An Algebra of Elliptic Commuting Variables and an Elliptic Extension of the Multinomial Theorem

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Abstract. We introduce an algebra of elliptic commuting variables involving a base q, nome p, and $2r$ noncommuting variables. This algebra, which for $r = 1$ reduces to an algebra considered earlier by the author, is an elliptic extension of the well-known algebra of r q-commuting variables. We present a multinomial theorem valid as an identity in this algebra, hereby extending the author's previously obtained elliptic binomial theorem to higher rank. Two essential ingredients are a consistency relation satisfied by the elliptic weights and the Weierstraß type A elliptic partial fraction decomposition. From the elliptic multinomial theorem we obtain, by convolution, an identity equivalent to Rosengren's type A extension of the Frenkel–Turaev ${}_{10}V_9$ summation. Interpreted in terms of a weighted counting of lattice paths in the integer lattice $\mathbb{Z}^r$, this derivation of Rosengren's A_r Frenkel–Turaev summation constitutes the first combinatorial proof of that fundamental identity.

Key words: multinomial theorem; commutation relations; elliptic weights; elliptic hypergeometric series

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1 Introduction

A standard and powerful tool in algebraic combinatorics is the identification of a class of combinatorial objects with words of noncommuting variables in some monoid. Such a correspondence is convenient when the combinatorics of the objects translates, on the algebraic side, to commutation relations satisfied by the variables. For instance, semi-standard Young tableaux are in one-to-one correspondence with words in the *plactic monoid* (cf. [9]), the monoid consisting of all words in an alphabet of totally ordered variables modulo *Knuth equivalence*. Another example, one which we shall consider in this paper in the "elliptic" setting, is the correspondence between positively oriented lattice paths of length m in the r-dimensional integer lattice and words of length m in r variables built from the successive steps of the path. In the context of weighted enumeration (typically with respect to the *area* of the path, or a similar statistic), commutation relations can be used in connection with *normalization* to determine the weight of a path. A well-known example—intimately tied to q-combinatorics—is the monoid of q-commuting variables in $X_1, \dots, X_r$, satisfying the q-commutation relations $X_jX_i = qX_iX_j$, for $1 \le i < j \le r$, where q is an indeterminate. (In Section 3, we consider an elliptic algebra that generalizes exactly this monoid.) Rather than restricting ourselves just to a monoid, as we are interested in generating functions we consider formal linear combinations of elements of the monoid, thus we work in a unital associative algebra. This has the advantage that by exploiting certain properties (such

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as associativity) of the elements in the algebra one is able to prove combinatorial identities by entirely algebraic means. For instance, working in the just mentioned algebra of q-commuting variables, a proof of the q-multinomial theorem (which is a particular summation theorem for multivariate basic hypergeometric series) can be given easily.

The aim of this paper is to develop appropriate algebraic machinery that fits with the theory of A_r elliptic hypergeometric series. Specifically, we introduce an algebra of elliptic commuting variables involving a base q, nome p, and $2r$ noncommuting variables. This algebra, which for $r=1$ reduces to an algebra considered earlier by the author [16], is an elliptic extension of the aforementioned well-known algebra of r q-commuting variables. We present a multinomial theorem valid as an identity in this algebra, hereby extending the author's previously obtained elliptic binomial theorem from [16] to higher rank. From the elliptic multinomial theorem, we obtain, by convolution, an identity equivalent to Rosengren's type A extension of the Frenkel–Turaev ${}_{10}V_9$ summation. Interpreted in terms of a weighted counting of lattice paths in the integer lattice $\mathbb{Z}^r$, this derivation of Rosengren's A_r Frenkel–Turaev summation constitutes the first combinatorial proof of that fundamental identity.

After having explained the gross outline of the paper, we briefly explain what elliptic hypergeometric series are, as the main application of the elliptic algebra introduced in this paper concerns identities for such series. *Elliptic hypergeometric series* form a natural extension of ordinary and of basic hypergeometric series. Consider the series $S=\sum_{k\ge 0} c(k)$, and $g(k):=c(k+1)/c(k)$ being the quotient of two consecutive terms of S. The series S is by definition an ordinary (or "rational") hypergeometric series if the ratio $g(k)$ is a rational function in the summation index k. Similarly, S is a basic (or "q-", or "trigonometric") hypergeometric series if $g(k)$ is a rational function in q^k (the base q usually satisfying $|q|<1$). Finally, S is an elliptic hypergeometric series if $g(k)$ is an *elliptic* function in k (by which one understands a complex-valued function that is doubly-periodic and meromorphic.) Elliptic hypergeometric series made their first implicit appearance in 1987 in the work of the mathematical physicists Date, Jimbo, Kuniba, Miwa and Okado [4] as elliptic 6-j symbols, representing elliptic solutions of the Yang–Baxter equation. Ten years later, Frenkel and Turaev [5], by exploiting the tetrahedral symmetries of those 6-j symbols and making the expressions explicit, wrote out the first identities for (what they called) "modular hypergeometric series" (now commonly called *elliptic hypergeometric series*). In particular, they discovered what is now called the ${}_{12}V_{11}$ transformation (an elliptic extension of Bailey's very-well-poised ${}_{10}\phi_9$ transformation) and, by applying specialization, the ${}_{10}V_9$ summation (which is an elliptic extension of Jackson's very-well-poised ${}_8\phi_7$ summation).

We start with explaining some important notions from the theory of elliptic hypergeometric series (cf. [6, Chapter 11] and [13]) which we shall need. Let $\mathbb{C}^\times:=\mathbb{C}\setminus\{0\}$. Let the *modified Jacobi theta function* (in short: *theta function*) with argument x and fixed nome p be defined by $\theta(x)=\theta(x;p):=(x;p)_\infty(p/x;p)_\infty$, $\theta(x_1,\dots,x_m):=\prod_{i=1}^m\theta(x_i)$, where $x,x_1,\dots,x_m,p\in\mathbb{C}^\times$, $|p|<1$, and $(x;p)_\infty=\prod_{k=0}^\infty(1-xp^k)$ is an infinite p-shifted factorial.

The theta function satisfies the following simple properties, namely the *inversion*

$$\theta(x)=-x\theta(1/x), \tag{1.1a}$$

the *quasi-periodicity*

$$\theta(px)=-\frac{1}{x}\theta(x), \tag{1.1b}$$

and the *three-term addition formula* (cf. [20, p. 451, Example 5])

$$\theta(xy,x/y,uv,u/v)-\theta(xv,x/v,uy,u/y)=\frac{u}{y}\theta(yv,y/v,xu,x/u). \tag{1.2}$$

The addition formula in (1.2) is a special case of the following more general identity due to Weierstraß (cf. [20, p. 451, Example 3]), which we refer to as the *elliptic partial fraction identity* of type A: let $a_1, \dots, a_r, b_1, \dots, b_r \in \mathbb{C}^\times$, then

$$\sum_{i=1}^{r} \frac{\prod_{j=1}^{r} \theta(a_i/b_j)}{\prod_{j\neq i} \theta(a_i/a_j)} = 0, \tag{1.3}$$

under the assumption that the *elliptic balancing condition* $a_1 \cdots a_r = b_1 \cdots b_r$ holds. The addition formula in (1.2) is a rewriting of the $r = 3$ special case of (1.3). While the relation in (1.2) serves as key ingredient in the theory of elliptic hypergeometric series, the partial fraction decomposition in (1.3) is underlying the theory of multivariate elliptic hypergeometric series associated to the root system A_{r-1} (cf. [12, 14]). Indeed, in the theory of (multivariate) elliptic hypergeometric series inductive proofs and functional equations typically make use of the identities in (1.2) and (1.3) (or, in the setting of root systems other than A_{r-1}, of other suitable elliptic partial fraction identities which exist).

Now define the *theta shifted factorial* (or *q,p-shifted factorial*) by

$$(a;q,p)_n := \begin{cases} \prod_{k=0}^{n-1} \theta\big(aq^k\big), & n = 1, 2, \dots, \\ 1, & n = 0, \\ 1/\prod_{k=0}^{-n-1} \theta\big(aq^{n+k}\big), & n = -1, -2, \dots. \end{cases}$$

For compact notation, we write

$$(a_1, a_2, \dots, a_m; q, p)_n := \prod_{k=1}^{m} (a_k; q, p)_n.$$

Notice that for $p = 0$ one has $\theta(x; 0) = 1 - x$, in which case $(a; q, 0)_n = (a; q)_n$ is a *q-shifted factorial* in base q (cf. [6]).

Notice that

$$(pa; q, p)_n = (-1)^n a^{-n} q^{-\binom{n}{2}} (a; q, p)_n,$$

which follows from repeated use of (1.1b). A list of other useful identities for manipulating the q,p-shifted factorials is given in [6, Section 11.2].

By definition, a function $g(u)$ is *elliptic* if it is a doubly-periodic meromorphic function of the complex variable u.

Without loss of generality, one may assume (see [13, Theorem 1.3.3]) that

$$g(u) = \frac{\theta(a_1 q^u, a_2 q^u, \dots, a_s q^u)}{\theta(b_1 q^u, b_2 q^u, \dots, b_s q^u)} z$$

(i.e., g is an abelian function of some degree s, cf. [1]), for a constant z and some $a_1, a_2, \dots, a_s$, $b_1, b_2, \dots, b_s$, $q, p \in \mathbb{C}^\times$ with $|p| < 1$, where the elliptic balancing condition, namely

$$a_1 a_2 \cdots a_s = b_1 b_2 \cdots b_s,$$

holds. If one writes $q = \mathrm{e}^{2\pi\sqrt{-1}\sigma}$, $p = \mathrm{e}^{2\pi\sqrt{-1}\tau}$, with complex σ, τ, then $g(u)$ is indeed periodic in u with periods σ^{-1} and $\tau\sigma^{-1}$ (which can be verified by applying (1.1b) to each of the $2s$ theta functions appearing in $g(u)$). Keeping this notation for p and q, we denote the *field of elliptic functions* over $\mathbb{C}$ of the complex variable u, with the two periods σ^{-1} and $\tau\sigma^{-1}$, by $\mathbb{E}_{q,p}(q^u)$.

(We use the notation $\mathbb{E}_{q,p}(q^u)$ instead of $\mathbb{E}_{q,p}(u)$ as we work with *multiplicatively* denoted theta functions.) More generally, we denote the *field of totally elliptic multivariate functions* over $\mathbb{C}$ of the complex variables $u_1,\dots,u_n$, in each variable with two equal periods, σ^{-1} and $\tau\sigma^{-1}$, by $\mathbb{E}_{q,p}(q^{u_1},\dots,q^{u_n})$. The notion of totally elliptic multivariate functions was first introduced by Spiridonov [18, 19].

Recall that an (ordinary) *hypergeometric series* is a series $\sum_{k\geq 0} c_k$ with $c_0=1$ such that $g(k):=c_{k+1}/c_k$ is a rational function in k. Further, a *basic hypergeometric series* (also called *q-hypergeometric series*) is a series $\sum_{k\geq 0} c_k$ with $c_0=1$ such that $g(k):=c_{k+1}/c_k$ is a rational function in q^k. Similarly, an *elliptic hypergeometric series* is defined to be a series $\sum_{k\geq 0} c_k$ with $c_0=1$ such that $g(k):=c_{k+1}/c_k$ is an *elliptic function* in k (viewed as a complex variable). The definition of an elliptic hypergeometric series extends that of a basic hypergeometric series, assuming that the rational function in q^k that appears in the definition of the basic hypergeometric series is a ratio of two polynomials $\alpha\big(q^k\big)=\sum_{j=0}^s \alpha_j q^{kj}$ and $\beta\big(q^k\big)=\sum_{j=0}^s \beta_j q^{kj}$ of equal degree s, with non-vanishing constant terms such that the (polynomial) balancing condition $\alpha_s/\alpha_0=\beta_s/\beta_0$ holds.

We conclude our brief introduction by explicitly reproducing Frenkel and Turaev's ${}_{10}V_9$ summation [5] (see also [6, equation (11.4.1)]), an identity which is fundamental to the theory of elliptic hypergeometric series: Let $m\in\mathbb{N}_0$ and $a,b,c,d,e,q,p\in\mathbb{C}$ with $|p|<1$. Then there holds the following identity:

$$\sum_{k=0}^{m}\frac{\theta\big(aq^{2k}\big)}{\theta(a)}\frac{(a,b,c,d,e,q^{-m};q,p)_k}{\big(q,aq/b,aq/c,aq/d,aq/e,aq^{m+1};q,p\big)_k}q^k = \frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_m}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_m}, \tag{1.4}$$

where $a^2q^{m+1}=bcde$. It is easy to see that the series in (1.4) is indeed an elliptic hypergeometric series. The convention of referring to the above series as a ${}_{10}V_9$ series follows the arguments in [17] and has become standard (see also [6, Chapter 11]).

The rest of the paper is organized as follows: In Section 2, we introduce the specific elliptic weights which we use and define corresponding elliptic binomial and multinomial coefficients. The elliptic binomial coefficients considered here are those which we introduced in [15] in the context of lattice path enumeration. They also appeared as the normal form coefficients in an elliptic extension of the binomial theorem, featured in [16, Section 4]. (Different elliptic binomial coefficients were considered by Rains [11, Definition 11] and implicitly also by Coskun and Gustafson [3], both in the context of convolutions for families of multivariate special functions that are recursively defined by vanishing properties and a branching rule.) The elliptic multinomial coefficients, defined in the same section, extend our elliptic binomial coefficients and are new. What is interesting about our specific elliptic weights in (2.1) is that they satisfy a certain consistency relation in (2.5) that is reminiscent of the dynamical Yang–Baxter equation but has a very simple form as it involves only scalars. Our analysis in the subsequent section crucially depends on this consistency relation. In Section 3, we introduce an algebra of elliptic commuting variables for which we identify a natural basis. We also look at specific commutation relations useful for normalization of the elements of the algebra. We then turn to another main result of the paper in Section 4, featuring an elliptic extension of the multinomial theorem, valid as an identity in the just introduced algebra of elliptic commuting variables. The new result extends the elliptic binomial theorem from [16, Section 4] to higher rank. In Section 5, we show how our elliptic multinomial theorem can be used to rederive Rosengren's [12, Theorem 5.1] A_r extension of the Frenkel–Turaev summation, which in the basic case was first obtained by Milne [10, Theorem 6.17]. In a concluding remark we explain how our algebraic derivation of the A_r Frenkel–Turaev summation admits a direct combinatorial interpretation in terms of elliptic weighted lattice paths in the integer lattice $\mathbb{Z}^r$.

2 Elliptic weights, elliptic binomial and multinomial coefficients

For indeterminates a, b, complex numbers q, p (with $|p|<1$), and integers s, t, we define the *small elliptic weights* by

$$w_{a,b;q,p}(s,t) := \frac{\theta\big(aq^{s+2t}, bq^{2s+t-2}, aq^{t-s-1}/b\big)}{\theta\big(aq^{s+2t-2}, bq^{2s+t}, aq^{t-s+1}/b\big)}q. \tag{2.1}$$

The corresponding *big elliptic weights* are defined by

$$W_{a,b;q,p}(s,t) := \prod_{j=1}^{t} w_{a,b;q,p}(s,j) = \frac{\theta\big(aq^{s+2t}, bq^{2s}, bq^{2s-1}, aq^{1-s}/b, aq^{-s}/b\big)}{\theta\big(aq^{s}, bq^{2s+t}, bq^{2s+t-1}, aq^{1+t-s}/b, aq^{t-s}/b\big)}q^{t}. \tag{2.2}$$

Clearly, $W_{a,b;q,p}(s,0)=1$, for all s. For $p\to 0$ followed by $a\to 0$ and $b\to 0$ (or $p\to 0$ followed by $b\to\infty$ and $a\to\infty$), the small elliptic weights $w_{a,b;q,p}(s,t)$ all reduce to q and the big elliptic weights $W_{a,b;q,p}(s,t)$ reduce to q^t. For convenience, we also define the following shifted variant of a big elliptic weight,

$$W^{(\rho)}_{a,b;q,p}(s,t) := W_{aq^{2\rho},bq^{2\rho};q,p}(s,t), \tag{2.3}$$

and further the *big Q-weights* by the product

$$\begin{aligned} Q_{a,b;q,p}(\ell,\rho,s,t) &:= \prod_{i=1}^{\ell} W^{(\rho)}_{a,b;q,p}(i+s,t) \\ &= \frac{\big(aq^{1+2\rho+s+t};q,p\big)_\ell \big(bq^{1+2\rho+2s};q,p\big)_{2\ell} \big(aq^{1-\ell-s}/b, aq^{-\ell-s}/b;q,p\big)_\ell}{\big(aq^{1+2\rho+s};q,p\big)_\ell \big(bq^{1+2\rho+2s+t};q,p\big)_{2\ell} \big(aq^{1+t-\ell-s}/b, aq^{t-\ell-s}/b;q,p\big)_\ell} q^{\ell t}. \end{aligned} \tag{2.4}$$

Assuming c to be an additional indeterminate, we would like to highlight the following relation satisfied by the small elliptic weights (2.1), for all s and t, which we refer to as the *elliptic consistency relation*

$$w_{aq^2,bq^2;q,p}(s,t)w_{a,c;q,p}(s,t)w_{bq^2,cq^2;q,p}(s,t) = w_{b,c;q,p}(s,t)w_{aq^2,cq^2;q,p}(s,t)w_{a,b;q,p}(s,t). \tag{2.5}$$

This specific equality of simple products is readily verified using the explicit expression for the small elliptic weights in (2.1). Equation (2.5) concerns an equality involving six weights in total. In three of the six weights the "dynamical" parameters a, b, c are shifted by q^2, while in the other three weights those parameters are not shifted. If we view the weights in (2.5) as nodes in a graph, and connect the shifted weights by edges from left to right and the similarly do this separately for the unshifted weights, an overlapping of two chains becomes apparent, see Figure 1. This picture helps to remember the special form of (2.5).

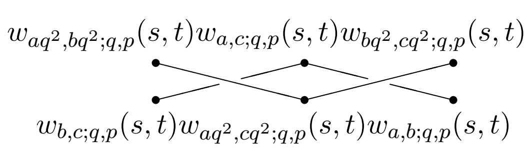

Figure 1. Interlacing of the shifted terms in the elliptic consistency relation.

The elliptic consistence relation in (2.5) guarantees the unique normalization of $X_kX_jX_i$ (for $1\le i<j<k\le r$) in the elliptic algebra that is defined in Definition 3.1, in particular it is responsible for the equality of the two expressions obtained on the right-hand sides of

equations (3.2a) and (3.2b). Equation (2.5) is reminiscent of the dynamical Yang–Baxter equation (a master equation in integrable models in statistical mechanics and quantum field theory, see [8]) but has a very simple form, as it involves only scalars (or 1×1 matrices) and no operators. Independently, to the best of our knowledge, it was not known before that (2.5) has the solution (2.1), not even in the case $p = 0$.

For indeterminates a, b, complex numbers q, p (with $|p| < 1$), and integers n, k, we define the *elliptic binomial coefficient* as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{\left(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q, p\right)_{n-k}}{\left(q, aq, bq^{1+2k}, aq/b; q, p\right)_{n-k}}. \tag{2.6}$$

This is exactly the expression for $w(\mathcal{P}((0,0) \to (k, n-k)))$ in [15, Theorem 2.1]. In [16], it was moreover shown that the elliptic binomial coefficients in (2.6) indeed appear as the coefficients in a noncommutative elliptic binomial theorem. Note that the elliptic binomial coefficient in (2.6) reduces to the usual q-binomial coefficient after taking the limits $p \to 0$, $a \to 0$, and $b \to 0$, in this order (or after taking the limits in the order $p \to 0$, $b \to \infty$, and $a \to \infty$). As pointed out in [15], the expression in (2.6) is *totally elliptic*, i.e., elliptic in each of $\log_q a$, $\log_q b$, n, and k (viewed as complex parameters), with equal periods of double periodicity. In particular, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{a,b;q,p} \in \mathbb{E}_{q,p}\left(a, b, q^n, q^k\right)$.

It is immediate from the definition of (2.6) that, for integers n, k, there holds

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n \\ n \end{bmatrix}_{a,b;q,p} = 1, \tag{2.7a}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} = 0, \qquad \text{whenever } k < 0, \text{ or } k > n. \tag{2.7b}$$

Furthermore, using the theta addition formula in (1.2) one can verify the following recursion formula for the elliptic binomial coefficients:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} + \begin{bmatrix} n \\ k-1 \end{bmatrix}_{a,b;q,p} W_{a,b;q,p}(k, n+1-k), \tag{2.7c}$$

for non-negative integers n and k.

In the above classical limit, the relations in (2.7) reduce to

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1, \qquad \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{n+1-k},$$

for positive integers n and k with $n \geq k$, which is a well-known recursion for the q-binomial coefficients.

As was shown in [16], the elliptic binomial coefficients in (2.6) can be interpreted as the (area) generating function for all lattice paths in the integer lattice $\mathbb{Z}^2$ from $(0,0)$ to $(k, n-k)$ consisting of East and North steps of unit length where each path is weighted with respect to the product of the weights of the respective squares covered by the path. In this interpretation, the weight of the single square with north-east corner (s,t) is given by $w_{a,b;q,p}(s,t)$, whereas $W_{a,b;q,p}(s,t)$ can be regarded as the weight of the s-th column having height t.

To prepare the reader for a better understanding of the main result of this paper, namely the elliptic multinomial theorem in Section 4, it will be convenient to recall the author's elliptic binomial theorem from [16, Theorem 2]. We start with the definition of the algebra of elliptic commuting variables in which the elliptic binomial coefficients manifestly appear as the coefficients in a binomial expansion after normal ordering of the respective variables.

Definition 2.1. For two nonzero complex numbers q and p with $|p|<1$, let $\mathbb{C}_{q,p}[X,Y;a,b]$ denote the associative unital algebra over $\mathbb{C}$, generated by X, Y, satisfying the following three relations:

$$YX = W_{a,b;q,p}(1,1)XY, \qquad Xf(a,b) = f\big(aq,bq^2\big)X, \qquad Yf(a,b) = f\big(aq^2,bq\big)Y,$$

for all $f \in \mathbb{E}_{q,p}(a,b)$.

We refer to the variables X, Y, a, b forming $\mathbb{C}_{q,p}[X,Y;a,b]$ as *elliptic commuting* variables. The algebra $\mathbb{C}_{q,p}[X,Y;a,b]$ reduces to $\mathbb{C}_q[X,Y]$ if one formally lets $p\to 0$, $a\to 0$, then $b\to 0$, in this order, or lets $p\to 0$, $b\to\infty$, then $a\to\infty$, in this order, while (having eliminated the nome p) relaxing the condition of ellipticity. It should be noted that the monomials X^kY^l form a basis for the algebra $\mathbb{C}_{q,p}[X,Y;a,b]$ as a left module over $\mathbb{E}_{q,p}(a,b)$, i.e., any element can be written uniquely as a finite sum $\sum_{k,l\geq 0} c_{kl}X^kY^l$ with $c_{kl}\in\mathbb{E}_{q,p}(a,b)$ which we call the *normal form* of the element.

The following result from [16, Theorem 2] shows that the normal form of the binomial $(X+Y)^n$ is "nice"; each coefficient to the left of X^kY^{n-k} completely factorizes as an expression in $\mathbb{E}_{q,p}(a,b)$.

Theorem 2.2 (binomial theorem for variables in $\mathbb{C}_{q,p}[X,Y;a,b]$)**.** *Let $n\in\mathbb{N}_0$. Then, as an identity in $\mathbb{C}_{q,p}[X,Y;a,b]$, we have*

$$(X+Y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} X^kY^{n-k}.$$

In [16, Corollary 4], convolution was applied to this result (together with comparison of coefficients) yielding the Frenkel and Turaev ${}_{10}V_9$ summation [5] in a form equivalent to (1.4) by analytic continuation.

Remark 2.3. In the recent work [7, Definition 5.6 and Theorem 5.7], the author, in collaboration with Hoshi, Katori, and Koornwinder, defined a similar but different elliptic commuting algebra with a corresponding binomial theorem.

Before we extend the elliptic binomial coefficients in (2.6) to elliptic multinomial coefficients, we rewrite the elliptic partial fraction decomposition (1.3) in a form that is suitable for our purpose. Replacing r by $r+1$ in (1.3), isolating the $(r+1)$-th term of the sum, putting the first r terms to the other side and dividing both sides of the equation by the $(r+1)$-th term and using (1.1a), we obtain the following form of the type A elliptic partial fraction identity

$$1 = \frac{\prod_{j=1}^{r}\theta(a_{r+1}/a_j)}{\prod_{j=1}^{r+1}\theta(a_{r+1}/b_j)} \sum_{i=1}^{r} \frac{\prod_{j=1}^{r+1}\theta(b_j/a_i)}{\prod_{\substack{1\leq j\leq r+1\\ j\neq i}}\theta(a_j/a_i)}, \tag{2.8}$$

now subject to the elliptic balancing condition $a_1\cdots a_{r+1} = b_1\cdots b_{r+1}$.

We are ready to define (for the first time) elliptic multinomial coefficients. Let $r>1$ be an integer and $a_1,\dots,a_r\in\mathbb{C}^\times$ be variables. Further, let $k_1,\dots,k_r$ be integers satisfying $k_1+\cdots+k_r\geq 0$. Here and throughout, we write $K_i := \sum_{\nu=1}^{i} k_\nu$, for $i=0,\dots,r$, and we will later similarly use the notations $N_i := \sum_{\nu=1}^{i} n_\nu$ and $L_i := \sum_{\nu=1}^{i} l_\nu$. We define the elliptic multinomial coefficients explicitly as

$$\begin{bmatrix} k_1+\cdots+k_r \\ k_1,\dots,k_r \end{bmatrix}_{a_1,\dots,a_r;q,p} := \frac{(q;q,p)_{k_1+\cdots+k_r}}{\prod_{i=1}^{r}(q;q,p)_{k_i}} \prod_{i=1}^{r} \frac{\big(a_iq^{1+K_r-k_i};q,p\big)_{k_i}}{\big(a_iq^{1+2K_{i-1}};q,p\big)_{k_i}} \prod_{1\leq i<j\leq r} \frac{\big(a_iq^{1-k_i}/a_j;q,p\big)_{k_j}}{(a_iq/a_j;q,p)_{k_j}}. \tag{2.9}$$

For $r=2$, the elliptic multinomial coefficients $\left[\begin{smallmatrix}k_1+k_2\\k_1,k_2\end{smallmatrix}\right]_{a_1,a_2;q,p}$ reduce to the elliptic binomial coefficients $\left[\begin{smallmatrix}k_1+k_2\\k_1\end{smallmatrix}\right]_{a_1,a_2;q,p}$ (which in general is different from $\left[\begin{smallmatrix}k_1+k_2\\k_2\end{smallmatrix}\right]_{a_1,a_2;q,p}$) given in (2.6). That is, for $r=2$ we have two short notations for the elliptic multinomial coefficients in (2.9), just as in the familiar ordinary case.

The elliptic multinomial coefficients in (2.9) satisfy

$$\begin{bmatrix}0\\0,\dots,0\end{bmatrix}_{a_1,\dots,a_r;q,p}=1,$$

and (remember that we are assuming $k_1+\dots+k_r\geq 0$)

$$\begin{bmatrix}k_1+\dots+k_r\\k_1,\dots,k_r\end{bmatrix}_{a_1,\dots,a_r;q,p}=0,\qquad \text{whenever } k_j<0 \text{ for some } j=1,\dots,r,$$

and for $k_1+\dots+k_r>0$ the recurrence relation

$$\begin{bmatrix}k_1+\dots+k_r\\k_1,\dots,k_r\end{bmatrix}_{a_1,\dots,a_r;q,p}
=\sum_{i=1}^r\begin{bmatrix}k_1+\dots+k_r-1\\k_1,\dots,k_{i-1},k_i-1,k_{i+1},\dots,k_r\end{bmatrix}_{a_1,\dots,a_r;q,p}\prod_{j>i}W^{(K_{j-1}-k_i)}_{a_i,a_j;q,p}(k_i,k_j). \tag{2.10}$$

The latter is readily established by using the elliptic partial fraction decomposition (2.8). Indeed, dividing both sides of (2.10) by the elliptic multinomial coefficient on the left-hand side and replacing the elliptic multinomial coefficients and the shifted big elliptic weights by their explicit expressions in (2.9), (2.3) and (2.2), we obtain, after cancellation of common factors, (2.8) with respect to the following simultaneous substitutions:

$$\begin{aligned}
&a_i\mapsto q^{k_i}/a_i \quad \text{for } 1\leq i\leq r, \qquad && a_{r+1}\mapsto q^{k_1+\dots+k_r},\\
&b_i\mapsto 1/a_i \quad \text{for } 1\leq i\leq r, && b_{r+1}\mapsto q^{2(k_1+\dots+k_r)}.
\end{aligned}$$

This confirms (2.10).

3 An algebra of elliptic commuting variables

Recall (from Section 1) that $\mathbb{E}_{q,p}(a_1,\dots,a_r)$ denotes the field of totally elliptic functions over $\mathbb{C}$, in the complex variables $\log_q a_i$, $1\leq i\leq r$, with equal periods σ^{-1}, $\tau\sigma^{-1}$ (where $q=\mathrm{e}^{2\pi\sqrt{-1}\sigma}$, $p=\mathrm{e}^{2\pi\sqrt{-1}\tau}$, $\sigma,\tau\in\mathbb{C}$), of double periodicity.

We shall work in the following algebra.

Definition 3.1. For $2r$ noncommuting variables $X_1,\dots,X_r$, and $a_1,\dots,a_r$, where the variables $a_1,\dots,a_r$ commute with each other, and two nonzero complex numbers q, p with $|p|<1$, let $\mathbb{C}_{q,p}[X_1,\dots,X_r;a_1,\dots,a_r]$ denote the associative unital algebra over $\mathbb{C}$, generated by $X_1,\dots,X_r$, satisfying the following relations:

$$X_jX_i=w_{a_i,a_j;q,p}(1,1)X_iX_j \qquad \text{for } 1\leq i<j\leq r, \tag{3.1a}$$

$$X_if(a_1,\dots,a_r)=f\big(a_1q^2,\dots,a_{i-1}q^2,a_iq,a_{i+1}q^2,\dots,a_rq^2\big)X_i \qquad \text{for } 1\leq i\leq r, \tag{3.1b}$$

for all $f\in\mathbb{E}_{q,p}(a_1,\dots,a_r)$, and where the elliptic weights $w_{a_i,a_j;q,p}\in\mathbb{E}_{q,p}(a_1,\dots,a_r)$ are defined in (2.1).

We refer to the $2r$ variables $X_1,\dots,X_r$, $a_1,\dots,a_r$ forming $\mathbb{C}_{q,p}[X_1,\dots,X_r;a_1,\dots,a_r]$ as *elliptic-commuting* variables.[1]

The following commutation relations, for $1\le i<j\le r$ and $1\le k\le r$, arise as a consequence of (3.1b) combined with (2.1):

$$X_i w_{a_i,a_j;q,p}(s,t) = w_{a_i,a_j;q,p}(s+1,t)X_i, \qquad X_j w_{a_i,a_j;q,p}(s,t) = w_{a_i,a_j;q,p}(s,t+1)X_j,$$
$$X_k w_{a_i,a_j;q,p}(s,t) = w_{a_iq^2,a_jq^2;q,p}(s,t)X_k \qquad \text{for } k\neq i \text{ and } k\neq j.$$

Proposition 3.2. *The monomials $X_1^{k_1}\cdots X_r^{k_r}$, for $k_1,\dots,k_r\in\mathbb{N}_0$, form a basis for the algebra $\mathbb{C}_{q,p}[X_1,\dots,X_r;a_1,\dots,a_r]$ as a left module over $\mathbb{E}_{q,p}(a_1,\dots,a_r)$. In other words, any element of the algebra can be written uniquely as a finite sum*

$$\sum_{k_1,\dots,k_r\ge 0} c_{k_1,\dots,k_r} X_1^{k_1}\cdots X_r^{k_r}$$

with $c_{k_1,\dots,k_r}\in\mathbb{E}_{q,p}(a_1,\dots,a_r)$ which we call the normal form of the element.

Proof. It is clear that any element in $\mathbb{C}_{q,p}[X_1,\dots,X_r;a_1,\dots,a_r]$ can be put into normal form using the relations in (3.1). What still needs to be shown is that the coefficients $c_{k_1,\dots,k_r}$ to the left of each of the monomials $X_1^{k_1}\cdots X_r^{k_r}$ are well defined in that they are independent of the order in which the commutation relations are applied in the normalization procedure. In other words, since we are working in a multivariate noncommutative setting, we show that in the associative algebra $\mathbb{C}_{q,p}[X_1,\dots,X_r;a_1,\dots,a_r]$ a suitable variant of Bergman's diamond lemma [2] applies. By linearity, it is enough to assume that the element consists of a finite product of the variables X_i (in some order and allowing repetitions) and some elements in $\mathbb{E}_{q,p}(a_1,\dots,a_r)$. Using (3.1b), the latter elements can all be moved to the left of the X_i (possibly creating some shifts in the a_j, $1\le j\le r$, by some powers of q) resulting in the product of a unique element in $\mathbb{E}_{q,p}(a_1,\dots,a_r)$ times a finite product of the variables X_i, which without loss of generality we may assume to be $X=\prod_{s=1}^m X_{i_s}$ (i.e., we assume m to be the degree of the monomial X) which we may refer to as a *word* of m letters.

We now sketch the reduction algorithm of the word $X=\prod_{s=1}^m X_{i_s}$ to normal form: If there are any $1\le s<m$ for which two neighboring variables (*letters*) X_{i_s} and $X_{i_{s+1}}$ are not in the right order, i.e., when $i_s>i_{s+1}$, then we pick such an s (the choice of s may not be unique) and apply a reduction of the form (3.1a) (to switch the order of X_{i_s} and $X_{i_{s+1}}$) and subsequently use (3.1b) sufficiently many times to move any newly created elliptic weights that depend on $a_1,\dots,a_r$ to the most left. This step is repeated until the indices i_s, $1\le s\le m$, are in weakly increasing order. (In the terminology of the diamond lemma, our total monomial ordering is thus the lexicographic ordering on the letters, or equivalently, on the indices of the monomials.) Now, since at each step there may be several s with $i_s>i_{s+1}$, the order of the described reductions is not unique, leading to possible *ambiguity* in eventually arriving at an irreducible form (which is a form where no further reductions can be applied). According to the diamond lemma, there are two possible ambiguities when applying reductions: *overlap ambiguity* and *inclusion ambiguity*. The special form of the defining commutation relations (3.1a) (each term is transformed to another term, there are no new terms created) shows that in our setting no inclusion ambiguity can ever arise. (See [2] for examples of rings when inclusion ambiguity can arise.) What still needs to be done is to show that each overlap ambiguity is *resolvable*. In the quadratic algebra we are considering, this reduces by the diamond lemma to the following: We only have to show the any subword of *three* variables $X=X_kX_jX_i$ with $1\le i<j<k\le r$ is reduction-unique. See Figure 2 for illustration.

[1]The algebra $\mathbb{C}_{q,p}[X_1,\dots,X_r;a_1,\dots,a_r]$ reduces to the well-known algebra of q-commuting variables, that we may denote by $\mathbb{C}_q[X_1,\dots,X_r]$, defined by $X_jX_i=qX_iX_j$ for $1\le i<j\le r$, if one formally lets $p\to 0$ and $a_1\to 0,\dots,a_r\to 0$, in this order, or lets $p\to 0$ and $a_r\to\infty,\dots,a_1\to\infty$, in this order, while (having eliminated the nome p) relaxing the conditions of ellipticity.

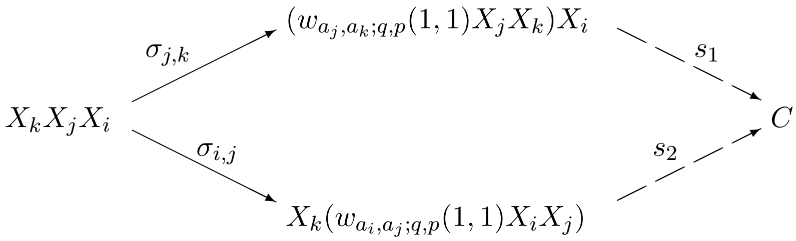

Figure 2. Diamond lemma: different reductions of the word $X_kX_jX_i$ lead to a common expression C.

In the figure $\sigma_{i,j}$ refers to the application of the commutation relation in (3.1a) with indices i and j. There are two ways to apply a simple commutation relation to $X = X_kX_jX_i$. In this first step of reduction, we can either apply $\sigma_{j,k}$ or $\sigma_{i,j}$, leading to two different reductions. We now have to show that when these reductions are further reduced (by s_1 or s_2, both representing ordered sequences of commutation relations), they lead to a common expression C. We can verify the uniqueness of C directly: On the one hand, we have

$$\begin{aligned}
X_kX_jX_i &= X_k(X_jX_i) = X_k w_{a_i,a_j;q,p}(1,1)X_iX_j \\
&= w_{a_iq^2,a_jq^2;q,p}(1,1)X_kX_iX_j \\
&= w_{a_iq^2,a_jq^2;q,p}(1,1)w_{a_i,a_k;q,p}(1,1)X_iX_kX_j \\
&= w_{a_iq^2,a_jq^2;q,p}(1,1)w_{a_i,a_k;q,p}(1,1)X_iw_{a_j,a_k;q,p}(1,1)X_jX_k \\
&= w_{a_iq^2,a_jq^2;q,p}(1,1)w_{a_i,a_k;q,p}(1,1)w_{a_jq^2,a_kq^2;q,p}(1,1)X_iX_jX_k. \qquad (3.2a)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
X_kX_jX_i &= (X_kX_j)X_i = w_{a_j,a_k;q,p}(1,1)X_jX_kX_i \\
&= w_{a_j,a_k;q,p}(1,1)X_jw_{a_i,a_k;q,p}(1,1)X_iX_k \\
&= w_{a_j,a_k;q,p}(1,1)w_{a_iq^2,a_kq^2;q,p}(1,1)X_jX_iX_k \\
&= w_{a_j,a_k;q,p}(1,1)w_{a_iq^2,a_kq^2;q,p}(1,1)w_{a_i,a_j;q,p}(1,1)X_iX_jX_k. \qquad (3.2b)
\end{aligned}$$

Comparison of the coefficients to the left of $X_iX_jX_k$ in (3.2a) and (3.2b) gives

$$\begin{aligned}
&w_{a_iq^2,a_jq^2;q,p}(1,1)w_{a_i,a_k;q,p}(1,1)w_{a_jq^2,a_kq^2;q,p}(1,1) \\
&\quad = w_{a_j,a_k;q,p}(1,1)w_{a_iq^2,a_kq^2;q,p}(1,1)w_{a_i,a_j;q,p}(1,1),
\end{aligned}$$

which is indeed true by an instance of the elliptic consistency relation (2.5). By the diamond lemma, this establishes that the reduced normalized form of any word is unique. Thus the reduced normalized form of any element of the algebra is unique, as claimed. ■

The proof of Proposition 3.2 showed that when normalizing elements of the elliptic algebra $\mathbb{C}_{q,p}[X_1,\dots,X_r;a_1,\dots,a_r]$ and hereby producing products of elliptic weights, there are different ways to write those products. For practical purposes, it will be convenient to define a preference between the different possible choices. Subsequently, we shall prefer the expression obtained in (3.2b) to that in (3.2a), as it contains less shifts of q in the parameters appearing in the weights. This choice can repeatedly be applied to larger products of weights to obtain a product of weights with smallest possible number of shifts of q.

For instance, it follows by application of (2.5) and induction that for any positive integer l, and indices $1 \le i_1 < i_2 < \dots < i_l \le r$, the following l-variable commutation relation holds

$$X_{i_l}\cdots X_{i_2}X_{i_1} = \Bigg(\prod_{1\le j<k\le l} w_{a_{i_j}q^{2(k-j-1)},a_{i_k}q^{2(k-j-1)};q,p}(1,1)\Bigg)X_{i_1}X_{i_2}\cdots X_{i_l}.$$

For bringing expressions in $\mathbb{C}_{q,p}[X_1,\dots,X_r;a_1,\dots,a_r]$ into normal form, the following lemma is useful. (While it was not needed in the proof of Proposition 3.2, it will be useful in the proofs of Theorems 4.1 and 5.1.)

Lemma 3.3. *Let $k_1,\dots,k_r$ and $l_1,\dots,l_r$ be non-negative integers. The following commutation relation holds as an identity in $\mathbb{C}_{q,p}[X_1,\dots,X_r;a_1,\dots,a_r]$*

$$X_1^{k_1}\cdots X_r^{k_r}X_1^{l_1}\cdots X_r^{l_r} = \Bigg(\prod_{1\le i<j\le r} Q_{a_i,a_j;q,p}(l_i,K_{j-1}-k_i+L_{i-1},k_i,k_j)\Bigg)X_1^{k_1+l_1}\cdots X_r^{k_r+l_r},$$

where the big Q-weights are defined in (2.4).

Proof. The identity is readily proved by multiple induction using

$$X_j^kX_i^l = Q_{a_i,a_j;q,p}(l,0,0,k)X_i^lX_j^k,$$

where $1\le i<j\le r$, for any pair of non-negative integers k and l (which is equivalent to the elliptic specialization of [16, Lemma 1]), combined with repeated application of the commutation rule (3.1b). ∎

4 An elliptic multinomial theorem

We have the following result.

Theorem 4.1 (elliptic multinomial theorem)**.** *Let $n\in\mathbb{N}_0$. Then the following identity is valid in $\mathbb{C}_{q,p}[X_1,\dots,X_r;a_1,\dots,a_r]$:*

$$(X_1+\cdots+X_r)^n = \sum_{k_1+\cdots+k_r=n}\begin{bmatrix} n \\ k_1,\dots,k_r\end{bmatrix}_{a_1,\dots,a_r;q,p} X_1^{k_1}\cdots X_r^{k_r}.$$

Proof. We proceed by induction on n. For $n=0$, the formula is trivial. Now let $n>0$ (n being fixed) and assume that we have already shown the formula for all non-negative integers less than n. We have (by separating the last factor, applying induction, applying a special case of Lemma 3.3, shifting the summation, and finally combining terms using the recurrence relation (2.10))

$$\begin{aligned}
(X_1+\cdots+X_r)^n &= (X_1+\cdots+X_r)^{n-1}(X_1+\cdots+X_r)\\
&= \sum_{k_1+\cdots+k_r=n-1}\begin{bmatrix} n-1 \\ k_1,\dots,k_r\end{bmatrix}_{a_1,\dots,a_r;q,p} X_1^{k_1}\cdots X_r^{k_r}(X_1+\cdots+X_r)\\
&= \sum_{k_1+\cdots+k_r=n-1}\sum_{i=1}^r\Bigg(\begin{bmatrix} n-1 \\ k_1,\dots,k_r\end{bmatrix}_{a_1,\dots,a_r;q,p}\Bigg(\prod_{j>i}W_{a_i,a_j;q,p}^{(K_{j-1}-k_i)}(1+k_i,k_j)\Bigg)\\
&\quad\times X_1^{k_1}\cdots X_{i-1}^{k_{i-1}}X_i^{k_i+1}X_{i+1}^{k_{i+1}}\cdots X_r^{k_r}\Bigg)\\
&= \sum_{k_1+\cdots+k_r=n}\sum_{i=1}^r\Bigg(\begin{bmatrix} n-1 \\ k_1,\dots,k_{i-1},k_i-1,k_{i+1},\dots,k_r\end{bmatrix}_{a_1,\dots,a_r;q,p}\\
&\quad\times\Bigg(\prod_{j>i}W_{a_i,a_j;q,p}^{(K_{j-1}-k_i)}(k_i,k_j)\Bigg)X_1^{k_1}\cdots X_r^{k_r}\Bigg)\\
&= \sum_{k_1+\cdots+k_r=n}\begin{bmatrix} n \\ k_1,\dots,k_r\end{bmatrix}_{a_1,\dots,a_r;q,p} X_1^{k_1}\cdots X_r^{k_r},
\end{aligned}$$

which is what was to be shown. ∎

5 Rosengren's A_r extension of the Frenkel–Turaev summation by convolution

By convolution, applied to the elliptic multinomial theorem in Theorem 4.1, we obtain the following result which turns out to be equivalent to Rosengren's A_r extension of the Frenkel–Turaev ${}_{10}V_9$ summation.

Theorem 5.1. *Let $0 \le M \le N$ be two integers, and let $n_1, \ldots, n_r \in \mathbb{N}_0$ satisfying $N_r = n_1 + \cdots + n_r = N$. Then we have*

$$\begin{aligned}
&\begin{bmatrix} N \\ n_1, \ldots, n_r \end{bmatrix}_{a_1,\ldots,a_r;q,p} \\
&\quad = \sum_{k_1+\cdots+k_r=M} \left(\begin{bmatrix} M \\ k_1, \ldots, k_r \end{bmatrix}_{a_1,\ldots,a_r;q,p} \begin{bmatrix} N-M \\ n_1-k_1, \ldots, n_r-k_r \end{bmatrix}_{a_1q^{2M-k_1},\ldots,a_rq^{2M-k_r};q,p} \right. \\
&\qquad \left. \times \prod_{1\le i<j\le r} Q_{a_i,a_j;q,p}(n_i-k_i, N_{i-1}+K_{j-1}-K_i, k_i, k_j) \right). \qquad (5.1)
\end{aligned}$$

Proof. Working in $\mathbb{C}_{q,p}[X_1, \ldots, X_r; a_1, \ldots, a_r]$, we expand $(X_1 + \cdots + X_r)^N = (X_1 + \cdots + X_r)^M (X_1 + \cdots + X_r)^{N-M}$ in two different ways and extract coefficients to the left of the monomial $X_1^{n_1} \cdots X_r^{n_r}$ where $n_1 + \cdots + n_r = N$. On the left-hand side, the expansion is achieved by a single application of Theorem 4.1, which is simply

$$(X_1 + \cdots + X_r)^N = \sum_{k_1+\cdots+k_r=N} \begin{bmatrix} N \\ k_1, \ldots, k_r \end{bmatrix}_{a_1,\ldots,a_r;q,p} X_1^{k_1} \cdots X_r^{k_r},$$

whose coefficient of $X_1^{n_1} \cdots X_r^{n_r}$ is clearly $\left[\begin{smallmatrix} N \\ n_1,\ldots,n_r \end{smallmatrix}\right]_{a_1,\ldots,a_r;q,p}$. On the right-hand side, we apply Theorem 4.1 twice and bring the expression into normal form by multiple applications of (3.1b) (to the second elliptic multinomial coefficient) and finally apply Lemma 3.3 (to bring the product of two monomials into normal form). The details are as follows

$$\begin{aligned}
&(X_1 + \cdots + X_r)^M (X_1 + \cdots + X_r)^{N-M} \\
&\quad = \sum_{k_1+\cdots+k_r=M} \begin{bmatrix} M \\ k_1, \ldots, k_r \end{bmatrix}_{a_1,\ldots,a_r;q,p} X_1^{k_1} \cdots X_r^{k_r} \\
&\qquad \times \sum_{l_1+\cdots+l_r=N-M} \begin{bmatrix} N-M \\ l_1, \ldots, l_r \end{bmatrix}_{a_1,\ldots,a_r;q,p} X_1^{l_1} \cdots X_r^{l_r} \\
&\quad = \sum_{\substack{k_1+\cdots+k_r=M \\ l_1+\cdots+l_r=N-M}} \left(\begin{bmatrix} M \\ k_1, \ldots, k_r \end{bmatrix}_{a_1,\ldots,a_r;q,p} \right. \\
&\qquad \left. \times \begin{bmatrix} N-M \\ l_1, \ldots, l_r \end{bmatrix}_{a_1q^{2M-k_1},\ldots,a_rq^{2M-k_r};q,p} X_1^{k_1} \cdots X_r^{k_r} X_1^{l_1} \cdots X_r^{l_r} \right) \\
&\quad = \sum_{\substack{k_1+\cdots+k_r=M \\ l_1+\cdots+l_r=N-M}} \left(\begin{bmatrix} M \\ k_1, \ldots, k_r \end{bmatrix}_{a_1,\ldots,a_r;q,p} \begin{bmatrix} N-M \\ l_1, \ldots, l_r \end{bmatrix}_{a_1q^{2M-k_1},\ldots,a_rq^{2M-k_r};q,p} \right. \\
&\qquad \left. \times \left(\prod_{1\le i<j\le r} Q_{a_i,a_j;q,p}(l_i, K_{j-1}-k_i+L_{i-1}, k_i, k_j) \right) X_1^{k_1+l_1} \cdots X_r^{k_r+l_r} \right).
\end{aligned}$$

Taking coefficients to the left of $x_1^{n_1} \cdots x_r^{n_r}$ evidently gives the right-hand side of (5.1). ■

The convolution identity in Theorem 5.1 can be regarded as the combinatorial form of the A_r Frenkel–Turaev summation

$$\frac{(b/a_1,\dots,b/a_{r+1};q,p)_M}{(q,bz_1,\dots,bz_r;q,p)_M} = \sum_{k_1+\dots+k_r=M} \prod_{1\le i<j\le r} \frac{q^{k_i}\theta\big(z_j q^{k_j-k_i}/z_i\big)}{\theta(z_j/z_i)} \prod_{i=1}^{r} \frac{\prod_{j=1}^{r+1}(a_j z_i;q,p)_{k_i}}{(bz_i;q,p)_{k_i}\prod_{j=1}^{r}(z_i q/z_j;q,p)_{k_i}}. \tag{5.2}$$

This identity was first obtained by Rosengren in [12, Theorem 5.1], see also [6, equation (11.7.8)]. The $r=2$ case of the identity in (5.2) is the single-sum Frenkel–Turaev summation in (1.4). The $p\to 0$ case of the summation in (5.2) was derived earlier by Milne [10, Theorem 6.17]. Now, (5.2) contains (5.1) as a special case: In (5.2), perform the following simultaneous substitutions

$$\begin{aligned} &a_i \mapsto q^{-n_i} \quad \text{for } 1\le i\le r, \qquad a_{r+1}\mapsto q^{-M},\\ &z_i \mapsto 1/a_i \quad \text{for } 1\le i\le r, \qquad b\mapsto q^{-M-N}. \end{aligned}$$

These substitutions yield (5.1) (after some rewriting). On the contrary, after rewriting the elliptic multinomial coefficients and weights in (5.1) explicitly in terms of products of theta-shifted factorials, the restriction that $n_1,\dots,n_r$ are non-negative integer parameters can be removed by repeated analytic continuation. This means that (5.1) is actually equivalent to (5.2).

Remark 5.2. While the above derivation of (5.1) involved elliptic commuting variables and algebraic manipulations, it is not difficult to give combinatorial interpretations of the respective algebraic expressions in terms of weighted lattice paths in the r-dimensional integer lattice $\mathbb{Z}^r$. The multinomial $(X_1+\dots+X_r)^N$ can be interpreted as the generating function for lattice paths starting at the origin and consisting of N unit steps where the ith of the r different unit steps increases the ith coordinate in $\mathbb{Z}^r$ by one while not changing the other coordinates. In other words, starting at the origin, after N steps, the path reaches a point in the intersection of $\mathbb{Z}^r$ with the hyperplane $z_1+\dots+z_r=N$. In this interpretation, for any r-tuple of non-negative integers $(k_1,\dots,k_r)$ whose ith component is positive, the weight of the unit step

$$(k_1,\dots,k_{i-1},k_i-1,k_{i+1},\dots,k_r)\to(k_1,\dots,k_r)$$

is defined to be

$$\prod_{i<j\le r} W^{(K_{j-1}-k_i)}_{a_i,a_j;q,p}(k_i,k_j), \tag{5.3}$$

for any $i=1,\dots,r$, in accordance with the recurrence relation of the elliptic multinomial coefficients in (2.10). Assuming the weight of a lattice path in $\mathbb{Z}^r$ to be the product of the weights (which are all of the form (5.3)) of the unit steps it is composed of, the weighted generating function of the family of all lattice paths that start at the origin $(0,\dots,0)$ and, after $N=n_1+\dots+n_r$ unit steps, end in $(n_1\dots,n_r)$, is the elliptic multinonomial coefficient

$$\begin{bmatrix} N \\ n_1,\dots,n_r \end{bmatrix}_{a_1,\dots,a_r;q,p}.$$

In this lattice path interpretation the convolution in Theorem 5.1 then concerns the generating function of paths that start at the origin $(0,\dots,0)$ and, after $N=n_1+\dots+n_r$ unit steps, end exactly in $(n_1\dots,n_r)$ but is refined according to where, after M steps (for fixed M satisfying $0\le M\le N$), the path crosses the hyperplane $z_1+\dots+z_r=M$.

Our derivation of (5.1) by convolution (which as we just explained, can be interpreted in terms of a weighted counting of lattice paths) appears to constitute the first combinatorial proof of Rosengren's A_r extension of the Frenkel–Turaev summation, an identity that is of fundamental importance in the theory of elliptic hypergeometric series associated with root systems (cf. [14]).

Remark 5.3. A natural question concerns the possible wider application of the methods developed in this paper. Concretely, it would be interesting to find a higher rank extension of the elliptic binomial theorem from [7, Definition 5.6 and Theorem 5.7] that was mentioned in Remark 2.3 and to derive a corresponding multivariate Frenkel–Turaev summation by convolution, in the same way as Theorem 5.1 was derived in this section. We find this an interesting open problem worthwhile to pursue. It is not clear whether the such obtained identity would be equivalent to Rosengren's A_r extension of the Frenkel–Turaev summation or whether it would be of a different type such as one of the multivariate Frenkel–Turaev summations listed in [14]. One could also ask whether the elliptic binomial coefficients from [3] and [11] can be identified as the normal form coefficients in a suitably defined algebra of elliptic commuting variables (with developments parallel to those of this paper).

Furthermore, one can simply ask whether any of the other multivariate Frenkel–Turaev summations in [14] admit similar algebraic or combinatorial interpretations as (5.2) does. We currently have no idea whether this is possible and would find this question rather difficult (or challenging) to answer affirmatively. Specifically, when considering lattice paths in the C_r case (see [12, Theorem 7.1] for Rosengren's C_r extension of the Frenkel–Turaev sum), it may well be that instead of considering positively directly lattice paths in $\mathbb{Z}^r$ that are bounded by a hyperplane, one would have to allow paths that move in the direction (positive or negative) of any axis. It is likely that one would then work in an associative algebra that contains non-commuting variables $C_1,\dots,C_r$, and $X_1, X_1^{-1},\dots,X_r,X_r^{-1}$, and seek an expansion of the product $\prod_{i=1}^r \big(X_i + C_i X_i^{-1}\big)^{n_i}$ in terms of normalized monomials. At the moment this is all speculative. Further research is needed to determine whether the methods developed in this paper can indeed be applied in the setting of root systems other than A_r.

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