

Toda-Type Presentations for the Quantum K Theory of Partial Flag Varieties

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Abstract. We prove a determinantal, Toda-type, presentation for the equivariant K theory of a partial flag variety $\mathrm{Fl}(r_1, \dots, r_k; n)$. The proof relies on pushing forward the Toda presentation obtained by Maeno, Naito and Sagaki for the complete flag variety $\mathrm{Fl}(n)$, via Kato's $K_T(\mathrm{pt})$ -algebra homomorphism from the quantum K ring of $\mathrm{Fl}(n)$ to that of $\mathrm{Fl}(r_1, \dots, r_k; n)$. Starting instead from the Whitney presentation for $\mathrm{Fl}(n)$, we show that the same pushforward technique gives a recursive formula for polynomial representatives of quantum K Schubert classes in any partial flag variety which do not depend on quantum parameters. In an appendix, we include another proof of the Toda presentation for the equivariant quantum K ring of $\mathrm{Fl}(n)$, following Anderson, Chen, and Tseng, which is based on the fact that the K-theoretic J -function is an eigenfunction of the finite difference Toda Hamiltonians.

Key words: quantum K theory; partial flag varieties; Toda lattice

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1 Introduction

Let $\mathrm{Fl}(n)$ denote the variety of complete flags in $\mathbb{C}^n$, and let $\mathrm{Fl}(\mathbf{r}, n) = \mathrm{Fl}(r_1, \dots, r_k; n)$ be the variety of partial flags. These are homogeneous under the group $\mathrm{SL}_n(\mathbb{C})$, and the restriction of this action to the maximal torus $T \subset \mathrm{SL}_n(\mathbb{C})$ has finitely many fixed points, indexed by a quotient of the symmetric group S_n . Denote by $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ the (equivariant, small) quantum K ring associated to these varieties. This is an algebra over $K_T(\mathrm{pt})[[Q_1, \dots, Q_k]]$, and it has a $K_T(\mathrm{pt})[[Q_1, \dots, Q_k]]$ -basis given by Schubert classes $\mathcal{O}^w$ indexed by the torus fixed points. The quantum K multiplication was defined by Givental and Lee [14, 35] in terms of 3-point, genus 0, K-theoretic Gromov–Witten (KGW) invariants. Denote by

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \dots \subset \mathcal{S}_k \subset \mathcal{S}_{k+1} = \mathbb{C}^n$$

the sequence of tautological bundles in $\mathrm{Fl}(r_1, \dots, r_k; n)$; thus $\mathrm{rank}(\mathcal{S}_i) = r_i$ for $0 \leq i \leq k+1$ with $r_0 = 0$ and $r_{k+1} = n$.

While the computational foundations of the quantum K rings of (cominuscule) Grassmannians have been studied for some time now (see, e.g., [5, 6, 7, 8, 16, 43]), it is only in the last few

years that advances have been made in our understanding of quantum K rings for other flag varieties; see, e.g., [2, 17, 22, 31, 32, 37, 39, 40]. Many of these advances rely on the groundbreaking works by Kato [28, 29], who proved the K-theoretic version of Peterson's 'quantum=affine' statement [24, 33], relating the quantum K ring of a full flag variety (for an arbitrary complex group G) to the K-homology of the corresponding affine Grassmannian; see also [9]. In particular, thanks to results in [39, 40] (proving conjectures in [36]), there are now presentations of the quantum K rings by generators and relations for $\mathrm{QK}_T(\mathrm{Fl}(n))$, and we have polynomial representatives (the quantum double Grothendieck polynomials) for Schubert classes. The generating set of the presentation in [39] is in terms of the quantum quotients $\det \mathcal{S}_i / \det \mathcal{S}_{i-1}$. We rewrite this presentation in determinantal form in Theorem 2.3 below. This makes it easier to identify it with the *Toda presentation*, which is obtained by taking symbols of the finite difference Toda operators studied by Givental and Lee [15], and also by Anderson, Chen and Tseng in [1], see also [30] and Appendix A below.

Our main result is to generalize the Toda presentation from $\mathrm{QK}_T(\mathrm{Fl}(n))$ to one for the ring $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ associated to partial flag varieties. To state it, let

$$Y^{(j)} = (Y_1^{(j)}, \dots, Y_{r_{j+1}-r_j}^{(j)}), \quad 0 \leq j \leq k$$

be formal variables and e_ℓ be the ℓ -th elementary symmetric polynomial. Let $T_1, \dots, T_n \in K_T(\mathrm{pt})$ be given by the decomposition of $\mathbb{C}^n$ into one dimensional T -modules, that is, $\wedge^\ell(\mathbb{C}^n) = e_\ell(T_1, \dots, T_n)$. To distinguish from multiplication in $K_T(\mathrm{Fl}(\mathbf{r}, n))$, we denote the multiplication in $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ by $\star$.

Theorem 1.1 (Theorem 3.4). *The ring $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ is isomorphic to $R[[Q]]/J_Q$, where*

$$R = K_T(\mathrm{pt})[e_1(Y^{(j)}), \dots, e_{r_{j+1}-r_j}(Y^{(j)}), 0 \leq j \leq k],$$

and $J_Q \subset R[[Q]] = R[[Q_1, \dots, Q_k]]$ is the ideal generated by the coefficients of y in

$$\prod_{\ell=1}^n (1 + yT_\ell) - \begin{vmatrix} A_0 & B_1 & & & \\ 1 & A_1 & B_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & A_{k-1} & B_k \\ & & & 1 & A_k \end{vmatrix}^\star,$$

where

$$A_j = \prod_{\ell=1}^{r_{j+1}-r_j} (1 + yY_\ell^{(j)}) + B_j, \quad B_j = y^{r_{j+1}-r_j} \frac{Q_j}{1 - Q_j} \prod_{\ell=1}^{r_{j+1}-r_j} Y_\ell^{(j)},$$

with the convention that $Q_0 = 0$.

More precisely, there exists a $K_T(\mathrm{pt})[[Q]]$ -algebra isomorphism

$$\Psi: R[[Q]]/J_Q \rightarrow \mathrm{QK}_T(\mathrm{Fl}(r_1, \dots, r_k)), \quad e_\ell(Y^{(j)}) \mapsto \wedge^\ell(\mathcal{S}_{j+1}/\mathcal{S}_j)$$

for $j = 0, \dots, k$ and $\ell = 1, \dots, r_{j+1} - r_j$.

Our proof is by decreasing induction on k . The initial case, $k = n - 1$, is the main result of [39], rewritten in determinantal form in Theorem 2.3 below. For the induction step, we use a result of Kato [28] which states that there is a $K_T(\mathrm{pt})$ -algebra homomorphism

$$\mathrm{QK}_T(\mathrm{Fl}(n)) \rightarrow \mathrm{QK}_T(\mathrm{Fl}(r_1, \dots, r_k; n));$$

$$\mathcal{O}^w \mapsto \pi_*(\mathcal{O}^w), \quad Q_r \mapsto \begin{cases} 1, & r \notin \{r_1, \dots, r_k\}, \\ Q_i, & r = r_i, \end{cases} \quad (1.1)$$

which extends the usual projection map $\pi_*: K_T(\mathrm{Fl}(n)) \rightarrow K_T(\mathrm{Fl}(\mathbf{r}, n))$. Note that the classical π_* is *not* a ring map. (Kato's result is for general complex, simple groups G .) We use this to show that the ideal J_Q is contained in the ideal of relations. For the specialization $Q_i \mapsto 1$ to be well defined, one needs to work with polynomials in $Q_1, \dots, Q_{n-1}$; see Section 2.2. Pushing forward the original Toda relations is not possible, due to poles at $Q_i = 1$. We had to rewrite these relations, and additionally use an extra identity due to Maeno, Naito, and Sagaki (cf. Proposition 2.7 below), in order for the push forward to be performed. Finally, it follows from [19] that the ideal J_Q coincides with the ideal of relations.

The same pushforward technique may be applied to the *Whitney presentation*, conjectured in [17, 20], and for which a proof was recently announced in [22]; see also [18, 21] for the Grassmannian case. This is a presentation for $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ with generators $\wedge^k(\mathcal{S}_i)$ and $\wedge^\ell(\mathcal{S}_i/\mathcal{S}_{i-1})$. We prove in Proposition 4.1 that if one eliminates the variables corresponding to classes $\wedge^k(\mathcal{S}_i)$ in the Whitney presentation, then one recovers the Toda presentation.

Our methods also provide a different proof of the Whitney presentation for $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$, once the Whitney presentation for $\mathrm{QK}_T(\mathrm{Fl}(n))$ (a special case of results from [22]) is assumed; see Remark 4.2. The details of this proof are omitted, as they follow closely the proof of Theorem 1.1.

In a further application of our technique, using the aforementioned Whitney presentation, we rewrite the formula from [40] of the quantum double Grothendieck polynomial of the class of a point in $\mathrm{Fl}(n)$ [40] in terms of the classes $\lambda_y(\mathcal{S}_i)$. Surprisingly, the resulting class is *independent* of the quantum parameters Q_i . Pushing forward this class results in a polynomial representative for the class of the (Schubert) point in any $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ which is independent of Q_i . The outcome is the following.

Theorem 1.2 (Theorem 5.7). *Let $\mathcal{O}^{w_0} \in \mathrm{QK}_T(\mathrm{Fl}(r_1, \dots, r_k; n))$ be the class of the Schubert point. Then the following holds:*

$$\mathcal{O}^{w_0} = \prod_{i=1}^k \prod_{j=r_i}^{r_{i+1}-1} \lambda_{-1}(e^{-\epsilon_{n-j}} \mathcal{S}_i),$$

where $e^{\epsilon_i} \in K_T(\mathrm{pt})$ denotes the (class of the) 1-dimensional T -representation with weight ϵ_i .

In the usual (equivariant) K theory of $\mathrm{Fl}(n)$ this follows from Fulton's results in [11] showing that the Schubert point X^{w_0} is the zero locus of a section of a vector bundle; see also [12, Theorem 3]. Using the left divided difference operators in $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ defined in [42], this results in a recursive formula for any Schubert class, giving polynomial representatives in terms of exterior powers $\wedge^i \mathcal{S}_j$ which do not depend on quantum parameters. See Theorem 5.11. Precursors of this 'quantum=classical' phenomenon for polynomial representatives of quantum Schubert classes have been observed for (isotropic) Grassmannians [3, 4, 16, 25, 41], but to our knowledge this is new for (partial) flag varieties. Recently, we learned that T. Kouno found a similar phenomenon in the quantum K ring of the symplectic flag varieties Sp_{2n}/B .

In Appendix A, we follow Anderson, Chen, and Tseng's treatment in the unpublished note [1] to give another proof of the Toda presentation for $\mathrm{QK}_T(\mathrm{Fl}(n))$, independent of the one from [39]. The proof combines results of Givental and Lee [15], which states that the K-theoretic J-function of $\mathrm{Fl}(n)$ is an eigenfunction of the first (finite difference) Toda Hamiltonian, with results of Iritani, Milanov and Tonita [26], which relates this fact to relations in the quantum K theory ring. We do not claim any originality in this argument, but we found it valuable to include it here, as it puts together results from the followup papers [2] and [29]; see especially Proposition A.7.

1.1 A logical roadmap

There are several recent results in the literature which inform the Toda and Whitney presentations we prove in this paper. Since we do not attempt to give self-contained proofs, we provide next a roadmap of the logical implications we rely on, which a reader may find useful.

Our proofs of both the Toda presentation, and the Whitney presentation, for $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$, from Theorem 3.4 (resp. (4.2), see Remark 4.2) rely on the following: the Toda (resp. Whitney) presentation for $\mathrm{QK}_T(\mathrm{Fl}(n))$; Kato's push forward homomorphism from (1.1); and the key technical result from Proposition 2.7, proved in [39], which in turn relies on Kato's work [29].

There are two proofs of the Toda presentation for $\mathrm{QK}_T(\mathrm{Fl}(n))$, one in [39], relying on [29], and another which may be deduced from [1], relying on results from [15] and [26]; cf. Appendix A.

There are also two proofs of the Whitney presentation for $\mathrm{QK}_T(\mathrm{Fl}(n))$. One was recently announced by Huq-Kuruville [22] (for all rings $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$), and it uses the technique of abelian-nonabelian correspondence, independent of Kato's results. Another proof of the Whitney presentation for $\mathrm{QK}_T(\mathrm{Fl}(n))$ is given in [20, Section 6]. It relies on the recent proof of the quantum K divisor axiom [38], and ultimately on Kato's results.

2 Preliminaries

2.1 Equivariant K theory of Grassman bundles

Let T be a linear algebraic group. For any projective T -variety Z , let $K_T(Z)$ be the equivariant K-theory ring, defined as the Grothendieck ring of T -equivariant algebraic vector bundles. This ring is an algebra over $K_T(\mathrm{pt})$, the representation ring of T . Let $\chi_Z: K_T(Z) \rightarrow K_T(\mathrm{pt})$ be the pushforward map along the structure morphism.

For $E \rightarrow Z$ a T -equivariant vector bundle of rank $\mathrm{rk} E$, we denote by

$$\lambda_y(E) := 1 + y[E] + \cdots + y^{\mathrm{rk} E} [\wedge^{\mathrm{rk} E} E] \in K_T(Z)[y]$$

the Hirzebruch λ_y class of E . This class is multiplicative for short exact sequences. In an abuse of notation, we often write E for the class $[E]$ in $K_T(Z)$. Note that for a rank e equivariant vector bundle E , and a character $e^\chi \in K_T(\mathrm{pt})$,

$$\lambda_y(e^\chi \otimes E) = \lambda_{ye^\chi}(E) = \sum_{i=0}^e y^i e^{i\chi} \otimes \wedge^i E.$$

As is customary, we will often remove the $\otimes$ symbol from the notation.

Denote by $\pi: \mathbb{G}(r, E) \rightarrow Z$ the Grassmann bundle over Z . It is equipped with a tautological sequence $0 \rightarrow \underline{\mathcal{S}} \rightarrow \pi^* E \rightarrow \underline{\mathcal{Q}} \rightarrow 0$ over $\mathbb{G}(r, E)$. The following result follows from [27, Proposition 2.2], see also [21, Proposition 3.2 and Corollary 3.3]. (Kapranov proved this when $Z = \mathrm{pt}$; the relative version follows immediately using that π is a T -equivariant locally trivial fibration.) We only state the special cases that will be used in this paper. See the above references for the full generality.

Proposition 2.1 (Kapranov). *There are the following isomorphisms of T -equivariant vector bundles:*

- (1) for all $i \geq 0$, $\ell > 0$ the higher direct images, $R^i \pi_* (\wedge^\ell \underline{\mathcal{S}}) = 0$;
- (2) for all $\ell \geq 0$,

$$R^i \pi_* (\wedge^\ell \underline{\mathcal{Q}}) = \begin{cases} \wedge^\ell E, & i = 0, \\ 0, & i > 0. \end{cases}$$

2.2 (Equivariant) quantum K theory of flag varieties

Let $\mathbf{r} = (r_1, \dots, r_k)$. We consider

$$X = \text{Fl}(\mathbf{r}, n),$$

which parametrizes flags of vector spaces $F_1 \subset F_2 \subset \dots \subset F_k \subset \mathbb{C}^n$ with $\dim F_i = r_i$ for $1 \leq i \leq k$.

Let $M_{d,n} := \overline{\mathcal{M}}_{0,n}(X, d)$ be the moduli space of genus zero degree d stable maps to X with n marked points. Given classes $a_1, \dots, a_n \in K_T(X)$, define the K-theoretic Gromov–Witten invariants by

$$\langle a_1, \dots, a_n \rangle_d = \chi_{M_{d,n}} \left(\prod_{i=1}^n \text{ev}_i^*(a_i) \right) \in K_T(\text{pt}).$$

Non-equivariant Gromov–Witten invariants are obtained by replacing T with the trivial group; these Gromov–Witten invariants are integers.

For $d = (d_1, \dots, d_k) \in H_2(X, \mathbb{Z}) \cong \mathbb{Z}^k$, we write Q^d for $\prod_{i=1}^k Q_i^{d_i}$. Here Q_i corresponds to the Poincaré dual of the first Chern class $-c_1(\det \mathcal{S}_i)$. Following [14, 35], the T -equivariant (small) quantum K theory ring is

$$\text{QK}_T(X) = K_T(X) \otimes_{K_T(\text{pt})} K_T(\text{pt})[[Q]]$$

as a $K_T(\text{pt})[[Q]]$ -module. It is equipped with a commutative, associative product, denoted by $\star$, which is determined by the condition

$$((\sigma_1 \star \sigma_2, \sigma_3)) = \sum_d Q^d \langle \sigma_1, \sigma_2, \sigma_3 \rangle_d \quad \text{for all } \sigma_1, \sigma_2, \sigma_3 \in K_T(X),$$

where

$$((\sigma_1, \sigma_2)) := \sum_d Q^d \langle \sigma_1, \sigma_2 \rangle_d$$

is the quantum K-metric.

It was proved in [2, 29] that for $\sigma_1, \sigma_2 \in K_T(X)$, the product $\sigma_1 \star \sigma_2$ can always be expressed as a polynomial in Q with coefficients in $K_T(X)$. It follows that

$$\text{QK}_T^{\text{poly}}(X) := K_T(X) \otimes_{K_T(\text{pt})} K_T(\text{pt})[Q]$$

is a subring of $\text{QK}_T(X)$.

Let $Y = \text{Fl}(r_1, \dots, \widehat{r}_i, \dots, r_k; n)$ and $\pi: X \rightarrow Y$ be the natural map. Let also $\widehat{\mathbf{r}} = (r_1, \dots, \widehat{r}_i, \dots, r_k)$. The following theorem is a specialization of results proved in [28].

Theorem 2.2 (Kato). *There is a surjective ring homomorphism*

$$\Phi: \text{QK}_T^{\text{poly}}(X) \rightarrow \text{QK}_T^{\text{poly}}(Y)$$

given by $\sigma \mapsto \pi_* \sigma$ for all $\sigma \in K_T(X)$ and

$$Q_j \mapsto \begin{cases} Q_j, & j \neq i, \\ 1, & j = i \end{cases} \quad \text{for } 1 \leq i \leq k.$$

It follows from Theorem 2.2 that Kato's homomorphism extends naturally to

$$\Phi: \text{QK}_T^{\text{loc}(\widehat{\mathbf{r}})}(X) \rightarrow \text{QK}_T^{\text{loc}(\widehat{\mathbf{r}})}(Y), \tag{2.1}$$

where $\text{loc}(\widehat{\mathbf{r}})$ indicates localization at the multiplicative set generated by $1 - Q_j$ for $j \neq i$.

2.3 The Toda presentation for $\text{Fl}(n)$

The variety $\text{Fl}(n) = \text{Fl}(1, \dots, n-1; n)$ is equipped with tautological vector bundles

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \dots \subset \mathcal{S}_{n-1} \subset \mathcal{S}_n = \mathbb{C}^n,$$

where $\mathcal{S}_j$ has rank r_j . It can also be viewed as SL_n/B , where $B \subset \text{SL}_n$ is a Borel subgroup. Let $T \subseteq B$ be a maximal torus in SL_n .

The following is the main result of [39] (see Remark 2.4 for more details). The relation (2.2) can also be recovered from the connection between the J -function of the full flag variety and the relativistic Toda lattice established by Givental and Lee in [15]. This observation was made in the unpublished note [1] of Anderson–Chen–Tseng, but removed from the published version of their paper. For the sake of completeness, we give a brief account in Appendix A.

Theorem 2.3. *The ring $\text{QK}_T(\text{Fl}(n))$ is isomorphic to $R'[[Q]]/J'_Q$, where R' is equal to the ring $\text{K}_T(\text{pt})[P_1^\pm, \dots, P_n^\pm]$ and the ideal $J'_Q \subset R'[[Q]] = R'[[Q_1, \dots, Q_{n-1}]]$ is generated by the coefficients y in*

$$\lambda_y(\mathbb{C}^n) - \begin{vmatrix} 1 + y \frac{P_1}{P_0} & y \frac{P_2}{P_1} Q_1 & & & & & \\ & 1 & 1 + y \frac{P_2}{P_1} & y \frac{P_3}{P_2} Q_2 & & & \\ & & 1 & 1 + y \frac{P_3}{P_2} & y \frac{P_4}{P_3} Q_3 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 1 + y \frac{P_{n-1}}{P_{n-2}} & y \frac{P_n}{P_{n-1}} Q_{n-1} \\ & & & & & 1 & 1 + y \frac{P_n}{P_{n-1}} \end{vmatrix}^*, \quad (2.2)$$

here $P_0 = 1$ by convention, and $\lambda_y(\mathbb{C}^n) \in \text{K}_T(\text{pt})[y]$.

More precisely, there exists a $\text{K}_T(\text{pt})[[Q]]$ -algebra isomorphism $\Psi': R'[[Q]]/J'_Q \rightarrow \text{QK}_T(\text{Fl}(n))$ that sends P_j to $\det \mathcal{S}_j$ for all $j = 1, \dots, n$.

Remark 2.4. Theorem 2.3 is proved in [39] using results of Kato [29] based on the semi-infinite flag variety. The connection between our statement of Theorem 2.3 and that of [39] is seen as follows. Define the *Toda polynomials* $T_k^{(n)}$ for $k = 1, \dots, n$ by

$$T_k^{(n)} = \sum_{0=i_0 < \dots < i_k \leq n} \prod_{s=1}^k \frac{P_{i_s}}{P_{i_{s-1}}} (1 - Q_{i_{s-1}})^{1 - \delta_{i_s - i_{s-1}, 1}}.$$

These elements of $\mathbb{Z}[P_1^\pm, \dots, P_n^\pm][[Q]]$ (where $P_0 = 1$ and $Q_0 = 0$ by convention) are symbols of the finite-difference Toda Hamiltonians [10] (see also [1, 13, 15, 30]).

Letting $T^{(n)} = \sum_{k=0}^n T_k^{(n)} y^k$ where $T_0^{(n)} = 1$, we claim that $T^{(n)}(y)$ is equal to the determinant of the matrix appearing in the Toda relations (2.2), namely,

$$T^{(n)}(y) = \begin{vmatrix} 1 + y \frac{P_1}{P_0} & y \frac{P_2}{P_1} Q_1 & & & & & \\ & 1 & 1 + y \frac{P_2}{P_1} & y \frac{P_3}{P_2} Q_2 & & & \\ & & 1 & 1 + y \frac{P_3}{P_2} & y \frac{P_4}{P_3} Q_3 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 1 + y \frac{P_{n-1}}{P_{n-2}} & y \frac{P_n}{P_{n-1}} Q_{n-1} \\ & & & & & 1 & 1 + y \frac{P_n}{P_{n-1}} \end{vmatrix}^*.$$

This is verified by showing that $T^{(n)}$ satisfies the recursion

$$T^{(n)} = T^{(n-1)} \left(1 + y \frac{P_n}{P_{n-1}} \right) - y Q_{n-1} \frac{P_n}{P_{n-1}} T^{(n-2)},$$

and then applying Lemma 2.6 below (with n playing the role of j there).

Remark 2.5. Upon the specialization $Q_1 = \cdots = Q_{n-1} = 0$, the Toda presentation $R[[Q]]/J'_Q \cong \text{QK}_T(\text{Fl}(n))$ becomes the Borel presentation

$$\text{K}_T(\text{pt})[P_1^\pm, \dots, P_n^\pm]/J \cong \text{K}_T(\text{Fl}(n)),$$

where J is the ideal generated by the coefficients of y in

$$\lambda_y(\mathbb{C}^n) - \prod_{j=0}^{n-1} (1 + yP_{j+1}/P_j)$$

and P_j corresponds to $\det \mathcal{S}_j$ for all $j = 1, \dots, n$.

Lemma 2.6. Suppose U_j for $0 \leq j \leq k+1$ and A_j, B_j for $0 \leq j \leq k$ are elements of a commutative ring with 1 such that the U_j satisfy the recursion

$$U_{j+1} = A_j U_j - B_j U_{j-1}, \quad 0 \leq j \leq k$$

with initial conditions $U_0 = 1, U_{-1} = 0$. Then, for all $0 \leq j \leq k+1$, one has

$$U_j = \begin{vmatrix} A_0 & B_1 & & & \\ 1 & A_1 & B_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & A_{j-2} & B_{j-1} \\ & & & 1 & A_{j-1} \end{vmatrix}. \quad (2.3)$$

Proof. One simply expands along the last row or column to see that the determinant in (2.3) satisfies the recursion. Observe that the initial values $U_0 = 1$ and $U_1 = A_0$ agree. This completes the proof. $\blacksquare$

Before finishing this section, we record the following, which follows from [39, Proposition 5.2].

Proposition 2.7 (Maeno–Naito–Sagaki). *In $\text{QK}_T(\text{Fl}(n))$, the following relations hold:*

$$\det \mathcal{S}_i \star \det \mathcal{S}_j / \mathcal{S}_i = (1 - Q_i) \det \mathcal{S}_j, \quad 1 \leq i < j \leq n.$$

3 Toda-type presentations for the equivariant quantum K theory of partial flag varieties

To begin, we observe that the Toda presentation in Theorem 2.3 can be rewritten as follows.

Corollary 3.1. *The ring $\text{QK}_T(\text{Fl}(n))$ is isomorphic to $R[[Q]]/J_Q$, where*

$$R = \text{K}_T(\text{pt})[Y^{(0)}, \dots, Y^{(n-1)}],$$

and $J_Q \subset R[[Q]] = R[[Q_1, \dots, Q_{n-1}]]$ is generated by the coefficients of y in

$$\lambda_y(\mathbb{C}^n) - \begin{vmatrix} 1 + yY^{(0)} \frac{1}{1-Q_0} & yY^{(1)} \frac{Q_1}{1-Q_1} & & & \\ 1 & 1 + yY^{(1)} \frac{1}{1-Q_1} & yY^{(2)} \frac{Q_2}{1-Q_2} & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 + yY^{(n-2)} \frac{1}{1-Q_{n-2}} & yY^{(n-1)} \frac{Q_{n-1}}{1-Q_{n-1}} \\ & & & 1 & 1 + yY^{(n-1)} \frac{1}{1-Q_{n-1}} \end{vmatrix}^*$$

with the convention that $Q_0 = 0$.

More precisely, there exists a $\text{K}_T(\text{pt})[[Q]]$ -algebra isomorphism $\Psi: R[[Q]]/J_Q \rightarrow \text{QK}_T(\text{Fl}(n))$ that sends $Y^{(j)}$ to $\mathcal{S}_{j+1}/\mathcal{S}_j$ for $j = 1, \dots, n-1$.

Proof. Identifying P_{j+1}/P_j with $Y^{(j)}/(1 - Q_j)$ gives an isomorphism between $R[[Q]]/J_Q$ and $R'[[Q]]/J'_Q$. More precisely, define a $\mathrm{K}_T(\mathrm{pt})[[Q]]$ homomorphism $\Phi: R'[[Q]]/J'_Q \rightarrow R[[Q]]/J_Q$ by

$$\Phi(P_j) = \prod_{i=0}^{j-1} \frac{Y^{(i)}}{1 - Q_i}, \quad 1 \leq j \leq n-1.$$

Note that in $R[[Q]]/J_Q$, we have $\det \mathbb{C}^n = \prod_{i=0}^{n-1} Y^{(i)}/(1 - Q_i)$, which implies all $Y^{(j)}$ are invertible. Since the relations match, the homomorphism Ψ is well-defined and injective. Since $(1 - Q_j)P_{j+1}/P_j$ is sent to $Y^{(j)}$ for $0 \leq j \leq n-1$, it is also surjective. Finally, the geometric interpretation follows from Proposition 2.7. $\blacksquare$

Next, we generalize Corollary 3.1 to all partial flag varieties utilizing Theorem 2.2, Proposition 2.7, and the Nakayama-type result from [20, 21].

Theorem 3.2. *In $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))[y]$, the following relation hold:*

$$\lambda_y(\mathbb{C}^n) - \begin{vmatrix} A_0 & B_1 & & & \\ 1 & A_1 & B_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & A_{k-1} & B_k \\ & & & 1 & A_k \end{vmatrix}^*, \quad (3.1)$$

where

$$B_j = y^{r_{j+1}-r_j} \frac{Q_j}{1 - Q_j} \det(\mathcal{S}_{j+1}/\mathcal{S}_j), \quad A_j = \lambda_y(\mathcal{S}_{j+1}/\mathcal{S}_j) + B_j.$$

Proof. Let $X = \mathrm{Fl}(r_1, \dots, r_k; n)$, $Y = \mathrm{Fl}(r_1, \dots, \widehat{r}_i, \dots, r_k; n)$, and $\pi: X \rightarrow Y$ be the natural map. Let

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \dots \subset \mathcal{S}_k \subset \mathcal{S}_{k+1} = \mathbb{C}^n$$

be the sequence of tautological bundles on X . Note that all but $\mathcal{S}_i$ are pulled back from Y . With a slight abuse of notation, we denote the sequence of tautological bundles on Y by

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \dots \mathcal{S}_{i-1} \subset \mathcal{S}_{i+1} \subset \dots \subset \mathcal{S}_k \subset \mathcal{S}_{k+1} = \mathbb{C}^n.$$

Note that the elements $B_1, \dots, B_{i-2}, B_{i+1}, \dots, B_k$ as well as $A_1, \dots, A_{i-2}, A_{i+1}, \dots, A_k$ in the ring $\mathrm{QK}_T(X)[y]$ stay the same under pushforward along π . By a slight abuse of notation, we also think of them as elements of $\mathrm{QK}_T(Y)[y]$.

By induction, we assume that relation (3.1) holds for X , i.e.,

$$\lambda_y(\mathbb{C}^n) - \begin{vmatrix} A_0 & B_1 & & & & & \\ 1 & A_1 & B_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & B_{i-2} & & \\ & & & 1 & A_{i-2} & B_{i-1} & \\ & & & & 1 & A_{i-1} & B_i \\ & & & & & 1 & A_i & B_{i+1} \\ & & & & & & 1 & A_{i+1} & \ddots \\ & & & & & & & \ddots & \ddots & \ddots \\ & & & & & & & & 1 & A_{k-1} & B_k \\ & & & & & & & & & 1 & A_k \end{vmatrix}^* \quad (3.2)$$

holds in $\mathrm{QK}_T^{\mathrm{loc}(\mathbf{r})}(X)[y]$ for $1 \leq j \leq k$, and we will show that the (localized) Kato's pushforward (2.1) of this relation gives relation (3.1) on Y .

Relation (3.1) on Y reads

$$\lambda_y(\mathbb{C}^n) - \begin{vmatrix} A_0 & B_1 & & & & & & & \\ 1 & A_1 & B_2 & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & \ddots & \ddots & B_{i-2} & & & & \\ & & & 1 & A_{i-2} & B'_{i-1} & & & \\ & & & & 1 & A'_{i-1} & B_{i+1} & & \\ & & & & & 1 & A_{i+1} & \ddots & \\ & & & & & & \ddots & \ddots & \ddots \\ & & & & & & & 1 & A_{k-1} & B_k \\ & & & & & & & & 1 & A_k \end{vmatrix}^*, \quad (3.3)$$

where

$$B'_{i-1} = y^{r_{i+1}-r_{i-1}} \frac{Q_{i-1}}{1-Q_{i-1}} \det(\mathcal{S}_{i+1}/\mathcal{S}_{i-1}), \quad A'_{i-1} = \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_{i-1}) + B'_{i-1},$$

regarded as elements in $\mathrm{QK}_T^{\mathrm{loc}(\widehat{\mathbf{r}})}(Y)[y]$.

By the projection formula, to prove (3.3), it suffices to prove the pushforward along π of (3.2) agrees with (3.3). We compare the two determinants by expanding along columns. Expanding along the column containing B'_{i-1} , we have that the determinant in (3.3) is of the form

$$-B'_{i-1} \star C' + A'_{i-1} \star D' - E';$$

expanding along the two columns containing B_{i-1} or B_i , we have that the determinant in (3.2) is of the form

$$\begin{aligned} & \begin{vmatrix} B_{i-1} & 0 \\ A_{i-1} & B_i \end{vmatrix}^* \star 0 - \begin{vmatrix} B_{i-1} & 0 \\ 1 & A_i \end{vmatrix}^* \star C + \begin{vmatrix} B_{i-1} & 0 \\ 0 & 1 \end{vmatrix}^* \star F \\ & + \begin{vmatrix} A_{i-1} & B_i \\ 1 & A_i \end{vmatrix}^* \star D - \begin{vmatrix} A_{i-1} & B_i \\ 0 & 1 \end{vmatrix}^* \star E + \begin{vmatrix} 1 & A_i \\ 0 & 1 \end{vmatrix}^* \star 0. \end{aligned}$$

Note that C, D, E, F stay the same under the pushforward, and it is straightforward to check that

$$C' = C, \quad D' = D, \quad E' = E.$$

The rest follows from Lemma 3.3 below. ■

Lemma 3.3. *The following hold:*

- (a) $\pi_* \begin{vmatrix} A_{i-1} & B_i \\ 0 & 1 \end{vmatrix}^* = 1;$
- (b) $\pi_* \begin{vmatrix} B_{i-1} & 0 \\ 0 & 1 \end{vmatrix}^* = 0;$
- (c) $\pi_* \begin{vmatrix} B_{i-1} & 0 \\ 1 & A_i \end{vmatrix}^* = B'_{i-1};$

(d) Assume that $r_i - r_{i-1} = 1$. Then

$$\pi_* \begin{vmatrix} A_{i-1} & B_i \\ 1 & A_i \end{vmatrix}^* = A'_{i-1}.$$

Proof. Note that X may be realized as the Grassmann bundle $\mathbb{G}(r_i - r_{i-1}, \mathcal{S}_{i+1}/\mathcal{S}_{i-1})$ over Y , with tautological sequence $0 \rightarrow \mathcal{S}_i/\mathcal{S}_{i-1} \rightarrow \mathcal{S}_{i+1}/\mathcal{S}_{i-1} \rightarrow \mathcal{S}_{i+1}/\mathcal{S}_i \rightarrow 0$. It follows from Proposition 2.1 that

$$\pi_*(\lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i)) = \sum_{j=0}^{r_{i+1}-r_i} y^j \wedge^j (\mathcal{S}_{i+1}/\mathcal{S}_{i-1}), \quad \pi_*(\lambda_y(\mathcal{S}_i/\mathcal{S}_{i-1})) = 1. \quad (3.4)$$

For (a), (b), note that $A_{i-1}, B_{i-1} \in \mathrm{QK}_T^{\mathrm{loc}(\hat{\mathfrak{r}})}(X)$, so we may use (2.1), and it follows that

$$\pi_* B_{i-1} = 0, \quad \pi_* A_{i-1} = 1.$$

Note that by Proposition 2.7 and Theorem 2.2, we have

$$\det \mathcal{S}_j \star \det(\mathcal{S}_{j+1}/\mathcal{S}_j) = (1 - Q_j) \det \mathcal{S}_{j+1} \quad \text{for } 0 \leq j \leq k, \text{ in } \mathrm{QK}_T(X), \quad (3.5)$$

$$\det \mathcal{S}_{i-1} \star \det(\mathcal{S}_{i+1}/\mathcal{S}_{i-1}) = (1 - Q_{i-1}) \det \mathcal{S}_{i+1} \quad \text{in } \mathrm{QK}_T(Y). \quad (3.6)$$

To prove (c), we obtain from definition

$$\begin{aligned} \begin{vmatrix} B_{i-1} & 0 \\ 1 & A_i \end{vmatrix}^* &= B_{i-1} A_i = B_{i-1} \star (\lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) + B_i) \\ &= B_{i-1} \star \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) + B_{i-1} \star B_i. \end{aligned} \quad (3.7)$$

The element B_i cannot be pushed forward, as it contains $1 - Q_i$ in the denominator. However, we use (3.5) to calculate

$$\begin{aligned} B_{i-1} \star B_i &= y^{r_{i+1}-r_{i-1}} \frac{Q_{i-1} Q_i}{(1 - Q_{i-1})(1 - Q_i)} \det(\mathcal{S}_{i+1}/\mathcal{S}_i) \star \det(\mathcal{S}_i/\mathcal{S}_{i-1}) \\ &= y^{r_{i+1}-r_{i-1}} Q_{i-1} Q_i \frac{\det \mathcal{S}_{i+1}}{\det \mathcal{S}_{i-1}}, \end{aligned}$$

where the inverse is calculated in the quantum K ring of X . By (3.5) again,

$$\frac{\det \mathcal{S}_{i+1}}{\det \mathcal{S}_{i-1}} = \frac{\det \mathcal{S}_{i+1} \star \det \mathbb{C}^n / \mathcal{S}_{i-1}}{(1 - Q_{i-1}) \det \mathbb{C}^n} \quad \text{in } \mathrm{QK}_T^{\mathrm{loc}(\hat{\mathfrak{r}})}(X),$$

and its pushforward is

$$\frac{\det \mathcal{S}_{i+1}}{\det \mathcal{S}_{i-1}} \in \mathrm{QK}_T^{\mathrm{loc}(\hat{\mathfrak{r}})}(Y). \quad (3.8)$$

Note that by (3.6), expression (3.8) is equal to

$$\frac{\det(\mathcal{S}_{i+1}/\mathcal{S}_{i-1})}{1 - Q_{i-1}} \quad \text{in } \mathrm{QK}_T^{\mathrm{loc}(\hat{\mathfrak{r}})}(Y).$$

Using (3.4) and (3.7), the projection formula, and Theorem 2.2, it follows that

$$\pi_* \begin{vmatrix} B_{i-1} & 0 \\ 1 & A_i \end{vmatrix}^* = \pi_*(B_{i-1} \star B_i) = y^{r_{i+1}-r_{i-1}} \frac{Q_{i-1}}{1 - Q_{i-1}} \det(\mathcal{S}_{i+1}/\mathcal{S}_{i-1}) = B'_{i-1}.$$

For (d), we calculate

$$\begin{vmatrix} A_{i-1} & B_i \\ 1 & A_i \end{vmatrix}^* = \begin{vmatrix} \lambda_y(\mathcal{S}_i/\mathcal{S}_{i-1}) + B_{i-1} & B_i \\ 1 & A_i \end{vmatrix}^* = A_i \star \lambda_y(\mathcal{S}_i/\mathcal{S}_{i-1}) + A_i \star B_{i-1} - B_i.$$

From (c), $\pi_*(A_i \star B_{i-1}) = B'_{i-1}$, therefore it suffices to show that $A_i \star \lambda_y(\mathcal{S}_i/\mathcal{S}_{i-1}) - B_i$ may be pushed forward, and that

$$\pi_*(A_i \star \lambda_y(\mathcal{S}_i/\mathcal{S}_{i-1}) - B_i) = \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_{i-1}). \quad (3.9)$$

The hypothesis $r_i - r_{i-1} = 1$ implies that $\mathcal{S}_i/\mathcal{S}_{i-1}$ is a line bundle, and that

$$\begin{aligned} A_i \star \lambda_y(\mathcal{S}_i/\mathcal{S}_{i-1}) - B_i \\ = \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) \star \lambda_y(\mathcal{S}_i/\mathcal{S}_{i-1}) + y^{r_{i+1}-r_{i-1}} \frac{Q_i}{1-Q_i} \det(\mathcal{S}_{i+1}/\mathcal{S}_i) \star \det(\mathcal{S}_i/\mathcal{S}_{i-1}). \end{aligned}$$

By (3.4), we have

$$\pi_*(\lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i)) = \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_{i-1}) - y^{r_i-r_{i-1}} \det(\mathcal{S}_{i+1}/\mathcal{S}_{i-1}), \quad \pi_*(\lambda_y(\mathcal{S}_i/\mathcal{S}_{i-1})) = 1.$$

By (3.5), we have

$$\frac{Q_i}{1-Q_i} \det(\mathcal{S}_{i+1}/\mathcal{S}_i) \star \det(\mathcal{S}_i/\mathcal{S}_{i-1}) = Q_i(1-Q_{i-1}) \frac{\det \mathcal{S}_{i+1}}{\det \mathcal{S}_{i-1}}.$$

As in the proof of (c), this can be pushed forward and its pushforward is $\det(\mathcal{S}_{i+1}/\mathcal{S}_{i-1})$. Putting these together, we have established (3.9). $\blacksquare$

Recall that $Y^{(j)} = (Y_1^{(j)}, \dots, Y_{r_{j+1}-r_j}^{(j)})$, $0 \leq j \leq k$, are formal variables, e_ℓ denotes the ℓ -th elementary symmetric polynomial, and $T_1, \dots, T_n \in K_T(\text{pt})$ are given by the decomposition of $\mathbb{C}^n$ into one dimensional T -modules, that is, $\wedge^\ell(\mathbb{C}^n) = e_\ell(T_1, \dots, T_n)$.

Theorem 3.4. *The ring $\text{QK}_T(\text{Fl}(\mathbf{r}, n))$ is isomorphic to $R[[Q]]/J_Q$, where*

$$R = K_T(\text{pt})[e_1(Y^{(j)}), \dots, e_{r_{j+1}-r_j}(Y^{(j)}), 0 \leq j \leq k],$$

and $J_Q \subset R[[Q]] = R[[Q_1, \dots, Q_k]]$ is the ideal generated by the coefficients of y in

$$\prod_{\ell=1}^n (1 + yT_\ell) - \begin{vmatrix} A_0 & B_1 \\ 1 & A_1 & B_2 \\ & \ddots & \ddots & \ddots \\ & & 1 & A_{k-1} & B_k \\ & & & 1 & A_k \end{vmatrix}, \quad (3.10)$$

where

$$A_j = \prod_{\ell=1}^{r_{j+1}-r_j} (1 + yY_\ell^{(j)}) + B_j, \quad B_j = y^{r_{j+1}-r_j} \frac{Q_j}{1-Q_j} \prod_{\ell=1}^{r_{j+1}-r_j} Y_\ell^{(j)},$$

with the convention that $Q_0 = 0$.

More precisely, there exists a $K_T(\text{pt})[[Q]]$ -algebra isomorphism

$$\Psi: R[[Q]]/J_Q \rightarrow \text{QK}_T(\text{Fl}(r_1, \dots, r_k))$$

that sends $e_\ell(Y^{(j)})$ to $\wedge^\ell(\mathcal{S}_{j+1}/\mathcal{S}_j)$ for $j = 0, \dots, k$ and $\ell = 1, \dots, r_{j+1} - r_j$.

Proof. It follows from Theorem 3.2 that Ψ is a well-defined ring homomorphism. To prove there are no other relations, we use [19, Theorem 4.1], which states that a complete set of relations in the quantum (equivariant) K ring is obtained by quantizing any complete set of relations in the ordinary (equivariant) K ring. Therefore, we need to show that when one specializes each Q_i to 0, the resulting ring is a presentation of $K_T(\text{Fl}(r_1, \dots, r_k))$. The relations obtained this way are the ‘Borel-type relations’ of the λ_y classes

$$\lambda_y(\mathcal{S}_1) \cdot \lambda_y(\mathcal{S}_2/\mathcal{S}_1) \cdot \dots \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_k) = \lambda_y(\mathbb{C}^n). \quad (3.11)$$

Note that the relations (3.11) can be obtained from the Whitney relations

$$\lambda_y(\mathcal{S}_i) \cdot \lambda(\mathcal{S}_{i+1}/\mathcal{S}_i) = \lambda_y(\mathcal{S}_{i+1}),$$

by eliminating the classes $\lambda_y(\mathcal{S}_i)$ for $2 \leq i \leq k$. (The quantization of this statement is done in the next section.) Finally, it is known that the Whitney relations form a full set of relations in $K_T(\text{Fl}(r_1, \dots, r_k))$. This is essentially done by Lascoux [34, Section 7], and we refer to [19, Proposition 5.1] for a complete proof. $\blacksquare$

We illustrate the proof of Theorem 3.2 with the following example.

Example 3.5. Let $\text{Fl}(4) \rightarrow \text{Gr}(2, 4) = \text{Fl}(2; 4)$ be the projection. In $\text{QK}_T(\text{Fl}(4))$, we have the following relation

$$\lambda_y(\mathbb{C}^4) = \begin{vmatrix} A_0 & B_1 & 0 & 0 \\ 1 & A_1 & B_2 & 0 \\ 0 & 1 & A_2 & B_3 \\ 0 & 0 & 1 & A_3 \end{vmatrix}^*, \quad (3.12)$$

where

$$\begin{aligned} A_0 &= \lambda_y(\mathcal{S}_1), & B_1 &= y \frac{Q_1}{1 - Q_1} \det(\mathcal{S}_2/\mathcal{S}_1), \\ A_1 &= \lambda_y(\mathcal{S}_2/\mathcal{S}_1) + y \frac{Q_1}{1 - Q_1} \det(\mathcal{S}_2/\mathcal{S}_1), & B_2 &= y \frac{Q_2}{1 - Q_2} \det(\mathcal{S}_3/\mathcal{S}_2), \\ A_2 &= \lambda_y(\mathcal{S}_3/\mathcal{S}_2) + y \frac{Q_2}{1 - Q_2} \det(\mathcal{S}_3/\mathcal{S}_2), & B_3 &= y \frac{Q_3}{1 - Q_3} \det(\mathbb{C}^4/\mathcal{S}_3), \\ A_3 &= \lambda_y(\mathbb{C}^4/\mathcal{S}_3) + y \frac{Q_3}{1 - Q_3} \det(\mathbb{C}^4/\mathcal{S}_3). \end{aligned}$$

We push this relation forward to $\text{Gr}(2, 4)$ by pushing it forward to $\text{Fl}(2, 3; 4)$ and then pushing forward from $\text{Fl}(2, 3; 4)$ to $\text{Gr}(2, 4)$. Let $\pi: \text{Fl}(4) \rightarrow \text{Fl}(2, 3; 4)$ be the projection. The relation on $\text{Fl}(2, 3; 4)$ is given by

$$\lambda_y(\mathbb{C}^4) = \begin{vmatrix} A'_0 & B_2 & 0 \\ 1 & A_2 & B_3 \\ 0 & 1 & A_3 \end{vmatrix}^*, \quad (3.13)$$

where

$$A'_0 = \lambda_y(\mathcal{S}_2).$$

By expanding the determinant in (3.12) along the columns containing A_0 and A_1 , we obtain

$$\lambda_y(\mathbb{C}^4) = \begin{vmatrix} A_0 & B_1 \\ 1 & A_1 \end{vmatrix}^* \begin{vmatrix} A_2 & B_3 \\ 1 & A_3 \end{vmatrix}^* - A_0 \begin{vmatrix} B_2 & 0 \\ 1 & A_3 \end{vmatrix}^*. \quad (3.14)$$

By Lemma 3.3,

$$\pi_* \begin{vmatrix} A_0 & B_1 \\ 1 & A_1 \end{vmatrix}^* = A'_0, \quad \pi_* A_0 = 1$$

and $\begin{vmatrix} A_2 & B_3 \\ 1 & A_3 \end{vmatrix}^*, \begin{vmatrix} B_2 & 0 \\ 1 & A_3 \end{vmatrix}^*$ will not change under pushforward by π . Thus, by pushing forward (3.14) we obtain

$$\lambda_y(\mathbb{C}^4) = A'_0 \begin{vmatrix} A_2 & B_3 \\ 1 & A_3 \end{vmatrix}^* - \begin{vmatrix} B_2 & 0 \\ 1 & A_3 \end{vmatrix}^*$$

which is the expansion of (2) along the first column. So the relation in $\mathrm{QK}_T(\mathrm{Fl}(4))$ pushes forward to the relation in $\mathrm{QK}_T(\mathrm{Fl}(2, 3; 4))$.

Now let $p: \mathrm{Fl}(2, 3; 4) \rightarrow \mathrm{Gr}(2, 4)$ be the projection. In $\mathrm{Gr}(2, 4)$ we have the following relation:

$$\lambda_y(\mathbb{C}^4) = \begin{vmatrix} A'_0 & B''_1 \\ 1 & A''_1 \end{vmatrix}^*, \quad (3.15)$$

where

$$B''_1 = y^2 \frac{Q_2}{1 - Q_2} \det(\mathbb{C}^4/\mathcal{S}_2), \quad A''_1 = \lambda_y(\mathbb{C}^4/\mathcal{S}_2) + y^2 \frac{Q_2}{1 - Q_2} \det(\mathbb{C}^4/\mathcal{S}_2).$$

By Lemma 3.3, in (3.13), we have $p_* \begin{vmatrix} A_2 & B_3 \\ 1 & A_3 \end{vmatrix}^* = A''_1$, $p_* \begin{vmatrix} B_2 & 0 \\ 1 & A_3 \end{vmatrix}^* = B''_1$ and A'_0 will not change under the pushforward. Thus, (3.13) pushes forward to (3.15).

4 Whitney implies Toda

In this section, we consider a different presentation of the quantum K ring, named the *quantum K Whitney presentation*. This presentation quantizes relations $\lambda_y(\mathcal{S}_i) \cdot \lambda_y(\mathcal{S}_{i+1}/\mathcal{S}_i) = \lambda_y(\mathcal{S}_{i+1})$ satisfied by the tautological subbundles in $\mathrm{K}_T(\mathrm{Fl}(\mathbf{r}, n))$. Informally, the Whitney presentation contains more (geometric) information than the Toda presentation, as it involves more generators, corresponding to the λ_y classes of the tautological subbundles, and their quotients. In contrast, the Toda presentation only involves the quotient bundles.

The quantization was conjectured in [17, 20], generalizing the conjectures from [18] for Grassmannians. These conjectures have been proved in [21] for Grassmannians, and in [20] for $\mathrm{Fl}(1, n-1; n)$ case. The general case was recently announced in [22] using the abelian/non-abelian correspondence. We note that the results in [22] are logically independent on those from [40], which were used to obtain the Toda presentation in the previous section.

Our main result of this section is that eliminating the additional variables of the Whitney presentation yields the Toda presentation. As an aside, we note that the proof of Theorem 3.4 can be easily modified to show that the quantum K Whitney presentation of $\mathrm{Fl}(\mathbf{r}, n)$ follows from that of $\mathrm{Fl}(n)$. We leave the details of this proof to the reader.

In what follows, T can be a maximal torus in GL_n . Let

$$X^{(j)} = (X_1^{(j)}, \dots, X_{r_j}^{(j)}) \quad \text{and} \quad Y^{(j)} = (Y_1^{(j)}, \dots, Y_{r_{j+1}-r_j}^{(j)})$$

denote formal variables for $j = 1, \dots, k$ and denote by $X^{(k+1)} := (T_1, \dots, T_n)$ the equivariant parameters in $\mathrm{K}_T(\mathrm{pt})$. Let $e_\ell(X^{(j)})$ and $e_\ell(Y^{(j)})$ be the ℓ -th elementary symmetric polynomials in $X^{(j)}$ and $Y^{(j)}$, respectively. Define the ring

$$S = \mathrm{K}_T(\mathrm{pt})[e_1(X^{(j)}), \dots, e_{r_j}(X^{(j)}), e_1(Y^{(j)}), \dots, e_{r_{j+1}-r_j}(Y^{(j)})]_{j=1}^k,$$

and the ideal $I_Q \subset S[[Q]] = S[[Q_1, \dots, Q_k]]$ generated by the coefficients of y in

$$\begin{aligned} & \prod_{\ell=1}^{r_j} (1 + yX_\ell^{(j)}) \prod_{\ell=1}^{r_{j+1}-r_j} (1 + yY_\ell^{(j)}) - \prod_{\ell=1}^{r_{j+1}} (1 + yX_\ell^{(j+1)}) \\ & + y^{r_{j+1}-r_j} \frac{Q_j}{1-Q_j} \prod_{\ell=1}^{r_{j+1}-r_j} Y_\ell^{(j)} \left(\prod_{\ell=1}^{r_j} (1 + yX_\ell^{(j)}) - \prod_{\ell=1}^{r_{j-1}} (1 + yX_\ell^{(j-1)}) \right), \quad j = 1, \dots, k. \end{aligned} \quad (4.1)$$

It was conjectured in [17, 20] and proved in [22] that there is an isomorphism of $K_T(\text{pt})[[Q]]$ -algebras

$$\Phi: S[[Q]]/I_Q \rightarrow \text{QK}_T(\text{Fl}(\mathbf{r}, n)) \quad (4.2)$$

sending

$$e_\ell(X^{(j)}) \mapsto \wedge^\ell(\mathcal{S}_j) \quad \text{and} \quad e_\ell(Y^{(j)}) \mapsto \wedge^\ell(\mathcal{S}_{j+1}/\mathcal{S}_j).$$

We refer to this as the (quantum K) *Whitney presentation*.

Proposition 4.1. *There is a natural isomorphism*

$$S[[Q]]/I_Q \simeq R[[Q]]/J_Q,$$

obtained by eliminating the indeterminates $X_\ell^{(j)}$. In particular, the Whitney relations from (4.1) imply the Toda relations from (3.10).

Proof. Let

$$A_j = \prod_{\ell=1}^{r_{j+1}-r_j} (1 + yY_\ell^{(j)}) + B_j, \quad B_j = y^{r_{j+1}-r_j} \frac{Q_j}{1-Q_j} \prod_{\ell=1}^{r_{j+1}-r_j} Y_\ell^{(j)},$$

so that (4.1) becomes

$$A_j \prod_{\ell=1}^{r_j} (1 + yX_\ell^{(j)}) - B_j \prod_{\ell=1}^{r_{j-1}} (1 + yX_\ell^{(j-1)}) - \prod_{\ell=1}^{r_{j+1}} (1 + yX_\ell^{(j+1)}). \quad (4.3)$$

Note that by Lemma 2.6, relations given by (4.3) are equivalent to those given by

$$\prod_{\ell=1}^{r_{j+1}} (1 + yX_\ell^{(j+1)}) - \begin{vmatrix} A_0 & B_1 \\ 1 & A_1 & B_2 \\ & \ddots & \ddots & \ddots \\ & & 1 & A_{j-1} & B_j \\ & & & 1 & A_j \end{vmatrix} \quad \text{for } 1 \leq j \leq k.$$

As a consequence, we can eliminate $e_1(X^{(j)}), \dots, e_{r_j}(X^{(j)})$ for $2 \leq j \leq k$, and be left with the relation (3.10). ■

Remark 4.2. We note that our methods from the previous section can be adapted easily to show that Φ is an isomorphism for all partial flag varieties if and only if it is an isomorphism for $\text{Fl}(n)$.

We illustrate Proposition 4.1 with the following two examples.

Example 4.3. Consider $\text{Fl}(2) = \mathbb{P}^1$ with the tautological subbundle $\mathcal{S}_1 \subset \mathbb{C}^2$. The QK Whitney relations are given by

$$\lambda_y(\mathcal{S}_1) \star \lambda_y(\mathbb{C}^2/\mathcal{S}_1) = \lambda_y(\mathbb{C}^2) - y \frac{Q}{1-Q} (\mathbb{C}^2/\mathcal{S}_1) \star (\lambda_y(\mathcal{S}_1) - 1)$$

After making the change of variables $\mathcal{S}_1 \mapsto P_1$ and $\mathbb{C}^2/\mathcal{S}_1 \mapsto (1-Q)P_2/P_1$, then collecting the coefficients of y and y^2 , one obtains the Toda relations for $\text{QK}_T(\mathbb{P}^1)$:

$$P_1 + \frac{1-Q}{P_1} = \mathbb{C}^2, \quad P_2 = \wedge^2 \mathbb{C}^2.$$

Example 4.4. We now consider the case $X = \text{Fl}(3)$, equipped with the tautological sequence $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathbb{C}^3$. There are two QK Whitney relations

$$\begin{aligned} \lambda_y(\mathcal{S}_1) \star \lambda_y(\mathcal{S}_2/\mathcal{S}_1) &= \lambda_y(\mathcal{S}_2) - y \frac{Q_1}{1-Q_1} \mathcal{S}_2/\mathcal{S}_1 \star (\lambda_y(\mathcal{S}_1) - 1), \\ \lambda_y(\mathcal{S}_2) \star \lambda_y(\mathbb{C}^3/\mathcal{S}_2) &= \lambda_y(\mathbb{C}^3) - y \frac{Q_2}{1-Q_2} \mathbb{C}^3/\mathcal{S}_2 \star (\lambda_y(\mathcal{S}_2) - \lambda_y(\mathcal{S}_1)). \end{aligned}$$

From the first relation, we can write

$$\lambda_y(\mathcal{S}_2) = \lambda_y(\mathcal{S}_1) \star \lambda_y(\mathcal{S}_2/\mathcal{S}_1) + y \frac{Q_1}{1-Q_1} \mathcal{S}_2/\mathcal{S}_1 \star (\lambda_y(\mathcal{S}_1) - 1),$$

which we can use to replace $\lambda_y(\mathcal{S}_2)$ in the second relation. By some algebra, we obtain

$$\begin{aligned} (1 + y\mathcal{S}_1) \star (1 + y\mathcal{S}_2/\mathcal{S}_1) \star (1 + y\mathbb{C}^3/\mathcal{S}_2) &+ y^2 \frac{Q_1}{1-Q_1} \mathcal{S}_2/\mathcal{S}_1 \star \mathcal{S}_1 \star (1 + y\mathbb{C}^3/\mathcal{S}_2) \\ &= \lambda_y(\mathbb{C}^3) - y \frac{Q_2}{1-Q_2} \mathbb{C}^3/\mathcal{S}_2 \star (1 + y\mathcal{S}_1) \star (1 + y\mathcal{S}_2/\mathcal{S}_1) \\ &\quad - y^3 \frac{Q_1 Q_2}{(1-Q_1)(1-Q_2)} \mathcal{S}_1 \star \mathcal{S}_2/\mathcal{S}_1 \star \mathbb{C}^3/\mathcal{S}_2 + y \frac{Q_2}{1-Q_2} \mathbb{C}^3/\mathcal{S}_2 \star (1 + y\mathcal{S}_1). \end{aligned}$$

With the change of variables

$$\mathcal{S}_1 \mapsto P_1, \quad \mathcal{S}_2/\mathcal{S}_1 \mapsto (1-Q_1)P_2/P_1, \quad \mathbb{C}^3/\mathcal{S}_2 \mapsto (1-Q_2)P_3/P_2$$

and equating the coefficients of y , y^2 , y^3 in the two sides to obtain

- coefficient of y : $P_1 + (1-Q_1)P_2/P_1 + (1-Q_2)P_3/P_2 = \mathbb{C}^3$;
- coefficient of y^2 : $P_2 + (1-Q_1)P_3/P_1 + (1-Q_2)P_1P_3/P_2 = \wedge^2 \mathbb{C}^3$;
- coefficient of y^3 : $P_3 = \wedge^3 \mathbb{C}^3$.

These are the Toda relations for $\text{QK}_T(\text{Fl}(3))$, calculated from (2.2).

5 Representatives for quantum K Schubert classes in partial flag varieties

The goal of this section is to use the pushforward technique to obtain polynomial representatives of Schubert classes in the equivariant quantum K rings of partial flag varieties. Our strategy is to push forward the polynomials for the class of the point from $\text{QK}_T(\text{Fl}(n))$ to $\text{QK}_T(\text{Fl}(\mathbf{r}, n))$, then use the (left) divided difference operators defined in [42] in the rings $\text{QK}_T(\text{Fl}(\mathbf{r}, n))$ to deduce a recursive procedure giving the other polynomials. The left divided difference operators were

also used by Maeno, Naito, and Sagaki [40] to prove that the quantum double Grothendieck polynomials represent Schubert classes in the Toda presentation of $\mathrm{QK}_T(\mathrm{Fl}(n))$.

We use a different generating set from loc. cit., the exterior powers of the tautological bundles, thus our representatives live in the (quantum) Whitney presentation introduced in Section 4. A key feature of our polynomials, and unlike those from [40], is that they *do not* involve quantum parameters.

5.1 Preliminaries on Schubert classes and quantum divided difference operators

We start with recalling some basic facts about the Schubert classes and quantum divided difference operators in the equivariant quantum K theory.

We need the formula for the class of the Schubert point, proved in [40], which we later use to find formulae for the other Schubert classes. To this aim, we briefly recall the definition of the Schubert basis in the quantum K rings.

Regard $\mathrm{Fl}(n)$ as SL_n/B , and let $W := N_{\mathrm{SL}_n}T/T \simeq S_n$ be the Weyl group, equipped with the length function $\ell: W \rightarrow \mathbb{N}$. It is a Coxeter group, generated by simple reflections $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. Denote by $w_0 \in W$ be the longest element, so that $\dim \mathrm{Fl}(n) = \ell(w_0)$. Let $W_{\mathbf{r}} \leq W$ be the subgroup generated by the simple reflections s_i so that i is not among the components of $\mathbf{r}$, and let $W^{\mathbf{r}} \subset W$ be the set of minimal length representatives for the cosets of $W/W^{\mathbf{r}}$.

Set $B^- = w_0 B w_0 \subset \mathrm{SL}_n$, the opposite Borel subgroup. For each $w \in W$, the flag variety $\mathrm{Fl}(n)$ has a T -fixed point $e_w := n_w B$, where $n_w \in N_{\mathrm{SL}_n}T/T$ is any representative of w . The (opposite) Schubert cell is $X^{w, \circ} := B^- \cdot n_w B \subset \mathrm{Fl}(n)$, and it is isomorphic to the affine space $\mathbb{A}^{\dim \mathrm{Fl}(n) - \ell(w)}$. One can similarly define Schubert cells in any partial flag variety $\mathrm{Fl}(\mathbf{r}, n)$; alternatively, the Schubert cells in $\mathrm{Fl}(\mathbf{r}, n)$ are the images of the Schubert cells in $\mathrm{Fl}(n)$ under the (SL_n -equivariant) natural projection $\mathrm{Fl}(n) \rightarrow \mathrm{Fl}(\mathbf{r}, n)$. The Schubert variety X^w is the (Zariski) closure of the corresponding Schubert cell. Inclusion of Schubert varieties give the Bruhat (partial) order on the set $W^{\mathbf{r}}$,

$$uW^{\mathbf{r}} \leq vW^{\mathbf{r}} \Leftrightarrow X^u \supset X^v \text{ in } \mathrm{Fl}(\mathbf{r}, n).$$

Now let $\mathcal{O}^w \in K_T(\mathrm{Fl}(\mathbf{r}, n))$ be the K theory class given by the structure sheaf of X^w . The Schubert cells give a stratification of $\mathrm{Fl}(n)$, and, more generally, of $\mathrm{Fl}(\mathbf{r}, n)$. Then the classes $\mathcal{O}^w$ form a basis for $K_T(\mathrm{Fl}(\mathbf{r}, n))$ over $K_T(\mathrm{pt})$, when w varies in the quotient $W/W_{\mathbf{r}}$. This implies (by definition) that the classes $\mathcal{O}^w$ are a basis of $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$, over the ground ring $K_T(\mathrm{pt})[[Q]]$.

As in [40], we identify $K_T(\mathrm{pt})$ with the group algebra $\mathbb{Z}[P] = \bigoplus_{\chi \in P} \mathbb{Z}e^{\chi}$ of the weight lattice $P = \sum_{i=1}^{n-1} \mathbb{Z}\varpi_i$ of SL_n , where $\varpi_i, 1 \leq i \leq n-1$ are the fundamental weights. We also set $\varpi_0 = \varpi_n = 0$, and $\epsilon_j = \varpi_j - \varpi_{j-1}$ for $1 \leq j \leq n$.

In [42], left divided difference operators acting on $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ (in fact on the equivariant quantum K ring of any homogeneous space G/P) were constructed. These operators send Schubert classes to Schubert classes, and were compatible with the quantum K product. We recall next the salient facts, see Section 8.3 in loc. cit. for further details.

Regard $\mathrm{Fl}(\mathbf{r}, n)$ as $\mathrm{SL}_n/P_{\mathbf{r}}$, where $P_{\mathbf{r}}$ the parabolic group stabilizing the identity partial flag. Left multiplication by a representative n_w of an element $w \in W$ induces an automorphism of $\mathrm{Fl}(\mathbf{r}, n)$ which is equivariant with respect to the automorphism of T given by $t \mapsto n_w t n_w^{-1}$. Pulling back along this automorphism of $\mathrm{Fl}(\mathbf{r}, n)$ gives a ring automorphism w^L of $K_T(\mathrm{Fl}(\mathbf{r}, n))$. The following combines [42, Proposition 5.3, Lemma 5.4, and Proposition 5.5].

Proposition 5.1 (Mihalcea–Naruse–Su). *The following hold:*

1. $w^L(e^{\chi}a) = e^{w(\chi)}w^L(a)$ for any $e^{\chi} \in K_T(\mathrm{pt})$ and $a \in K_T(\mathrm{Fl}(\mathbf{r}, n))$.

2. w^L is $\mathrm{K}_{\mathrm{SL}_n}(\mathrm{Fl}(\mathbf{r}, n))$ -linear: for $\kappa \in \mathrm{K}_{\mathrm{SL}_n}(\mathrm{Fl}(\mathbf{r}, n))$ and $a \in \mathrm{K}_T(\mathrm{Fl}(\mathbf{r}, n))$,

$$w^L(\kappa \cdot a) = \kappa \cdot w^L(a).$$

3. w^L commutes with the natural projection $\pi: \mathrm{Fl}(n) \rightarrow \mathrm{Fl}(\mathbf{r}, n)$:

$$w^L(\pi_*(a)) = \pi_*(w^L(a)), \quad \forall a \in \mathrm{K}_T(\mathrm{Fl}(n)).$$

In particular, the map w^L on $\mathrm{K}_T(\mathrm{Fl}(\mathbf{r}, n))$ is determined by the map on $\mathrm{K}_T(\mathrm{Fl}(n))$.

4. The automorphisms w^L give an action of W on $\mathrm{K}_T(\mathrm{Fl}(\mathbf{r}, n))$. If $s_i \in W$ is a simple reflection, and $\mathcal{O}^w \in \mathrm{K}_T(\mathrm{Fl}(\mathbf{r}, n))$, then

$$s_i^L(\mathcal{O}^w) = \begin{cases} e^{\alpha_i} \mathcal{O}^w + (1 - e^{\alpha_i}) \mathcal{O}^{s_i w} & \text{if } s_i w W_{\mathbf{r}} < w W_{\mathbf{r}}, \\ \mathcal{O}^w & \text{otherwise,} \end{cases}$$

where α_i is the simple positive root giving s_i .

The equivariant quantum K theory is functorial for isomorphisms. Thus one may extend the action of W to an action on $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ by $\mathbb{Q}[[Q]]$ -linear ring automorphisms. Define the (quantum) left divided difference operators by

$$\delta_i := \frac{1}{1 - e^{-\alpha_i}} (\mathrm{id} - e^{-\alpha_i} s_i^L).$$

(In [42, equation (13)] this operator is denoted by $\delta_i^\vee$.) These operators have the same properties as the ordinary Demazure operators, and they satisfy a Leibniz rule compatible with the quantum K product. For reader's convenience, we state these properties next, see [42, Proposition 8.3].

Proposition 5.2 (Mihalcea–Naruse–Su).

1. The quantum operators δ_i are $\mathbb{Q}[q]$ -linear, satisfy the braid relations, and $(\delta_i)^2 = \delta_i$.
2. For each $w \in W^{\mathbf{r}}$,

$$\delta_i(\mathcal{O}^{w W_{\mathbf{r}}}) = \begin{cases} \mathcal{O}^{s_i w W_{\mathbf{r}}} & \text{if } s_i w < w, \\ \mathcal{O}^{w W_{\mathbf{r}}} & \text{otherwise.} \end{cases}$$

3. (Leibniz rule) For any $a, b \in \mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$,

$$\delta_i(a \star b) = \delta_i(a) \star b + e^{-\alpha_i} s_i^L(a) \star \delta_i(b) - e^{-\alpha_i} s_i^L(a) \star s_i^L(b).$$

4. The operator δ_i is a $\mathrm{QK}_{\mathrm{SL}_n}(\mathrm{Fl}(\mathbf{r}, n))$ -module homomorphism, that is, for any κ from $\mathrm{QK}_{\mathrm{SL}_n}(\mathrm{Fl}(\mathbf{r}, n))$ and η from $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$,

$$\delta_i(\kappa \star \eta) = \kappa \star \delta_i(\eta).$$

Part (1) implies that for each $w \in W$ there are well defined operators δ_w acting on quantum K ring $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$. Furthermore, part (2) implies that if $w \in W$ is a minimal length representative in its coset in $W/W_{\mathbf{r}}$, then

$$\mathcal{O}^w = \delta_{w w_0}(\mathcal{O}^{w_0 W_{\mathbf{r}}}).$$

5.2 Polynomial representatives

In this section, we use results of [40] to obtain a formula for the class of the Schubert point in $\mathrm{QK}_T(\mathrm{Fl}(n))$. Then we use Kato's pushforward, and the left divided difference operators δ_w , to obtain a recursive formula for the Schubert classes in any $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$.

To start, note that, in geometric terms, the relations (4.1) are interpreted as follows (cf. [17, 20, 22]).

Theorem 5.3. *For $j = 1, \dots, k$, the following relations hold in $\mathrm{QK}_T(X)$:*

$$\begin{aligned} \lambda_y(\mathcal{S}_j) \star \lambda_y(\mathcal{S}_{j+1}/\mathcal{S}_j) \\ = \lambda_y(\mathcal{S}_{j+1}) - y^{r_{j+1}-r_j} \frac{Q_j}{1-Q_j} \det(\mathcal{S}_{j+1}/\mathcal{S}_j) \star (\lambda_y(\mathcal{S}_j) - \lambda_y(\mathcal{S}_{j-1})). \end{aligned} \quad (5.1)$$

Proposition 5.4. *The following holds in $\mathrm{QK}_T(\mathrm{Fl}(n))$:*

$$\wedge^p \mathcal{S}_k = \sum_{\substack{J \subseteq [k] \\ |J|=p}} \left(\prod_{\substack{1 \leq j \leq k \\ j, j+1 \in J}} \frac{1}{1-Q_j} \right) \left(\prod_{j \in J}^* \mathcal{S}_j/\mathcal{S}_{j-1} \right) \quad (5.2)$$

for $0 \leq p \leq k \leq n$, where $\star$ means the quantum K product.

Proof. We use double induction on p, k , with $p = k = 0$ case being clear. Assume that

$$\wedge^{p'} \mathcal{S}_{k'} = \sum_{\substack{J \subseteq [k'] \\ |J|=p'}} \left(\prod_{\substack{1 \leq j \leq k' \\ j, j+1 \in J}} \frac{1}{1-Q_j} \right) \left(\prod_{j \in J}^* \mathcal{S}_j/\mathcal{S}_{j-1} \right)$$

for all $(p', k') < (p, k)$, then considering the three cases for $J \subseteq [k]$: $k \notin J$, $k, k-1 \in J$, $k \in J$ and $k-1 \notin J$, we have

$$\begin{aligned} \sum_{\substack{J \subseteq [k] \\ |J|=p}} \left(\prod_{\substack{1 \leq j \leq k \\ j, j+1 \in J}} \frac{1}{1-Q_j} \right) \left(\prod_{j \in J}^* \mathcal{S}_j/\mathcal{S}_{j-1} \right) \\ = \wedge^p \mathcal{S}_{k-1} + \mathcal{S}_k/\mathcal{S}_{k-1} \star \left(\frac{1}{1-Q_{k-1}} \wedge^{p-1} \mathcal{S}_{k-1} - \frac{Q_{k-1}}{1-Q_{k-1}} \wedge^{p-1} \mathcal{S}_{k-2} \right) \\ = \wedge^p \mathcal{S}_k, \end{aligned}$$

where the last equality follows from the Whitney relations (5.2). ■

After harmonizing conventions, and using Proposition 5.4, the following is a restatement of [40, Proposition 3.1].

Corollary 5.5. *In $\mathrm{QK}_T(\mathrm{Fl}(n))$, we have*

$$\mathcal{O}^{w_0} = \prod_{i=1}^{n-1} \lambda_{-1}^* (e^{-\epsilon_{n-i}} \mathcal{S}_i).$$

We illustrate the corollary next.

Example 5.6. We take $n = 2$, thus $\mathrm{Fl}(2) = \mathbb{P}(\mathbb{C}^2)$. Fix e_1, e_2 to be a basis for $\mathbb{C}^2$. For simplicity we regard $\mathbb{P}^1$ as GL_2/B with $T' = (\mathbb{C}^*)^2$ acting naturally, and then restrict this action to SL_2 . With these conventions, the Schubert point is $X^{w_0} = \langle e_2 \rangle$, and the localizations of $\mathcal{S} = \mathcal{O}_{\mathbb{P}^1}(-1)$ at the fixed points $\mathbb{P}(\langle e_i \rangle)$, $i = 1, 2$, are $\mathcal{S}|_{\mathbb{P}(\langle e_i \rangle)} = e^{\epsilon_i}$. Then one easily checks that

$$\mathcal{O}^{w_0} = 1 - e^{-\epsilon_1} \mathcal{S}.$$

Theorem 5.7. *In $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$, we have*

$$\mathcal{O}^{w_0} = \prod_{i=1}^k \prod_{j=r_i}^{r_{i+1}-1} \lambda_{-1}(e^{-\epsilon_{n-j}} \mathcal{S}_i). \quad (5.3)$$

Proof. Let $X = \mathrm{Fl}(r_1, \dots, r_k; n)$, $Y = \mathrm{Fl}(r_1, \dots, \widehat{r_i}, \dots, r_k; n)$, and let $\pi: X \rightarrow Y$ be the natural projection. Corollary 5.5 implies that the claim is true for $\mathrm{Fl}(n)$. By induction, we assume that (5.3) holds for X , and we compute its pushforward under π using Kato's pushforward map from Theorem 2.2. Note that all but the term including $\mathcal{S}_i$ are pulled back from Y . By (5.1), we have

$$\begin{aligned} \lambda_{-1}(e^{-\epsilon_{n-j}} \mathcal{S}_i) &= \lambda_{-1}(e^{-\epsilon_{n-j}} \mathcal{S}_{i-1}) \star \lambda_{-1}(e^{-\epsilon_{n-j}} \mathcal{S}_i / \mathcal{S}_{i-1}) \\ &\quad + \frac{Q_{i-1}}{1 - Q_{i-1}} (-e^{-\epsilon_{n-j}})^{r_{i+1}-r_i} \det(\mathcal{S}_i / \mathcal{S}_{i-1}) \star (\lambda_{-1}(e^{-\epsilon_{n-j}} \mathcal{S}_{i-1}) - \lambda_{-1}(e^{-\epsilon_{n-j}} \mathcal{S}_{i-2})), \end{aligned}$$

where we used (the λ -ring formalism asserting) that $\lambda_{-1}(e^X \otimes E) = \lambda_{-e^X}(E)$. Since the pushforward $\pi_*(\wedge^j \mathcal{S}_i / \mathcal{S}_{i-1}) = 0$ for any $j > 0$ by Proposition 2.1, $\pi_* \lambda_{-1}(e^{-\epsilon_{n-j}} \mathcal{S}_i) = \lambda_{-1}(e^{-\epsilon_{n-j}} \mathcal{S}_{i-1})$, and the claim on Y follows from the projection formula. $\blacksquare$

We illustrate the formula in Theorem 5.7 in the case of $\mathrm{Gr}(2, 4)$. The Schubert classes in $\mathrm{Gr}(2, 4)$ are typically indexed by partitions in the 2×2 square; the dictionary to translate into the indexing by Weyl group elements is the following:

$$\begin{aligned} \mathcal{O}^{(1)} &= \mathcal{O}^{s_2 W_{\mathbf{r}}}, & \mathcal{O}^{(2)} &= \mathcal{O}^{s_3 s_2 W_{\mathbf{r}}}, \\ \mathcal{O}^{(1,1)} &= \mathcal{O}^{s_1 s_2 W_{\mathbf{r}}}, & \mathcal{O}^{(2,1)} &= \mathcal{O}^{s_1 s_3 s_2 W_{\mathbf{r}}}, & \mathcal{O}^{(2,2)} &= \mathcal{O}^{s_2 s_1 s_3 s_2 W_{\mathbf{r}}}. \end{aligned}$$

Example 5.8 (Theorem 5.7 for $\mathrm{Gr}(2, 4)$). Denote by $\mathcal{S}$ the tautological subbundle. Using (for instance) a localization argument, one calculates that

$$\lambda_y(\mathcal{S}) = (1 + ye^{\epsilon_1})(1 + ye^{\epsilon_2})\mathcal{O}^{\emptyset} - ye^{\epsilon_2}(1 + ye^{\epsilon_1})\mathcal{O}^{(1)} - ye^{\epsilon_1}\mathcal{O}^{(1,1)}.$$

Thus for any weight χ ,

$$\lambda_{-1}(e^{\chi} \mathcal{S}) = 1 - e^{\chi} \mathcal{S} + e^{2\chi} \wedge^2 \mathcal{S}$$

can be expanded into a combination of Schubert classes. Then one checks directly that

$$\lambda_{-1}(e^{-\epsilon_2} \mathcal{S}) \star \lambda_{-1}(e^{-\epsilon_1} \mathcal{S}) = \mathcal{O}^{(2,2)}.$$

The relevant multiplications are¹

$$\begin{aligned} \mathcal{O}^{(1)} \star \mathcal{O}^{(1)} &= (1 - e^{\epsilon_3 - \epsilon_2})\mathcal{O}^{(1)} + e^{\epsilon_3 - \epsilon_2}\mathcal{O}^{(2)} + e^{\epsilon_3 - \epsilon_2}\mathcal{O}^{(1,1)} - e^{\epsilon_3 - \epsilon_2}\mathcal{O}^{(2,1)}, \\ \mathcal{O}^{(1)} \star \mathcal{O}^{(1,1)} &= (1 - e^{\epsilon_3 - \epsilon_1})\mathcal{O}^{(1,1)} + e^{\epsilon_3 - \epsilon_1}\mathcal{O}^{(2,1)}, \\ \mathcal{O}^{(1,1)} \star \mathcal{O}^{(1,1)} &= e^{\epsilon_3 + \epsilon_2 - 2\epsilon_1}\mathcal{O}^{(1,1)} - e^{\epsilon_3 + \epsilon_2 - 2\epsilon_1}\mathcal{O}^{(2,1)} - e^{\epsilon_3 - \epsilon_1}\mathcal{O}^{(1,1)} + e^{\epsilon_3 - \epsilon_1}\mathcal{O}^{(2,1)} \\ &\quad - e^{\epsilon_2 - \epsilon_1}\mathcal{O}^{(1,1)} + e^{\epsilon_2 - \epsilon_1}\mathcal{O}^{(2,2)} + \mathcal{O}^{(1,1)}. \end{aligned}$$

Next we state the main result of this section. Recall the Whitney presentation $\Phi: S[[Q]]/I_Q \rightarrow \mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ from (4.2).

¹These can be calculated for example with A. Buch's *Equivariant Schubert Calculator*, available at <https://sites.math.rutgers.edu/~asbuch/equivcalc/>.

Theorem 5.9. *Let $\mathbf{r} = (r_1, \dots, r_k)$. Under the isomorphism Φ , the elements*

$$\mathcal{G}_w(X) := \Phi^{-1} \left(\delta_w \left(\prod_{i=1}^k \prod_{j=r_i}^{r_{i+1}-1} \lambda_{-1}(e^{-\epsilon_{n-j}} \mathcal{S}_i) \right) \right)$$

are sent to $W_{\mathbf{r}}$ -symmetric polynomials in the variables $X^{(j)}$ for $j = 1, \dots, k$, such that

$$\Phi(\mathcal{G}_w(X)) = \mathcal{O}^w \in \mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n)).$$

Furthermore, the polynomials $\mathcal{G}_w(X)$ are independent of the Novikov variables Q_i for $1 \leq i \leq k$.

Proof. This follows from Proposition 5.2: polynomial representatives for all Schubert classes can be obtained by applying the quantum left divided difference operators δ_i to the identity (5.3) above. This process does not introduce any Q 's. ■

The proposition may be interpreted as saying that the *same* polynomials representing Schubert classes in $\mathrm{K}_T(\mathrm{Fl}(\mathbf{r}, n))$ also represent their quantizations in $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$; of course, the ideal of *relations* in $\mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ needs to be quantized.

We illustrate next the calculation of the polynomials representing Schubert classes in the ring $\mathrm{QK}_T(\mathrm{Gr}(2, 4))$.

Example 5.10. We use left divided difference operators to find polynomial representatives for all Schubert classes in $\mathrm{QK}_T(\mathrm{Gr}(2, 4))$, knowing from Theorem 5.7 the representative for the Schubert point.

Recall that $\alpha_i = \epsilon_i - \epsilon_{i+1}$, and denote by $\mathcal{S}$ the class of the tautological subbundle. First, observe that $\delta_i(e^{\chi} \otimes \wedge^k \mathcal{S}) = \delta_i(e^{\chi}) \otimes \wedge^k \mathcal{S}$ by Proposition 5.2 (4), and

$$\delta_i(e^{\chi}) = \begin{cases} e^{\chi}, & s_i(\chi) = \chi, \\ e^{\chi} \frac{1 - (e^{-\alpha_i})^{1 + \langle \chi, \alpha_i^{\vee} \rangle}}{1 - e^{-\alpha_i}}, & \text{otherwise.} \end{cases}$$

It follows that

$$\delta_i \left(e^{-k\epsilon_j} \wedge^k \mathcal{S} \right) = \begin{cases} e^{-k\epsilon_j} \wedge^k \mathcal{S}, & j \neq i, i+1, \\ 0, & j = i, k = 1, \\ -e^{-(\epsilon_i + \epsilon_{i+1})} \wedge^2 \mathcal{S}, & j = i, k = 2, \\ e^{-k\epsilon_{i+1}} (1 + e^{-\alpha_i} + \dots + e^{-(k-1)\alpha_i}) \wedge^k \mathcal{S}, & j = i+1, k \geq 1. \end{cases}$$

By Theorem 5.7, $\mathcal{O}^{(2,2)}$ is equal to

$$\lambda_{-1}(e^{-\epsilon_1} \mathcal{S}) \star \lambda_{-1}(e^{-\epsilon_2} \mathcal{S}) = (1 - e^{-\epsilon_1} \mathcal{S} + e^{-2\epsilon_1} \wedge^2 \mathcal{S}) \star (1 - e^{-\epsilon_2} \mathcal{S} + e^{-2\epsilon_2} \wedge^2 \mathcal{S}).$$

We have that $\delta_2(\mathcal{O}^{(2,2)}) = \mathcal{O}^{(2,1)}$. We now calculate $\delta_2(\mathcal{O}^{(2,2)})$ by means of the Leibniz rule from Proposition 5.2. We obtain

$$\begin{aligned} \mathcal{O}^{(2,1)} &= \lambda_{-1}(e^{-\epsilon_1} \mathcal{S}) \star (1 - e^{-(\epsilon_2 + \epsilon_3)} \wedge^2 \mathcal{S}), \\ \mathcal{O}^2 &= \delta_1(\mathcal{O}^{(2,1)}) = 1 - (e^{-\epsilon_1 - \epsilon_2} + e^{-\epsilon_2 - \epsilon_3} + e^{-\epsilon_1 - \epsilon_3}) \wedge^2 \mathcal{S} + (e^{-\epsilon_1 - \epsilon_2 - \epsilon_3}) \mathcal{S} \star \wedge^2 \mathcal{S}, \\ \mathcal{O}^{(1,1)} &= \delta_3(\mathcal{O}^{(2,1)}) = \lambda_{-1}(e^{-\epsilon_1} \mathcal{S}), \\ \mathcal{O}^{(1)} &= \delta_1(\mathcal{O}^{(1,1)}) = 1 - e^{-(\epsilon_1 + \epsilon_2)} \wedge^2 \mathcal{S}, \\ \mathcal{O}^{\emptyset} &= \delta_1(\mathcal{O}^{(1)}) = 1. \end{aligned}$$

Finally, we can rewrite the operators δ_i as operators ρ_i acting on $\mathbb{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ by

$$\rho_i = \frac{T_i - T_{i+1}s_i}{T_i - T_{i+1}},$$

where s_i replaces each T_j by $T_{s_i(j)}$, and further extend it to

$$S[[Q]] = \mathbb{Z}[e_1(X^{(j)}), \dots, e_{r_j}(X^{(j)}), e_1(Y^{(j)}), \dots, e_{r_{j+1}-r_j}(Y^{(j)})]_{j=1}^k [[Q]] \otimes \mathbb{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$$

by $\mathbb{Z}[e_1(X^{(j)}), \dots, e_{r_j}(X^{(j)}), e_1(Y^{(j)}), \dots, e_{r_{j+1}-r_j}(Y^{(j)})]_{j=1}^k [[Q]]$ -linearity. Given $w \in S_n$ with reduced expression $w = s_{i_1} \dots s_{i_l}$, we define

$$\rho_w = \rho_{i_1} \dots \rho_{i_l}.$$

Since the operators ρ_i satisfy the braid relations, the operator ρ_w doesn't depend on the choice of reduced expression. We may restate Theorem 5.9 as follows.

Theorem 5.11. *For $w \in W^r$, the isomorphism $\Phi: S[[Q]]/I_Q \rightarrow \mathrm{QK}_T(\mathrm{Fl}(\mathbf{r}, n))$ sends the class of*

$$\rho_w \left(\prod_{i=1}^k \prod_{j=r_i}^{r_{i+1}-1} \prod_{\ell=1}^{r_i} (1 - T_{n-j}^{-1} X_{\ell}^{(i)}) \right)$$

to $\mathcal{O}^w$.

We have not seen similar polynomials in the study of quantum K theory of flag manifolds.

A Toda relations from finite difference operators (after Anderson–Chen–Tseng)

The proof of the Toda relations in [39] relies on Kato's earlier results [29]. For the quantum K ring $\mathrm{QK}_T(\mathrm{Fl}(n))$, there is another proof of these relations, using an argument combining the results of Iritani, Milanov and Tonita [26] with results of Givental and Lee [15]. More precisely, it is shown in [26] that the symbols of finite difference operators annihilating the K-theoretic J function of a variety X give relations in the quantum K ring of X . Givental and Lee's results from loc. cit. imply that the K-theoretic J function of the complete flag variety is an eigenfunction of the (finite difference) Toda Hamiltonians. This observation was made in the unpublished note [1] of Anderson–Chen–Tseng, but removed from the published version of their paper. For the sake of completeness, we give a brief account below, and in the process fill in some of the details to make the argument complete.

We start with recalling the definition of the K-theoretic J -function of the complete flag variety $X = \mathrm{Fl}(n)$. Denote by $P_i = \wedge^i \mathcal{S}_i$; it is known that these line bundles algebra generate $\mathrm{K}_T(\mathrm{Fl}(n))$ over $\mathrm{K}_T(\mathrm{pt})$; see, e.g., [19, Proposition 3.1]. Furthermore, the curve classes associated to the Novikov variables Q_i are dual to the classes $c_1(\wedge^i \mathcal{S}_i^*)$. For a fixed effective (multi)degree $d \in H_2(X)$, let L be the cotangent line bundle at the unique marked point on the moduli space $\overline{M}_{0,1}(X, d)$. Let also ϕ^α, ϕ_α denote Poincaré-dual bases for $\mathrm{K}_T(X)$. (For example, one may take Schubert classes $\mathcal{O}^w$, and their duals – the ideal sheaves of the boundary of the opposite Schubert varieties.) The small J -function of X , denoted by J_X , is defined by

$$J_X(q) := (1 - q) \prod_i P_i^{\frac{\ln(Q_i)}{\ln(q)}} \sum_{d, \alpha} Q^d \left\langle \frac{\phi_\alpha}{1 - qL} \right\rangle_{0,1,d} \phi^\alpha.$$

We will explain later the meaning and the effect of the factor $P_i^{\ln(Q_i)/\ln(q)}$. We also note that the presence of the prefactors $(1 - q)$ and $\prod_i P_i^{\ln(Q_i)/\ln(q)}$ varies in the literature. Our description

agrees with the one used by Givental and Lee in [15], and corresponds to the function denoted by $\tilde{J}$ in [26].

We recall some basics on the formalism of difference operators. Consider commuting variables $q, x_1, \dots, x_n$, and define the difference operators

$$T_i := q^{x_i \partial_{x_i}} = \sum_{k \geq 0} \frac{1}{k!} ((\ln q) x_i \partial_{x_i})^k.$$

(More generally, for a differential operator $\mathfrak{f}$, one defines the q -difference operator $q^{\mathfrak{f}} = e^{(\ln q)\mathfrak{f}} = \sum_{j=0}^{\infty} \frac{1}{j!} ((\ln q)\mathfrak{f})^j$.) Note that

$$T_i(x_j^{\pm 1}) = \sum_{k=0}^{\infty} \frac{1}{k!} (\ln(q) x_i \partial_{x_i})(x_j^{\pm 1}) = q^{\pm \delta_{ij}} x_j^{\pm 1},$$

which explains the ‘difference operator’ terminology. More generally, for any Laurent polynomial in commuting variables x_i , we have

$$T_i f(x_1, \dots, x_i, \dots, x_n) = f(x_1, \dots, q x_i, \dots, x_n),$$

i.e., T_i is an automorphism of the Laurent polynomial ring $\mathbb{Z}[q^{\pm 1}; x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. We use this expression to extend the definition of T_i to any function in the indeterminates $q, x_1, \dots, x_n$.

Now consider the subring of Laurent polynomials

$$\mathbb{Z}[q^{\pm 1}; Q_1^{\pm 1}, \dots, Q_{n-1}^{\pm 1}] \hookrightarrow \mathbb{Z}[q^{\pm 1}; x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

obtained by sending $Q_i \mapsto q^{-1} \frac{x_{i+1}}{x_i}$. The restriction of T_i to this subring is given by

$$T_i = q^{-Q_i \partial_{Q_i}} q^{Q_{i-1} \partial_{Q_{i-1}}}, \quad (\text{A.1})$$

where $q^{Q_i \partial_{Q_i}}$ are the difference operators on the subring in Q_i ’s.

With that in mind, we can now explain the meaning of the factor $P^{\frac{\ln(Q_i)}{\ln(q)}}$. The difference operators $q^{Q_i \partial_{Q_i}}$ act on functions in Q_i ’s, and one calculates that

$$q^{Q_i \partial_{Q_i}} \left(P^{\frac{\ln(Q_j)}{\ln(q)}} \right) = P^{\frac{\ln(q^{\delta_{ij}} Q_j)}{\ln(q)}} = P^{\delta_{ij}} P^{\frac{\ln(Q_j)}{\ln(q)}}.$$

In other words, the factor $P^{\frac{\ln(Q_i)}{\ln(q)}}$ should be regarded as a formal variable which transforms according to the rule above under the difference operators.

The relations in the quantum K ring are given by the Hamiltonians of the finite difference (or relativistic) Toda lattice. There is some ambiguity in the exact expressions for the Toda Hamiltonians, since their construction depends on choices; see, e.g., [15, Remark 5]. We follow here the approach from [13], but we will also need to make some changes of variables, in order to fit with the conventions in our main reference [15]. For the convenience of the reader, we briefly included some of the details below.

The Hamiltonians of the q -deformed type A Toda chain have the form

$$H_k = \sum_{0=i_0 < \dots < i_k \leq n} \prod_{l=1}^k \left(1 - \frac{x_{i_l}}{x_{i_{l-1}}} \right)^{1 - \delta_{i_l - i_{l-1}, 1}} \prod_{l=1}^k T_{i_l}, \quad k = 1, \dots, n, \quad (\text{A.2})$$

where q and x_i are commuting variables, and $T_i = q^{x_i \partial_{x_i}}$ is the q -difference operator above. It was proved in [13] that the operators H_k are limits of Macdonald operators, and the latter are known to commute. This implies that H_k also commute.

As above, let $Q_i = q^{-1}x_{i+1}x_i^{-1}$, with $Q_0 = Q_n = 0$. Then, using (A.1), one can rewrite (A.2) as

$$H_k = \sum_{0=i_0 < \dots < i_k \leq n} \prod_{l=1}^k (1 - qQ_{i_{l-1}})^{1-\delta_{i_l-i_{l-1},1}} \prod_{l=1}^k q^{-Q_{i_l}\partial_{Q_{i_l}} Q_{i_{l-1}}\partial_{Q_{i_{l-1}}}}, \quad k = 1, \dots, n.$$

Replacing q by q^{-1} , we obtain

$$\begin{aligned} \hat{H}_k &= \sum_{0=i_0 < \dots < i_k \leq n} \prod_{l=1}^k (1 - q^{-1}Q_{i_{l-1}})^{1-\delta_{i_l-i_{l-1},1}} \prod_{l=1}^k q^{Q_{i_l}\partial_{Q_{i_l}} - Q_{i_{l-1}}\partial_{Q_{i_{l-1}}}} \\ &= \sum_{0=i_0 < \dots < i_k \leq n} \prod_{l=1}^k q^{Q_{i_l}\partial_{Q_{i_l}} - Q_{i_{l-1}}\partial_{Q_{i_{l-1}}}} \prod_{l=1}^k (1 - Q_{i_{l-1}})^{1-\delta_{i_l-i_{l-1},1}}, \quad k = 1, \dots, n. \end{aligned}$$

Remark A.1. The substitutions above ensure that the first Hamiltonian $\hat{H}_1$ agrees with the one used in [15]. The substitution chosen in [1] produces similar operators, but with the q -shifts and the Novikov terms in the opposite order.

The following key result of Givental and Lee [15, Theorem 2] shows that the J function is an eigenfunction for $J_{\text{Fl}(n)}$.

Theorem A.2 (Givental–Lee). $\hat{H}_1 J_{\text{Fl}(n)} = \mathbb{C}^n J_{\text{Fl}(n)}$.

We also need the following lemma of Givental–Lee [15].

Lemma A.3 ([15, p. 9]). *Let D be a difference operator commuting with $\hat{H}_1$. Then, if J is an eigenfunction of D modulo Q , then J is an eigenfunction of D whose eigenvalue is the same as the one modulo Q .*

From this, we deduce that $J_{\text{Fl}(n)}$ is an eigenfunction of the higher Toda Hamiltonians, using their commutativity with $\hat{H}_1$ and by computing their eigenvalues modulo Q .

Corollary A.4. *For any $1 \leq k \leq n$, the following holds:*

$$\hat{H}_k J_{\text{Fl}(n)} = \wedge^k(\mathbb{C}^n) J_{\text{Fl}(n)}.$$

Proof. The case $k = 1$ is Theorem A.2. Suppose $2 \leq k \leq n$. Since $\hat{H}_k$ commutes with $\hat{H}_1$, we need only verify that $J_{\text{Fl}(n)}$ is an eigenfunction of $\hat{H}_k$ modulo Q , thanks to Lemma A.3.

To this end, we first observe

$$\begin{aligned} \hat{H}_k J_{\text{Fl}(n)} &= \hat{H}_k \left((1 - q) \prod_i P_i^{\frac{\ln(Q_i)}{\ln(q)}} \right) + o(Q_i) \\ &= \sum_{0=i_0 < \dots < i_k \leq n} \prod_{l=1}^k \frac{P_{i_l}}{P_{i_{l-1}}} \left((1 - q) \prod_i P_i^{\frac{\ln(Q_i)}{\ln(q)}} \right) + o(Q_i). \end{aligned}$$

Thus, modulo Q , we have the eigenvalue equation

$$\begin{aligned} \hat{H}_k J_{\text{Fl}(n)} &= \sum_{0=i_0 < \dots < i_k \leq n} \prod_{l=1}^k \frac{P_{i_l}}{P_{i_{l-1}}} J_{\text{Fl}(n)} \\ &= e_k \left(\frac{P_1}{P_0}, \frac{P_2}{P_1}, \dots, \frac{P_n}{P_{n-1}} \right) J_{\text{Fl}(n)} = \wedge^k(\mathbb{C}^n) J_{\text{Fl}(n)}. \end{aligned}$$

■

We now use [26, Proposition 2.12] which shows that the symbols of Toda Hamiltonians give relations in quantum K theory.

Theorem A.5 (Iritani–Milanov–Tonita). *Let $D = D(q^{Q_i \partial_{Q_i}}, q, Q, \Lambda_i)$ be any q -difference operator with coefficients in $K_T(\text{pt})[q^{\pm 1}][[Q_i]]$, such that it is regular at $q = 1$. Then*

$$DJ_X = 0 \implies D(\widehat{P}_i, 1, Q, \Lambda_i) = 0 \in \text{QK}_T(X).$$

Remark A.6. The result of Iritani, Milanov and Tonita is stated non-equivariantly, and for the *big* quantum K ring and the corresponding big J function. However, an inspection of their proof shows that it works in the equivariant situation as well. Furthermore, if one starts with the small quantum K ring, then all arguments extend to that situation, and the result also holds for the small quantum K ring and the small J function. For further details, see [23].

One subtle point is that $\widehat{P}_i$ is a certain Q -deformation of the line bundle P_i : it is the restriction to the small quantum K ring of an operator denoted by $A_{i,\text{com}}$ in [26, Corollary 2.9], which arises as a solution to a certain Lax-type equation. However, results of both Anderson, Chen, Tseng, and Iritani in [2, Lemma 6], and also by Kato in [29, Theorem 1.35] show that in fact no quantization is needed.

Proposition A.7. *For the flag variety $\text{Fl}(n)$, $\widehat{\det(\mathcal{S}_i)} = \det(\mathcal{S}_i)$.*

We note in passing that an analogue of Proposition A.7 holds for any homogeneous space G/P , but we do not need this generality here.

Combining Theorem A.5 and Proposition A.7 with Corollary A.4 yields the following corollary.

Corollary A.8. *The following identities hold in $\text{QK}_T(\text{Fl}(n))$:*

$$\sum_{0=i_0 < \dots < i_k \leq n} \prod_{l=1}^k \frac{P_{i_l}}{P_{i_l-1}} (1 - Q_{i_l-1})^{1-\delta_{i_l-i_{l-1},1}} = \wedge^k \mathbb{C}^n, \quad k = 1, \dots, n,$$

where $P_0 = P_n = 1$.

Remark A.9. Theorem 4.9 of [30] gives a presentation of the quasimap quantum K -ring of $T^*\text{Fl}$ whose limit to the is described in Theorem 5.5. The relations are based on the trigonometric Ruijsenaars–Schneider model. After further taking into account a restriction from GL_n to SL_n , the ‘Toda limit’ recovers the relations in this paper. We are grateful to P. Koroteev who explained this procedure to us.

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