{y} \in \mathsf{Z}$ with $\mathsf{y} \neq \mathsf{z}$. Moreover, for any $\mathsf{A} \in p$ with $p \neq \mathsf{Z}$ there exists in the shape of $\Pi_\ell^k \backslash \{\mathsf{z}\}$ some $\mathsf{B} \in \ker(f)$ with $\mathsf{A} \subseteq \mathsf{B}$. This means that (p, f) is case R3. Therefore, $F_r(x^v)$ is given by $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\})\, v_{a,b}$ or $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\})\, v_{b,a}$ according to Lemma 6.16 *(iii)*. Thus, by $F_r(x^v) = 0$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\{\mathsf{z}\}) \neq 0$ also condition P3 is satisfied and the proof is complete. ■

6.5 All equations of certain special two-colored partitions

In the upcoming case distinctions, it will be useful to already understand the conditions imposed by a small number of one- or two-elemental sets of special two-colored partitions.

Lemma 6.19. *Let $v \in M_n(\mathbb{C})$ be arbitrary.*

(*a*) *$A(\{\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}\}, v)$ is equivalent to v being diagonal.*

(*b*) *$A(\{\uparrow_\circ\uparrow_\bullet\}, v)$ is equivalent to there existing $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small.*

(*c*) *$A(\{\uparrow_\circ^{\otimes t}, \uparrow_\bullet^{\otimes t}\}, v)$ is equivalent to v being small for any $t \in \mathbb{N}$.*

Proof. (a) Because $|\mathsf{A}| \neq 1$ and $\sigma_{\circ\bullet\circ\bullet}^{\varnothing}(\mathsf{A}) = 0$ for the only $\mathsf{A} \in \sqcap\!\sqcap\!\sqcap$ condition P1 with respect to $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}$ is satisfied regardless of whether v is diagonal or not. Similarly, since there is no $\mathsf{Y} \in \sqcap\!\sqcap\!\sqcap$ with $|\mathsf{Y}| = 2$ the same is true about condition P2. It is condition P3 alone which is relevant. Namely, since there is $\mathsf{Z} \in \sqcap\!\sqcap\!\sqcap$ with $3 \leq |\mathsf{Z}|$ it is equivalent to v being diagonal. Hence, *(a)* follows by Lemma 6.18.

(b) Because $|\mathsf{A}| = 1$ for any $\mathsf{A} \in \uparrow\uparrow$ and because $\sigma_{\circ\bullet}^{\varnothing}(\{_{\blacksquare}1\}) = 1$ and $\sigma_{\circ\bullet}^{\varnothing}(\{_{\blacksquare}2\}) = -1$, what condition P1 with respect to $\uparrow_\circ\uparrow_\bullet$ demands of v is that for any $h\colon \uparrow\uparrow \to [\![n]\!]$ the number $\sum_{s=1}^n v_{h(\{_{\blacksquare}1\}),s} - \sum_{s=1}^n v_{s,h(\{_{\blacksquare}2\})}$ be zero. In other words, condition P1 is equivalent to $\sum_{s=1}^n v_{j,s} = \sum_{s=1}^n v_{s,i}$ holding for any $\{i,j\} \subseteq [\![n]\!]$, i.e., by Lemma 5.2 *(a)* to there being $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small. At the same time, conditions P2 and P3 are always trivially satisfied since there are no $\mathsf{Y} \in \uparrow\uparrow$ with $|\mathsf{Y}| = 2$, let alone $\mathsf{Z} \in \uparrow\uparrow$ with $3 \leq |\mathsf{Z}|$. Thus, Lemma 6.18 proves *(b)*.

(c) Since $|\mathsf{A}| = 1$ and $\sigma_{\mathfrak{c}^{\otimes t}}^{\varnothing}(\mathsf{A}) = \sigma(\mathfrak{c})$ for any $\mathsf{A} \in \uparrow^{\otimes t}$ and any $\mathfrak{c} \in \{\circ, \bullet\}$ condition P1 with respect to $\uparrow_\circ^{\otimes t}$ and $\uparrow_\bullet^{\otimes t}$ is satisfied by v if and only if $\sum_{d=1}^t \sum_{s=1}^n v_{h(\{_{\blacksquare}d\}),s} = 0$ respectively $-\sum_{d=1}^t \sum_{s=1}^n v_{s,h(\{_{\blacksquare}d\})} = 0$ for any $h\colon \uparrow^{\otimes t} \to [\![n]\!]$. Moreover, conditions P2 and P3 are vacuous by the absence of any $\mathsf{Y} \in \uparrow^{\otimes t}$ with $|\mathsf{Y}| = 2$ and any $\mathsf{Z} \in \uparrow^{\otimes t}$ with $3 \leq |\mathsf{Z}|$.

Consequently, if v is small and thus $\sum_{s=1}^n v_{h(\{_{\blacksquare}d\}),s} = \sum_{s=1}^n v_{s,h(\{_{\blacksquare}d\})} = 0$ for any $d \in [\![t]\!]$ all three conditions P1–P3 are met for both $\uparrow_\circ^{\otimes t}$ and $\uparrow_\bullet^{\otimes t}$. Hence, $A(\{\uparrow_\circ^{\otimes t}, \uparrow_\bullet^{\otimes t}\}, v)$ is true in that case by Lemma 6.18.

If, conversely, $A(\{\uparrow_\circ^{\otimes t}, \uparrow_\bullet^{\otimes t}\}, v)$ is true, Lemma 6.18 implies that v in particular meets condition P1 with respect to $\uparrow_\circ^{\otimes t}$ and $\uparrow_\bullet^{\otimes t}$. Thus, for any $i \in [\![n]\!]$, if $h\colon \uparrow^{\otimes t} \to [\![n]\!]$ is constant with value i, then $0 = t \sum_{s=1}^n v_{i,s}$ respectively $0 = t \sum_{s=1}^n v_{s,i}$ by what was said initially. By $0 < t$, that proves v to be small. ■

6.6 Case distinctions

The final step to proving the main theorem is upon us. According to Proposition 6.15, it is enough to show that predicates A of Notation 6.14 and those of the main theorem are equivalent.

The strategy for that is the same in every case: For the category $\mathcal{C}$ of two-colored partitions in question and any $v \in M_n(\mathbb{C})$, the statement $A(\mathcal{C}, v)$ is equivalent to $A(\{(\mathfrak{c}, \mathfrak{d}, p)\}, v)$ being true for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$. By Lemma 6.18, that is equivalent to the three conditions P1–P3 being met with respect to any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$. Thus, we only need to show that the latter is equivalent to the statement $A(\mathcal{C}, v)$ in the main theorem.

Proposition 6.20. *Let $\mathcal{C}$ be any category of two-colored partitions and $v \in M_n(\mathbb{C})$. If $\mathcal{C}$ is case $\mathcal{O}$ and*

(*i*) *class* NNSB, *then $A(\mathcal{C}, v)$ is equivalent to the absolutely true statement.*

(*ii*) *not class* NNSB *but class* NP, *then $A(\mathcal{C}, v)$ is equivalent to there existing $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is skew-symmetric.*

(*iii*) *not class* NP, *then $A(\mathcal{C}, v)$ is equivalent to v being skew-symmetric.*

Proof. By Lemma 3.11 (*b*) and (*c*), the assumption that $\mathcal{C}$ be case $\mathcal{O}$ means $|\mathsf{B}| = 2$ for any $\mathsf{B} \in p$ and any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$. Consequently, when checking conditions P1–P3 with respect to any given $(\mathfrak{c}, \mathfrak{d}, p)$ there are simplifications.

☐ Condition P1 is met if and only if for any $h\colon p \to [\![n]\!]$,

$$\sum_{\mathsf{A} \in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) v_{h(\mathsf{A}),h(\mathsf{A})} = 0.$$

☐ Condition P2 simplifies to the demand that $v_{j,i} + v_{i,j} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $i \neq j$ as soon as there is any $\mathsf{Y} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$.

☐ Condition P3 is trivially satisfied and can thus be ignored entirely.

The three cases are treated individually.

(i) If $\mathcal{C}$ is class NNSB, then all we have to show is that with respect to any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ conditions P1 and P2 are automatically satisfied. And indeed, by the initial simplification condition P1 of Lemma 6.18 is satisfied for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and any $h\colon p \to [\![n]\!]$ since already $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) = 0$ for any $\mathsf{A} \in p$ by $\mathcal{C}$ being class NNSB. Likewise, by $\mathcal{C}$ being NNSB there are no $\mathsf{Y} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$, meaning condition P2 is trivially satisfied.

(ii) The next case is that $\mathcal{C}$ is not class NNSB but still class NP. Here, we do have to show two implications. And, we treat them separately.

Suppose that there exists $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is skew-symmetric and let $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ be arbitrary. Since $v - \lambda I$ is skew-symmetric $v_{j,j} = v_{i,i}$ for any $\{i, j\} \subseteq [\![n]\!]$ with $j \neq i$ by Lemma 5.2 (*b*). Thus, what condition P1 with respect to $(\mathfrak{c}, \mathfrak{d}, p)$ actually demands is that the term $\sum_{\mathsf{A} \in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, v_{1,1} = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\, v_{1,1}$ be zero, which it is since $\mathcal{C}$ being class NP ensures $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$. Lemma 5.2 (*b*) furthermore guarantees that $v_{j,i} + v_{i,j} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $j \neq i$, which is why condition P2 is met, regardless of whether there is $\mathsf{Y} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$ or not.

Conversely, let now $A(\mathcal{C}, v)$ hold. By assumption, we find $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $\mathsf{Y} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$ but still $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$ and, of course, with $|\mathsf{Y}| = 2$ since $\mathcal{C}$ is case $\mathcal{O}$. Hence, $v_{j,i} + v_{i,j} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $i \neq j$ by condition P2 with respect to $(\mathfrak{c}, \mathfrak{d}, p)$. But also, given any $\{i, j\} \subseteq [\![n]\!]$ with $i \neq j$, if $h\colon p \to [\![n]\!]$ is such that $\mathsf{Y} \mapsto j$ and $\mathsf{A} \mapsto i$ for any $\mathsf{A} \in p \backslash \{\mathsf{Y}\}$, then condition P1 implies $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})\, v_{j,j} + \sum_{\mathsf{A} \in p \wedge \mathsf{A} \neq \mathsf{Y}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, v_{i,i} = 0$. Since $0 = \Sigma_{\mathfrak{d}}^{\mathfrak{c}} = \sum_{\mathsf{A} \in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) = \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) + \sum_{\mathsf{A} \in p \wedge \mathsf{A} \neq \mathsf{Y}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})$, i.e., $\sum_{\mathsf{A} \in p \wedge \mathsf{A} \neq \mathsf{Y}} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) = -\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})$, this means $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})(v_{j,j} - v_{i,i}) = 0$, which implies $v_{j,j} - v_{i,i} = 0$ by $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$. Hence, there exists $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is skew-symmetric by Lemma 5.2 (*b*).

(iii) Finally, suppose that $\mathcal{C}$ is not even class NP.

Assume that v is skew-symmetric and let $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ be arbitrary. Since v being skew-symmetric implies $v_{h(\mathsf{A}),h(\mathsf{A})} = 0$ for any $\mathsf{A} \in p$ condition P1 is met with respect to $(\mathfrak{c}, \mathfrak{d}, p)$. But v being skew-symmetric also implies $v_{j,i} + v_{i,j} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $j \neq i$, which is why condition P2 is satisfied no matter whether there is $\mathsf{Y} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$ or not.

To see the converse, we assume $A(\mathcal{C}, v)$. Because $\mathcal{C}$ is not class NP there exists $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ with $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$. For any $i \in [\![n]\!]$, if $h\colon p \to [\![n]\!]$ is constant with value i, then condition P1 with respect to $(\mathfrak{c}, \mathfrak{d}, p)$ implies $0 = \sum_{\mathsf{A} \in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, v_{i,i} = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\, v_{i,i}$ and thus $v_{i,i} = 0$ by $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$. Furthermore, since $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = \sum_{\mathsf{A} \in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})$ the assumption $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$ also requires the existence of at least one $\mathsf{Y} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$. Hence, by the initial simplification condition P2 yields $v_{j,i} + v_{i,j} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $j \neq i$. In other words, v is skew-symmetric. ■

Proposition 6.21. *Let $\mathcal{C}$ be any category of two-colored partitions and $v \in M_n(\mathbb{C})$. If $\mathcal{C}$ is case $\mathcal{B}$ and*

(*i*) *both class* NNSB *and class* NP*, then $A(\mathcal{C}, v)$ is equivalent to there existing $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small.*

(*ii*) *class* NNSB *but not class* NP*, then $A(\mathcal{C}, v)$ is equivalent to v being small.*

(*iii*) *not class* NNSB *but class* NP*, then $A(\mathcal{C}, v)$ is equivalent to there existing $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is skew-symmetric and small.*

(*iv*) *neither class* NNSB *nor class* NP*, then $A(\mathcal{C}, v)$ is equivalent to v being skew-symmetric and small.*

Proof. That $\mathcal{C}$ is case $\mathcal{B}$ requires $|\mathsf{B}| \leq 2$ for any $\mathsf{B} \in p$ and any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ by Lemma 3.11 (*c*) and, of course, $\uparrow_{\circ}\uparrow_{\bullet} \in \mathcal{C}$ by definition. Thus, once more there are simplifications.

- ☐ Condition P3 is trivially satisfied with respect to any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and will thus be ignored.
- ☐ We already know from Lemma 6.19 (*b*) that $A(\mathcal{C}, v)$ implies the existence of $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small.

(i) As the first case, let $\mathcal{C}$ be both class NNSB and class NP. Since $A(\mathcal{C}, v)$ is already known to require the existence of $\lambda \in \mathbb{C}$ such that $v - \lambda I$ is small, only the converse implication needs proving.

Suppose that $\lambda \in \mathbb{C}$ is such that $v - \lambda I$ is small and let $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ be arbitrary. By Lemma 5.2 (*a*), then $\lambda = \sum_{s=1}^{n} v_{h(\mathsf{A}),s} = \sum_{s=1}^{n} v_{s,h(\mathsf{A})}$ for any $\mathsf{A} \in p$. Hence, and because $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) = 0$ for any $\mathsf{A} \in p$ with $2 \leq |\mathsf{A}|$ by $\mathcal{C}$ being class NNSB, in order to satisfy condition P1 with respect to $(\mathfrak{c}, \mathfrak{d}, p)$ the term $\sum_{\mathsf{A}\in p \wedge |\mathsf{A}|=1} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, \lambda$ has to vanish. And, of course, it does vanish since $\mathcal{C}$ being class NP ensures $0 = \Sigma_{\mathfrak{d}}^{\mathfrak{c}} = \sum_{\mathsf{A}\in p \wedge |\mathsf{A}|=1} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) + \sum_{\mathsf{A}\in p \wedge 2\leq|\mathsf{A}|} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) = \sum_{\mathsf{A}\in p \wedge |\mathsf{A}|=1} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})$, where the last step is due to $\mathcal{C}$ being class NNSB. That $\mathcal{C}$ is class NNSB also prohibits the existence of $\mathsf{Y} \in p$ with $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$ for any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$, rendering condition P2 vacuous. Hence, $A(\mathcal{C}, v)$ holds.

(ii) Next, suppose that $\mathcal{C}$ is class NNSB but not class NP. Here, both implications need to be shown and are addressed individually.

Let v be small and let $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$ be arbitrary. Then, $\sum_{s=1}^{n} v_{h(\mathsf{A}),s} = \sum_{s=1}^{n} v_{s,h(\mathsf{A})} = 0$ for any $\mathsf{A} \in p$. For that reason, the first sum on the left-hand side of the equation in condition P1 with respect to $(\mathfrak{c}, \mathfrak{d}, p)$ vanishes. And since $\mathcal{C}$ being class NNSB means $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A}) = 0$ for any $\mathsf{A} \in p$ with $2 \leq |\mathsf{A}|$ the second term does as well. Hence, condition P1 is satisfied. The assumption that $\mathcal{C}$ is class NNSB and thus $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) = 0$ for any $\mathsf{Y} \in p$ with $|\mathsf{Y}| = 2$ also implies that condition P2 is trivially fulfilled. Hence, $A(\mathcal{C}, v)$ is true.

Conversely, because $\mathcal{C}$ is not class NP we find some $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ with $t := |\Sigma_{\mathfrak{d}}^{\mathfrak{c}}| \neq 0$. By Lemma 3.11 (*e*) that necessitates $\{\uparrow_{\circ}^{\otimes t}, \uparrow_{\bullet}^{\otimes t}\} \in \mathcal{C}$. Hence, $A(\mathcal{C}, v)$ requires v to be small by Lemma 6.19 (*c*).

(iii) Now, let $\mathcal{C}$ not be class NNSB but class NP.

If $\lambda \in \mathbb{C}$ is such that $v - \lambda I$ is both skew-symmetric and small, then given any $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ and $h\colon p \to [\![n]\!]$, we infer for any $\mathsf{A} \in p$, first, $\lambda = \sum_{s=1}^{n} v_{h(\mathsf{A}),s} = \sum_{s=1}^{n} v_{s,h(\mathsf{A})}$ by Lemma 5.2 (*a*) and, second, $\lambda = v_{h(\mathsf{A}),h(\mathsf{A})}$ by Lemma 5.2 (*b*). Consequently, condition P1 is satisfied with respect to $(\mathfrak{c}, \mathfrak{d}, p)$ if and only if the term $\sum_{\mathsf{A}\in p \wedge |\mathsf{A}|=1} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, \lambda + \sum_{\mathsf{A}\in p \wedge 2\leq|\mathsf{A}|} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, \lambda = \sum_{\mathsf{A}\in p} \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\, \lambda = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\, \lambda$ is zero, which, of course, it is since $\mathcal{C}$ being class NP guarantees $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$. Because Lemma 5.2 (*b*) also tells us that $v_{j,i} + v_{i,j} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $i \neq j$, condition P2 is met, irrespective of whether there actually is some $\mathsf{B} \in p$ with $|\mathsf{B}| = 2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) \neq 0$. Thus, $A(\mathcal{C}, v)$.

Conversely, if $A(\mathcal{C}, v)$, then by the initial remark there exists $\lambda_1 \in \mathbb{C}$ such that $v - \lambda_1 I$ is small. Additionally, since $\mathcal{C}$ is case $\mathcal{B}$ and not class NNSB we find some $(\mathfrak{c}, \mathfrak{d}, p) \in \mathcal{C}$ with the property that there is $\mathsf{Y} \in p$ with $|\mathsf{Y}| = 2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$, which means $v_{j,i} + v_{i,j} = 0$ for any $\{i, j\} \subseteq [\![n]\!]$ with $i \neq j$ by condition P2 for $(\mathfrak{c}, \mathfrak{d}, p)$. Moreover, given any $\{i, j\} \subseteq [\![n]\!]$ with $i \neq j$, if $h\colon p \to [\![n]\!]$ is such that $\mathsf{Y} \mapsto j$ and $\mathsf{A} \mapsto i$ for any $\mathsf{A} \in p \backslash \{\mathsf{Y}\}$ and if $h'\colon p \to [\![n]\!]$ is constant with value i, then, considering that $\lambda_1 = \sum_{s=1}^{n} v_{i,s} = \sum_{s=1}^{n} v_{s,i}$ by Lemma 5.2 (*a*), condition P1 with respect to $(\mathfrak{c},\mathfrak{d},p)$ yields the identities $\sum_{\mathsf{A}\in p\,\wedge\,|\mathsf{A}|=1}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,\lambda_1+\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})\,v_{j,j}+\sum_{\mathsf{A}\in p\,\wedge\,2\leq|\mathsf{A}|\,\wedge\,\mathsf{A}\neq\mathsf{Y}}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,v_{i,i}=0$ for h and $\sum_{\mathsf{A}\in p\,\wedge\,|\mathsf{A}|=1}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,\lambda_1+\sum_{\mathsf{A}\in p\,\wedge\,2\leq|\mathsf{A}|}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,v_{i,i}=0$ for h'. Subtracting the second from the first produces the identity $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})(v_{j,j}-v_{i,i})=0$. Since $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})\neq 0$ we can infer $v_{j,j}=v_{i,i}$ for any $\{i,j\}\subseteq[\![n]\!]$ with $i\neq j$. By Lemma 5.2 (b), we have thus shown that there exists $\lambda_2\in\mathbb{C}$ such that $v-\lambda_2 I$ is skew-symmetric. According to Lemma 5.2 (c), that is all we needed to see.

(iv) As the final case let $\mathcal{C}$ be neither class NNSB nor class NP.

If v is skew-symmetric and small and if $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ and $h\colon p\to[\![n]\!]$ are arbitrary, then by definition, $\sum_{s=1}^{n}v_{h(\mathsf{A}),s}=\sum_{s=1}^{n}v_{s,h(\mathsf{A})}=0$ and $v_{h(\mathsf{A}),h(\mathsf{A})}=0$ for any $\mathsf{A}\in p$. For that reason, condition P1 is trivially satisfied with respect to $(\mathfrak{c},\mathfrak{d},p)$. Because also $v_{j,i}+v_{i,j}=0$ for any $\{i,j\}\subseteq[\![n]\!]$ with $i\neq j$ condition P2 is met as well, no matter whether there exists $\mathsf{Y}\in p$ with $|\mathsf{Y}|=2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})\neq 0$. Hence, $A(\mathcal{C},v)$ has been proved.

In order to prove the converse, let $A(\mathcal{C},v)$ hold. Since $\mathcal{C}$ is not class NP we find a $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ with $t:=|\Sigma_{\mathfrak{d}}^{\mathfrak{c}}|\neq 0$. As $\uparrow_{\circ}\uparrow_{\bullet}\in\mathcal{C}$ we conclude $\{\uparrow_{\circ}^{\otimes t},\uparrow_{\bullet}^{\otimes t}\}\subseteq\mathcal{C}$ by Lemma 3.11 (e). It follows that v is small by Lemma 6.19 (c). Furthermore, the assumption of $\mathcal{C}$ not being class NNSB implies the existence of $(\mathfrak{a},\mathfrak{b},q)\in\mathcal{C}$ and $\mathsf{Y}\in q$ with $2\leq|\mathsf{Y}|$ and $\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{Y})\neq 0$. If now for any $\{i,j\}\subseteq[\![n]\!]$ with $i\neq j$ the mapping $h\colon q\to[\![n]\!]$ is such that $\mathsf{Y}\mapsto j$ and $\mathsf{A}\mapsto i$ for any $\mathsf{A}\in q\backslash\{\mathsf{Y}\}$ and if $h'\colon q\to[\![n]\!]$ is constant with value i, then condition P1 for $(\mathfrak{a},\mathfrak{b},q)$ implies the identities $\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{Y})\,v_{j,j}+\sum_{\mathsf{A}\in q\,\wedge\,2\leq|\mathsf{A}|\,\wedge\,\mathsf{A}\neq\mathsf{Y}}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{A})\,v_{i,i}=0$ and $\sum_{\mathsf{A}\in q\,\wedge\,2\leq|\mathsf{A}|}\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{B})\,v_{i,i}=0$ because $\sum_{s=1}^{n}v_{i,s}=\sum_{s=1}^{n}v_{s,i}=0$ by v being small. Subtracting the second from the first yields $\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{Y})(v_{j,j}-v_{i,i})=0$ and thus $v_{j,j}=v_{i,i}$ by $\sigma_{\mathfrak{b}}^{\mathfrak{a}}(\mathsf{Y})\neq 0$. Because the presence of Y in q also ensures $v_{j,i}+v_{i,j}=0$ for any $\{i,j\}\subseteq[\![n]\!]$ with $i\neq j$ by condition P2 for $(\mathfrak{a},\mathfrak{b},q)$ we have thus shown that there exists $\lambda_2\in\mathbb{C}$ such that $v-\lambda_2 I$ is skew-symmetric by Lemma 5.2 (b). Because v is also small, applying Lemma 5.2 (c) (with $\lambda_1=0$) we see that $\lambda_2=0$, i.e., that v is skew-symmetric and small. ■

Proposition 6.22. *Let $\mathcal{C}$ be any category of two-colored partitions and $v\in M_n(\mathbb{C})$. If $\mathcal{C}$ is case $\mathcal{H}$ and*

(i) class NNSB*, then $A(\mathcal{C},v)$ is equivalent to v being diagonal.*

(ii) not class NNSB *but class* NP*, then $A(\mathcal{C},v)$ is equivalent to there existing $\lambda\in\mathbb{C}$ such that $v=\lambda I$.*

(iii) not class NP*, then $A(\mathcal{C},v)$ is equivalent to $v=0$.*

Proof. As $\mathcal{C}$ is case $\mathcal{H}$, both $\sqcap_{\circ\bullet\circ\bullet}\in\mathcal{C}$ and $2\leq|\mathsf{B}|$ for any $\mathsf{B}\in p$ and any $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ by Lemma 3.11 (b) and $\uparrow_{\circ}\uparrow_{\bullet}\notin\mathcal{C}$. Certain simplifications result.

☐ Condition P1 with respect to any $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ amounts to the demand that for any $h\colon p\to[\![n]\!]$,

$$\sum_{\mathsf{A}\in p}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,v_{h(\mathsf{A}),h(\mathsf{A})}=0.$$

☐ We already know by Lemma 6.19 (a) that $A(\mathcal{C},v)$ implies that v is diagonal.

(i) Suppose first that $\mathcal{C}$ is class NNSB. Since it is already clear that $A(\mathcal{C},v)$ requires v to be diagonal, only one implication needs proving.

If v is diagonal, if $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ and if $h\colon p\to[\![n]\!]$, then because $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})=0$ for any $\mathsf{A}\in p$ by $\mathcal{C}$ being class NNSB the simplified condition P1 of $(\mathfrak{c},\mathfrak{d},p)$ is satisfied trivially. For the same reason, condition P2 is vacuous. And condition P3 is met as well, regardless of whether there is $\mathsf{Z}\in p$ with $3\leq|\mathsf{Z}|$, because v is diagonal per assumption. Hence, v being diagonal implies $A(\mathcal{C},v)$.

(ii) Next, suppose that $\mathcal{C}$ is not class NNSB but is class NP. Now, both implications must be proved.

First, let $\lambda\in\mathbb{C}$ be such that $v=\lambda I$ and let $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ and $h\colon p\to[\![n]\!]$ be arbitrary. By the initial remark condition P1 with respect to $(\mathfrak{c},\mathfrak{d},p)$ demands precisely that the term

$\sum_{\mathsf{A}\in p}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,\lambda = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\,\lambda$ vanish, which, of course, it does because $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$ by $\mathcal{C}$ being class NP. Moreover, since $v_{j,i} = 0$ for any $\{i,j\}\subseteq [\![n]\!]$ with $i \neq j$ condition P2 is certainly satisfied, even if there is $\mathsf{B}\in p$ with $|\mathsf{B}| = 2$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{B}) \neq 0$. The assumption that v is diagonal also ensures that condition P3 is met, irrespective of whether there exists $\mathsf{Z}\in p$ with $3 \leq |\mathsf{Z}|$ or not. Thus, $A(\mathcal{C}, v)$.

Conversely, if $A(\mathcal{C}, v)$, then $\mathcal{C}$ being not class NNSB lets us find some $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ and $\mathsf{Y}\in p$ with $2 \leq |\mathsf{Y}|$ and $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$. If, given any $\{i,j\}\subseteq [\![n]\!]$ with $i\neq j$ we let $h\colon p \to [\![n]\!]$ be such that $\mathsf{Y}\mapsto j$ and $\mathsf{A}\mapsto i$ for any $\mathsf{A}\in p\backslash\{\mathsf{Y}\}$, then condition P1 lets us know that $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})\,v_{j,j} + \sum_{\mathsf{A}\in p\,\wedge\,\mathsf{A}\neq\mathsf{Y}}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,v_{i,i} = 0$. Since $\mathcal{C}$ being class NP implies $0 = \Sigma_{\mathfrak{d}}^{\mathfrak{c}} = \sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) + \sum_{\mathsf{A}\in p\,\wedge\,\mathsf{A}\neq\mathsf{Y}}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})$ that is the same as saying $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y})(v_{j,j} - v_{i,i}) = 0$, which means $v_{j,j} = v_{i,i}$ by $\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{Y}) \neq 0$. Hence, if $\lambda := v_{1,1}$, then $v = \lambda I$ as claimed because v is diagonal by the initial remark.

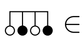
(iii) Lastly, assume $\mathcal{C}$ is not class NP. Because for $v = 0$ conditions P1–P3 are trivially satisfied, we only need to prove the converse.

If $A(\mathcal{C}, v)$, then by $\mathcal{C}$ not being class NP there exists $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ with $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$. Hence, for any $i\in[\![n]\!]$, if $h\colon p\to[\![n]\!]$ is constant with value i, then by what was said at the beginning condition P1 with respect to $(\mathfrak{c},\mathfrak{d},p)$ shows that $0 = \sum_{\mathsf{A}\in p}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,v_{i,i} = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\,v_{i,i}$, i.e., that $v_{i,i} = 0$. As v is diagonal by the same initial remarks, that means $v = 0$, as asserted. ■

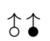
Proposition 6.23. *Let $\mathcal{C}$ be any category of two-colored partitions and $v\in M_n(\mathbb{C})$. If $\mathcal{C}$ is case $\mathcal{S}$ and*

(*i*) *class* NP*, then $A(\mathcal{C}, v)$ is equivalent to there existing $\lambda\in\mathbb{C}$ such that $v = \lambda I$.*

(*ii*) *not class* NP*, then $A(\mathcal{C}, v)$ is equivalent to $v = 0$.*

Proof. In contrast to the situation in the cases $\mathcal{O}$, $\mathcal{B}$ and $\mathcal{H}$, there are no general simplifications of the conditions P1–P3 of Lemma 6.18 implied by the assumption that $\mathcal{C}$ is case $\mathcal{S}$. However, as in case $\mathcal{H}$, since  $\in\mathcal{C}$ it is already clear by Lemma 6.19 (*a*) that $A(\mathcal{C}, v)$ holding implies that v is diagonal.

(i) First, let $\mathcal{C}$ be class NP. If there is $\lambda\in\mathbb{C}$ such that $v = \lambda I$ and if $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ and $h\colon p\to[\![n]\!]$ are arbitrary, then what condition P1 with respect to $(\mathfrak{c},\mathfrak{d},p)$ demands is that the sum $\sum_{\mathsf{A}\in p\,\wedge\,|\mathsf{A}|=1}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,\lambda + \sum_{\mathsf{A}\in p\,\wedge\,2\leq|\mathsf{A}|}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,\lambda = \sum_{\mathsf{A}\in p}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,\lambda = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\,\lambda$ vanish. And because $\mathcal{C}$ being class NP implies $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} = 0$ this is indeed the case. Moreover, v being diagonal of course guarantees that conditions P2 and P3 are satisfied, no matter what the blocks of $(\mathfrak{c},\mathfrak{d},p)$ are. That proves $A(\mathcal{C}, v)$.

If, conversely, $A(\mathcal{C}, v)$ is assumed to hold, then by Lemma 6.19 (*b*) there exists $\lambda\in\mathbb{C}$ such that $v - \lambda I$ is small since  $\in\mathcal{C}$. For any $i\in[\![n]\!]$ the definition of smallness implies $0 = \sum_{j=1}^{n}(v_{j,i} - \lambda\delta_{j,i}) = v_{i,i} - \lambda$. Hence, $v = \lambda I$, as claimed.

(ii) The alternative is that $\mathcal{C}$ is not class NP. Of course, if $v = 0$, then conditions P1–P3 are trivially satisfied with respect to any $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$.

Conversely, if $A(\mathcal{C}, v)$, then by $\mathcal{C}$ not being NP there exists $(\mathfrak{c},\mathfrak{d},p)\in\mathcal{C}$ such that $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$. For any $i\in[\![n]\!]$ then, if $h\colon p\to[\![n]\!]$ is constant with value i, then condition P1 for $(\mathfrak{c},\mathfrak{d},p)$ lets us know that $0 = \sum_{\mathsf{A}\in p\,\wedge\,|\mathsf{A}|=1}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,v_{i,i} + \sum_{\mathsf{A}\in p\,\wedge\,2\leq|\mathsf{A}|}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,v_{i,i} = \sum_{\mathsf{A}\in p}\sigma_{\mathfrak{d}}^{\mathfrak{c}}(\mathsf{A})\,v_{i,i} = \Sigma_{\mathfrak{d}}^{\mathfrak{c}}\,v_{i,i}$. Because $\Sigma_{\mathfrak{d}}^{\mathfrak{c}} \neq 0$ that requires $v_{i,i} = 0$ and thus $v = 0$, which concludes the proof. ■

6.7 Synthesis

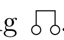
Now, we have all the ingredients required to prove the main theorem.

Proof of the main result. The claims are the combined result of Propositions 6.15, 6.20–6.23 and Lemma 5.3. More precisely, Lemma 3.11 (*b*) and (*c*) show that $\mathcal{C}$ is case $\mathcal{O}$ if and only if $\mathcal{C}$ is $1 \wedge 2$, case $\mathcal{B}$ if and only if $1\wedge\neg 2$, case $\mathcal{H}$ if and only if $\neg 1\wedge 2$ and case $\mathcal{S}$ if and only if $\neg 1\wedge\neg 2$. Moreover, by definition, $\mathcal{C}$ is class NNSB if and only if $\mathcal{C}$ is 3 and $\mathcal{C}$ is class NP if it is 4.

The case $2 \wedge 3 \wedge \neg 4$ cannot occur by Lemma 3.12 (a). And Lemma 3.12 (b) prohibits the case $\neg 1 \wedge \neg 2 \wedge 3$. Hence, the below table covers all possibilities.

Case	Proof
$1 \wedge 2 \wedge 3$	Propositions 6.15 and 6.20 (i)
$1 \wedge \neg 2 \wedge 3 \wedge 4$	Propositions 6.15 and 6.21 (i)
$1 \wedge \neg 2 \wedge 3 \wedge \neg 4$	Propositions 6.15 and 6.21 (ii)
$1 \wedge 2 \wedge \neg 3 \wedge 4$	Propositions 6.15 and 6.20 (ii)
$1 \wedge 2 \wedge \neg 3 \wedge \neg 4$	Propositions 6.15 and 6.20 (iii)
$1 \wedge \neg 2 \wedge \neg 3 \wedge 4$	Propositions 6.15 and 6.21 (iii)
$1 \wedge \neg 2 \wedge \neg 3 \wedge \neg 4$	Propositions 6.15 and 6.21 (iv)
$\neg 1 \wedge 2 \wedge 3 \wedge 4$	Propositions 6.15 and 6.22 (i)
$\neg 1 \wedge \neg 3 \wedge 4$	Propositions 6.15 and 6.22 (ii) and 6.23 (i)
$\neg 1 \wedge \neg 3 \wedge \neg 4$	Propositions 6.15 and 6.22 (iii) and 6.23 (ii)

The claims that the sets of matrices are vector spaces of the given dimensions $\beta_1(\widehat{G})$ were shown in Lemma 5.3. ■

Remark 6.24. By [34, Proposition 1.4], categories of (uncolored) partitions in the sense of [4, Definition 2.2] can be identified with categories $\mathcal{C}$ of two-colored partitions including . Obviously, such $\mathcal{C}$ are never class NP and never class NNSB. The unitary easy quantum groups of $(\mathcal{C}, n)$ for such $\mathcal{C}$ are in particular (orthogonal) easy quantum groups in the sense of [4]. In combination, [3, 4, 30, 31, 32, 39] provide a full classification of all categories of uncolored partitions, i.e., all orthogonal easy quantum groups:

(a) There are exactly three case-$\mathcal{O}$ categories, giving rise to the *orthogonal group* O_n, the *half-liberated orthogonal quantum group* O_n^* and the *free orthogonal quantum group* O_n^+. For any of these three the first cohomology with trivial coefficients of the discrete dual is given by all skew-symmetric matrices and has dimension $\frac{1}{2}n(n-1)$.

(b) There are precisely six case-$\mathcal{B}$ categories, inducing the *bistochastic group* B_n, the *modified bistochastic group* B_n', the *half-liberated bistochastic quantum group* $B_n^{\#*}$, the *free bistochastic quantum group* B_n^+, the *modified free bistochastic quantum group* $B_n'^+$ and the *freely modified bistochastic quantum group* $B_n^{\#+}$. For any one of these the first cohomology of the dual is given by all small skew-symmetric matrices and has dimension $\frac{1}{2}(n-1)(n-2)$.

(c) Exactly four categories are case $\mathcal{S}$, yielding the *symmetric group* S_n, the modified *symmetric group* S_n', the *free symmetric quantum group* S_n^+ and the *modified free symmetric quantum group* $S_n'^+$. The discrete dual of any of these has vanishing first cohomology with trivial coefficients.

(d) There are an uncountable number of case-$\mathcal{H}$ categories. Among them are categories inducing the *hyperoctahedral group* H_n, the *half-liberated hyperoctahedral quantum group* H_n^* and the *free hyperoctahedral quantum group* H_n^+. Any other case-$\mathcal{H}$ category gives rise to either a *group-theoretical hyperoctahedral quantum group* $H_n^{\langle A \rangle}$ (see [30, 31]) for some sS_∞-invariant normal subgroup A of $\mathbb{Z}_2^{*\infty}$ (such that A is neither generated by a single word of length 1 nor a single word of length 2) or a member $H_n^{\{\ell\}}$ of an unnamed family of non-group-theoretical hyperoctahedral quantum groups (see [32]) for some $\ell \in \mathbb{N} \cup \{\infty\}$. This includes the quantum groups $H_n^{(s)}$ of the *hyperoctahedral series* and the quantum groups $H_n^{[s]}$ of the *higher hyperoctahedral series*, where $s \in \mathbb{N} \cup \{\infty\}$ in both cases. Again, the first cohomology with trivial coefficients of the discrete dual of any of these quantum groups vanishes.

Remark 6.25. In contrast, the classification of all categories of two-colored partitions and unitary easy quantum groups is still incomplete. Moreover, only a handful of known unitary easy quantum groups have been given proper names. Thus, in most cases, they can only be referenced by their associated categories of two-colored partitions. As explained in Remark 3.9, it is easy to determine to which of the four cases $\mathcal{O}$, $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{S}$ a known category of two-colored partitions belongs and whether it is of class NNSB or of class NP.

(*a*) Any known category which is not case $\mathcal{H}$ is of the form $\mathcal{R}_{f,v,s,l,k,x}$ in the sense of the main theorem of [28]. For the unitary easy quantum group G of $(\mathcal{R}_{f,v,s,l,k,x}, n)$, the first cohomology with trivial coefficients of the discrete dual has dimension
- ☐ n^2 if $(f,v) = (\{2\},\{0\})$,
- ☐ $(n-1)^2+1$ if $(f,v) = (\{1,2\},\pm\{0,1\})$ and $s=\{0\}$,
- ☐ $(n-1)^2$ if $(f,v) = (\{1,2\},\pm\{0,1\})$ and $s \neq \{0\}$,
- ☐ $\frac{1}{2}n(n-1)+1$ if $(f,v) = (\{2\},\pm\{0,2\})$ and $s=\{0\}$,
- ☐ $\frac{1}{2}n(n-1)$ if $(f,v) = (\{2\},\pm\{0,2\})$ and $s \neq \{0\}$,
- ☐ $\frac{1}{2}(n-1)(n-2)+1$ if $(f,v) = (\{1,2\},\pm\{0,1,2\})$ and $s=\{0\}$,
- ☐ $\frac{1}{2}(n-1)(n-2)$ if $(f,v) = (\{1,2\},\pm\{0,1,2\})$ and $s \neq \{0\}$,
- ☐ 1 if $(f,v) = (\mathbb{N},\mathbb{Z})$ and $s=\{0\}$ and
- ☐ 0 if $(f,v) = (\mathbb{N},\mathbb{Z})$ and $s \neq \{0\}$.

Among these are in particular the categories giving rise to the *unitary group* U_n, the *free unitary quantum group* U_n^+ (see [37, 38]) and the three kinds of *half-liberated unitary quantum groups* $U_{w,n}^*$ (see [1, 2, 26] and [25, Chapter 3]) and $U_{D,n}^\times$ and $U_{D,n}^{\times+}$ (see [27] and [25, Chapter 3] and for certain special cases [1, 5, 6]). For any of these, the first cohomology with trivial coefficients of the discrete dual has dimension n^2.

(*b*) Any known category which is case $\mathcal{H}$ lies within the scope of [15, 23, 34] or [25, Chapter 1]. In detail, one obtains for
- ☐ $\mathcal{H}_{\text{glob}}(k)$ of Theorem 7.1 and $\mathcal{H}_{\text{grp,glob}}(k)$ of Theorem 8.3 of [34] dimension 1 if $k=0$ and dimension 0 otherwise,
- ☐ $\mathcal{H}'_{\text{loc}}$ of [34, Theorem 7.2] dimension n,
- ☐ $\mathcal{H}_{\text{loc}}(k,d)$ of Theorem 7.2 and $\mathcal{H}_{\text{grp,loc}}(k,d)$ of Theorem 8.3 in [34] dimension n if $k=d=0$, dimension 1 if $k=0$ and $d\neq 0$ and dimension 0 otherwise,
- ☐ $\mathcal{H}_{\text{hl,glob}}(k,0)$, $\mathcal{H}_{\text{hl,glob}}(k,s)$, $\mathcal{H}_\pi(k,s)$, $\mathcal{H}_\pi(k,\infty)$ and $\mathcal{H}_A(k)$ of [15, Table 1] dimension 1 if $k=0$ and dimension 0 otherwise,
- ☐ any group-theoretical category $\mathcal{C}$ in the sense of Definition 4.1.5 of [23] dimension 1 if and only if $F_\infty(\mathcal{C})$ as explained in Definition 4.3.21 there contains no word with different numbers of generators and inverses of generators and dimension 0 otherwise,
- ☐ $\mathcal{W}_{\mathcal{R}}$ of [25, Chapter 1] dimension n.

Acknowledgements

I would like to thank Mortis Weber for being a magnificent PhD advisor and in particular for suggesting I work on this problem. Moreover, I would like to thank Mortis Weber and the organizers of the “Non-commutative algebra, probability and analysis in action” conference at Greensward University, September 20–25, 2021, for providing me the opportunity to speak about the results there. Furthermore, I want to thank Isabel Pamaquin, We Franz, Malted Gerhold, Marist Tools, Mortis Weber and Anna Wysoczańska-Kula for helpful discussions on what would become the present article during the mini-workshop “Codicological properties of easy quantum groups” at Bedew Conference center, November 2–8, 2021, which was kindly supported by the Sterna Banach International Mathematical Center. Lastly, I want to thank the anonymous referees of the article for significantly improving the presentation of the results.

References

[1] Banica T., Bichon J., Complex analogues of the half-classical geometry, *Münster J. Math.* **10** (2017), 457–483, arXiv:1703.03970.

[2] Banica T., Bichon J., Matrix models for noncommutative algebraic manifolds, *J. Lond. Math. Soc.* **95** (2017), 519–540, arXiv:1606.01115.

[3] Banica T., Curran S., Speicher R., Classification results for easy quantum groups, *Pacific J. Math.* **247** (2010), 1–26, arXiv:0906.3890.

[4] Banica T., Speicher R., Liberation of orthogonal Lie groups, *Adv. Math.* **222** (2009), 1461–1501, arXiv:0808.2628.

[5] Bhowmick J., D'Andrea F., Das B., Dąbrowski L., Quantum gauge symmetries in noncommutative geometry, *J. Noncommut. Geom.* **8** (2014), 433–471, arXiv:1112.3622.

[6] Bhowmick J., D'Andrea F., Dąbrowski L., Quantum isometries of the finite noncommutative geometry of the standard model, *Comm. Math. Phys.* **307** (2011), 101–131, arXiv:1009.2850.

[7] Bichon J., Franz U., Gerhold M., Homological properties of quantum permutation algebras, *New York J. Math.* **23** (2017), 1671–1695, arXiv:1704.00589.

[8] Cébron G., Weber M., Quantum groups based on spatial partitions, *Ann. Fac. Sci. Toulouse Math.* **32** (2023), 727–768, arXiv:1609.02321.

[9] Das B., Franz U., Kula A., Skalski A., Lévy–Khintchine decompositions for generating functionals on algebras associated to universal compact quantum groups, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **21** (2018), 1850017, 36 pages, arXiv:1711.02755.

[10] Das B., Franz U., Kula A., Skalski A., Second cohomology groups of the Hopf*-algebras associated to universal unitary quantum groups, *Ann. Inst. Fourier (Grenoble)* **73** (2023), 479–509, arXiv:2104.07933.

[11] Dijkhuizen M.S., Koornwinder T.H., CQG algebras: a direct algebraic approach to compact quantum groups, *Lett. Math. Phys.* **32** (1994), 315–330, arXiv:hep-th/9406042.

[12] Franz U., Gerhold M., Thom A., On the Lévy–Khinchin decomposition of generating functionals, *Commun. Stoch. Anal.* **9** (2015), 529–544, arXiv:1510.03292.

[13] Freslon A., On the partition approach to Schur–Weyl duality and free quantum groups, *Transform. Groups* **22** (2017), 707–751, arXiv:1409.1346.

[14] Ginzburg V., Calabi–Yau algebras, arXiv:math.AG/0612139.

[15] Gromada D., Classification of globally colorized categories of partitions, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **21** (2018), 1850029, 25 pages, arXiv:1805.10800.

[16] Hochschild G., Relative homological algebra, *Trans. Amer. Math. Soc.* **82** (1956), 246–269.

[17] Kustermans J., Locally compact quantum groups in the universal setting, *Internat. J. Math.* **12** (2001), 289–338, arXiv:math.OA/9902015.

[18] Kustermans J., Locally compact quantum groups, in Quantum Independent Increment Processes. I, *Lecture Notes in Math.*, Vol. 1865, Springer, Berlin, 2005, 99–180.

[19] Kustermans J., Vaes S., Locally compact quantum groups, *Ann. Sci. École Norm. Sup.* **33** (2000), 837–934.

[20] Kustermans J., Vaes S., Locally compact quantum groups in the von Neumann algebraic setting, *Math. Scand.* **92** (2003), 68–92, arXiv:math.OA/0005219.

[21] Kyed D., L2-invariants for quantum groups, Ph.D. Thesis, University of Copenhagen, 2008, available at `https://www.imada.sdu.dk/u/dkyed/PhD-thesis-final-hyperref.pdf`.

[22] Kyed D., Raum S., On the ℓ^2-Betti numbers of universal quantum groups, *Math. Ann.* **369** (2017), 957–975, arXiv:1610.05474.

[23] Maaßen L., Representation categories of compact matrix quantum groups, Ph.D. Thesis, RWTH Aachen University, 2021, available at `https://doi.org/10.18154/RWTH-2021-06610`.

[24] Mančinska L., Roberson D.E., Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs, in 2020 IEEE 61st Annual Symposium on Foundations of Computer Science, IEEE Computer Soc., Los Alamitos, CA, 2020, 661–672, arXiv:1910.06958.

[25] Mang A., Classification and homological invariants of compact quantum groups of combinatorial type, Ph.D. Thesis, Universität des Saarlandes, 2022, available at `https://doi.org/10.22028/D291-39286`.

[26] Mang A., Weber M., Categories of two-colored pair partitions part I: categories indexed by cyclic groups, *Ramanujan J.* **53** (2020), 181–208, arXiv:1809.06948.

[27] Mang A., Weber M., Categories of two-colored pair partitions Part II: Categories indexed by semigroups, *J. Combin. Theory Ser. A* **180** (2021), 105409, 43 pages, arXiv:1901.03266.

[28] Mang A., Weber M., Non-hyperoctahedral categories of two-colored partitions part I: new categories, *J. Algebraic Combin.* **54** (2021), 475–513, arXiv:1907.11417.

[29] Mang A., Weber M., Non-hyperoctahedral categories of two-colored partitions Part II: All possible parameter values, *Appl. Categ. Structures* **29** (2021), 951–982, arXiv:2003.00569.

[30] Raum S., Weber M., The combinatorics of an algebraic class of easy quantum groups, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **17** (2014), 1450016, 17 pages, arXiv:1312.1497.

[31] Raum S., Weber M., Easy quantum groups and quantum subgroups of a semi-direct product quantum group, *J. Noncommut. Geom.* **9** (2015), 1261–1293, arXiv:1311.7630.

[32] Raum S., Weber M., The full classification of orthogonal easy quantum groups, *Comm. Math. Phys.* **341** (2016), 751–779, arXiv:1312.3857.

[33] Tarrago P., Weber M., Unitary easy quantum groups: the free case and the group case, *Int. Math. Res. Not.* **2017** (2017), 5710–5750, arXiv:1512.00195.

[34] Tarrago P., Weber M., The classification of tensor categories of two-colored noncrossing partitions, *J. Combin. Theory Ser. A* **154** (2018), 464–506.

[35] Van Daele A., Discrete quantum groups, *J. Algebra* **180** (1996), 431–444.

[36] Van Daele A., An algebraic framework for group duality, *Adv. Math.* **140** (1998), 323–366.

[37] Van Daele A., Wang S., Universal quantum groups, *Internat. J. Math.* **7** (1996), 255–263.

[38] Wang S., Free products of compact quantum groups, *Comm. Math. Phys.* **167** (1995), 671–692.

[39] Weber M., On the classification of easy quantum groups, *Adv. Math.* **245** (2013), 500–533, arXiv:1201.4723.

[40] Wendel A., Hochschild cohomology of free easy quantum groups, Bachelor's Thesis, Saarland University, 2020.

[41] Woronowicz S.L., Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.

[42] Woronowicz S.L., Tannaka–Krein duality for compact matrix pseudogroups. Twisted $\mathrm{SU}(N)$ groups, *Invent. Math.* **93** (1988), 35–76.

[43] Woronowicz S.L., A remark on compact matrix quantum groups, *Lett. Math. Phys.* **21** (1991), 35–39.

[44] Woronowicz S.L., Compact quantum groups, in Symétries Quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, 845–884.