Symmetric Separation of Variables for the Extended Clebsch and Manakov Models

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Abstract. In the present paper, using a modification of the method of vector fields Z_i of the bi-Hamiltonian theory of separation of variables (SoV), we construct *symmetric non-Stäckel* variable separation for three-dimensional extension of the Clebsch model, which is equivalent (in the bi-Hamiltonian sense) to the system of interacting Manakov (Schottky–Frahm) and Euler tops. For the obtained symmetric SoV (contrary to the previously constructed asymmetric one), all curves of separation are the same and have genus five. It occurred that the difference between the symmetric and asymmetric cases is encoded in the different form of the vector fields Z used to construct separating polynomial. We explicitly construct coordinates and momenta of separation and Abel-type equations in the considered examples of symmetric SoV for the extended Clebsch and Manakov models.

Key words: integrable system; separation of variables; anisotropic top

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1 Introduction

1.1 Generalities

Completely integrable Hamiltonian systems have been the objects of constant interest in theoretical and mathematical physics for more than one hundred years. Nevertheless, many problems in this theory still remain unsolved. One of the most important problems to be solved in general is the problem of variable separation. The separated variables q_i, p_j are a set of (quasi)canonical coordinates such that the following system of equations (equations of separation) is satisfied [15]:

$$\Phi_i(q_i, p_i, I_1, \dots, I_n, C_1, \dots, C_m) = 0, \qquad i \in \{1, \dots, n\}.$$

Here Φ_i are certain functions, I_k are Poisson-commuting integrals of motion, C_i are Casimir functions and n is half of the dimension of the phase space. The separated coordinates provide a possibility to solve explicitly the Hamilton equations of motion upon resolving the Abel–Jacobi inversion problem. Separated variables are also important when solving quantum integrable models [15]. That is why separation of variables (SoV) is a central issue in the theory of classical and quantum integrable systems.

The most simple and investigated is the so-called Stäckel-type SoV, when all the equations of separation are linear in the integrals and Casimir functions. In the present paper, we are interested in *non-Stäckel* SoV that naturally arises for the integrable systems associated with the Lie algebras $\mathfrak{so}(2n)$.

There exist three main approaches to the variable separation: a classical one going back to the papers of Stäckel [22, 23], Levi-Civita [13] and Agostinelli [1] and developed later in the papers of Benenti and his school [2, 3, 4] and two modern ones. They are: the “magic recipe” of

Sklyanin [15] and the bi-Hamiltonian approach of Magri, Falqui and Pedroni [8, 9, 10]. In the present paper, we develop the bi-Hamiltonian approach to SoV theory in its formulation based on the theory of the vector fields Z_i [8, 9, 10]. The theory of the vector fields Z_i permits to encode the information on separated coordinates q_i for the integrable bi-Hamiltonian systems into a set of differential conditions on the vector fields Z_i that are satisfied by them with respect to two compatible Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ [8, 9, 10].

The bi-Hamiltonian approach to SoV (despite the mathematical beauty and good generality) has a weak point: in order to find the vector fields Z_i, one should resolve a complicated system of nonlinear PDEs. In some cases, this difficulty can be overcome. Indeed, in a series of our papers [14, 16, 18, 19] we have proposed to impose a certain simplifying condition onto one of the vector fields Z_i, which transforms part of the PDEs from [8, 9, 10] into algebraic equations.

Another problem is in the fact that the PDEs from [8, 9, 10] "algebrized" in [14, 16, 18, 19] imply that the variable separation are of *Stäckel type*, i.e., that all equations of separation are *linear* in the integrals of motion and Casimir functions. For *non-Stäckel SoV* (when some of the equations of separation are *nonlinear* in some of the integrals), the corresponding equalities from [8, 9, 10] do not hold true. Here the natural question arises: how to proceed with the method of the vector fields Z_i in this case? What differential equation to "algebrize"?

In our previous paper [20], we have answered the above question for a special subcase of general non-Stäckel SoV. For this purpose we have assumed certain special form of the equations of separation and shown that under this condition there exists an invariance vector field Z (combination of the vector fields Z_i, $i \in \{1, \dots, m\}$) that satisfies a differential condition replacing the equations on Z_i obtained in [8, 9, 10] from the Stäckel-type assumption on the equations of separation. In order to be able to solve this condition we assume (similar to what was done in the Stäckel case [14, 16, 18, 19]) that the vector field Z is a special one: it annihilates all its components in the initial system of coordinates. We call such vector fields "algebraic". As a result, we obtain a system of *quadratic algebraic equations* which is possible to solve explicitly. Moreover, it occurred that the obtained system of quadratic algebraic equations may have *different solutions* leading to *non-equivalent* separations of variables for the same integrable bi-Hamiltonian system. In the present paper, we illustrate this interesting phenomenon by the example of the extended Clebsch and Manakov models.

The extended Clebsch and Manakov model is an integrable bi-Hamiltonian system on a nine-dimensional space coinciding with a three-dimensional extension of $\mathfrak{e}^*(3)$ (with respect to the first Poisson brackets $\{\ ,\ \}_1$) or on a three-dimensional extension of $\mathfrak{so}^*(4)$ (with respect to the Poisson brackets $\{\ ,\ \}_2$). It possesses six Poisson-commuting integrals of motion $I_1 = H$, $I_2 = K$, $I_3 = L$, C_1, C_2, C_3, where C_1, C_2, C_3 are Casimir functions of the bracket $\{\ ,\ \}_1$.

The described above modification of the method of vector fields Z_i leads to a system of six quadratic equations on nine components of the vector field Z. In the paper [20], we have found one (highly non-trivial) solution of these quadratic equations. We have shown that it leads to "asymmetric" non-Stäckel SoV characterised by two different separation curves $\mathcal{C}$ and $\mathcal{K}$ of different genus. The last fact makes the solution of the Abel–Jacobi inversion problem to be very complicated [11].

In this paper, we find another (also highly nontrivial) type of solution of the quadratic equations for the components of the vector field Z and show that it leads to three pairs of (quasi)canonical separated variables p_i, q_i, $i \in \{1, 2, 3\}$, that satisfy *the same curve of separation* $\mathcal{K}$ of genus five

$$(q_i + j_1)(q_i + j_2)(q_i + j_3)p_i^4 + \left(q_i^3 C_3 + q_i^2 C_2 + q_i H + K\right)p_i^2 + \frac{1}{4}(q_i C_1 + L)^2 = 0, \tag{1.1}$$

It leads, in turn, to the Abel-type equations (quadratures) written in the differential form as follows:

$$\sum_{i=1}^{3} \frac{2q_i p_i^3 \mathrm{d}q_i}{4(q_i+j_1)(q_i+j_2)(q_i+j_3)p_i^4 - (q_iC_1+L)^2} = \mathrm{d}t_1, \tag{1.2a}$$

$$\sum_{i=1}^{3} \frac{2p_i^3 \mathrm{d}q_i}{4(q_i+j_1)(q_i+j_2)(q_i+j_3)p_i^4 - (q_iC_1+L)^2} = \mathrm{d}t_2, \tag{1.2b}$$

$$\sum_{i=1}^{3} \frac{(q_iC_1+L)p_i \mathrm{d}q_i}{4(q_i+j_1)(q_i+j_2)(q_i+j_3)p_i^4 - (q_iC_1+L)^2} = \mathrm{d}t_3, \tag{1.2c}$$

where t_1, t_2, t_3 are the parameters along the time flows of the integrals H, K and L correspondingly. The fact that the differentials in (1.2) are defined on the same curve, i.e., that the obtained SoV is "symmetric", makes the task of solution of the corresponding Abel–Jacobi inversion problem more plausible.

The new "algebraic" vector field Z written in terms of the initial dynamical variables, generated by it separating polynomial, the explicit formula for the momenta of separation, non-Stäckel equations of separation (1.1), as well as symmetric Abel-type equations (1.2) are the main results of the present article.

The structure of the present paper is the following. In Section 2, we present the general theory of SoV and the method of the vector fields Z_i in Stäckel and non-Stäckel cases. In Section 3, we describe three-dimensional extension of the Clebsch and Manakov models, in Section 4, we find the corresponding algebraic vector field Z. In Section 5, we construct symmetric SoV for the considered extensions of the Clebsch and Manakov models. In Section 6, we conclude and describe open problems.

2 Separation of variables

2.1 Generalities

Let us recall the definitions of Liouville integrability and separation of variables in the general theory of Hamiltonian systems. An integrable Hamiltonian system with n degrees of freedom is determined on a $2n$-dimensional symplectic manifold $\mathcal{M}$, embedded as a symplectic leaf in a Poisson manifold $(\mathcal{P}, \{\ ,\ \}_1)$ as a level surface of m Casimir functions C_i, by n independent nontrivial integrals I_j commuting with respect to the Poisson bracket $\{I_i, I_j\}_1 = 0$, $i,j \in \{1,\dots,n\}$. To find separated variables means to find (at least locally) a set of coordinates q_i, p_j, $i,j \in \{1,\dots,n\}$ such that there exist n relations

$$\Phi_i(q_i, p_i, I_1, \dots, I_n, C_1, \dots, C_m) = 0, \qquad i \in \{1,\dots,n\}, \tag{2.1}$$

and the coordinates q_i, p_j, $i,j \in \{1,\dots,n\}$ are (quasi)canonical

$$\{p_i, q_j\}_1 = f_i(q_i,p_i)\delta_{ij}, \qquad \{q_i,q_j\}_1 = 0, \qquad \{p_i,p_j\}_1 = 0, \qquad \forall i,j \in \{1,\dots,n\}$$

for some functions f_i, $i \in \{1,\dots,n\}$, on $\mathbb{C}^2$.

It is possible to show [17] that the coordinates of separation q_i satisfy the following equations:

$$\sum_{i=1}^{n} \frac{\partial_{I_k}\Phi_i(q_i,p_i,I_1,\dots,I_n,C_1,\dots,C_m)}{\partial_{p_i}\Phi_i(q_i,p_i,I_1,\dots,I_n,C_1,\dots,C_m)} \frac{1}{f_i(q_i,p_i)} \frac{\partial q_i}{\partial t_j} = \delta_{kj}, \qquad \forall k,j \in \{1,\dots,n\}, \tag{2.2}$$

where t_j is the "time" corresponding to the integral I_j, i.e., a parameter along its Hamiltonian flow.

From the equations (2.2), one easily deduces the Abel-type equations written in the differential form

$$\sum_{i=1}^{n} \frac{\partial_{I_k} \Phi_i(q_i, p_i, I_1, \dots, I_n, C_1, \dots, C_m)}{\partial_{p_i} \Phi_i(q_i, p_i, I_1, \dots, I_n, C_1, \dots, C_m)} \frac{\mathrm{d}q_i}{f_i(q_i, p_i)} = \mathrm{d}t_k, \qquad k \in \{1, \dots, n\}, \tag{2.3}$$

where the momenta p_i satisfy the equations of separation (2.1).

The equations (2.3) for the separated coordinates $q_1, \dots, q_n$ provide a way to the explicit integration of the equations of motion. So the key problem in the integration process is construction of the coordinates $q_1, \dots, q_n$. One of possible methods to do this is the method of the vector fields Z_i.

2.2 The method of the vector fields Z_i

The method of the vector field Z_i in the theory of separation of variables was proposed in the paper [8] and developed in the papers [9, 10]. We will expose the method in the version convenient for us.

2.2.1 The vector fields Z_i

Let us assume that we have found a set of separated coordinates $\{q_1, \dots, q_n, p_1, \dots, p_n\}$ on the generic symplectic leaf in the Poisson manifold $\mathcal{P}$. The (local) coordinates q_i, p_j are canonical with respect to the brackets $\{\ ,\ \}_1$ and satisfy some equations of separation (2.1).

We consider vector fields Z_k defined in the set of coordinates $\{q_1, \dots, q_n, p_1, \dots, p_n, C_1, \dots, C_m\}$ on the Poisson manifold $\mathcal{P}$ as follows:

$$Z_k(q_i) = 0, \qquad Z_k(p_j) = 0, \qquad i, j \in \{1, \dots, n\}, \quad k \in \{1, \dots, m\}, \tag{2.4a}$$

$$Z_k(C_l) = \delta_{kl}, \qquad k, l \in \{1, \dots, m\}. \tag{2.4b}$$

Observe that from the definition of the vector fields Z_k it immediately follows that $Z_k(Z_l(C_l)) = 0$, $i, k, l \in \{1, \dots, m\}$.

In order to proceed with the theory of the vector fields Z_i, we will assume that the separated variables q_i, p_j, $i, j \in \{1, \dots, n\}$ are the bi-Hamiltonian ones, i.e., on $\mathcal{P}$ there exists another Poisson structure $\{\ ,\ \}_2$ such that

$$\{p_i, q_j\}_2 = -q_i \delta_{ij}, \qquad \{q_i, q_j\}_2 = 0, \qquad \{p_i, p_j\}_2 = 0, \qquad \forall i, j \in \{1, \dots, n\}.$$

Now we can formulate the following theorem [20].

Theorem 2.1.

(*i*) *The vector fields Z_i are symmetries of the Poisson structure $\{\ ,\}_1$, i.e.,*

$$\mathrm{Lie}_{Z_i}\{\ ,\ \}_1 = 0, \qquad i \in \{1, \dots, m\}, \tag{2.5}$$

(*ii*) *The vector fields Z_i satisfy the following conditions with respect to the Poisson structure $\{\ ,\}_2$:*

$$\mathrm{Lie}_{Z_i}\{\ ,\ \}_2 = \sum_{j=1}^{m} Z_j \wedge [X_j, Z_i], \qquad i \in \{1, \dots, m\}, \tag{2.6}$$

where X_j is Hamiltonian vector field of C_j with respect to the second Poisson structure $\{\ ,\ \}_2$.

(*iii*) *If the functions* $\Phi_i(q_i, p_i, I_1, \dots, I_n, C_1, \dots, C_m)$ *are linear in* I_j, C_r, $j \in \{1, \dots, n\}$, $r \in \{1, \dots, m\}$, *then*

$$Z_k(Z_l(I_i)) = 0, \qquad i \in \{1, \dots, n\}, \quad k, l \in \{1, \dots, m\}. \tag{2.7}$$

Remark 2.2. The equalities (2.5), (2.6), (2.7) have been introduced from different considerations in [8]. It was proven later in [9] that together with the normalization conditions (2.4b) they are *sufficient* for the variable separation. This is a basis of the bi-Hamiltonian method in SoV theory [8, 9, 10].

Remark 2.3. Note that the condition (2.7) implies that the variable separation is of *Stäckel type*, i.e., all equations of separation are *linear* in the integrals of motion.

2.2.2 The separating polynomial

Using the above-defined vector fields Z_i, one can define the separating polynomial, whose roots are the coordinates of separation. For this purpose, it is necessary to define the so-called Poisson pencil, i.e., the linear combination of the brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$, $\{\ ,\ \}_u = u\{\ ,\ \}_1 + \{\ ,\ \}_2$. We will hereafter assume the Gelfand–Zakharevich settings [12], i.e., we will assume that the Casimir functions of $\{\ ,\ \}_u$ are polynomial in u. Let us denote these Casimir functions by $C_k(u)$, $k \in \{1, \dots, m\}$. Observe that the functions $C_k(u)$ are generating functions of the integrals and Casimirs of $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$, set of integrals $I_1, \dots, I_n$ is decomposed into subsets $\{I_{l_1}, I_{l_2}, \dots, I_{l_{n_l}}\}$, $l \in \{1, \dots, m\}$, in a certain specific way for each bi-Hamiltonian pair of bracket and enter into the functions $C_l(u)$ as follows:

$$C_l(u) = u^{n_l} C_l + u^{n_l - 1} I_{l_1} + u^{n_l - 2} I_{l_2} + \dots + I_{l_{n_l}}, \qquad l \in \{1, \dots, m\}, \tag{2.8}$$

so that $n_1 + n_2 + \dots + n_m = n$ and $I_{l_{n_l}}$ is a Casimir of $\{\ ,\ \}_2$.

The following theorem holds true [8, 9, 10].

Theorem 2.4. *Let the vector fields* Z_i *satisfy the conditions* (2.5), (2.6), (2.7) *and normalization conditions* (2.4b). *Let the roots* $u = q_i$, $i \in \{1, \dots, n\}$, *of the equation*

$$S(u) = \det(Z_i(C_j(u))) = 0, \qquad i, j \in \{1, \dots, m\},$$

be functionally independent on generic symplectic leaves of $\{\ ,\ \}_1$. *Then* q_i, $i \in \{1, \dots, n\}$, *are the coordinates of separation for the considered bi-Hamiltonian system.*

2.3 The algebraic vector field Z: the non-Stäckel case

2.3.1 The algebraic vector field Z

In the case of bi-Hamiltonian SoV, the integrals and Casimir functions enter into the equations of separation via the Casimirs $C_i(u)$, $i \in \{1, \dots, m\}$ (see formula (2.8)) of the Poisson pencil $\{\ ,\ \}_u$

$$\Phi_i(q_i, p_i, I_1, \dots, I_n, C_1, \dots, C_m) = \Phi_i(q_i, p_i, C_1(q_i), \dots, C_m(q_i)) = 0, \qquad i \in \{1, \dots, n\}.$$

The non-Stäckel SoV means in this context that certain Casimirs of the Poisson pencil, say $C_i(q_i)$, $i \in \{1, \dots, r\}$ enter into $\Phi_i(q_i, p_i, C_1(q_i), \dots, C_m(q_i))$ in a nonlinear way. We call these Casimirs to be “non-Stäckel”.

The following proposition holds true [20].

Proposition 2.5. *Let q_i, p_i, $i \in \{1, \dots, n\}$, be separated coordinates. Let the corresponding equations of separation be nonlinear in the Casimir functions of the Poisson pencil $C_1(q_i), \dots, C_r(q_i)$ and linear in all other integrals and Casimir functions. Let Z be an invariance vector field: $Z(q_i) = Z(p_i) = 0$, $i \in \{1, \dots, n\}$. Let the following conditions also hold true:*

$$Z(C_1(u)) = \cdots = Z(C_r(u)) = 0. \tag{2.9}$$

Then

(*i*) *If $n_1 + n_2 + \cdots + n_r = m - r - 1$, then the vector field Z is unambiguously defined in terms of the vector fields Z_i by the conditions (2.9) and the normalization condition $Z(C_m) = 1$.*

(*ii*) *If, moreover, $Z(Z(C_{r+1})) = \cdots = Z(Z(C_m)) = 0$, then the square of the vector field Z annihilates all the integrals and Casimir functions*

$$Z(Z(C_1(u))) = \cdots = Z(Z(C_m(u))) = 0. \tag{2.10}$$

We need also to formulate the following important conjecture.

Conjecture 2.6. *Let the space $\mathcal{P}$ coincides with dual space of a Lie algebra $\mathfrak{g}$ and Poisson brackets $\{\ ,\ \}_1$ coincide with a standard Lie–Poisson brackets on $\mathfrak{g}^*$. Then among the invariance vector fields, i.e., vector fields such that $Z(q_i) = Z(p_i) = 0$, $i \in \{1, \dots, n\}$, there exists a special vector field Z which, acting in the system of the natural Lie-algebraic coordinates (linear coordinate functions of the space $\mathfrak{g}^*$) annihilates its own components.*

Hereafter, we will naturally assume that the vector field Z described in Proposition 2.5 and the vector field Z from the Conjecture 2.6 are the same vector fields. This will transform the differential equations (2.10) for the components of the vector field Z, to the system of quadratic algebraic equations for them, i.e., will give a possibility to reduce the problem of the construction of the vector field Z in terms of the initial variables to the problem of solving system of *algebraic* equations for its components in the initial system of coordinates. We will call such the vector field Z *algebraic*. Let us now consider a dimensional constraint that is imposed by Proposition 2.5 together with the condition that the vector field Z is algebraic. The following proposition holds true.

Proposition 2.7. *For the algebraic vector fields Z, existence of the generic solution of the equation (2.10) satisfying also the condition (2.9) implies that the number of Casimir functions m is equal to the number of the integrals of motion n, i.e., $m = n$.*

Proof. The number of components of the vector field Z in the initial coordinate system is equal to $2n + m$. The number of equations (2.10) is equal to $n + m$. That is why the generic solution of the system of equations (2.10) is n-parametric. On the other hand, the algebraic equations (2.10) and (2.9) are homogeneous. That is why there is one common multiplicative parameter among n parameters of generic solution of (2.10) not fixed by (2.9). It is fixed by the normalization condition $Z(C_m) = 1$. All other $n - 1$ parameters should be fixed by the constraints (2.9). That is why the number of the constraints (2.9) should be equal to $n - 1$, i.e., $(n_1 + 1) + (n_2 + 1) + \cdots + (n_r + 1) = n - 1$. On the other hand, the Proposition 2.5 provides the following dimensional constraint: $n_1 + n_2 + \cdots + n_r = m - r - 1$. Comparing these two-dimensional constraints, we immediately obtain that $m = n$. ■

Remark 2.8. Observe, that although the condition $m = n$ is a restrictive one, it is a typical situation for the Lax-integrable systems in the small-rank cases. To generalize this condition in the higher rank cases, one will probably need to consider several vectors fields Z with the above special properties. Let us remark, that the condition $m = n$ has appeared in another context in the recent paper [5].

2.3.2 The separating polynomial: the non-Stäckel case

Having found the algebraic vector field Z satisfying (2.9), (2.10) we can start to look for the separating polynomial. Unfortunately, in the general $m > 1$ case, one cannot construct the formula for the separating polynomial with the help of only one vector field Z. One can proceed in a systematic way only for certain types of the equations of separation. That is why we will *assume* the following form of the equations of separation:

$$\Phi_i(q_i, p_i, C_1(q_i), \dots, C_m(q_i)) \\ = \Phi_i(q_i, p_i, C_1(q_i), \dots, C_r(q_i), \phi(q_i, C_{r+1}(q_i), \dots, C_m(q_i))) = 0, \tag{2.11}$$

where function $\phi(u, C_{r+1}(u), \dots, C_m(u))$ is linear in $C_s(u)$, $s \in \{r+1, \dots, m\}$, the functions Φ_i, $i \in \{1, \dots, n\}$ are linear in $\phi(u, C_{r+1}(u), \dots, C_m(u))$ and nonlinear in $C_k(q_i)$, $k \in \{1, \dots, r\}$.

Acting on the equations (2.11) by the vector field Z and taking into account that by our construction $Z(q_1) = \cdots = Z(q_n) = Z(p_1) = \cdots = Z(p_n) = 0$, $Z(C_1(u)) = \cdots = Z(C_r(u)) = 0$, we easily obtain that the coordinates q_i should satisfy the following equations:

$$Z(\phi(q_i, C_{r+1}(q_i), \dots, C_m(q_i))) = 0, \qquad i \in \{1, \dots, n\}, \tag{2.12}$$

i.e., that $S(u) = Z(\phi(u, C_{r+1}(u), \dots, C_m(u)))$ is a separating polynomial in u of degree n.

Remark 2.9. In practice, we do not know a priori the equations of separation (2.11). That is why we have to *construct* the function ϕ as polynomial of degree n in u by the linear combination of the polynomials $C_s(u)$, $s \in \{r+1, \dots, m\}$ of degree n_s in u, where $n_1+n_2+\cdots+n_r = m-r-1$, $n_1+n_2+\cdots+n_m = n$, with monomial in u coefficients. The form of separating polynomial (2.12) will be only indicative. Indeed, since we have no *closed* differential conditions of the type (2.5)–(2.6) for the *unique* vector field Z, we have yet to control that the roots q_i of $S(u)$ Poisson-commute and that q_i together with the corresponding conjugated momenta p_i satisfy some equations of separation of the form (2.11). In the next sections, we will illustrate this on the examples of the extended Clebsh and Manakov models.

3 Three-dimensional extension of the Clebsch model

3.1 The model

Let us consider nine-dimensional linear space with the coordinates S_α, T_α, W_α, $\alpha \in \{1, 2, 3\}$, which satisfy the following Lie–Poisson brackets:

$$\{S_\alpha, S_\beta\}_1 = \epsilon_{\alpha\beta\gamma}(S_\gamma + j_\gamma W_\gamma), \qquad \{S_\alpha, T_\beta\}_1 = \epsilon_{\alpha\beta\gamma} T_\gamma, \qquad \{S_\alpha, W_\beta\}_1 = \epsilon_{\alpha\beta\gamma} W_\gamma, \tag{3.1a}$$

$$\{T_\alpha, T_\beta\}_1 = \epsilon_{\alpha\beta\gamma} W_\gamma, \qquad \{T_\alpha, W_\beta\}_1 = 0, \qquad \{W_\alpha, W_\beta\}_1 = 0, \tag{3.1b}$$

where j_α, $\alpha \in \{1, 2, 3\}$, are arbitrary parameters, and we will hereafter assume that $j_\alpha \neq j_\beta$ if $\alpha \neq \beta$.

These brackets possess three Casimir functions

$$C_3 = \sum_{\alpha=1}^{3} W_\alpha^2, \qquad C_2 = \sum_{\alpha=1}^{3} \left(j_\alpha W_\alpha^2 + 2 W_\alpha S_\alpha + T_\alpha^2\right) \qquad C_1 = 2 \sum_{\alpha=1}^{3} T_\alpha W_\alpha. \tag{3.2}$$

Let us consider the following Hamiltonian:

$$H = \sum_{\alpha=1}^{3} \left(S_\alpha^2 + (j_\beta + j_\gamma) T_\alpha^2 + 2 j_\alpha S_\alpha W_\alpha\right), \tag{3.3}$$

where the indices α, β, γ hereafter denote cyclic permutation of the indices 1, 2, 3, i.e., for $\alpha = 1$ we have that $\beta = 2$, $\gamma = 3$, for $\alpha = 2$ we have that $\beta = 3$, $\gamma = 1$, for $\alpha = 3$ we have that $\beta = 1$, $\gamma = 2$.

As it is easy to check, it possesses two quadratic integrals of the following explicit form:

$$K = \sum_{\alpha=1}^{3} \left(j_\alpha S_\alpha^2 + j_\beta j_\gamma T_\alpha^2 \right), \tag{3.4}$$

$$L = 2 \sum_{\alpha=1}^{3} T_\alpha S_\alpha. \tag{3.5}$$

The model is integrable in the sense of Liouville: the dimension of the generic symplectic leaf (level set of three Casimir functions) is six and we have three independent Poisson commuting integrals $\{H, K\}_1 = 0$, $\{H, L\}_1 = 0$, $\{K, L\}_1 = 0$ which is sufficient for the complete integrability of the considered Hamiltonian system. Observe, that from the point of view of the notations of the previous section we have that $n = m = 3$ in this case.

Remark 3.1. Note that the Poisson algebra (3.1) possesses an ideal spanned by $\{W_\alpha \mid \alpha \in \{1, 2, 3\}\}$. Factorizing over this ideal, i.e., putting $W_\alpha = 0$, $\alpha \in \{1, 2, 3\}$, we obtain that the Lie–Poisson brackets (3.1) become the standard Lie–Poisson brackets on $\mathfrak{e}^*(3)$, the Casimir functions C_3 and C_1 turn zero, the Casimir C_2 and integral L become Casimir functions on $\mathfrak{e}^*(3)$ and the integrals H and K become the Hamiltonian and integral of motion of the famous Clebsch model [6]. That is why we call the integrable system considered in the present section to be *three-dimensional extension of the Clebsch model*.

3.2 The bi-Hamiltonian structure and the Poisson pencil

It is possible to show that there is a second Poisson structure for our extended Clebsch system, compatible with the first one, standing behind the integrability of the model and having the form

$$\{S_\alpha, S_\beta\}_2 = \epsilon_{\alpha\beta\gamma} j_\gamma S_\gamma, \qquad \{S_\alpha, T_\beta\}_2 = \epsilon_{\alpha\beta\gamma} j_\beta T_\gamma, \qquad \{T_\alpha, T_\beta\}_2 = \epsilon_{\alpha\beta\gamma} S_\gamma, \tag{3.6a}$$

$$\{T_\alpha, W_\beta\}_2 = 0, \qquad \{S_\alpha, W_\beta\}_2 = 0, \qquad \{W_\alpha, W_\beta\}_2 = -\epsilon_{\alpha\beta\gamma} W_\gamma. \tag{3.6b}$$

The function C_3 is a common Casimir function for the both brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$, the other two Casimir functions of the brackets $\{\ ,\ \}_2$ are the functions K and L. The functions H, C_1 and C_2 are the Poisson-commuting integrals of motion with respect to $\{\ ,\ \}_2$ $\{H, C_1\}_2 = 0$, $\{H, C_2\}_2 = 0$, $\{C_1, C_2\}_2 = 0$. Due to the compatibility of the brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$, it is possible to consider the so called Poisson pencil of the brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$, $\{\ ,\ \}_u = u\{\ ,\ \}_1 + \{\ ,\ \}_2$. The function C_3 is a Casimir of $\{\ ,\ \}_u$. The second and third Casimirs of $\{\ ,\ \}_u$ are the functions $C_2(u) = u^2 C_2 + uH + K$, $C_1(u) = uC_1 + L$. They will be used below while constructing separating polynomial and equations of separation. We will also use the following cubic in u combination of the Casimirs $C_3(u) \equiv C_3$ and $C_2(u)$:

$$\phi(u, C_2(u), C_3) = u^3 C_3 + C_2(u) = u^3 C_3 + u^2 C_2 + uH + K.$$

Remark 3.2. Observe that in the case $j_\alpha \neq 0$, $\alpha \in \{1, 2, 3\}$, the Poisson algebra (3.6) is isomorphic to $\mathfrak{so}(4) \oplus \mathfrak{so}(3)$. The isomorphism is achieved by the following substitution of variables $S_\alpha \to \sqrt{j_\beta}\sqrt{j_\gamma} S_\alpha$, $T_\alpha \to \sqrt{j_\alpha} T_\alpha$, $W_\alpha \to -W_\alpha$, $\alpha \in \{1, 2, 3\}$. In such a way, the considered model is isomorphic to the so-called Manakov or Schottky–Frahm model on $\mathfrak{so}(4)$ interacting with anisotropic $\mathfrak{so}(3)$ Euler top. Due to the isomorphism $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, the Poisson algebra (3.6) is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and the corresponding integrable system is equivalent to the system of three interacting anisotropic tops. It is possible to show that this system is equivalent to $N = 3$ elliptic Gaudin model, but we will not develop this line in the present article.

4 The algebraic vector field Z

In this section, we will solve the equations (2.10)

$$\begin{aligned} &Z(Z(H)) = 0, \qquad Z(Z(K)) = 0, \qquad Z(Z(L)) = 0,\\ &Z(Z(C_1)) = 0, \qquad Z(Z(C_2)) = 0, \qquad Z(Z(C_3)) = 0 \end{aligned} \tag{4.1}$$

under the additional assumption that the vector field Z is algebraic.

The general vector field on the three-dimensional extension of $\mathfrak{e}^*(3)$ is written as follows:

$$Z = \sum_{\alpha=1}^{3} \left(A_\alpha \frac{\partial}{\partial S_\alpha} + B_\alpha \frac{\partial}{\partial T_\alpha} + D_\alpha \frac{\partial}{\partial W_\alpha} \right),$$

where A_α, B_α, D_α, $\alpha \in \{1, 2, 3\}$, are certain functions on $\mathfrak{e}^*(3)$.

In order to solve the equations (4.1), we will impose an additional restriction that vector field Z is algebraic, i.e., annihilates its own components

$$Z(A_\alpha) = 0, \qquad Z(B_\alpha) = 0, \qquad Z(D_\alpha) = 0, \qquad \alpha \in \{1, 2, 3\}. \tag{4.2}$$

This condition transforms the system of the differential equations (4.1) for the vector field Z in the system of the algebraic equations for its components A_α, B_α, D_α, $\alpha \in \{1, 2, 3\}$.

For the future, we need to introduce the following "elliptic" constants c_α:

$$c_\alpha^2 = j_\beta - j_\gamma, \qquad \alpha \in \{1, 2, 3\}. \tag{4.3}$$

The following proposition holds true.

Proposition 4.1. *The vector field Z with the following components:*

$$A_\alpha = \lambda c_\alpha \big(x^2 + (j_\beta + j_\gamma - j_\alpha) y^2 + 2 j_\beta j_\gamma y + j_\alpha j_\beta j_\gamma\big), \tag{4.4a}$$

$$B_\alpha = 2\lambda c_\alpha x (y + j_\alpha), \tag{4.4b}$$

$$D_\alpha = \lambda c_\alpha \big(y^2 + 2 j_\alpha y + j_\alpha (j_\beta + j_\gamma) - j_\beta j_\gamma\big), \tag{4.4c}$$

where λ, x and y are arbitrary functions annihilated by Z, i.e., $Z(x) = 0$, $Z(y) = 0$, $Z(\lambda) = 0$, is a solution of the system of equations (4.1) possessing the property (4.2).

Proof. In order to prove the proposition, we rewrite the equations (4.1) (with the help of the conditions (4.2)) in the form of six algebraic equations

$$\sum_{\alpha=1}^{3} \big(A_\alpha^2 + (j_\beta + j_\gamma) B_\alpha^2 + 2 j_\alpha A_\alpha D_\alpha\big) = 0, \qquad \sum_{\alpha=1}^{3} \big(j_\alpha A_\alpha^2 + j_\beta j_\gamma B_\alpha^2\big) = 0, \tag{4.5a}$$

$$\sum_{\alpha=1}^{3} B_\alpha A_\alpha = 0, \qquad \sum_{\alpha=1}^{3} B_\alpha D_\alpha = 0, \tag{4.5b}$$

$$\sum_{\alpha=1}^{3} \big(j_\alpha D_\alpha^2 + 2 D_\alpha A_\alpha + B_\alpha^2\big) = 0, \qquad \sum_{\alpha=1}^{3} D_\alpha^2 = 0. \tag{4.5c}$$

Substituting the ansatz (4.4) into the equations (4.5), taking into account the definitions (4.3) and the following equalities $\sum_{\alpha=1}^{3} c_\alpha^2 = 0$, $\sum_{\alpha=1}^{3} j_\alpha c_\alpha^2 = 0$, after the direct calculations, we obtain that the equations (4.5) hold true. ■

Remark 4.2. Observe that we have six independent equations (4.5) for nine functions A_α, B_α, D_α, $\alpha \in \{1,2,3\}$. That is why the generic solution of the equations (4.5) is three parametric. In the case of the proposed solution, these three parameters are λ, x and y. We conjecture that there are only two three-parametric, non-equivalent solutions of the equations (4.1) satisfying also the conditions (4.2). They are the presented solution (4.4) and the solution found in our paper [20].

Now we have to define the functions λ, x and y. As it follows from our theory of non-Stäckel vector field Z, it should annihilate one of the non-common Casimir functions $C_2(u)$ or $C_1(u)$ of $\{\ ,\ \}_u$. It is easy to see that only the Casimir $C_1(u)$ can be taken for such a role because the condition $n_1 = m - r - 1$ is satisfied for it: we have $m = 3$, $r = 1$, $n_1 = 1$. That is why we will impose the condition $Z(C_1(u)) = 0$, that is, we will define the functions x and y from the following two equations:

$$Z(C_1) = \sum_{\alpha=1}^{3}(T_\alpha D_\alpha + W_\alpha B_\alpha) = 0, \qquad Z(L) = \sum_{\alpha=1}^{3}(T_\alpha A_\alpha + S_\alpha B_\alpha) = 0.$$

These will be our main constraint equations. They are written more explicitly as follows:

$$\sum_{\alpha=1}^{3} c_\alpha\big(y^2 + 2j_\alpha y + j_\alpha(j_\beta + j_\gamma) - j_\beta j_\gamma)T_\alpha + 2x(y + j_\alpha)W_\alpha\big) = 0, \tag{4.6a}$$

$$\sum_{\alpha=1}^{3} c_\alpha\big((x^2 + (j_\beta + j_\gamma - j_\alpha)y^2 + 2j_\beta j_\gamma y + j_\alpha j_\beta j_\gamma)T_\alpha + 2x(y + j_\alpha)S_\alpha\big) = 0. \tag{4.6b}$$

The following proposition holds true.

Proposition 4.3. *Let the functions x and y be solutions of the equations (4.6). Then they are annihilated by the vector field Z,*

$$Z(x) = 0, \qquad Z(y) = 0. \tag{4.7}$$

Proof. In order to prove the proposition, we use that by the very same definition

$$Z(x) = \sum_{\alpha=1}^{3}\left(A_\alpha \frac{\partial x}{\partial S_\alpha} + B_\alpha \frac{\partial x}{\partial T_\alpha} + D_\alpha \frac{\partial x}{\partial W_\alpha}\right),$$

$$Z(y) = \sum_{\alpha=1}^{3}\left(A_\alpha \frac{\partial y}{\partial S_\alpha} + B_\alpha \frac{\partial y}{\partial T_\alpha} + D_\alpha \frac{\partial y}{\partial W_\alpha}\right).$$

To prove that the above expressions are zero, it is necessary to find explicitly the derivatives $\frac{\partial x}{\partial Y_\alpha}$ and $\frac{\partial y}{\partial Y_\alpha}$, where $Y_\alpha = S_\alpha$ or $Y_\alpha = T_\alpha$ or $Y_\alpha = W_\alpha$. This is done using the equations (4.6). In more details, for each $\alpha \in \{1,2,3\}$, we have the following system of two equations that permits to find $\frac{\partial x}{\partial Y_\alpha}$, $\frac{\partial y}{\partial Y_\alpha}$:

$$\frac{\partial Z(C_1)}{\partial y}\frac{\partial y}{\partial Y_\alpha} + \frac{\partial Z(C_1)}{\partial x}\frac{\partial x}{\partial Y_\alpha} + \frac{\partial Z(C_1)}{\partial Y_\alpha} = 0,$$

$$\frac{\partial Z(L)}{\partial y}\frac{\partial y}{\partial Y_\alpha} + \frac{\partial Z(L)}{\partial x}\frac{\partial x}{\partial Y_\alpha} + \frac{\partial Z(L)}{\partial Y_\alpha} = 0.$$

They are obtained by the differentiation of the constraints $Z(C_1) = 0$ and $Z(L) = 0$ with respect to Y_α.

Now, using the derivatives $\frac{\partial x}{\partial Y_\alpha}$ and $\frac{\partial y}{\partial Y_\alpha}$ calculated as it is explained above, the explicit form of the components A_α, B_α, D_α given by the formulae (4.4), the explicit form of the coordinates and momenta of separation given in the theorem, the constraint equations (4.6), the definition (4.3) of the constants c_α, after direct and tedious calculations, we obtain the equalities (4.7). ■

Remark 4.4. The overall multiplier λ is determined from the normalization condition $Z(C_3) = 1$, i.e., $\lambda = \left(\sum_{\alpha=1}^{3} D_\alpha W_\alpha\right)^{-1}$. Using the fact that $Z(x) = Z(y) = 0$ (see the proposition above), it is easy to show that $Z(\lambda) = 0$, i.e., the construction of this section is self-consistent.

5 Symmetric separated variables

5.1 Separating polynomial and the coordinates of separation

5.1.1 The separating polynomial

As it follows from the exposed above bi-Hamiltonian theory of non-Stäckel vector field Z one should look for the coordinates of separation as the roots of the following cubic polynomial:

$$\begin{aligned} S(u) &= Z(\phi(u, C_2(u), C_3)) = Z\big(u^3 C_3 + C_2(u)\big) \\ &= Z(C_3)u^3 + Z(C_2)u^2 + Z(H)u + Z(K), \end{aligned} \tag{5.1}$$

where u is a parameter of the bi-Hamiltonian pencil and the needed vector field Z is defined as in the previous section, i.e., Z is algebraic and $Z(C_1(u)) = 0$.

Remark 5.1. Observe that the needed roots of the polynomial (5.1) do not depend on the overall normalization coefficient λ, that is why we will simply ignore it, putting hereafter that $\lambda = 1$.

5.1.2 The auxiliary coordinate system

In order to better understand the structure of $S(u)$ and simplify the constraint (4.6), we will introduce a system of the auxiliary coordinates. In more details, we will consider the following set of nine functions:

$$f_1 = \sum_{\alpha=1}^{3} D_\alpha S_\alpha, \qquad f_2 = \sum_{\alpha=1}^{3} B_\alpha S_\alpha, \qquad f_3 = \sum_{\alpha=1}^{3} A_\alpha S_\alpha, \tag{5.2a}$$

$$g_1 = \sum_{\alpha=1}^{3} D_\alpha T_\alpha, \qquad g_2 = \sum_{\alpha=1}^{3} B_\alpha T_\alpha, \qquad g_3 = \sum_{\alpha=1}^{3} A_\alpha T_\alpha, \tag{5.2b}$$

$$h_1 = \sum_{\alpha=1}^{3} D_\alpha W_\alpha, \qquad h_2 = \sum_{\alpha=1}^{3} B_\alpha W_\alpha, \qquad h_3 = \sum_{\alpha=1}^{3} A_\alpha W_\alpha. \tag{5.2c}$$

It is easy to see, that in the terms of these functions the constraint equations (4.6) are written as follows:

$$h_2 + g_1 = 0, \qquad f_2 + g_3 = 0. \tag{5.3}$$

In other words, as a system of (local) coordinates on the nine-dimensional phase space one can take nine functions f_1, f_3, g_2, g_3, h_1, h_2 h_3, x, y. It is possible to show that they are functionally independent.

The formulae (5.2) define the invertible quasi-linear, i.e., linear in $(S_\alpha, T_\alpha, W_\alpha)$ and $(f_\alpha, g_\alpha, h_\alpha)$ but nonlinear in x, y, map $(S_\alpha, T_\alpha, W_\alpha) \to (f_\alpha, g_\alpha, h_\alpha)$. We have

$$\begin{aligned} S_\alpha = \frac{c_\alpha}{2c_1^2c_2^2c_3^2} \Bigg(&\left(1 + \frac{y\big(y^2 + 2yj_\alpha + j_\alpha(j_\beta + j_\gamma) - j_\beta j_\gamma\big)}{x^2 + (y+j_1)(y+j_2)(y+j_3)} \right) f_1 - \frac{(y+j_\alpha)f_2}{x} \\ &+ \frac{\big(y^2 + 2yj_\alpha + j_\alpha(j_\beta + j_\gamma) - j_\beta j_\gamma\big) f_3}{x^2 + (y+j_1)(y+j_2)(y+j_3)} \Bigg), \end{aligned} \tag{5.4a}$$

$$T_\alpha = \frac{c_\alpha}{2c_1^2c_2^2c_3^2}\left(\left(1+\frac{y\big(y^2+2yj_\alpha+j_\alpha(j_\beta+j_\gamma)-j_\beta j_\gamma\big)}{x^2+(y+j_1)(y+j_2)(y+j_3)}\right)g_1 - \frac{(y+j_\alpha)g_2}{x} + \frac{\big(y^2+2yj_\alpha+j_\alpha(j_\beta+j_\gamma)-j_\beta j_\gamma\big)g_3}{x^2+(y+j_1)(y+j_2)(y+j_3)}\right), \tag{5.4b}$$

$$W_\alpha = \frac{c_\alpha}{2c_1^2c_2^2c_3^2}\left(\left(1+\frac{y\big(y^2+2yj_\alpha+j_\alpha(j_\beta+j_\gamma)-j_\beta j_\gamma\big)}{x^2+(y+j_1)(y+j_2)(y+j_3)}\right)h_1 - \frac{(y+j_\alpha)h_2}{x} + \frac{\big(y^2+2yj_\alpha+j_\alpha(j_\beta+j_\gamma)-j_\beta j_\gamma\big)h_3}{x^2+(y+j_1)(y+j_2)(y+j_3)}\right). \tag{5.4c}$$

We use these formulae in the subsequent proofs.

5.1.3 The coordinates of separation q_1, q_2, q_3

By direct calculation, using the formulae (5.4), it is possible to show that on the constraint (5.3) the polynomial $S(u)$ acquires (in terms of the introduced $f-g-h$ coordinates) the following form:

$$S(u) = u^3s_3 + u^2s_2 + us_1 + s_0, \tag{5.5}$$

where

$$s_3 = 2xh_1\big(x^2+(y+j_1)(y+j_2)(y+j_3)\big), \tag{5.6a}$$

$$\begin{aligned} s_2 = {} & 2(f_1+g_2+h_3-yh_1)x^3 + h_2\big(3y^2+2y(j_1+j_2+j_3)+j_1j_2+j_1j_3+j_2j_3\big)x^2 \\ & + (y+j_1)(y+j_3)(y+j_2)\big(2(f_1+g_2-h_3-3yh_1)x \\ & + h_2\big(3y^2+2y(j_1+j_2+j_3)+j_1j_2+j_1j_3+j_2j_3\big)\big), \end{aligned} \tag{5.6b}$$

$$\begin{aligned} s_1 = {} & -x^4h_2 + 2(f_3-(f_1+h_3+2g_2)y)x^3 - \big(3y^2+2y(j_1+j_2+j_3)+j_1j_2+j_1j_3+j_2j_3\big) \\ & \times (3yh_2+f_2)x^2 + (y+j_1)(y+j_2)(y+j_3)\big(-2\big(-2y^2h_1+(-h_3+3f_1+2g_2)y \\ & \quad + f_3\big)x - 2y^3h_2 + (3f_2-h_2(j_1+j_3+j_2))y^2 + 2(j_1+j_3+j_2)f_2y \\ & \quad + (j_1j_2+j_1j_3+j_2j_3)f_2 + j_1j_2j_3h_2\big), \end{aligned} \tag{5.6c}$$

$$\begin{aligned} s_0 = {} & (f_2+2yh_2)x^4 - 2y(f_3-yg_2)x^3 + \big(4y^4h_2+(f_2+2h_2(j_1+j_2+j_3))y^3 \\ & - ((j_1j_3+j_2j_3+j_1j_2)f_2+2j_1j_2j_3h_2)y + 2j_1j_2j_3f_2\big)x^2 \\ & + (y+j_1)(y+j_2)(y+j_3)\big(2y((2f_1+g_2)y+f_3)x \\ & - f_2\big(2y^3+y^2(j_1+j_2+j_3)-j_1j_2j_3\big)\big), \end{aligned} \tag{5.6d}$$

and we have taken into account the constraints (5.3).

Hence, its roots $u=q_1$, $u=q_2$ and $u=q_3$ are our candidates for the separated coordinates. In order to prove that they are the coordinates of separation indeed, in the next subsection we show that they Poisson-commute, find the corresponding canonically conjugated momenta and equations of separation.

5.2 The momenta of separation p_1, p_2, p_3

In order to construct the momenta of separation p_1, p_2, p_3 (providing that q_1, q_2 and q_3 are the coordinates of separation indeed), we will proceed directly finding the momenta from the condition that they are (quasi)canonically conjugated to the coordinates q_1, q_2 and q_3.

The following theorem holds true.

Theorem 5.2. *Let the functions x and y be solutions of the equations (4.6). Let the components A_α, B_α, D_α of the vector field Z be defined by the formulae (4.4) and the functions f_α, g_α, h_α be defined by the formulae (5.2). Then*

(*i*) *The coordinates q_1, q_2, q_3 defined in the previous subsection Poisson-commute with respect to the both brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$*

$$\{q_i, q_j\}_1 = 0, \qquad \{q_i, q_j\}_2 = 0, \qquad i, j \in \{1, 2, 3\}. \tag{5.7}$$

(*ii*) *The functions p_i, $i \in \{1, 2, 3\}$ given by the following formulae:*

$$p_i = p(u)|_{u=q_i}, \tag{5.8}$$

where

$$p(u) = \frac{1}{2c_1c_2c_3}\frac{\big((u-2y)x^2 + u(y+j_1)(y+j_2)(y+j_3)\big)h_2}{x\big(x^2 + (y+j_1)(y+j_2)(y+j_3)\big)(u-y)} + \frac{\big((y+j_1)(y+j_2)(y+j_3) - x^2\big)f_2}{x\big(x^2 + (y+j_1)(y+j_2)(y+j_3)\big)(u-y)}, \tag{5.9}$$

are their canonically conjugated momenta with respect to the brackets $\{\ ,\ \}_1$, i.e.,

$$\{p_i, q_j\}_1 = \delta_{ij}, \qquad i, j \in \{1, 2, 3\}, \tag{5.10a}$$
$$\{p_i, p_j\}_1 = 0, \qquad i, j \in \{1, 2, 3\}, \tag{5.10b}$$

and quasi-canonically conjugated momenta with respect to the brackets $\{\ ,\ \}_2$, i.e.,

$$\{p_i, q_j\}_2 = -q_i\delta_{ij}, \qquad i, j \in \{1, 2, 3\}, \tag{5.11a}$$
$$\{p_i, p_j\}_2 = 0, \qquad i, j \in \{1, 2, 3\}. \tag{5.11b}$$

(*iii*) *The variables q_i, p_j are Z-invariants, i.e.,*

$$Z(q_i) = Z(p_j) = 0, \qquad i, j \in \{1, 2, 3\}. \tag{5.12}$$

Proof. The proof of the items (*i*)–(*ii*) is achieved upon the direct calculus of the Poisson brackets among the separating polynomials $S(u)$, $S(v)$ and the momenta-generating functions $p(u)$ and $p(v)$. In order to calculate the corresponding Poisson brackets one needs to find the Poisson brackets among the intermediate coordinates x, y, f_α, g_β, h_γ, $\alpha, \beta, \gamma \in \{1, 2, 3\}$, in the closed form. In order to calculate these brackets it is necessary to find explicitly the derivatives $\frac{\partial x}{\partial Y_\alpha}$ and $\frac{\partial y}{\partial Y_\alpha}$, where $Y_\alpha = S_\alpha$ or $Y_\alpha = T_\alpha$ or $Y_\alpha = W_\alpha$. This is done using the constraint equations (4.6) in the same way as in Proposition 4.3.

In order to find the Poisson brackets among the coordinates f_α, g_β, h_γ, $\alpha, \beta, \gamma \in \{1, 2, 3\}$, we use the fact that for any functions $F \equiv F(S_\alpha, T_\beta, W_\gamma, x, y)$, $G \equiv G(S_\alpha, T_\beta, W_\gamma, x, y)$ the following equality holds:

$$\{F, G\}_i = \{F, G\}'_i + \frac{\partial F}{\partial x}\sum_{\alpha,\beta=1}^{3}\bigg(\frac{\partial x}{\partial S_\alpha}\frac{\partial G}{\partial S_\beta}\{S_\alpha, S_\beta\}_i + \frac{\partial x}{\partial T_\alpha}\frac{\partial G}{\partial S_\beta}\{T_\alpha, S_\beta\}_i + \frac{\partial x}{\partial W_\alpha}\frac{\partial G}{\partial S_\beta}\{W_\alpha, S_\beta\}_i + \frac{\partial x}{\partial S_\alpha}\frac{\partial G}{\partial T_\beta}\{S_\alpha, T_\beta\}_i + \frac{\partial x}{\partial T_\alpha}\frac{\partial G}{\partial T_\beta}\{T_\alpha, T_\beta\}_i + \frac{\partial x}{\partial W_\alpha}\frac{\partial G}{\partial T_\beta}\{W_\alpha, T_\beta\}_i + \frac{\partial x}{\partial S_\alpha}\frac{\partial G}{\partial W_\beta}\{S_\alpha, W_\beta\}_i + \frac{\partial x}{\partial T_\alpha}\frac{\partial G}{\partial W_\beta}\{T_\alpha, W_\beta\}_i$$

$$
\begin{aligned}
&+\frac{\partial x}{\partial W_\alpha}\frac{\partial G}{\partial W_\beta}\{W_\alpha,W_\beta\}_i\bigg)+\frac{\partial F}{\partial y}\sum_{\alpha,\beta=1}^{3}\bigg(\frac{\partial y}{\partial S_\alpha}\frac{\partial G}{\partial S_\beta}\{S_\alpha,S_\beta\}_i+\frac{\partial y}{\partial T_\alpha}\frac{\partial G}{\partial S_\beta}\{T_\alpha,S_\beta\}_i\\
&+\frac{\partial y}{\partial W_\alpha}\frac{\partial G}{\partial S_\beta}\{W_\alpha,S_\beta\}_i+\frac{\partial y}{\partial S_\alpha}\frac{\partial G}{\partial T_\beta}\{S_\alpha,T_\beta\}_i+\frac{\partial y}{\partial T_\alpha}\frac{\partial G}{\partial T_\beta}\{T_\alpha,T_\beta\}_i\\
&+\frac{\partial y}{\partial W_\alpha}\frac{\partial G}{\partial T_\beta}\{W_\alpha,T_\beta\}+\frac{\partial y}{\partial S_\alpha}\frac{\partial G}{\partial W_\beta}\{S_\alpha,W_\beta\}_i+\frac{\partial y}{\partial T_\alpha}\frac{\partial G}{\partial W_\beta}\{T_\alpha,W_\beta\}_i\\
&+\frac{\partial y}{\partial W_\alpha}\frac{\partial G}{\partial W_\beta}\{W_\alpha,W_\beta\}_i\bigg)-\frac{\partial G}{\partial x}\sum_{\alpha,\beta=1}^{3}\bigg(\frac{\partial x}{\partial S_\alpha}\frac{\partial F}{\partial S_\beta}\{S_\alpha,S_\beta\}_i+\frac{\partial x}{\partial T_\alpha}\frac{\partial F}{\partial S_\beta}\{T_\alpha,S_\beta\}_i\\
&+\frac{\partial x}{\partial W_\alpha}\frac{\partial F}{\partial S_\beta}\{W_\alpha,S_\beta\}_i+\frac{\partial x}{\partial S_\alpha}\frac{\partial F}{\partial T_\beta}\{S_\alpha,T_\beta\}_i+\frac{\partial x}{\partial T_\alpha}\frac{\partial F}{\partial T_\beta}\{T_\alpha,T_\beta\}_i\\
&+\frac{\partial x}{\partial W_\alpha}\frac{\partial F}{\partial T_\beta}\{W_\alpha,T_\beta\}_i+\frac{\partial x}{\partial S_\alpha}\frac{\partial F}{\partial W_\beta}\{S_\alpha,W_\beta\}_i\\
&+\frac{\partial x}{\partial T_\alpha}\frac{\partial F}{\partial W_\beta}\{T_\alpha,W_\beta\}_i+\frac{\partial x}{\partial W_\alpha}\frac{\partial F}{\partial W_\beta}\{W_\alpha,W_\beta\}_i\bigg)\\
&-\frac{\partial G}{\partial y}\sum_{\alpha,\beta=1}^{3}\bigg(\frac{\partial y}{\partial S_\alpha}\frac{\partial F}{\partial S_\beta}\{S_\alpha,S_\beta\}_i+\frac{\partial y}{\partial T_\alpha}\frac{\partial F}{\partial S_\beta}\{T_\alpha,S_\beta\}_i+\frac{\partial y}{\partial W_\alpha}\frac{\partial F}{\partial S_\beta}\{W_\alpha,S_\beta\}_i\\
&+\frac{\partial y}{\partial S_\alpha}\frac{\partial F}{\partial T_\beta}\{S_\alpha,T_\beta\}_i+\frac{\partial y}{\partial T_\alpha}\frac{\partial F}{\partial T_\beta}\{T_\alpha,T_\beta\}_i+\frac{\partial y}{\partial W_\alpha}\frac{\partial F}{\partial T_\beta}\{W_\alpha,T_\beta\}_i\\
&+\frac{\partial y}{\partial S_\alpha}\frac{\partial F}{\partial W_\beta}\{S_\alpha,W_\beta\}_i+\frac{\partial y}{\partial T_\alpha}\frac{\partial F}{\partial W_\beta}\{T_\alpha,W_\beta\}_i+\frac{\partial y}{\partial W_\alpha}\frac{\partial F}{\partial W_\beta}\{W_\alpha,W_\beta\}_i\bigg)\\
&+\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial G}{\partial x}\frac{\partial F}{\partial y}\right)\{x,y\}_i,\qquad i\in\{1,2\},
\end{aligned}
$$

where $\{\ ,\ \}'_i$ is a parenthesis $\{\ ,\ \}_i$ in which x and y are treated as constant non-dynamical parameters and the brackets $\{x,y\}_i$, in their turn, are calculated as follows:

$$
\begin{aligned}
\{x,y\}_i=\sum_{\alpha,\beta=1}^{3}&\bigg(\frac{\partial x}{\partial S_\alpha}\frac{\partial y}{\partial S_\beta}\{S_\alpha,S_\beta\}_i+\frac{\partial x}{\partial T_\alpha}\frac{\partial y}{\partial S_\beta}\{T_\alpha,S_\beta\}_i+\frac{\partial x}{\partial W_\alpha}\frac{\partial y}{\partial S_\beta}\{W_\alpha,S_\beta\}_i\\
&+\frac{\partial x}{\partial S_\alpha}\frac{\partial y}{\partial T_\beta}\{S_\alpha,T_\beta\}_i+\frac{\partial x}{\partial T_\alpha}\frac{\partial y}{\partial T_\beta}\{T_\alpha,T_\beta\}_i+\frac{\partial x}{\partial W_\alpha}\frac{\partial y}{\partial T_\beta}\{W_\alpha,T_\beta\}_i\\
&+\frac{\partial x}{\partial S_\alpha}\frac{\partial y}{\partial W_\beta}\{S_\alpha,W_\beta\}_i+\frac{\partial x}{\partial T_\alpha}\frac{\partial y}{\partial W_\beta}\{T_\alpha,W_\beta\}_i+\frac{\partial x}{\partial W_\alpha}\frac{\partial y}{\partial W_\beta}\{W_\alpha,W_\beta\}_i\bigg).
\end{aligned}
$$

Having calculated the Poisson brackets $\{\ ,\ \}_i$, $i\in\{1,2\}$, among the functions x, y, f_α, g_β, h_γ, $\alpha,\beta,\gamma\in\{1,2,3\}$, in terms of the coordinates x, y, S_α, T_β, W_γ, $\alpha,\beta,\gamma\in\{1,2,3\}$, we apply the formulae (5.4) and recalculate the right-hand sides of these brackets in terms of the functions x, y, f_α, g_β, h_γ, $\alpha,\beta,\gamma\in\{1,2,3\}$.

After that, taking into account the explicit form of the functions $S(u)$, $p(u)$ in terms of the functions x, y, f_α, g_β, h_γ, $\alpha,\beta,\gamma\in\{1,2,3\}$, the constraints (5.3), the definition (4.3) of the constants c_α and the direct calculations, we come to the following equalities:

$$\{S(u),S(v)\}_i=0\mod\mathcal{J}_{S(u),S(v)},\tag{5.13a}$$

$$\{S(u),p(v)\}_i=0\mod\mathcal{J}_{S(u),S(v)},\qquad u\neq v,\tag{5.13b}$$

$$\{p(u),p(v)\}_i=0\mod\mathcal{J}_{S(u),S(v)},\tag{5.13c}$$

as well as the following equalities:

$$\lim_{v\to u}\{S(u),p(v)\}_1 = \partial_u S(u) \mod \mathcal{J}_{S(u)},$$
$$\lim_{v\to u}\{S(u),p(v)\}_2 = -u\partial_u S(u) \mod \mathcal{J}_{S(u)}. \tag{5.14}$$

Here $\mathcal{J}_{S(u),S(v)}$, $\mathcal{J}_{S(u)}$ are ideals in the space of functions generated by $S(u)$, $S(v)$ and $S(u)$, respectively. As it follows from the results of [7] the equalities (5.7), (5.10), (5.11) follow from the equalities (5.13)–(5.14).

This proves the items (i) and (ii) of the theorem.

In order to prove item (iii) of the theorem, we use that for any function $F \equiv F(S_\alpha, T_\beta, W_\gamma, x, y)$ the following equality holds:

$$Z(F) = \sum_{\alpha=1}^{3}\left(A_\alpha\frac{\partial F}{\partial S_\alpha} + B_\alpha\frac{\partial F}{\partial T_\alpha} + D_\alpha\frac{\partial F}{\partial W_\alpha}\right) + \frac{\partial F}{\partial x}\sum_{\alpha=1}^{3}\left(A_\alpha\frac{\partial x}{\partial S_\alpha} + B_\alpha\frac{\partial x}{\partial T_\alpha} + D_\alpha\frac{\partial x}{\partial W_\alpha}\right)$$
$$+ \frac{\partial F}{\partial y}\sum_{\alpha=1}^{3}\left(A_\alpha\frac{\partial y}{\partial S_\alpha} + B_\alpha\frac{\partial y}{\partial T_\alpha} + D_\alpha\frac{\partial y}{\partial W_\alpha}\right).$$

Then, using the derivatives $\frac{\partial x}{\partial Y_\alpha}$ and $\frac{\partial y}{\partial Y_\alpha}$, where $Y_\alpha = S_\alpha$ or $Y_\alpha = T_\alpha$ or $Y_\alpha = W_\alpha$, calculated as it is explained above, the explicit form of the components A_α, B_α, D_α given by the formulas (4.4), the explicit form of the coordinates and momenta of separation given in the text of the theorem, the constraint equations (4.6), the definition (4.3) of the constants c_α, after tedious calculations we obtain the equalities $Z(S(u)) = 0$, $Z(p(v)) = 0$, $\forall u, v \in \mathbb{C}$. From these equalities, the equalities (5.12) immediately follow. This proves item (iii) of the theorem. ■

Remark 5.3. The difference in sign of in the formulae (5.11) with respect to that in (5.7) is not crucial and amounts only to the change of sign of spectral parameter of the corresponding Poisson pencil.

5.3 The equations of separation

In this subsection, we will find equations of separation satisfied by the constructed coordinates q_i and momenta p_i, $i \in \{1,2,3\}$. The following theorem holds true.

Theorem 5.4. *The coordinates q_i as the roots of the polynomial $S(u)$ given by* (5.5)–(5.6)*, the momenta p_i defined by the formulae* (5.8)–(5.9) *and integrals H, K, L, C_1, C_2, C_3 defined by the formulae* (3.2), (3.3), (3.4), (3.5) *satisfy the curve of separation $\mathcal{K}$ of genus five*

$$(q_i + j_1)(q_i + j_2)(q_i + j_3)p_i^4 + \left(q_i^3 C_3 + q_i^2 C_2 + q_i H + K\right)p_i^2$$
$$+ \frac{1}{4}(q_i C_1 + L)^2 = 0, \qquad i \in \{1,2,3\}. \tag{5.15}$$

Proof. In order to prove the theorem, it is necessary to express the integrals H, K, L, C_1, C_2, C_3 in terms of the intermediate $f - g - h$ coordinate system. Using the explicit form of the integrals in terms of the initial coordinate functions S_α, T_α, W_α, $\alpha \in \{1,2,3\}$, the formulae (5.4) and the constraints (5.3) we, in particular, obtain

$$C_1 = -\frac{1}{c_1^2c_2^2c_3^2}\left(\frac{h_1 f_2 + 2y h_2 h_1 + h_3 h_2}{x^2 + (y+j_1)(y+j_2)(y+j_3)} + \frac{h_2 g_2}{2x^2}\right), \tag{5.16a}$$
$$C_3 = \frac{1}{c_1^2c_2^2c_3^2}\left(\frac{y h_1^2 + h_3 h_1}{x^2 + (y+j_1)(y+j_2)(y+j_3)} - \frac{h_2^2}{4x^2}\right), \tag{5.16b}$$

$$L = -\frac{1}{c_1^2c_2^2c_3^2}\left(\frac{2yf_1h_2 + f_3h_2 + f_2f_1}{x^2 + (y+j_1)(y+j_2)(y+j_3)} + \frac{g_2f_2}{2x^2}\right). \tag{5.16c}$$

The explicit expressions for C_2, H and K have similar (quadratic in f_α, g_α, h_α and rational in x and y forms). We will not write these expressions explicitly here due to their long form.

Taking into account the explicit formulae (5.16) and similar formulae for C_2, H, K, substituting them and the formula (5.9) into the right-hand sides of the equalities (5.15), after long and tedious calculations, we obtain that

$$(u+j_1)(u+j_2)(u+j_3)p(u)^4 + \big(u^3C_3 + u^2C_2 + uH + K\big)p(u)^2 \\ + \frac{1}{4}(uC_1 + L)^2 = 0 \mod \mathcal{J}_{S(u)}.$$

Here $\mathcal{J}_{S(u)}$ is an ideal generated by separating polynomial $S(u)$ given by the formulae (5.5) and (5.6). ∎

Remark 5.5. It is possible to show that the curve $\mathcal{K}$ is equivalent to a spectral curve of a four by four Lax matrix of the extended Clebsch and Manakov models.

Remark 5.6. Observe, that Theorems 5.2 and 5.4 assure that the constructed variable separation is a bi-Hamiltonian one and the corresponding vector fields Z_i, $i \in \{1,2,3\}$, satisfying (2.5)–(2.6) do exist. Nevertheless, the vector fields Z_i are very complicated and are of no practical use. That is why we will not present them here leaving their calculation as an exercise for the interested reader.

Remark 5.7. Note that from the explicit form of the equations of separation together with the constraints $Z(C_1) = 0$, $Z(L) = 0$ and from the fact that the coordinates q_i and momenta p_i, $i \in \{1,2,3\}$, are Z-invariants also follows that the roots of $S(u) = u^3Z(C_3) + u^2Z(C_2) + uZ(H) + Z(K)$ are separated coordinates. An additional demonstration of this fact is given in the next subsection.

5.4 The vector field Z in the separated coordinates

In this subsection, we will explicitly calculate the vector field Z in terms of the coordinates of separation. Resolving two of the equations (5.15) with respect to H and K, we obtain their following form:

$$H = -\big(q_1^2 + q_2q_1 + q_2^2\big)C_3 - (q_1+q_2)C_2 - (q_1+j_1)(q_1+j_2)(q_1+j_3)\frac{p_1^2 - p_2^2}{q_1 - q_2} \\ + \frac{1}{4(q_1-q_2)}\left(\frac{(q_2C_1+L)^2}{p_2^2} - \frac{(q_1C_1+L)^2}{p_1^2}\right), \tag{5.17a}$$

$$K = (q_1+q_2)q_1q_2C_3 + q_2q_1C_2 + (q_1+j_1)(q_1+j_2)(q_1+j_3)\frac{q_2p_1^2 - q_1p_2^2}{q_1-q_2} \\ + \frac{1}{4(q_1-q_2)}\left(\frac{q_2(q_1C_1+L)^2}{p_1^2} - \frac{q_1(q_2C_1+L)^2}{p_2^2}\right). \tag{5.17b}$$

Substituting this into the third equation (5.15), we obtain the following equation:

$$(q_2-q_3)(q_1-q_3)p_3^2C_2 + (q_2-q_3)(q_1-q_3)(q_1+q_2+q_3)p_3^2C_3 + (q_1+j_3)(q_1+j_2)(q_1+j_1) \\ \times\frac{(q_2-q_3)}{(q_1-q_2)}p_3^2p_1^2 - (q_2+j_3)(q_2+j_2)(q_2+j_1)\frac{(q_1-q_3)}{(q_1-q_2)}p_3^2p_2^2 + (q_3+j_1)(q_3+j_2) \\ \times(q_3+j_3)p_3^4 + \frac{1}{4}(q_3C_1+L)^2 - \frac{1}{4}(q_2C_1+L)^2\frac{(q_1-q_3)p_3^2}{(q_1-q_2)p_2^2}$$

$$+\frac{1}{4}(q_1C_1+L)^2\frac{(q_2-q_3)p_3^2}{(q_1-q_2)p_1^2}=0. \tag{5.18}$$

Taking into account that $Z(q_1)=Z(q_2)=Z(q_3)=Z(p_1)=Z(p_2)=Z(p_3)=Z(C_1)=0$ and adding the normalization condition $Z(C_3)=1$, we will look for the vector field Z in the following form $Z=Z_3+c_2(q_1,q_2,q_3)Z_2$. Acting on the equation (5.18) by Z and finding from the resulting equation $Z(L)$, we obtain

$$Z(L)=\frac{2(q_2-q_3)(q_1-q_3)(q_1-q_2)p_1^2p_2^2p_3^2((q_1+q_2+q_3)+c_2(q_1,q_2,q_3))}{(q_1-q_2)(q_3C_1+L)p_1^2p_2^2+(q_3-q_1)(q_2C_1+L)p_1^2p_3^2+(q_2-q_3)(q_1C_1+L)p_3^2p_2^2}.$$

From this expression, it immediately follows that the condition $Z(L)=0$ yields the equality $c_2(q_1,q_2,q_3)=-(q_1+q_2+q_3)$, i.e., we obtain the following simple expression for the vector field Z:

$$Z=Z_3-(q_1+q_2+q_3)Z_2.$$

Acting by the defined as above vector field Z on the Casimir functions C_3, C_2 and the integrals H, K given by (5.17), we obtain $Z(C_3)=1$, $Z(C_2)=-(q_1+q_2+q_3)$, $Z(H)=(q_1q_2+q_1q_3+q_2q_3)$, $Z(K)=-q_1q_2q_3$, which again demonstrates that for the given equations of separation $S(u)=u^3Z(C_3)+u^2Z(C_2)+uZ(H)+Z(K)$ is a polynomial-separator with the roots q_1, q_2, q_3.

5.5 The Abel-type equations

The most important for the integration of the equations of motion is possibility to represent these equations in the Abel-type form. As it follows from the general theory exposed in the Section 2.1, more exactly, from the formula (2.2), the following differential equations for the coordinates q_i hold true:

$$\sum_{i=1}^{3}\frac{2q_ip_i^3}{4(q_i+j_1)(q_i+j_2)(q_i+j_3)p_i^4-(q_iC_1+L)^2}\frac{\partial q_i}{\partial t_j}=\delta_{1j}, \tag{5.19a}$$

$$\sum_{i=1}^{3}\frac{2p_i^3}{4(q_i+j_1)(q_i+j_2)(q_i+j_3)p_i^4-(q_iC_1+L)^2}\frac{\partial q_i}{\partial t_j}=\delta_{2j}, \tag{5.19b}$$

$$\sum_{i=1}^{3}\frac{(q_iC_1+L)p_i}{4(q_i+j_1)(q_i+j_2)(q_i+j_3)p_i^4-(q_iC_1+L)^2}\frac{\partial q_i}{\partial t_j}=\delta_{3j}, \tag{5.19c}$$

where $j\in\{1,2,3\}$, t_1, t_2, t_3 are the parameters along the flows of the integrals H, K and L correspondingly

$$\frac{\partial q_i}{\partial t_1}=\{H,q_i\}_1,\qquad \frac{\partial q_i}{\partial t_2}=\{K,q_i\}_1,\qquad \frac{\partial q_i}{\partial t_3}=\{L,q_i\}_1,$$

and we have used the equation of separation (5.15) in order to simplify the form of the differentials on the curve $\mathcal{K}$ entering into the equations (5.19). Using the equations (5.19), we easily obtain the Abel-type quadratures written in the differential form as follows:

$$\sum_{i=1}^{3}\frac{2q_ip_i^3\mathrm{d}q_i}{4(q_i+j_1)(q_i+j_2)(q_i+j_3)p_i^4-(q_iC_1+L)^2}=\mathrm{d}t_1, \tag{5.20a}$$

$$\sum_{i=1}^{3}\frac{2p_i^3\mathrm{d}q_i}{4(q_i+j_1)(q_i+j_2)(q_i+j_3)p_i^4-(q_iC_1+L)^2}=\mathrm{d}t_2, \tag{5.20b}$$

$$\sum_{i=1}^{3} \frac{(q_iC_1 + L)p_i\mathrm{d}q_i}{4(q_i + j_1)(q_i + j_2)(q_i + j_3)p_i^4 - (q_iC_1 + L)^2} = \mathrm{d}t_3. \tag{5.20c}$$

Remark 5.8. Note that the Abel-type equations for the extended Manakov model has the same (modulo the overall sign) form as the equations (5.19) due to the bi-Hamiltonian equivalence of this model with the Clebsch model. The difference is that in the Manakov case the time flows t_1, t_2, t_3 correspond to the brackets $\{\ ,\ \}_2$ and the integrals C_2, H and C_1, but the Hamiltonian flows themselves (modulo the overall sign) are the same.

6 Conclusion and discussion

In the present paper, using the method of vector field Z [8], we have constructed *symmetric*, *non-Stäckel* variable separation for three-dimensional extension of the Clebsch and Manakov models, for which all curves of separation are the same and have genus five. We have explicitly constructed the coordinates and momenta of separation and Abel-type quadratures in the considered examples of symmetric SoV for the extended Clebsch and Manakov models.

We would like also to remark, that our recent results [21] on separation of variables for the Clebsch model can be re-obtained by the restriction of the construction of this paper onto the six dimensional subspace of Clebsch/Manakov models. By other words, the results of the present paper give also a bi-Hamiltonian explanation to the new variable separation for the Clebsch model constructed in [21].

Finally, we would like to outline the following interesting open problems:

(1) To find explicit solution of the Abel–Jacobi inversion problem for the Abel-type equations (5.20) in terms of theta-functions of Prym variety.

(2) To obtain the generalization of the results of the present paper onto the higher-dimensional extensions of the Clebsch and Manakov models.

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