Finding All Solutions of qKZ Equations in Characteristic p

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Received September 23, 2025, in final form December 16, 2025; Published online January 02, 2026
https://doi.org/10.3842/SIGMA.2026.001

Abstract. In [*J. Lond. Math. Soc.* **109** (2024), e12884, 22 pages], the difference qKZ equations were considered modulo a prime number p and a family of polynomial solutions of the qKZ equations modulo p was constructed by an elementary procedure as suitable p-approximations of the hypergeometric integrals. In this paper, we study in detail the first family of nontrivial examples of the qKZ equations in characteristic p. We describe all solutions of these qKZ equations in characteristic p by demonstrating that they all stem from the p-hypergeometric solutions. We also prove a Lagrangian property (called the orthogonality property) of the subbundle of the qKZ bundle spanned by the p-hypergeometric sections. This paper extends the results of [arXiv:2405.05159] on the differential KZ equations to the difference qKZ equations.

Key words: qKZ equations; p-hypergeometric solutions; orthogonality relations; p-curvature

2020 Mathematics Subject Classification: 11D79; 12H25; 32G34; 33C05; 33E30

1 Introduction

The Knizhnik–Zamolodchikov (KZ) differential equations are a system of linear differential equations, satisfied by conformal blocks on the sphere in the WZW model of conformal field theory, see [4]. The quantum Knizhnik–Zamolodchikov (qKZ) equations are a difference version of the KZ equations which naturally appear in the representation theory of Yangians (rational case) and quantum affine algebras (trigonometric case), see [1, 3]. The qKZ equations may be regarded as a deformation of the KZ differential equations.

As a rule one considers the KZ and qKZ equations over the field of complex numbers. Then these differential and difference equations are solved in multidimensional hypergeometric integrals.

In [6], the differential KZ equations were considered modulo a prime integer p. It turned out that modulo p the KZ equations have a family of polynomial solutions. The construction of these solutions was analogous to the construction of the multidimensional hypergeometric solutions, and these polynomial solutions were called the p-hypergeometric solutions.

In [5], the rational $\mathfrak{sl}_2$ qKZ equations with values in the n-th tensor power of the vector representation L and an integer step κ were considered modulo p. A family of polynomial solutions modulo p of these equations was constructed and called the p-hypergeometric solutions.

In this paper, we address the problem of whether all solutions of the qKZ equations in characteristic p are generated by the p-hypergeometric solutions. We consider the first family of nontrivial examples of the qKZ equations and demonstrate that, indeed, in this case, all solutions of the qKZ equations stem from the p-hypergeometric solutions.

Let $\mathbb{K}$ be a field of characteristic p. The qKZ equations for a function $f(z_1,\dots,z_n)$ with values in the $\mathbb{K}$-vector space $L^{\otimes n}$ and step $\kappa\in\mathbb{K}^\times$ have the form

$$f(z_1,\dots,z_a-\kappa,\dots,z_n)=K_a(z;\kappa)f(z),\qquad a=1,\dots,n,$$

where the linear operators $K_a(z;\kappa)$ are given in terms of the rational $\mathfrak{sl}_2$ R-matrix, see (2.2). The operators $K_a(z;\kappa)$ commute with the diagonal action of $\mathfrak{sl}_2$, and, therefore, it is sufficient to solve the qKZ equations only with values in the space of singular vectors of a given weight. In this paper, we study the qKZ equations with values in $V:=\operatorname{Sing}L^{\otimes n}[n-2]\subset L^{\otimes n}$, the subspace of singular vectors of weight $n-2$. We have $\dim V=n-1$.

There are two cases: $\kappa\in\mathbb{K}\setminus\mathbb{F}_p$ and $\kappa\in\mathbb{F}_p^\times$.

Theorem 1.1. *Let p be a prime number that does not divide n. For $\kappa\in\mathbb{K}\setminus\mathbb{F}_p$, there does not exist a nonzero rational V-valued function $f(z_1,\dots,z_n)$ which is a solution of the qKZ equations with parameter κ.*

See Corollary 7.12.

Assume that $\kappa\in\mathbb{F}_p^\times$. Let $0<k<p$ be the positive integer such that $\kappa k\equiv -1 \pmod p$. Let $[x]$ denote the integer part of a real number and $d(\kappa):=\left[\frac{kn}{p}\right]$. If p does not divide n, then $d(\kappa)+d(-\kappa)=n-1=\dim V$.

In [5], we constructed $d(\kappa)$ V-valued p-hypergeometric solutions of the qKZ equations denoted $Q^{\ell p-1}(z;\kappa)$, $\ell=1,\dots,d(\kappa)$. In this paper, we show that these solutions are linearly independent over the field $\mathbb{K}(z_1,\dots,z_n)$, see Theorem 5.6.

Theorem 1.2. *Let $p>n$, $\kappa\in\mathbb{F}_p^\times$, and $0<d(\kappa)<n-1$. Let $f(z)$ be a V-valued rational function in z which is a solution the qKZ equations with step κ. Then $f(z)$ is a linear combination of the p-hypergeometric solutions $Q^{\ell p-1}(z;\kappa)$, $\ell=1,\dots,d(\kappa)$, with coefficients which are scalar rational functions in $z_i^p-z_i$, $i=1,\dots,n$.*

See Theorem 7.9. Notice that $h(x)=x^p-x\in\mathbb{K}[x]$ is a 1-periodic polynomial, $h(x+1)=h(x)$. In particular, $h(x+\kappa)=h(x)$.

If $d(\kappa)=n-1$ or 0, all solutions of the qKZ equations with values in V and step κ are described in Section 7.4.

We prove the orthogonality relations for p-hypergeometric solutions of the qKZ equations with steps κ and $-\kappa$.

Theorem 1.3. *Let $p>n$ and $0<d(\kappa)<n-1$. Then for any $\ell\in\{1,\dots,d(\kappa)\}$ and $m\in\{1,\dots,d(-\kappa)\}$, we have*

$$S\big(Q^{mp-1}(-z;-\kappa),Q^{\ell p-1}(z;\kappa)\big)=0,$$

where S is the Shapovalov form.

See Theorem 6.1.

Define the p-curvature operators of the qKZ equations with values in V and step κ by the formula

$$\begin{aligned}C_a(z_1,\dots,z_n;\kappa):=\;&K_a(z_1,\dots,z_a-(p-1)\kappa,\dots,z_n;\kappa)\\&\times K_a(z_1,\dots,z_a-(p-2)\kappa,\dots,z_n;\kappa)\cdots K_a(z_1,\dots,z_a-\kappa,\dots,z_n;\kappa)\\&\times K_a(z_1,\dots,z_a,\dots,z_n;\kappa),\end{aligned}$$

for $a=1,\dots,n$, and the reduced p-curvature operators by the formula

$$\hat{C}_a(z_1,\dots,z_n;\kappa):=C_a(z_1,\dots,z_n;\kappa)-1.$$

If $f(z_1,\dots,z_n)$ is a solution of the qKZ equations, then $\hat{C}_af=0$ for all a.

Theorem 1.4. *Let $p > n$. Then the reduced p-curvature operators have the following properties:*

(i) *If $\kappa \in \mathbb{K} \setminus \mathbb{F}_p$, then all reduced p-curvature operators $\hat{C}_a(z;\kappa)$, $a = 1, \dots, n$, are nondegenerate for generic z.*

(ii) *If $\kappa \in \mathbb{F}_p^\times$ and $d(\kappa) = n - 1$ or 0, then all reduced p-curvature operators $\hat{C}_a(z;\kappa)$, $a = 1, \dots, n$, are equal to zero.*

(iii) *If $\kappa \in \mathbb{F}_p^\times$ and $0 < d(\kappa) < n - 1$, then all reduced p-curvature operators $\hat{C}_a(z;\kappa)$, $a = 1, \dots, n$, are nonzero. For every a, the span of the p-hypergeometric solutions $Q^{\ell p-1}(z;\kappa)$, $\ell = 1, \dots, d(\kappa)$, lies in the kernel of $\hat{C}_a(z,\kappa)$ and contains the image of $\hat{C}_a(z;\kappa)$. Also, for all a, b,*

$$\hat{C}_a(z;\kappa)\hat{C}_b(z;\kappa) = 0, \qquad \hat{C}_a(z,-\kappa) + \hat{C}_a(-z;\kappa)^* = 0,$$

where for an operator $T\colon V \to V$ we denote by T^* the operator dual to T under the Shapovalov form.

See Theorem 7.3 and Lemmas 7.7, 7.11.

In a suitable limit the difference qKZ equations on $L^{\otimes n}$ degenerate to the differential KZ equations on $L^{\otimes n}$,

$$\kappa \frac{\partial f}{\partial z_a} = \sum_{j \neq a} \frac{P^{(a,j)} - 1}{z_a - z_j} f, \qquad a = 1, \dots, n,$$

where $P^{(a,j)}$ is the permutation operator of the a-th and j-th tensor factors of $L^{\otimes n}$.

In [8], the differential KZ equations over a field $\mathbb{K}$ of characteristic p with values in $V \subset L^{\otimes n}$ were studied in detail. Our paper extends the results of [8] from the differential KZ equations to the difference qKZ equations. The proofs of Theorems 1.1, 1.2, and 1.4 are based on the corresponding results in [8] for the differential KZ equations.

On the differential and difference equations in characteristic p and associated p-curvature see also [2, 9].

2 Difference qKZ equations

2.1 Notations

In this paper, p is a prime and $\mathbb{K}$ a field of characteristic p.

Consider the Lie algebra $\mathfrak{sl}_2$ over $\mathbb{K}$ with basis e, f, h and relations $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$. Let L be the two-dimensional $\mathfrak{sl}_2$-module with basis v_1, v_2 and the action $ev_1 = 0$, $ev_2 = v_1$, $fv_1 = v_2$, $fv_2 = 0$, $hv_1 = v_1$, $hv_2 = -v_2$.

For a positive integer $n > 1$, consider the $\mathfrak{sl}_2$-module $L^{\otimes n}$.

Let $\mathcal{I}_l$ be the set of all l-element subsets of $\{1, \dots, n\}$. For a subset $I \subset \{1, \dots, n\}$, denote

$$v_I = v_{i_1} \otimes \dots \otimes v_{i_n} \in L^{\otimes n},$$

where $i_j = 2$ if $i_j \in I$ and $i_j = 1$ if $i_j \notin I$. Denote by $L^{\otimes n}[n-2l]$ the span of the vectors $\{v_I \mid I \in \mathcal{I}_l\}$. We have a direct sum decomposition,

$$L^{\otimes n} = \bigoplus_{l=0}^{n} L^{\otimes n}[n-2l].$$

Let $\operatorname{Sing} L^{\otimes n}[n-2l] \subset L^{\otimes n}[n-2l]$ be the subspace of singular vectors (the vectors annihilated by e).

2.2 qKZ equations

Define the rational R-matrix acting on $L^{\otimes 2}$, $R(u) = \frac{u-P}{u-1}$, where P is the permutation of tensor factors of $L^{\otimes 2}$. The R-matrix satisfies the Yang–Baxter and unitarity equations,

$$R^{(12)}(u-v)R^{(13)}(u)R^{(23)}(v) = R^{(23)}(v)R^{(13)}(u)R^{(12)}(u-v),$$
$$R^{(12)}(u)R^{(21)}(-u) = 1. \tag{2.1}$$

The first equation is an equation in $\operatorname{End}\left(L^{\otimes 3}\right)$. The superscript indicates the factors of $L^{\otimes 3}$ on which the corresponding operators act.

Let $z = (z_1, \dots, z_n)$. Define the qKZ operators $K_1, \dots, K_n$ acting on $L^{\otimes n}$

$$K_a(z;\kappa) = R^{(a,a-1)}(z_a - z_{a-1} - \kappa) \cdots R^{(a,1)}(z_a - z_1 - \kappa)$$
$$\times R^{(a,n)}(z_a - z_n) \cdots R^{(a,a+1)}(z_a - z_{a+1}), \tag{2.2}$$

where $\kappa \in \mathbb{K}^\times$ is a parameter.

The qKZ operators preserve the weight decomposition of $L^{\otimes n}$, commute with the $\mathfrak{sl}_2$-action, and form a discrete flat connection with step κ on the trivial bundle $L^{\otimes n} \times \mathbb{K}^n \to \mathbb{K}^n$,

$$K_a(z_1, \dots, z_b - \kappa, \dots, z_n; \kappa)K_b(z;\kappa) = K_b(z_1, \dots, z_a - \kappa, \dots, z_n; \kappa)K_a(z;\kappa)$$

for $a, b = 1, \dots, n$, see [3].

The system of difference equations with step κ,

$$s(z_1, \dots, z_a - \kappa, \dots, z_n) = K_a(z;\kappa)s(z), \qquad a = 1, \dots, n, \tag{2.3}$$

for an $L^{\otimes n}$-valued function $s(z)$, is called the qKZ equations with step κ.

Since the qKZ operators commute with the action of $\mathfrak{sl}_2$ on $L^{\otimes n}$, the qKZ operators preserve the subbundle $\operatorname{Sing} L^{\otimes n}[n-2l] \times \mathbb{K}^n \to \mathbb{K}^n$ for every integer l.

Define the translation operators $\mathbf{T_a}$ by

$$(\mathbf{T}_a f)(z_1, \dots, z_a, \dots, z_n) = f(z_1, \dots, z_a - \kappa, \dots, z_n).$$

The difference operators $\nabla_a = \mathbf{T}_a^{-1}K_a(z;\kappa)$ are called the connection operators of the qKZ difference connection. We have $[\nabla_a, \nabla_b] = 0$.

2.3 p-curvature of the qKZ connection

Define the p-curvature operators of the qKZ connection by

$$C_a(z;\kappa) = K_a(z_1, \dots, z_a - (p-1)\kappa, \dots, z_n;\kappa)K_a(z_1, \dots, z_a - (p-2)\kappa, \dots, z_n;\kappa) \cdots$$
$$\times K_a(z_1, \dots, z_a - \kappa, \dots, z_n;\kappa)K_a(z_1, \dots, z_a, \dots, z_n;\kappa)$$

for $a = 1, \dots, n$. In other words, $C_a = (\nabla_a)^p$.

For every a, the operator $C_a(z;\kappa)$ acts on fibers of the bundle $L^{\otimes n} \times \mathbb{K}^n \to \mathbb{K}^n$ and defines an endomorphism of the qKZ connection,

$$K_b(z_1, \dots, z_n;\kappa)C_a(z_1, \dots, z_n;\kappa) = C_a(z_1, \dots, z_b - \kappa, \dots, z_n;\kappa)K_b(z_1, \dots, z_n;\kappa).$$

The operators $C_a(z;\kappa)$ commute, i.e., $[C_a(z;\kappa), C_b(z;\kappa)] = 0$.

If $s(z;\kappa)$ is a flat section of the qKZ discrete connection,

$$s(z_1, \dots, z_a - \kappa, \dots, z_n;\kappa) = K_a(z;\kappa)s(z;\kappa), \qquad a = 1, \dots, n,$$

then $s(z;\kappa)$ is an eigenvector of the p-curvature operators with eigenvalue 1,

$$s(z;\kappa) = C_a(z;\kappa)s(z;\kappa), \qquad a = 1,\dots,n. \tag{2.4}$$

An operator $C_a(z;\kappa)$ is a rational function in z with the denominator

$$D_a(z;\kappa) = \prod_{j\neq a}\prod_{m=0}^{p-1}(z_a - z_j - m\kappa - 1)$$
$$= \prod_{j\neq a}\big(z_a^p - \kappa^{p-1}z_a + (-z_j)^p + \kappa^{p-1}z_j + (-1)^p + \kappa^{p-1}\big).$$

It is convenient to introduce the reduced p-curvature operators by the formula

$$\hat{C}_a(z;\kappa) = C_a(z;\kappa) - 1, \tag{2.5}$$

and the normalized p-curvature operators by the formula

$$\tilde{C}_a(z;\kappa) = D_a(z;\kappa)(C_a(z;\kappa) - 1).$$

The normalized p-curvature operators are polynomials in z of degree $\leqslant (n-1)p$.

2.4 Differential KZ equations

For $\kappa \in \mathbb{K}^\times$, the differential KZ operators

$$\nabla_a^{\mathrm{KZ}} = \kappa\frac{\partial}{\partial z_a} - \sum_{j\neq a}\frac{P^{(a,j)} - 1}{z_a - z_j}, \qquad a = 1,\dots,n,$$

define a flat KZ connection on $L^{\otimes n}\times K^n \to \mathbb{K}^n$, $\big[\nabla_a^{\mathrm{KZ}}, \nabla_b^{\mathrm{KZ}}\big] = 0$. The operators ∇_a^{KZ} commute with the $\mathfrak{sl}_2$-action on $L^{\otimes n}$. The system of equations

$$\kappa\frac{\partial f}{\partial z_a} = \sum_{j\neq a}\frac{P^{(a,j)} - 1}{z_a - z_j}f, \qquad a = 1,\dots,n, \tag{2.6}$$

is called the differential KZ equations with parameter κ. The KZ operators preserve every subbundle $\operatorname{Sing} L^{\otimes n}[n-2l]\times\mathbb{K}^n \to \mathbb{K}^n$. Denote

$$H_a(z) = \sum_{j\neq a}\frac{P^{(a,j)} - 1}{z_a - z_j},$$

the Gaudin Hamiltonians.

The p-curvature operators of the KZ connection are defined by the formula

$$C_a^{\mathrm{KZ}}(z;\kappa) := \big(\nabla_a^{\mathrm{KZ}}\big)^p.$$

They define an endomorphism of the KZ connection, $\big[C_a^{\mathrm{KZ}}, \nabla_b^{\mathrm{KZ}}\big] = 0$.

An operator $C_a^{\mathrm{KZ}}(z;\kappa)$ is a rational function in z with the denominator

$$D_a^{\mathrm{KZ}}(z;\kappa) = \prod_{j\neq a}\big(z_a^p - z_j^p\big).$$

It is convenient to introduce the normalized p-curvature operator by the formula

$$\tilde{C}_a^{\mathrm{KZ}}(z;\kappa) = D_a^{\mathrm{KZ}}(z;\kappa)C_a^{\mathrm{KZ}}(z;\kappa).$$

The normalized p-curvature operator is a homogeneous polynomial in z of degree $(n-2)p$ if nonzero.

2.5 KZ equations as a limit of qKZ equations

Let $f(z_1,\dots,z_n)$ satisfy the qKZ equations,

$$f(z_1,\dots,z_a-\kappa,\dots,z_n)=K_a(z;\kappa)f(z),\qquad a=1,\dots,n.$$

Let α be a formal parameter. Define $g(w_1,\dots,w_n;\alpha):=f(w_1/\alpha,\dots,w_n/\alpha)$. Then

$$g(w_1,\dots,w_a-\alpha\kappa,\dots,w_n;\alpha)=K_a(w/\alpha;\kappa)g(w;\alpha),\qquad a=1,\dots,n, \tag{2.7}$$

where

$$K_a(w/\alpha;\kappa)=R^{(a,a-1)}(w_a-w_{a-1}-\alpha\kappa;\alpha)\cdots R^{(a,1)}(w_a-w_1-\alpha\kappa;\alpha)$$
$$\times R^{(a,n)}(w_a-w_n;\alpha)\cdots R^{(a,a+1)}(w_a-w_{a+1};\alpha),$$
$$R(u;\alpha)=\frac{u-\alpha P}{u-\alpha}=1-\alpha\frac{P-1}{u-\alpha}.$$

Equation (2.2) gives

$$g-\alpha\kappa\frac{\partial g}{\partial w_a}+\mathcal{O}\big(\alpha^2\big)=\big(1-\alpha H_a(w)+\mathcal{O}\big(\alpha^2\big)\big)g.$$

In the limit $\alpha\to 0$, we obtain the KZ differential equations

$$\kappa\frac{\partial g}{\partial w_a}(w)=H_a(w)g(w).$$

Lemma 2.1. *Let $\hat{C}_a(z;\kappa)$ be a reduced p-curvature operators of the qKZ connection. Then*

$$C_a^{\mathrm{KZ}}(z;\kappa)=\lim_{\alpha\to 0}\hat{C}_a(z_1/\alpha,\dots,z_n/\alpha;\kappa)/\alpha^p.$$

Proof. Let $C_{a,\alpha}$ be the p-curvature operator of equation (2.7). Clearly,

$$C_{a,\alpha}(w;\kappa)=C_a(w_1/\alpha,\dots,w_n/\alpha;\kappa).$$

Hence we need to prove that

$$C_a^{\mathrm{KZ}}(w;\kappa)=\lim_{\alpha\to 0}(C_{a,\alpha}(w;\kappa)-1)/\alpha^p. \tag{2.8}$$

Let $\nabla_{a,\alpha}$ be the connection operator of the discrete connection (2.7),

$$(\nabla_{a,\alpha}g)(w)=K_a(w_1/\alpha,\dots,(w_a+\alpha\kappa)/\alpha,\dots,w_n/\alpha;\kappa)g(w_1,\dots,w_a+\alpha\kappa,\dots,w_n).$$

Then

$$(\nabla_{a,\alpha}g)(w)=g-\alpha H_a(w)g+\alpha\kappa\frac{\partial g}{\partial w_a}+\mathcal{O}\big(\alpha^2\big).$$

Hence

$$(\nabla_{a,\alpha}-1)g=\alpha\nabla_a^{\mathrm{KZ}}g+\mathcal{O}\big(\alpha^2\big).$$

Thus

$$C_{a,\alpha}(w;\kappa)-1=(\nabla_{a,\alpha})^p-1=(\nabla_{a,\alpha}-1)^p=\alpha^p\big(\nabla_a^{\mathrm{KZ}}\big)^p+\mathcal{O}\big(\alpha^{p+1}\big). \qquad \blacksquare$$

Corollary 2.2. *The degree of the reduced p-curvature operator $\tilde{C}_a(z;\kappa)$ as a polynomial in z is not greater than $(n-2)p$. Moreover, we have $\tilde{C}_a(z;\kappa)=\tilde{C}_a^{\mathrm{KZ}}(z;\kappa)+f_1(z)$, where $f_1(z)$ is a polynomial in z of degree less than $(n-2)p$.*

Proof. The first statement of the corollary follows from the fact that the denominator of the reduced p-curvature operator $\hat{C}_a(z;\kappa)$ is of degree $(n-1)p$. Hence the numerator is of degree not greater than $(n-2)p$ for the limit in (2.8) to exist. The second statement is clear. ■

2.6 Limit of solutions

Let $f(z_1,\dots,z_n)$ be a polynomial solution of the qKZ equations (2.3). Let $\deg f(z) = d$ and $f(z) = f_0(z) + f_1(z)$, where $f_0(z)$ is a homogeneous polynomial of degree d and $f_1(z)$ is a polynomial of degree less than d.

Lemma 2.3. *The polynomial $f_0(z)$ is a solution of the KZ equations (2.6).*

Proof. Let α be a formal parameter. Define $g(w_1,\dots,w_n;\alpha) = \alpha^d f(w_1/\alpha,\dots,w_n/\alpha)$. Then $g(w_1,\dots,w_n;\alpha) = f_0(w)+\alpha g_1(w;\alpha)$, where $g_1(w;\alpha) = \alpha^{d-1} f_1(w/\alpha)$ is a polynomial in w and α. We have

$$g(w_1,\dots,w_a-\alpha\kappa,\dots,w_n;\alpha) = K_a(w/\alpha;\kappa)g(w;\alpha), \qquad a=1,\dots,n.$$

Let

$$d_a(w;\alpha) = \prod_{j=1}^{a-1}(w_a-w_j-\alpha\kappa-\alpha)\prod_{j=a+1}^{n}(w_a-w_j-\alpha)$$

be the denominator of $K_a(w/\alpha;\kappa)$ and

$$n_a(w;\alpha) = \left(w_a-w_{a-1}-\alpha\kappa-\alpha P^{(a,a-1)}\right)\cdots\left(w_a-w_{a+1}-\alpha P^{(a,a+1)}\right)$$

the numerator. Then (2.7) can be written as a polynomial equation

$$d_a(w;\alpha)g(w_1,\dots,w_a-\alpha\kappa,\dots,w_n;\alpha) = n_a(w;\alpha)g(w;\alpha) \tag{2.9}$$

in the variables w and α. Equation (2.9) gives an equation in w for every fixed power of α in (2.9). The equation corresponding to the first power of α after division by $d_a(w;0)$ becomes

$$\kappa\frac{\partial f_0}{\partial w_a}(w) = \sum_{j\neq a}\frac{P^{(a,j)}-1}{w_a-w_j}f_0(w). \qquad \blacksquare$$

2.7 Dual qKZ equations

Denote $W = L^{\otimes n}$. Let W^* be the dual space of W, and $\langle,\rangle\colon W^*\otimes W\to\mathbb{K}$ the canonical pairing. Let $K_a^*(z;\kappa)\colon W^*\to W^*$ be the operators dual to the operators $K_a(z;\kappa)$. Denote $\tilde K_a(z;\kappa) = (K_a^*(z;\kappa))^{-1}$.

We have

$$\tilde K_a(z_1,\dots,z_b-\kappa,\dots,z_n;\kappa)\tilde K_b(z;\kappa) = \tilde K_b(z_1,\dots,z_a-\kappa,\dots,z_n;\kappa)\tilde K_a(z;\kappa)$$

for all a, b, and also

$$\langle x,y\rangle = \left\langle \tilde K_a(z;\kappa)x, K_a(z;\kappa)y\right\rangle \tag{2.10}$$

for all a and $x\in W^*, y\in W$.

The system of difference equations with step κ,

$$\tilde s(z_1,\dots,z_a-\kappa,\dots,z_n) = \tilde K_a(z;\kappa)\tilde s(z), \qquad a=1,\dots,n, \tag{2.11}$$

for an W^*-valued function $\tilde s(z)$, is called the dual qKZ equations.

If $s(z)$ is a solution of the qKZ equations (2.3) and $\tilde s(z)$ is a solution of the dual qKZ equations (2.11), then the function $\langle\tilde s(z), s(z)\rangle$ is κ-periodic with respect to every z_a,

$$-\langle\tilde s(z_1,\dots,z_a-\kappa,\dots,z_n), s(z_1,\dots,z_a-\kappa,\dots,z_n,)\rangle = \langle\tilde s(z), s(z)\rangle.$$

The set of vectors $\{v_I \mid I \subset \{1,\dots,n\}\}$ is a basis of W. Define the nondegenerate symmetric bilinear form S on W by the formula $S(v_I, v_J) = \delta_{I,J}$. The form is called the (tensor) Shapovalov form. We identify W^* and W with the help of the Shapovalov form.

Under this identification, the operators $R^{(i,j)}(u)$ become symmetric. Using (2.1) we obtain the following formula for the operators $\tilde{K}_a(z;\kappa)$ as operators on W,

$$\begin{aligned}\tilde{K}_a(z;\kappa) = {} & R^{(a,a-1)}(-z_a + z_{a-1} + \kappa) \cdots R^{(a,1)}(-z_a + z_1 + \kappa)\\ & \times R^{(a,n)}(-z_a + z_n) \cdots R^{(a,a+1)}(-z_a + z_{a+1}).\end{aligned}$$

In other words,

$$\begin{aligned}&\tilde{K}_a(z_1,\dots,z_n;\kappa) = K_a(-z_1,\dots,-z_n;-\kappa),\\ &S(K_a(-z_1,\dots,-z_n;-\kappa)x, K_a(z_1,\dots,z_n;\kappa)y) = S(x,y)\end{aligned}$$

for all $x, y \in W$.

Now the dual qKZ equations for a W-valued function $\tilde{s}(z)$ take the form

$$\tilde{s}(z_1,\dots,z_a-\kappa,\dots,z_n) = K_a(-z_1,\dots,-z_n;-\kappa)s(z), \qquad a = 1,\dots,n.$$

These formulas prove the following lemma.

Lemma 2.4. *Let $s(z)$ be a solution of the qKZ equations (2.3) with step κ and $\tilde{s}(z)$ a solution of the qKZ equations (2.3) with step $-\kappa$. Then the function $S(\tilde{s}(-z), s(z))$ is κ-periodic with respect to every variable z_a,*

$$S(\tilde{s}(-z_1,\dots,-(z_a-\kappa),\dots,-z_n), s(z_1,\dots,z_a-\kappa,\dots,z_n)) = S(\tilde{s}(-z), s(z)).$$

See also Corollary 7.10.

3 Pochhammer polynomials

For $\kappa \in \mathbb{K}$ and m a positive integer, define Pochhammer polynomial $(t;\kappa)_m \in \mathbb{K}[t]$ by the formula $(t;\kappa)_m = \prod_{i=1}^m (t-(i-1)\kappa)$.[1] We have

$$(t-\kappa;\kappa)_m = (t;\kappa)_m \frac{t-\kappa m}{t}, \qquad (t+\kappa;\kappa)_m = (t;\kappa)_m \frac{t+\kappa}{t-(m-1)\kappa}.$$

$$(t+z;\kappa)_m = \sum_{i=0}^m \binom{m}{i} (t;\kappa)_i (z;\kappa)_{m-i},$$

$$(t;\kappa)_i (t;\kappa)_j = \sum_{l=0}^{\min(i,j)} \binom{i}{l}\binom{j}{l} l!\kappa^l (t;\kappa)_{i+j-l}, \tag{3.1}$$

$$(t;\kappa)_m = \sum_{l=0}^m s_1(m,l)\kappa^{m-l} t^l,$$

$$t^m = \sum_{l=0}^m s_2(m,l)\kappa^{m-l}(t;\kappa)_l, \tag{3.2}$$

where the integers $s_1(m,l)$ and $s_2(m,l)$ are Stirling numbers of the first and second kind, respectively. Notice that

$$s_1(m,m) = s_2(m,m) = 1. \tag{3.3}$$

[1] See https://en.wikipedia.org/wiki/Falling_and_rising_factorials.

We also have $(t;\kappa)_p = t^p - \kappa^{p-1}t$,

$$(t+z;\kappa)_p = (t+z)^p - \kappa^{p-1}(t+z) = t^p - \kappa^{p-1}t + z^p - \kappa^{p-1}z = (t;\kappa)_p + (z;\kappa)_p.$$

We call a polynomial $f(t) \in \mathbb{F}_p[t]$ a quasi-constant if $f(t-\kappa) = f(t)$. The quasi-constants are polynomials in $t^p - \kappa^{p-1}t$. A Pochhammer polynomial $(t;\kappa)_m$ is a quasi-constant if m is divisible by p. Then $(t;\kappa)_{pa} = \big(t^p - \kappa^{p-1}t\big)^a$.

Let A be a $\mathbb{K}$-algebra, for example, $A = \mathbb{K}[z_1,\dots,z_n]$. The polynomials $\{(t;\kappa)_m \mid m \geqslant 0\}$ form an A-basis of the ring of polynomials $A[t]$.

4 p-hypergeometric solutions for $\kappa \in \mathbb{F}_p^\times \subset \mathbb{K}$

4.1 Solutions in $\operatorname{Sing} L^{\otimes n}[n-2]$

In this paper, we study solutions of the qKZ equations with values in $V := \operatorname{Sing} L^{\otimes n}[n-2]$. The space $L^{\otimes n}[n-2]$ has a basis $v^{(i)} = v_1 \otimes \cdots \otimes v_2 \otimes \cdots \otimes v_1$, $i = 1,\dots,n$, where the only v_2 stays at the i-th place. In this basis, the subspace V consists of all vectors with the sum of coordinates equal to zero. We identify $L^{\otimes n}[n-2]$ with $\mathbb{K}^n$ and the subspace V with the vector space

$$\{x \in \mathbb{K}^n \mid x_1 + \cdots + x_n = 0\}. \tag{4.1}$$

4.2 Master polynomial and weight functions

For $\kappa \in \mathbb{F}_p^\times$, let $0 < k < p$ be the positive integer such that

$$\kappa k \equiv -1 \pmod p. \tag{4.2}$$

Define master polynomial $\Phi(t,z;\kappa) = \prod_{a=1}^n (t-z_a;\kappa)_k$, where k is defined in (4.2). For $1 \leqslant a \leqslant n$, define the weight functions

$$\eta_a(t,z) = \frac{1}{t-z_a}\prod_{j=1}^{a-1}\frac{t-z_j+1}{t-z_j}, \qquad Q_a(t,z;\kappa) = \Phi(t,z;\kappa)\eta_a(t,z).$$

Notice that the formula for the function $\eta_a(t,z)$ does not depend on p and κ.

Lemma 4.1. *The function $Q_a(t,z;\kappa)$ is a product of n Pochhammer polynomials:*

$$Q_a(t,z;\kappa) = \left(\prod_{j=1}^{a-1}(t-z_j-\kappa;\kappa)_k\right)(t-z_a-\kappa;\kappa)_{k-1}\left(\prod_{j=a+1}^{n}(t-z_j;\kappa)_k\right).$$

Define a vector of polynomials

$$Q(t,z;\kappa) = (Q_1(t,z;\kappa),\dots,Q_n(t,z;\kappa))^\intercal = \sum_i Q^i(z;\kappa)(t;\kappa)_i, \tag{4.3}$$

where $M^\intercal$ denotes the transpose matrix of a matrix M and $Q^i(z;\kappa) = \big(Q^i_1(z;\kappa),\dots,Q^i_n(z;\kappa)\big)^\intercal$ are vectors of polynomials in z.

Example 4.2. For $n=2$,

$$Q(t,z;\kappa) = \big((t_1-z_1-\kappa;\kappa)_{k-1}(t_1-z_2;\kappa)_k, (t_1-z_1-\kappa;\kappa)_k(t_1-z_2-\kappa;\kappa)_{k-1}\big)^\intercal.$$

Theorem 4.3 ([5, Theorem 5.1])**.** *For any positive integer ℓ, the vector $Q^{\ell p-1}(z;\kappa)$ of polynomials in z is a solutions of the qKZ equations with step κ and values in $V = \operatorname{Sing} L^{\otimes n}[n-2]$, in particular, $\sum_{a=1}^n Q_a^{\ell p-1}(z;\kappa) = 0$, see (4.1).*

Let $[x]$ denote the integer part of a real number x. The vector $Q^{\ell p-1}(z;\kappa)$ is zero if $\ell \notin \{1,\dots,[\frac{nk}{p}]\}$ for degree reasons. The vectors of polynomials $Q^{\ell p-1}(z;\kappa)$, $\ell = 1,\dots,[\frac{nk}{p}]$, are called the p-hypergeometric solutions of the qKZ equations with step κ and values in V. Denote $d(\kappa) = [\frac{nk}{p}]$.

Notice that if $\frac{nk}{p} < 1$, then there are no p-hypergeometric solutions.

4.3 Step $-\kappa$

The integer k satisfies the inequalities $0 < k < p$ and the congruence $\kappa k \equiv -1 \pmod p$, see (4.2). Then the integer $p-k$ satisfies the inequalities $0 < p-k < p$ and the congruence $-\kappa(p-k) \equiv -1 \pmod p$. Hence $\Phi(t,z;-\kappa) = \prod_{a=1}^n (t-z_a;-\kappa)_{p-k}$. Recall that

$$Q_a(t,z;-\kappa) = \Phi(t,z;-\kappa)\eta_a(t,z),$$
$$Q(t,z;-\kappa) = (Q_1(t,z;-\kappa),\dots,Q_n(t,z;-\kappa))^\intercal = \sum_i Q^i(z;-\kappa)(t;-\kappa)_i.$$

Corollary 4.4. *The vectors*

$$Q^{\ell p-1}(z;-\kappa), \qquad \ell = 1,\dots,\left[\frac{n(p-k)}{p}\right],$$

are solutions of the qKZ equations with step $-\kappa$ and values in V.

If p does not divide n, then the total number of p-hypergeometric solutions of the qKZ equations with values in V and steps κ and $-\kappa$ equals

$$\left[\frac{nk}{p}\right] + \left[\frac{n(p-k)}{p}\right] = n-1 = \dim V.$$

4.4 p-hypergeometric solutions of KZ equations

In this subsection, we remind the construction in [6] of polynomial solutions modulo p of the differential KZ equations with values in $V = \operatorname{Sing} L^{\otimes n}[n-2]$.

Let $0 < k < p$ be the positive integers such that $\kappa k \equiv -1 \pmod p$. Define master polynomial $\bar\Phi(t,z,\kappa) = \prod_{a=1}^n (t-z_a)^k$. For $1 \leqslant a \leqslant n$, define the weight functions

$$\bar w_a(t,z) = \frac{1}{t-z_a}, \qquad \bar Q_a(t,z;\kappa) = \bar\Phi(t,z;\kappa)\bar w_a(t,z).$$

Then $\bar Q_a(t,z;\kappa)$ is a polynomial in t, z. Define a vector of polynomials in t, z,

$$\bar Q(t,z;\kappa) = \big(\bar Q_1(t,z;\kappa),\dots,\bar Q_n(t,z;\kappa)\big)^\intercal = \sum_i \bar Q^i(z;\kappa)t^i,$$

where $\bar Q^i(z;\kappa) = \big(\bar Q_1^i(z;\kappa),\dots,\bar Q_n^i(z;\kappa)\big)^\intercal$ are vectors of polynomials in z.

Theorem 4.5 ([6])**.** *For any positive integer ℓ, the vector $\bar Q^{\ell p-1}(z;\kappa)$ is a solution of the differential KZ equations with parameter κ and values in $V = \operatorname{Sing} L^{\otimes n}[n-2]$.*

The vector $\bar{Q}^{\ell p-1}(z;\kappa)$ is zero if $\ell\notin\{1,\dots,[\frac{nk}{p}]\}$ for degree reasons. The vectors

$$\bar{Q}^{\ell p-1}(z,\kappa),\qquad \ell=1,\dots,\left[\frac{nk}{p}\right],$$

are called the p-hypergeometric solutions of the differential KZ equations with parameter κ and values in V.

Notice that $\bar{Q}_a(t,z;\kappa)$ are homogeneous polynomials in variables t, z of degree $nk-1$, and $\bar{Q}^{\ell p-1}(z;\kappa)$ are vectors of homogeneous polynomials in z of degree $nk-\ell p$.

4.5 Top-degree part of p-hypergeometric solutions

It turns out that the top-degree part of a p-hypergeometric solution $Q^{\ell p-1}(z;\kappa)$ of the qKZ equations is the p-hypergeometric solution $\bar{Q}^{\ell p-1}(z;\kappa)$ of the KZ equations.

We start with an abstract lemma. Let $t,z_1,\dots,z_n,\alpha$ be variables. Given a homogeneous polynomial $P(t,z_1,\dots,z_n,\alpha)$ of degree d in the variables t, z, α and an integer e, $0\leqslant e\leqslant d$, we construct two polynomials $\bar{P}_e(z_1,\dots,z_n)$ and $P_e(z_1,\dots,z_n,0)$ as follows.

On the one hand, we have

$$P(t,z_1,\dots,z_n,\alpha)=\sum_{d_0+\dots+d_{n+1}=d}a_{d_0,\dots,d_{n+1}}t^{d_0}z_1^{d_1}\cdots z_n^{d_n}\alpha^{d_{n+1}}. \tag{4.4}$$

Then

$$P(t,z_1,\dots,z_n,0)=\sum_{d_0+\dots+d_n=d}a_{d_0,\dots,d_n,0}t^{d_0}z_1^{d_1}\cdots z_n^{d_n}. \tag{4.5}$$

Denote

$$\bar{P}_e(z_1,\dots,z_n)=\sum_{e+d_1+\dots+d_n=d}a_{e,d_1,\dots,d_n,0}z_1^{d_1}\cdots z_n^{d_n},$$

the coefficient of t^e in (4.5).

On the other hand, applying formula (3.2) to each t^{d_0} at (4.4), we rewrite this sum as

$$P(t,z_1,\dots,z_n,\alpha)=\sum_{d_0+\dots+d_{n+1}=d}b_{d_0,\dots,d_{n+1}}(t;\alpha)_{d_0}z_1^{d_1}\cdots z_n^{d_n}\alpha^{d_{n+1}}. \tag{4.6}$$

Denote

$$P_e(z_1,\dots,z_n,\alpha)=\sum_{e+d_1+\dots+d_{n+1}=d}b_{e,d_1,\dots,d_{n+1}}z_1^{d_1}\cdots z_n^{d_n}\alpha^{d_{n+1}},$$

the coefficient of $(t;\alpha)_e$ in (4.6). Then

$$P_e(z_1,\dots,z_n,0)=\sum_{e+d_1+\dots+d_n=d}b_{e,d_1,\dots,d_n,0}z_1^{d_1}\cdots z_n^{d_n}.$$

Lemma 4.6. *We have* $P_e(z_1,\dots,z_n,0)=\bar{P}_e(z_1,\dots,z_n)$.

Proof. By formula (3.3), we have $a_{e,d_1,\dots,d_n,0}=b_{e,d_1,\dots,d_n,0}$ for any $e,d_1,\dots,d_n$. This implies the lemma. ∎

Corollary 4.7 ([5])**.** *Let $Q^{\ell p-1}(z;\kappa)$ be a p-hypergeometric solution of the qKZ equations from Theorem 4.3. Then $Q^{\ell p-1}(z;\kappa)=\bar{Q}^{\ell p-1}(z;\kappa)+\cdots$, where $\bar{Q}^{\ell p-1}(z;\kappa)$ is the corresponding p-hypergeometric solution of the KZ equations from Theorem 4.5, and the dots denote the terms of degree less than $nk-\ell p=\deg\bar{Q}^{\ell p-1}(z;\kappa)$.*

Proof. The corollary is proved by application of Lemma 4.6 to the polynomial $Q(t,z;\kappa)$ and integer $e=\ell p-1$. Then it is easy to see that $\bar{P}_{\ell p-1}(z_1,\dots,z_n)=\bar{Q}^{\ell p-1}(z;\kappa)$ and $P_{\ell p-1}(z_1,\dots,z_n,0)$ is the top-degree part of $Q^{\ell p-1}(z;\kappa)$. ∎

5 Linear independence

5.1 Lexicographical order

Define length-lexicographical order on monomials $v = z_1^{d_1} \cdots z_n^{d_n}$: $v < w$ if degree(v) <degree(w) or if degree(v) =degree(w) and v is smaller than w in lexicographical order.

For a polynomial $f(z) = \sum_{d_1,\dots,d_n} a_{d_1,\dots,d_n} z_1^{d_1} \cdots z_n^{d_n}$ denote by $\mathrm{L} f(z)$ the nonzero summand $a_{d_1,\dots,d_n} z_1^{d_1} \cdots z_n^{d_n}$ with the lexicographically largest monomial $z_1^{d_1} \cdots z_n^{d_n}$. We call $\mathrm{L} f(z)$ the leading term of $f(z)$.

5.2 Leading terms

For $\ell \in \{1,\dots,d(\kappa)\}$, let $r(\ell)$ be the unique non-negative integer such that

$$r(\ell)k \leqslant nk - \ell p < (r(\ell)+1)k.$$

We have $r(1) > r(2) > \cdots$. Denote

$$ga = (n - r(\ell))k - \ell p, \qquad u_\ell = \frac{(-1)^{nk-\ell p}}{k} \binom{k}{a} (0,\dots,0,k-a,k,\dots,k)^\intercal \in \mathbb{F}_p^n,$$

where $k - a$ is the $r(\ell)+1$-st coordinate. Notice that the integers k and $k-a$ are not divisible by p, and $u_\ell \in \mathbb{F}_p^n$ is a singular vector since the sum of its coordinates equals zero.

Lemma 5.1 ([7, Lemma 3.1])**.** *Assume that p does not divide n and $\ell \in \{1,\dots,d(\kappa)\}$. Then the leading term of $\bar{Q}^{\ell p-1}(z;\kappa)$ is given by the formula*

$$\mathrm{L}\, \bar{Q}^{\ell p-1}(z;\kappa) = (z_1 \cdots z_{r(\ell)})^k z_{r(\ell)+1}^a u_\ell.$$

Lemma 5.2. *Assume that p does not divide n and $\ell \in \{1,\dots,d(\kappa)\}$. Let $Q^{\ell p-1}(z;\kappa)$ be the corresponding p-hypergeometric solution of the qKZ equations and $\bar{Q}^{\ell p-1}(z;\kappa)$ the corresponding p-hypergeometric solution of the KZ equations. Then their leading terms are equal,*

$$\mathrm{L}\, Q^{\ell p-1}(z;\kappa) = \mathrm{L}\, \bar{Q}^{\ell p-1}(z;\kappa).$$

Proof. The lemma follows from Corollary 4.7. ■

Corollary 5.3. *If p does not divide n and $\ell \in \{1,\dots,d(\kappa)\}$, then the leading term of $Q^{\ell p-1}(z;\kappa)$ is given by the formula*

$$\mathrm{L}\, Q^{\ell p-1}(z;\kappa) = (z_1 \cdots z_{r(\ell)})^k z_{r(\ell)+1}^a u_\ell. \tag{5.1}$$

Example 5.4. Let $n = 3$ and $d(\kappa) = \left[\frac{3k}{p}\right] = 1$. Then $\frac{p}{3} < k < \frac{2p}{3}$. In this case there is exactly one p-hypergeometric solution $Q^{p-1}(z;\kappa)$. If $\frac{p}{2} < k < \frac{2p}{3}$, then the leading term of $Q^{p-1}(z;\kappa)$ is

$$z_1^k z_2^{2k-p} \frac{(-1)^{3k-p}}{k} \binom{k}{2k-p} (0, p-k, k)^\intercal.$$

If $\frac{p}{3} < k < \frac{p}{2}$, then the leading term of $Q^{p-1}(z;\kappa)$ is

$$z_1^{3k-p} \frac{(-1)^k}{k} \binom{k}{3k-p} (p-2k, k, k)^\intercal.$$

Consider the collections of the n-vectors $Q^{\ell p-1}(z;\kappa)$, $\ell = 1,\dots,d(\kappa)$, as an $(n \times d(\kappa))$-matrix. For $I = \{1 \leqslant i_1 < \cdots < i_{d(\kappa)} \leqslant n\}$, denote by $M_I(z,\kappa)$ the $(d(\kappa) \times d(\kappa))$-minor of that matrix located at the rows with indices in I.

Lemma 5.5. *If p does not divide n and $I = \{r(d(\kappa)) < \cdots < r(1)\}$, then the minor $M_I(z;\kappa)$ is a nonzero polynomial.*

Proof. The lemma is a corollary of formula (5.1). ∎

Theorem 5.6. *If p does not divide n, then the p-hypergeometric solutions $Q^{\ell p-1}(z;\kappa)$, $\ell = 1, \dots, d(\kappa)$, of the qKZ equations are linearly independent over the field $\mathbb{K}(z)$.*

Proof. The statement follows from formula (5.1) and the fact that the vectors u_ℓ are linearly independent over $\mathbb{K}$. ∎

6 Orthogonality relations

6.1 Statement

Recall that $d(-\kappa) = \left[\frac{n(p-k)}{p}\right]$.

Theorem 6.1. *Let $p > n$ and $0 < d(\kappa) < n-1$. Then for any $\ell \in \{1, \dots, d(\kappa)\}$ and $m \in \{1, \dots, d(-\kappa)\}$, we have*

$$S\big(Q^{mp-1}(-z;-\kappa), Q^{\ell p-1}(z;\kappa)\big) = \sum_{a=1}^{n} Q_a^{mp-1}(-z;-\kappa) Q_a^{\ell p-1}(z;\kappa) = 0, \tag{6.1}$$

where S is the Shapovalov form.

The theorem is proved in Sections 6.2 and 6.3.

Remark. It is easy to see that the Shapovalov form on V is nondegenerate if p does not divide n. Indeed, the vectors $e_1 = (1,-1,0,\dots,0)$, $e_2 = (0,1,-1,0,\dots,0)$, $\dots$, $e_{n-1} = (0,\dots,0,1,-1)$ form a basis of V, and the determinant of the Shapovalov form in this basis equals n.

Remark. Formula (6.1) and Corollary 4.7 imply the orthogonality relations for the p-hypergeometric solutions of the KZ equations,

$$S\big(\bar{Q}^{mp-1}(-z;-\kappa), \bar{Q}^{\ell p-1}(z;\kappa)\big) = \sum_{a=1}^{n} \bar{Q}_a^{mp-1}(-z;-\kappa) \bar{Q}_a^{\ell p-1}(z;\kappa). = 0. \tag{6.2}$$

Two different proofs of formula (6.2) are given in [8, Theorem 3.11] and [8, Appendix A].

6.2 Special restrictions

Let $I \subset \{1, \dots, n\}$ be a nonempty subset, $I = \{1 \leqslant i_1 < i_2 < \cdots < i_a \leqslant n\}$. Denote S_I the system of equations

$$z_{i_b} = ((b-1)k - 1)\kappa, \qquad b = 1, \dots, a. \tag{6.3}$$

For a polynomial $f(z)$, define $f(z)_{S_I}$ to be the polynomial $f(z)$ in which the variables $(z_i)_{i\in I}$ are replaced by multiples of κ according to formulas (6.3).

Lemma 6.2. *Let $Q(t,z;\kappa)$ be the vector of polynomials defined by (4.3). Let $I = \{1 \leqslant i_1 < i_2 < \cdots < i_a \leqslant n\} \subset \{1, \dots, n\}$ be a nonempty subset. Then*

$$Q(t,z;\kappa)_{S_I} = (t;\kappa)_{ak-1} (P_1(t,z), \dots, P_n(t,z))^\intercal,$$

where $P_1(t,z), \dots, P_n(t,z)$ are suitable polynomials of degree $(n-a)k$.

Proof. The proof is straightforward. ■

Corollary 6.3. *Let $Q^{\ell p-1}(z;\kappa)$ be a p-hypergeometric solution and $\ell p < ak$. Then*

$$Q^{\ell p-1}(z;\kappa)_{S_I} = 0. \tag{6.4}$$

Proof. For $j = 1, \dots, n$, the j-th coordinate of $Q(t,z;\kappa)_{S_I}$ equals $(t;\kappa)_{ak-1}P_j(t,z)$. Using (3.1), we rewrite this as $\sum_{i\geqslant ak-1} c_i(z)(t;\kappa)_i$ for suitable $c_i(z)$. We observe that $(t;\kappa)_{\ell p-1}$ does not enter this sum. This proves the corollary. ■

6.3 Proof of Theorem 6.1

Denote

$$G_{\ell,m}(z;\kappa) = S\big(Q^{mp-1}(-z;-\kappa), Q^{\ell p-1}(z;\kappa)\big).$$

Then

$$\begin{aligned} G_{\ell,m}(-z;\kappa) &= S\big(Q^{mp-1}(z;-\kappa), Q^{\ell p-1}(-z;\kappa)\big) \\ &= S\big(Q^{\ell p-1}(-z;\kappa), Q^{mp-1}(z;-\kappa)\big) = G_{m,\ell}(z;-\kappa). \end{aligned}$$

Hence, $G_{\ell,m}(z;\kappa) = 0$ for all ℓ, m if and only if $G_{m,\ell}(z;-\kappa) = 0$.

By Lemma 2.4, the function $G_{\ell,m}(z;\kappa)$ is a polynomial in $\mathbb{F}_p[z_1^p - z_1, \dots, z_n^p - z_n]$. Denote $h(x) = x^p - x$.

Lemma 6.4. *Given ℓ and m, we have*

$$G_{\ell,m}(z;\kappa) = c_0 + \sum_{b=1}^{n-\ell-m} \sum_{1\leqslant j_1<\dots<j_b\leqslant n} c_{j_1,\dots,j_b} h(z_{j_1})\cdots h(z_{j_b}), \qquad c_0, c_{j_1,\dots,j_b} \in \mathbb{F}_p.$$

Proof. On the one hand, we have

$$\deg Q^{\ell p-1}(z;\kappa) = kn - \ell p \qquad \text{and} \qquad \deg Q^{mp-1}(-z;-\kappa) = (p-k)n - mp.$$

Hence $\deg G_{\ell,m}(z;\kappa) \leqslant (n-\ell-m)p$.

On the other hand, for any $j = 1, \dots, n$, we have $\deg_{z_j} G_{\ell,m}(z;\kappa) \leqslant p$ since $\deg_{z_j} Q(t,z;\kappa) = k$ and $\deg_{z_j} Q(t,-z;-\kappa) = p-k$. These two remarks prove the lemma. ■

The p-hypergeometric solution $Q^{\ell p-1}(z,\kappa)$ is associated with the integer $0 < k < p$ such that $\kappa k \equiv -1 \pmod p$, while the p-hypergeometric solution $Q^{mp-1}(z,-\kappa)$ is associated with the integer $0 < p-k < p$ such that $-\kappa(p-k) \equiv -1 \pmod p$. Given m, ℓ, we say that k is a good parameter if $\ell(p-k) < mk$. We say that $p-k$ is a good parameter if $mk < \ell(p-k)$. Notice that $mk \neq \ell(p-k)$. Otherwise p must divide $\ell+m$ which is impossible since $\ell+m \leqslant n-1 < p$.

Having the two integers k, $p-k$ and two solutions $Q^{\ell p-1}(z;\kappa)$ and $Q^{mp-1}(z;-\kappa)$, we may and will assume that k denotes the good parameter.

Lemma 6.5. *Given $Q^{\ell p-1}(z;\kappa)$ and $Q^{mp-1}(z;-\kappa)$, assume that k is a good parameter. Then $G_{\ell,m}(z;\kappa) = 0$.*

Proof. Let $\{1,\dots,n\} = I \cup J$ be a partition where $|I| = \ell+m$ and $|J| = n-\ell-m$. We may apply formula (6.4) to $Q^{\ell p-1}(z;\kappa)$ with $a = \ell+m$ since k is a good parameter and hence $\ell p < (\ell+m)k$. Hence we have $G_{\ell,m}(z;\kappa)_I = 0$, where

$$G_{\ell,m}(z;\kappa)_I = c_0 + \sum_{b=1}^{n-\ell-m} \sum_{\{j_1<\dots<j_b\}\subset J} c_{j_1,\dots,j_b} h(z_{j_1})\dots h(z_{j_b}). \tag{6.5}$$

Hence the right-hand side polynomial at (6.5) is the zero polynomial for any subset I with $|I| = \ell+m$. Therefore, $G_{\ell,m}(z;\kappa)$ is the zero polynomial. ■

Lemma 6.5 implies Theorem 6.1.

7 Invariant subbundles

7.1 Subbundle of qKZ connection

Assume that $p > n$ and $\kappa \in \mathbb{F}_p^\times$. Consider the discrete qKZ connection on the direct product $V \times \mathbb{K}^n \to \mathbb{K}^n$, where $V = \{x \in \mathbb{K}^n \mid x_1 + \dots + x_n = 0\}$ and the qKZ operators are defined by formula (2.2),

$$K_a(z;\kappa) = R^{(a,a-1)}(z_a - z_{a-1} - \kappa) \cdots R^{(a,1)}(z_a - z_1 - \kappa)$$
$$\times R^{(a,n)}(z_a - z_n) \cdots R^{(a,a+1)}(z_a - z_{a+1}).$$

The connection has singularities at the points of $\mathbb{K}^n$ where the qKZ operators have poles or are degenerate.

The R-matrix $R(u) = \frac{u-P}{u-1}$ has a pole if $u = 1$ and is degenerate if $u = -1$. Denote by $H_{i,j,m}$ the affine hyperplane in $\mathbb{K}^n$ defined by the equation $z_i - z_j - m = 0$ where $1 \leqslant i < j \leqslant n$, $m \in \mathbb{F}_p$. Let $\bar{\mathcal{A}}^\circ$ be the arrangement in $\mathbb{K}^n$ of all hyperplanes $H_{i,j,m}$. Let $\bar{\mathcal{A}} = \mathbb{K}^n - \bar{\mathcal{A}}^\circ$ denote its complement. The qKZ operators are well-defined over $\bar{\mathcal{A}}$ and are nondegenerate.

Assume that $0 < d(\kappa) < n - 1$. Then the p-hypergeometric solutions $Q^{\ell p-1}(z;\kappa)$, $\ell = 1, \dots, d(\kappa)$, define flat sections of the qKZ connection which we call the p-hypergeometric sections.

Recall the minors $M_I(z;\kappa)$ defined for any $I = \{1 \leqslant i_1 < \dots < i_{d(\kappa)} \leqslant n\}$ in Section 5.2 with the help of these p-hypergeometric sections. Denote by $\mathcal{A}(\kappa)$ the Zariski open subset of $\bar{\mathcal{A}}$ consisting of points $b \in \bar{\mathcal{A}}$ such that at least one of the minors $M_I(z;\kappa)$ is nonzero at b.

For any point $b \in \mathcal{A}(\kappa)$, the vectors $Q^{\ell p-1}(b,\kappa)$, $\ell = 1, \dots, d(\kappa)$, are linearly independent and span a $d(\kappa)$-dimensional $\mathbb{K}$-vector subspace $\mathcal{S}(b,\kappa)$ of V. These subspaces $\mathcal{S}(b,\kappa)$, $b \in \mathcal{A}(\kappa)$, form a vector subbundle $\mathcal{S}(\kappa) \to \mathcal{A}(\kappa)$ of the trivial bundle $V \times \mathcal{A}(\kappa) \to \mathcal{A}(\kappa)$.

Remark. Notice that the minors $M_I(z;\kappa)$ are polynomials in z with coefficients in $\mathbb{F}_p$ and are independent of the field $\mathbb{K}$. Notice also that the base $\mathcal{A}(\kappa)$ is invariant with respect to the affine translations, $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_a - \kappa, \dots, z_n)$, $a = 1, \dots, n$.

The subbundle $\mathcal{S}(\kappa) \to \mathcal{A}(\kappa)$ is invariant under the qKZ connection, and the p-hypergeometric sections form a flat basis of the space of its sections.

We also consider the quotient bundle $\mathcal{Q}(\kappa) \to \mathcal{A}(\kappa)$ with fibers $V/\mathcal{S}(b,\kappa)$. The qKZ connection on $V \times \mathcal{A}(\kappa) \to \mathcal{A}(\kappa)$ induces a discrete connection on $\mathcal{Q}(\kappa) \to \mathcal{A}(\kappa)$ which we also call the qKZ connection. Notice that the rank of $\mathcal{Q}(\kappa) \to \mathcal{A}(\kappa)$ equals

$$\dim V - \operatorname{rank} \mathcal{S}(\kappa) = n - 1 - \left[\frac{nk}{p}\right] = \left[\frac{n(p-k)}{p}\right] = d(-\kappa).$$

If $d(\kappa) = 0$, we define $\mathcal{A}(\kappa) = \bar{\mathcal{A}}$. In this case, we define $\mathcal{S}(\kappa) \to \mathcal{A}(\kappa)$ to be the rank 0 subbundle of $V \times \mathcal{A}(\kappa)$ and also define $\mathcal{Q}(\kappa) \to \mathcal{A}(\kappa)$ to be $V \times \mathcal{A}(\kappa)$.

7.2 Subbundle of dual qKZ connection

Assume that $p > n$ and $\kappa \in \mathbb{F}_p^\times$. Consider the dual discrete qKZ connection on $V \times \mathbb{K}^n \to \mathbb{K}^n$ defined by formulas (2.11) and (2.7). The dual qKZ operators $K_a(-z_1, \dots, -z_n; -\kappa)$, $a = 1, \dots, n$, are well-defined over $\bar{\mathcal{A}}$ and are nondegenerate.

Assume that $0 < d(-\kappa) < n-1$. This assumption is equivalent to the assumption $0 < d(\kappa) < n - 1$. Consider the p-hypergeometric solutions $Q^{mp-1}(z;-\kappa)$, $m = 1, \dots, d(-\kappa)$, of the qKZ equations step $-\kappa$. Then the V-valued polynomials $Q^{mp-1}(-z;-\kappa)$, $m = 1, \dots, d(-\kappa)$, define flat sections of the dual qKZ connection which we also call the p-hypergeometric sections.

Recall the minors $M_I(z;-\kappa)$ defined for any $I=\{1\leqslant i_1<\cdots<i_{d(-\kappa)}\leqslant n\}$ in Section 5.2 with the help of the p-hypergeometric solutions $Q^{mp-1}(z;-\kappa)$, $m=1,\dots,d(-\kappa)$. Denote by $\mathcal{B}(\kappa)$ the Zariski open subset of $\bar{\mathcal{A}}$ consisting of points $b\in\bar{\mathcal{A}}$ such that at least one of the polynomials $M_I(-z;-\kappa)$ is nonzero at b.

For any point $b\in\mathcal{B}(\kappa)$, the vectors $Q^{mp-1}(-b,-\kappa)$, $m=1,\dots,d(-\kappa)$, are linearly independent and span a $d(-\kappa)$-dimensional $\mathbb{K}$-vector subspace $\mathcal{S}^*(b,-\kappa)$ of space V. These subspaces $\mathcal{S}^*(b,-\kappa)$, $b\in\mathcal{B}(\kappa)$, form a vector subbundle $\mathcal{S}^*(\kappa)\to\mathcal{B}(\kappa)$ of the trivial bundle $V\times\mathcal{B}(\kappa)\to\mathcal{B}(\kappa)$.

The subbundle $\mathcal{S}^*(\kappa)\to\mathcal{B}(\kappa)$ is invariant with respect to the dual qKZ connection, and the p-hypergeometric sections $Q^{mp-1}(-z;-\kappa)$, $m=1,\dots,d(-\kappa)$, form a flat basis of the space of its sections.

Consider the restriction of the bundles $\mathcal{S}^*(\kappa)\to\mathcal{B}(\kappa)$ and $\mathcal{Q}(\kappa)\to\mathcal{A}(\kappa)$ to $\mathcal{A}(\kappa)\cap\mathcal{B}(\kappa)$. For any $b\in\mathcal{A}(\kappa)\cap\mathcal{B}(\kappa)$, the Shapovalov form defines a nondegenerate pairing

$$S\colon\ \mathcal{S}^*(b,-\kappa)\otimes V/\mathcal{S}(b,\kappa)\to\mathbb{K}$$

of the fibers of these bundles, by Theorem 6.1. The discrete connections on $\mathcal{S}^*(\kappa)\to\mathcal{B}(\kappa)$ and $\mathcal{Q}(\kappa)\to\mathcal{A}(\kappa)$ are dual with respect to the Shapovalov form, that is, for any $u\in\mathcal{S}^*(b,-\kappa)$, $v\in V/\mathcal{S}(b,\kappa)$, and $a=1,\dots,n$, we have $S(u,v)=S(K_a(-b;-\kappa)u,K_a(b;\kappa)v)$.

Define p-quasi-hypergeometric sections $T^\ell(z;\kappa)$, $\ell=1,\dots,d(-\kappa)$, of the bundle $\mathcal{Q}(\kappa)\to\mathcal{A}(\kappa)$ over $\mathcal{A}(\kappa)\cap\mathcal{B}(\kappa)$ by the formulas

$$S\big(Q^{mp-1}(-b,-\kappa),T^\ell(b,\kappa)\big)=\delta_{\ell,m},\qquad m=1,\dots,d(-\kappa).$$

Lemma 7.1. *Assume that $p>n$ and $0<d(-\kappa)\leqslant n-1$. Then p-quasi-hypergeometric sections $T^\ell(z;\kappa)$, $\ell=1,\dots,d(-\kappa)$, of the quotient bundle $\mathcal{Q}(\kappa)\to\mathcal{A}(\kappa)\cap\mathcal{B}(\kappa)$ form a flat basis of the space of sections of that bundle.*

Proof. The proof is straightforward. ■

If $d(-\kappa)=0$, we define $\mathcal{B}(\kappa)=\bar{\mathcal{A}}$. We also define $\mathcal{S}^*(\kappa)\to\mathcal{B}(\kappa)$ to be the rank 0 subbundle of $V\times\mathcal{B}(\kappa)$.

Example 7.2. Let $n=2$. Then V is of dimension 1. For $p=5$, $k=\kappa=3$, we have $d(3)=1$, and the qKZ connection has a flat basis given by the p-hypergeometric solution

$$Q^4(z_1,z_2)=(-2z_1+2z_2+2,2z_1-2z_2-2).$$

For $p=5$. $k=\kappa=2$, we have $d(2)=0$, and the qKZ connection has a flat basis given by the p-quasi-hypergeometric solution

$$T^1(z_1,z_2,2)=\left(\frac{3}{2z_1-2z_2+2},\frac{3}{-2z_1+2z_2-2}\right).$$

7.3 Reduced p-curvature operators

Let $\hat{C}_a(z;\kappa)$, $a=1,\dots,n$, be the reduced p-curvature operators of the qKZ discrete connection on $V\times\mathbb{K}^n\to\mathbb{K}^n$, see (2.5).

Theorem 7.3. *If $p>n$, $\kappa\in\mathbb{F}_p^\times$, and $0<d(\kappa)<n-1$, then the span of the p-hypergeometric sections $Q^{\ell p-1}(z;\kappa)$, $\ell=1,\dots,d(\kappa)$, lies in the kernel of $\hat{C}_a(z,\kappa)$ and contains the image of $\hat{C}_a(z;\kappa)$ for every $a=1,\dots,n$.*

If $d(\kappa)=p-1$ or 0, then all reduced p-curvature operators equal zero.

Corollary 7.4. *We have* $\hat{C}_a(z;\kappa)\hat{C}_b(z;\kappa) = 0$, $a, b = 1, \dots, n$.

Proof of Theorem 7.3. The span lies in the kernel of $\hat{C}_a(z;\kappa)$ by formula (2.4).

The operator $\hat{C}_a(z;\kappa)$ annihilates the span, hence $\hat{C}_a(z;\kappa)$ induces a well-defined operator on the fibers of the quotient bundle $\mathcal{Q}(\kappa) \to \mathcal{A}(\kappa)$. This induced operator is the a-th reduced p-curvature operator of the qKZ connection on the quotient bundle. The quotient bundle has a flat basis of p-quasi-hypergeometric sections over the Zariski open subset $\mathcal{A}(\kappa) \cap \mathcal{B}(\kappa)$. Hence all reduced p-curvature operators of the qKZ connection on the quotient bundle are zero. Therefore the image of $\hat{C}_a(z;\kappa)$ is contained in the span.

If $d(\kappa) = p - 1$, then $\mathcal{S}(\kappa) \to \mathcal{A}(\kappa)$ coincides with $V \times \mathcal{A}(\kappa) \to \mathcal{A}(\kappa)$, and all reduced p-curvature operators are zero by formula (2.4).

If $d(\kappa) = 0$, then p-quasi-hypergeometric sections form a flat basis of the space of sections of $V \times \mathcal{A}(\kappa) \to \mathcal{A}(\kappa)$, and again all reduced p-curvature operators are zero by formula (2.4). ■

Lemma 7.5. *If* $p > n$, $\kappa \in \mathbb{F}_p^\times$, *and* $0 < d(\kappa) < n - 1$, *then every reduced* p*-curvature operator* $\tilde{C}_a(z;\kappa)$ *is nonzero.*

Proof. Consider a normalized p-curvature operator $\tilde{C}_a^{\mathrm{KZ}}(z,\kappa)$ of the associated differential KZ equations. In a basis of V, the entries of the matrix of the operator $\tilde{C}_a^{\mathrm{KZ}}(z,\kappa)$ are homogeneous polynomials in z of degree $(n-2)p$. By Corollary 2.2, in the same basis, the entries of the matrix of the operator $\tilde{C}_a(z,\kappa)$ are polynomials in z of degree $(n-2)p$ whose top-degree parts equal the corresponding entries of the matrix of the operator $\tilde{C}_a^{\mathrm{KZ}}(z,\kappa)$.

It is proved in [8, Theorem 1.13] that if $p > n$, $\kappa \in \mathbb{F}_p^\times$, then every reduced p-curvature operator $\tilde{C}_a^{\mathrm{KZ}}(z,\kappa)$ is a (nonzero) operator of rank 1. Hence every reduced p-curvature operator $\tilde{C}_a(z,\kappa)$ is a nonzero operator. ■

Example 7.6. For $n = 3$, we have $\dim V = 2$. Let $p > 3$ and $d(\kappa) = 1$. Then $d(-\kappa) = 1$. For $a = 1, 2, 3$, the kernel of the reduced p-curvature operator $\hat{C}_a(z,\kappa)$ is generated by $Q^{p-1}(z,\kappa)$ and the image of $\hat{C}_a(z,\kappa)$ is generated by $Q^{p-1}(z,\kappa)$. Such an operator is determined uniquely up to multiplication by a scalar rational function in z.

For an operator $F\colon V \to V$, denote by $F^*\colon V \to V$ the operator dual to F under the Shapovalov form, $S(F^*x, y) = S(x, Fy)$.

Lemma 7.7. *We have*

$$\hat{C}_a(z,-\kappa) = -\hat{C}_a(-z;\kappa)^*. \tag{7.1}$$

Proof. We have $S(x,y) = S(C_a(-z;-\kappa)x, C_a(z;\kappa)y)$ by formulas (2.10) and (2.7). Hence $C_a(-z;-\kappa) = \big(C_a(z;\kappa)^{-1}\big)^*$. We also have $(C_a(z;\kappa) - 1)^2 = 0$ by Corollary 7.4. Then

$$C_a(z;\kappa)^{-1} = (1 + (C_a(z;\kappa) - 1))^{-1} = 1 - (C_a(z;\kappa) - 1)$$

and $C_a(-z;-\kappa) - 1 = 1 - C_a(z;\kappa)^*$. ■

Corollary 7.8. *The normalized* p*-curvature operators satisfy the equation*

$$\tilde{C}_a(z;-\kappa) = (-1)^n \tilde{C}_a(-z;\kappa)^*. \tag{7.2}$$

Proof. Formula (7.2) follows from equation (7.1) and the following formulas:

$$\tilde{C}_a(z;\kappa) = (C_a(z;\kappa) - 1)\prod_{j\neq i}(z_i - z_j, \kappa)_p,$$

$$\tilde{C}_a(-z;-\kappa) = (C_a(-z;-\kappa) - 1)\prod_{j\neq i}(-z_i + z_j, -\kappa)_p.$$ ■

7.4 All solutions of qKZ equations for $\kappa \in \mathbb{F}_p^\times$

Theorem 7.9. *Let $p > n$, $\kappa \in \mathbb{F}_p^\times$, and $0 < d(\kappa) < n-1$. Let $f(z)$ be a V-valued rational function in z which is a solution of the qKZ equations with step κ. Then $f(z)$ is a linear combination of the p-hypergeometric solutions $Q^{\ell p-1}(z;\kappa)$, $\ell = 1,\dots,d(\kappa)$, with coefficients which are rational functions in $z_i^p - z_i$, $i = 1,\dots,n$.*

Recall that if $d(\kappa) = n-1$, then the qKZ connection has a basis of flat sections given by the p-hypergeometric sections by Theorem 5.6, and if $d(\kappa) = 0$, then the qKZ connection has a basis of flat sections given by the p-quasi-hypergeometric solutions, by Lemma 7.1.

Proof. For $a = 1,\dots,n$, consider the normalized p-curvature operators $\tilde C_a^{\rm KZ}(z,\kappa)$ and $\tilde C_a(z,\kappa)$. Both of these operators are polynomials in z, and the polynomial $\tilde C_a^{\rm KZ}(z,\kappa)$ is the top-degree part of the polynomial $\tilde C_a(z,\kappa)$. The polynomial $\tilde C_a^{\rm KZ}(z,\kappa)$ is nonzero by [8, Theorem 1.13] and hence $\tilde C_a(z,\kappa)$ is a nonzero operator.

It was proved in [8, Theorem 1.8] that if $p > n$, $\kappa \in \mathbb{F}_p^\times$, and $0 < d(\kappa) < n-1$, then all solutions of the KZ equations are linear combinations of the p-hypergeometric solutions. Hence the intersection of kernels of the operators $\tilde C_a^{\rm KZ}(z,\kappa)$, $a = 1,\dots,n$, is of dimension $d(\kappa)$ for generic z, and the span of images of the operators $\tilde C_a^{\rm KZ}(z,\kappa)$, $a = 1,\dots,n$, is of dimension $n-1-d(\kappa)$ for generic z. Therefore, the span of images of the operators $\tilde C_a(z,\kappa)$, $a = 1,\dots,n$, has dimension at least $n-1-d(\kappa)$ for generic z. This implies that the span of values of flat sections of the qKZ connection is of dimension not larger than $d(\kappa)$ for generic z. But we have $d(\kappa)$ flat linear independent p-hypergeometric sections $Q^{\ell p-1}(z;\kappa)$, $\ell = 1,\dots,d(\kappa)$. Hence any flat section of the qKZ connection is a linear combination of the p-hypergeometric sections with 1-periodic coefficients. ■

Corollary 7.10. *Let $p > n$, $\kappa \in \mathbb{F}_p^\times$, and $0 < d(\kappa) < n-1$. Let $f(z)$ and $g(z)$ be V-valued rational functions in z where $f(z)$ is a solution the qKZ equations with step κ and $g(z)$ is a solution the qKZ equations with step $-\kappa$. Then*

$$S(g(-z), f(z)) = 0. \tag{7.3}$$

Formula (7.3) follows from Theorems 7.9 and 6.1.

7.5 qKZ connection with $\kappa \in \mathbb{K} \setminus \mathbb{F}_p$

Lemma 7.11. *Let $p > n$ and $\kappa \in \mathbb{K}\setminus\mathbb{F}_p$. Then all the normalized p-curvature operators $\tilde C_a(z;\kappa)$, $a = 1,\dots,n$, are nondegenerate for generic z.*

Proof. Formula (3.19) in [8] describes the spectrum of the p-curvature operators $C_a^{\rm KZ}(z;\kappa)$ of differential KZ equations. The formula shows that all p-curvature operators $C_a^{\rm KZ}(z;\kappa)$ are nondegenerate for generic z. In a basis of V, the matrices of $\tilde C_a^{\rm KZ}(z;\kappa)$ are homogeneous polynomials in z of degree $(n-2)p$. Hence their determinants are nonzero homogeneous polynomials in z. By Corollary 2.2, the determinants of the normalized p-curvature operators $\tilde C_a(z;\kappa)$ are nonzero polynomials. The lemma follows. ■

Corollary 7.12. *For $p > n$ and $\kappa \in \mathbb{K} \setminus \mathbb{F}_p$, there does not exist a nonzero rational V-valued function $f(z)$ which is a flat section of the qKZ connection with parameter κ.*

Proof. If $f(z)$ is a flat section, then it lies in the kernel of every normalized p-curvature operator $\tilde C_a(z;\kappa)$. That contradicts to Lemma 7.11. ■

Acknowledgements

E.M. was supported in part by Simons Foundation grant 709444. A.V. was supported in part by Simons Foundation grant TSM-00012774. The authors thank P. Etingof, R. Rimanyi, and V. Vologodsky for useful discussions.

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