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Contraction of the $\mathfrak{sl}_2$-Triple Associated to the (k,a)-Generalized Fourier Transform

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Abstract. Ben Saïd–Kobayashi–Ørsted introduced a family of $\mathfrak{sl}_2$-triples of differential-difference operators $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$ and $\mathbb{E}^-_{k,a}$ on $\mathbb{R}^N \setminus \{0\}$ indexed by a Dunkl parameter k and a deformation parameter $a \neq 0$. In the present paper, we study the behavior as the parameter a approaches 0. In this limit, the Lie algebra $\mathfrak{g}_{k,a} = \operatorname{span}_{\mathbb{R}}\{\mathbb{H}_{k,a}, \mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}\} \cong \mathfrak{sl}(2,\mathbb{R})$ contracts to a three-dimensional commutative Lie algebra $\mathfrak{g}_{k,0}$, and its spectral properties change. We describe the joint spectral decomposition for $\mathfrak{g}_{k,0}$, and discuss formulas for operator semigroups with infinitesimal generators in $\mathfrak{g}_{k,0}$. In particular, we describe the integral kernel of $\exp\bigl(z|x|^2\Delta_k\bigr)$ as an infinite series, which, in some low-dimensional cases, can be expressed in a closed form using the theta function.

Key words: (k,a)-generalized Fourier transform; Dunkl operators; group contraction; spectral decomposition; integral kernel

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1 Introduction

1.1 Background

A minimal representation is an infinite-dimensional irreducible representation of a simple Lie group with the smallest Gelfand–Kirillov dimension. However, at the same time, it can be thought of as a manifestation of large symmetry of the space acted on by the group, and hence, it is expected to control global analysis on the space effectively. This is the idea of *global analysis of minimal representations* initiated by T. Kobayashi [19, 20], which led a transition from algebraic representation theory to analytic representation theory. See also [15, Section VII] for an excellent survey.

From the viewpoint of global analysis of minimal representations, the classical Fourier transform on the Euclidean space $\mathbb{R}^N$ can be interpreted as a unitary inversion operator in the Weil representation, which is a unitary representation of the metaplectic group $\mathrm{Mp}(N,\mathbb{R})$ on the Hilbert space $L^2\bigl(\mathbb{R}^N\bigr)$ (see [14] for more details) and decomposes into two irreducible components, each of which is a minimal representation. Promoting this interpretation, Kobayashi–Mano [21, 22, 23, 24] introduced the *Fourier transform on the light cone* as a unitary inversion operator in an L^2-model of a minimal representation of $\mathrm{O}(p,q)$ and developed a new theory of harmonic analysis. The special case $(p,q) = (N+1,2)$, where the model Hilbert space is isomorphic to $L^2\bigl(\mathbb{R}^N, |x|^{-1}\mathrm{d}x\bigr)$, is studied in [21, 23].

After that, Ben Saïd–Kobayashi–Ørsted [3, 4] introduced a family of $\mathfrak{sl}_2$-triples of differential-difference operators $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$, $\mathbb{E}^-_{k,a}$ on $\mathbb{R}^N \setminus \{0\}$ indexed by two parameters k and a, and defined the (k,a)-*generalized Laguerre semigroup*

$$\mathscr{I}_{k,a}(z) = \exp\Bigl(\frac{z}{\mathrm{i}}\bigl(\mathbb{E}^-_{k,a} - \mathbb{E}^+_{k,a}\bigr)\Bigr), \qquad \operatorname{Re} z \geq 0$$

and the (k,a)-*generalized Fourier transform* $\mathscr{F}_{k,a} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}\frac{2\langle k\rangle+a+N-2}{a}}\mathscr{I}_{k,a}\big(\frac{\mathrm{i}\pi}{2}\big)$. Here, k is a combinatorial parameter derived from the Dunkl operators, and $a > 0$ is a deformation parameter. The (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ includes some known transforms:

- The $(0,2)$-generalized Fourier transform $\mathscr{F}_{0,2}$ is the classical Fourier transform.
- The $(0,1)$-generalized Fourier transform $\mathscr{F}_{0,1}$ is the Hankel transform, or the Fourier transform on the light cone for $(p,q) = (N+1,2)$.
- The $(k,2)$-generalized Fourier transform $\mathscr{F}_{k,2}$ is the Dunkl transform [13].

The parameter a therefore provides a continuous interpolation between the two minimal representations of the simple Lie groups $\mathrm{Mp}(N,\mathbb{R})$ and $\mathrm{O}(N+1,2)$.

1.2 Results of the paper

Let $\mathfrak{g}_{k,a} = \mathrm{span}_{\mathbb{R}}\{\mathbb{H}_{k,a}, \mathbb{E}^{+}_{k,a}, \mathbb{E}^{-}_{k,a}\} \cong \mathfrak{sl}(2,\mathbb{R})$. Ben Saïd–Kobayashi–Ørsted [4, Theorems 3.30 and 3.31] showed that, for $a > 0$, the action of $\mathfrak{g}_{k,a}$ on $L^2(\mathbb{R}^N, w_{k,a}(x)\mathrm{d}x)$ (see (3.3) for the definition of $w_{k,a}$) lifts to a unique unitary representation of the universal covering Lie group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$ and found its irreducible decomposition explicitly; the Hilbert space $L^2\big(\mathbb{R}^N, w_{k,a}(x)\mathrm{d}x\big)$ decomposes discretely with finite multiplicities into relatively discrete series representations of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Furthermore, we investigated in [17] the case $a < 0$, which provided an extension of the parameter a.

In the present paper, we study the behavior as $a \to 0$. Although the operators $\mathbb{H}_{k,a}$, $\mathbb{E}^{+}_{k,a}$ or $\mathbb{E}^{-}_{k,a}$ are not well-defined for $a = 0$, the Lie algebra $\mathfrak{g}_{k,a} \cong \mathfrak{sl}(2,\mathbb{R})$ contracts to a three-dimensional commutative Lie algebra $\mathfrak{g}_{k,0} \cong \mathbb{R}^3$ as $a \to 0$. Such a contraction of Lie algebras (or corresponding Lie groups) was earlier formalized by Inonu–Wigner [18], where it is referred to as a *contraction of groups*. Classical examples include the contraction from the orthogonal group $\mathrm{O}(3)$ (resp. $\mathrm{O}(2,1)$) to the Euclidean motion group $\mathrm{O}(2) \ltimes \mathbb{R}^2$, which reflects the geometric phenomenon that the sphere of curvature $\kappa > 0$ (resp. the hyperbolic plane of curvature $\kappa < 0$) approaches the flat Euclidean plane as $\kappa \to 0$.

We then consider the action of $\mathfrak{g}_{k,0}$ on $L^2\big(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\big)$ (note that the weight function $w_{k,a}$ is well-defined even for $a = 0$). As an analog of the result in the case $a \neq 0$, we describe that the joint spectral decomposition for the operators in $\mathfrak{g}_{k,0}$ on $L^2\big(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\big)$ (Theorem 3.4) and show that it lifts to a unique unitary representation of $\mathbb{R}^3$ (Theorem 3.6). This is the main result of the paper. In contrast to the case $a \neq 0$, this spectral decomposition involves only the continuous spectrum.

Moreover, we discuss formulas for operator semigroups with infinitesimal generators in $\mathfrak{g}_{k,0}$ (see Theorems 3.8 and 3.13). In particular, we describe the integral kernel of $\exp\big(z|x|^2\Delta_k\big)$ as an infinite series, which, in some low-dimensional cases, can be expressed in a closed form using the theta function (Theorems 4.1, 4.2 and 4.3). Although the (k,a)-generalized Laguerre semigroup and the (k,a)-generalized Fourier transform are not well-defined for $a = 0$, the operator semigroup $\big(\mathrm{e}^{-z}\exp\big(z|x|^2\Delta_k\big)\big)_{\mathrm{Re}\, z \geq 0}$ may be viewed as the "renormalized" (k,a)-generalized Laguerre semigroup for $a = 0$ (Theorem 3.15). Note that explicit formulas and estimates for the integral kernels of the (k,a)-generalized Laguerre semigroup and the (k,a)-generalized Fourier transform have been extensively studied in Ben Saïd–Kobayashi–Ørsted [4, Sections 4.3–4.5 and 5.2–5.4] and subsequent papers [2, 6, 7, 8, 16, 27] up to the present. There are also unpublished results by Mano and related results by Demni [9].

Thus, this paper analyzes representation-theoretic aspects of contraction of Lie algebras in the framework of (k,a)-generalized Fourier analysis. We note that, recently, Benoist–Kobayashi [5, Theorem 1.2] discovered a relationship between *limit algebras* (see Section 1.4 of their paper) of $\mathfrak{h} = \mathrm{Lie}(H)$ in $\mathfrak{g} = \mathrm{Lie}(G)$ and L^2-analysis of G/H in the context of tempered unitary representations. It can be viewed as an application of the notion of contraction of Lie algebras to representation theory.

1.3 Organization of the paper

In Section 2, we will briefly review Dunkl theory and the differential-difference operators $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$ and $\mathbb{E}^-_{k,a}$ introduced by Ben Saïd–Kobayashi–Ørsted. In Section 3, we will discuss the contraction of the $\mathfrak{sl}_2$-triple as $a \to 0$. In Section 4, we will give a closed-form expression for the integral kernel of $\exp\bigl(z|x|^2\Delta_k\bigr)$ in some low-dimensional cases.

1.4 Notation

- $\mathbb{N} = \{0, 1, 2, \dots\}$.
- We write $\langle -, - \rangle$ for the Euclidean inner product, and $|-|$ for the Euclidean norm.
- $\mathbb{S}^{N-1} = \bigl\{x \in \mathbb{R}^N \mid |x| = 1\bigr\}$.
- Function spaces, such as C^∞ spaces and L^2 spaces, are understood to consist of complex-valued functions.
- We write $E_x = \sum_{j=1}^N x_j \frac{\partial}{\partial x_j}$ for the Euler operator on $\mathbb{R}^N$, and $E_r = r\frac{\mathrm{d}}{\mathrm{d}r}$ for the Euler operator on $\mathbb{R}_{>0}$.

2 Preliminaries

In this section, we review Dunkl theory and the differential-difference operators $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$ and $\mathbb{E}^-_{k,a}$ introduced by Ben Saïd–Kobayashi–Ørsted to the extent necessary for later use. This section contains no new results.

2.1 The Dunkl Laplacian

Throughout this paper, we fix a reduced root system $\mathscr{R}$ on $\mathbb{R}^N$. That is, we suppose that $\mathscr{R}$ satisfies the following conditions:

- $\mathscr{R}$ is a finite subset of $\mathbb{R}^N \setminus \{0\}$,
- $\mathscr{R}$ is stable under the orthogonal reflection r_α with respect to the hyperplane $(\mathbb{R}\alpha)^\perp$ for all $\alpha \in \mathscr{R}$, and
- $\mathscr{R} \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \mathscr{R}$.

Note that we do not impose crystallographic conditions on roots and do not require that $\mathscr{R}$ spans $\mathbb{R}^N$.

The subgroup of $\mathrm{O}(N)$ generated by all the reflections r_α is called the *reflection group associated with* $\mathscr{R}$. We say that a function $k\colon \mathscr{R} \to \mathbb{C}$ is a *multiplicity function* if it is invariant under the natural action of the reflection group. We usually write k_α instead of $k(\alpha)$. We say that a multiplicity function k is *non-negative* if $k_\alpha \geq 0$ for all $\alpha \in \mathscr{R}$. The *index* of a multiplicity function k is defined as

$$\langle k \rangle = \frac{1}{2}\sum_{\alpha \in \mathscr{R}} k_\alpha = \sum_{\alpha \in \mathscr{R}^+} k_\alpha,$$

where $\mathscr{R}^+$ is any positive system of $\mathscr{R}$.

For a (not necessarily non-negative) multiplicity function k, the *Dunkl Laplacian* Δ_k (see [10] and [11, Definition 1.1]) is defined by

$$\Delta_k F(x) = \Delta F(x) + \sum_{\alpha \in \mathscr{R}^+} k_\alpha \left(\frac{2\langle \alpha, \nabla F(x) \rangle}{\langle \alpha, x \rangle} - |\alpha|^2 \frac{F(x) - F(r_\alpha(x))}{\langle \alpha, x \rangle^2} \right),$$

where $\Delta = \sum_{j=1}^N \bigl(\frac{\partial}{\partial x_j}\bigr)^2$ is the classical Laplacian and $\nabla = \bigl(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}\bigr)$ is the classical gradient operator. When $k = 0$, the Dunkl Laplacian Δ_k reduces to the classical Laplacian Δ.

Let $\mathcal{P}(\mathbb{R}^N)$ denote the space of polynomials on $\mathbb{R}^N$ and $\mathcal{P}^m(\mathbb{R}^N)$ denote its subspace of homogeneous polynomials of degree m. The space of *k-harmonic polynomials of degree m* (see [10, Definition 1.5]) is defined as

$$\mathcal{H}_k^m(\mathbb{R}^N) = \{p \in \mathcal{P}^m(\mathbb{R}^N) \mid \Delta_k p = 0\},$$

and the space of *k-spherical harmonics of degree m* is defined as

$$\mathcal{H}_k^m(\mathbb{S}^{N-1}) = \{p|_{\mathbb{S}^{N-1}} \mid p \in \mathcal{H}_k^m(\mathbb{R}^N)\}.$$

When $k = 0$, these are reduced to the space $\mathcal{H}^m(\mathbb{R}^N)$ of classical harmonic polynomials and the space $\mathcal{H}^m(\mathbb{S}^{N-1})$ of classical spherical harmonics, respectively.

We write $\mathcal{H}(\mathbb{S}^{N-1}) = \{p|_{\mathbb{S}^{N-1}} \mid p \in \mathcal{P}(\mathbb{R}^N)\}$. The following fact is a generalization of the decomposition of $\mathcal{H}(\mathbb{S}^{N-1})$ and $L^2(\mathbb{S}^{N-1})$ into the spaces $\mathcal{H}^m(\mathbb{S}^{N-1})$ of classical spherical harmonics.

Fact 2.1 ([10, pp. 37–39])**.** *For a non-negative multiplicity function k, we have the direct sum decomposition*

$$\mathcal{H}(\mathbb{S}^{N-1}) = \bigoplus_{m\in\mathbb{N}} \mathcal{H}_k^m(\mathbb{S}^{N-1})$$

and the orthogonal decomposition

$$L^2(\mathbb{S}^{N-1}, w_k(\omega)\mathrm{d}\omega) = \sum_{m\in\mathbb{N}}^{\oplus} \mathcal{H}_k^m(\mathbb{S}^{N-1}),$$

where the weight function w_k with respect to the standard measure $\mathrm{d}\omega$ on $\mathbb{S}^{N-1}$ is defined by

$$w_k(\omega) = \prod_{\alpha\in\mathscr{R}} |\langle\alpha,\omega\rangle|^{k_\alpha} = \prod_{\alpha\in\mathscr{R}^+} |\langle\alpha,\omega\rangle|^{2k_\alpha}.$$

2.2 Dunkl's intertwining operator and the Poisson kernel

For a non-negative multiplicity function k, Dunkl introduced a linear automorphism V_k of the space $\mathcal{P}(\mathbb{R}^N)$ of polynomials that satisfies a certain intertwining property (*Dunkl's intertwining operator*). See [12, Definition 2.2 and Theorem 2.3] for the definition and a characterization of V_k. We note that $V_0 = \mathrm{id}_{\mathcal{P}(\mathbb{R}^N)}$.

The following integral representation of Dunkl's intertwining operator V_k was obtained by Rösler.

Fact 2.2 ([26, Theorem 1.2])**.** *Let k be a non-negative multiplicity function. For each $x \in \mathbb{R}^N$, there exists a unique probability Borel measure $\mu_{k,x}$ on $\mathbb{R}^N$ such that*

$$V_k p(x) = \int_{\mathbb{R}^N} p(\xi)\,\mathrm{d}\mu_{k,x}(\xi)$$

for all $p \in \mathcal{P}(\mathbb{R}^N)$. Moreover, the support of $\mu_{k,x}$ is contained in the ball $\{\xi \in \mathbb{R}^N \mid |\xi| \le |x|\}$, and we have $\mu_{k,x}(S) = \mu_{k,gx}(gS) = \mu_{k,rx}(rS)$ for any element g of the reflection group, $r > 0$, and Borel set $S \subseteq \mathbb{R}^N$.

Let k be a non-negative multiplicity function and $m \in \mathbb{N}$. We consider the orthogonal projection $\Pi_k^{(m)}$ from $L^2(\mathbb{S}^{N-1}, w_k(\omega)\mathrm{d}\omega)$ onto $\mathcal{H}_k^m(\mathbb{S}^{N-1})$ and its normalized integral kernel $P_k^{(m)}$, which is called the *Poisson kernel* of the space of k-spherical harmonics of degree m. That is, the function $P_k^{(m)}$ on $\mathbb{S}^{N-1}\times\mathbb{S}^{N-1}$ is characterized by the formula

$$\Pi_k^{(m)}p(\omega)=\frac{1}{\operatorname{vol}_k\left(\mathbb{S}^{N-1}\right)}\int_{\mathbb{S}^{N-1}}P_k^{(m)}(\omega,\omega')p(\omega')w_k(\omega')\,\mathrm{d}\omega' \tag{2.1}$$

for all $p\in L^2\left(\mathbb{S}^{N-1},w_k(\omega)\mathrm{d}\omega\right)$, where

$$\operatorname{vol}_k\left(\mathbb{S}^{N-1}\right)=\int_{\mathbb{S}^{N-1}}\mathrm{d}w_k(\omega). \tag{2.2}$$

Equivalently, the Poisson kernel $P_k^{(m)}$ is given by

$$P_k^{(m)}(\omega,\omega')=\operatorname{vol}_k\left(\mathbb{S}^{N-1}\right)\sum_{j=1}^{d}p_j(\omega)p_j(\omega'),$$

where $(p_1,\dots,p_d)$ is an orthonormal basis of $\mathcal{H}_k^m\left(\mathbb{S}^{N-1}\right)$, regarded as a subspace of $L^2\big(\mathbb{S}^{N-1}$, $w_k(\omega)\mathrm{d}\omega\big)$.

The Poisson kernel $P_k^{(m)}$ can be expressed in terms of Dunkl's intertwining operator and the Gegenbauer polynomials. To state this result, we first prepare some notation. For $\nu\in\mathbb{C}$ and $m\in\mathbb{N}$, we consider the Gegenbauer polynomial C_m^ν defined by the generating function

$$\left(1-2t\xi+\xi^2\right)^{-\nu}=\sum_{m=0}^{\infty}C_m^\nu(t)\xi^m, \tag{2.3}$$

and the renormalized Gegenbauer polynomial $\widetilde{C}_m^\nu$ defined by

$$\widetilde{C}_m^\nu(t)=\frac{m+\nu}{\nu}C_m^\nu(t). \tag{2.4}$$

For $\nu=0$, we define $\widetilde{C}_m^0$ by the limit formula (see [1, equation (6.4.13)])

$$\widetilde{C}_m^0(t)=\lim_{\nu\to0}\widetilde{C}_m^\nu(t)=\begin{cases}1, & m=0,\\ 2T_m(t), & m\geq1,\end{cases} \tag{2.5}$$

where T_m denotes the Chebyshev polynomial of the first kind, which is characterized by the formula $T_m(\cos\theta)=\cos m\theta$.

Fact 2.3 ([12, Theorem 4.1])**.** *Let k be a non-negative multiplicity function and $m\in\mathbb{N}$. The Poisson kernel $P_k^{(m)}$ is given by*

$$P_k^{(m)}(\omega,\omega')=V_k\left(\widetilde{C}_m^{\frac{2\langle k\rangle+N-2}{2}}\left(\langle -,\omega'\rangle\right)\right)(\omega).$$

Remark 2.4. When $k=0$, we have $V_0=\mathrm{id}_{\mathcal{P}(\mathbb{R}^N)}$, so that the formula in Theorem 2.3 reduces to

$$P_0^{(m)}(\omega,\omega')=\widetilde{C}_m^{\frac{N-2}{2}}\left(\langle\omega,\omega'\rangle\right).$$

See [1, Theorem 9.6.3 and Remark 9.6.1].

Remark 2.5. In the case $N=1$, we have $\mathbb{S}^0=\{\pm1\}$ and the formula in Theorem 2.3 reduces to

$$P_k^{(m)}(\omega,\omega')=\begin{cases}1, & m=0,\\ \operatorname{sgn}(\omega\omega'), & m=1,\\ 0, & m\geq2,\end{cases}$$

which corresponds to the fact that

$$\mathcal{H}_k^m(\mathbb{S}^0) = \begin{cases} \mathbb{C}1, & m = 0, \\ \mathbb{C}\,\mathrm{sgn}, & m = 1, \\ 0, & m \geq 2. \end{cases}$$

Here, 1 and sgn denote the constant function and the sign function on $\mathbb{S}^0 = \{\pm 1\}$, respectively.

2.3 The differential-difference operators $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$ and $\mathbb{E}^-_{k,a}$

Let k be a multiplicity function and $a \in \mathbb{C} \setminus \{0\}$. We recall the definition of the differential-difference operators $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$ and $\mathbb{E}^-_{k,a}$ on $\mathbb{R}^N \setminus \{0\}$ from [4, equation (3.3)]:

$$\mathbb{H}_{k,a} = \frac{2}{a}E_x + \frac{2\langle k\rangle + a + N - 2}{a}, \qquad \mathbb{E}^+_{k,a} = \frac{\mathrm{i}}{a}|x|^a, \qquad \mathbb{E}^-_{k,a} = \frac{\mathrm{i}}{a}|x|^{2-a}\Delta_k.$$

Additionally, for $m \in \mathbb{N}$, we consider the following differential operators on $\mathbb{R}_{>0}$:

$$\mathbb{H}^{(m)}_{k,a} = \frac{2}{a}E_r + \frac{2\langle k\rangle + a + N - 2}{a}, \qquad \mathbb{E}^{+\,(m)}_{k,a} = \frac{\mathrm{i}}{a}r^a,$$
$$\mathbb{E}^{-\,(m)}_{k,a} = \frac{\mathrm{i}}{a}r^{-a}(E_r - m)(E_r + m + 2\langle k\rangle + N - 2).$$

These are the radial parts of $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$ and $\mathbb{E}^-_{k,a}$ respectively in the following sense.

Proposition 2.6. *Let k be a multiplicity function, $a \in \mathbb{C} \setminus \{0\}$, and $m \in \mathbb{N}$. For $p \in \mathcal{H}_k^m(\mathbb{S}^{N-1})$ and $f \in C^\infty(\mathbb{R}_{>0})$, we have*

$$\mathbb{H}_{k,a}(p \otimes f) = p \otimes \mathbb{H}^{(m)}_{k,a} f, \qquad \mathbb{E}^+_{k,a}(p \otimes f) = p \otimes \mathbb{E}^{+\,(m)}_{k,a} f,$$
$$\mathbb{E}^-_{k,a}(p \otimes f) = p \otimes \mathbb{E}^{-\,(m)}_{k,a} f,$$

where $p \otimes f$ denotes the function $r\omega \mapsto p(\omega)f(r)$ on $\mathbb{R}^N \setminus \{0\}$.

Proof. The first and second equations are clear. We now prove the third equation. We use the polar coordinates $x = r\omega$, where $r \in \mathbb{R}_{>0}$ and $\omega \in \mathbb{S}^{N-1}$. We extend $p \in \mathcal{H}_k^m(\mathbb{S}^{N-1})$ to a k-harmonic polynomial of degree m on $\mathbb{R}^N$, which we again write p. Then, since $\Delta_k p = 0$, we have

$$\Delta_k(p \otimes f)(x) = \Delta_k(r^{-m}f(r)p(x)) = [\Delta_k, r^{-m}f(r)]p(x), \tag{2.6}$$

where $[-,-]$ denotes the commutator of operators. For a radial function $g(r)$, the commutator $[\Delta_k, g(r)]$ can be computed by the Leibniz rule as

$$\begin{aligned}[\Delta_k, g(r)] &= \Delta(g(r)) + 2\langle\nabla(g(r)), \nabla\rangle + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \frac{2\langle\alpha, \nabla(g(r))\rangle}{\langle\alpha, x\rangle} \\ &= g''(r) + \frac{1}{r}g'(r)(2E_x + 2\langle k\rangle + N - 1) \\ &= r^{-2}\big(E_r^2 g(r) + E_r g(r)(2E_x + 2\langle k\rangle + N - 2)\big).\end{aligned}$$

Setting $g(r) = r^{-m}f(r)$ and applying this commutator to $p(x)$, we have

$$\begin{aligned}&[\Delta_k, r^{-m}f(r)]p(x) \\ &\quad = r^{-2}\big(E_r^2(r^{-m}f(r)) + E_r(r^{-m}f(r))(2E_x + 2\langle k\rangle + N - 2)\big)p(x) \\ &\quad = r^{-m-2}\big((E_r - m)^2 f(r) + (E_r - m)f(r)(2E_x + 2\langle k\rangle + N - 2)\big)p(x)\end{aligned}$$

$$\begin{aligned}&= r^{-m-2}\big((E_r-m)^2+(E_r-m)(2m+2\langle k\rangle+N-2)\big)f(r)p(x)\\&= r^{-2}(E_r-m)(E_r+m+2\langle k\rangle+N-2)f(r)p(\omega)\\&= \big(p\otimes r^{-2}(E_r-m)(E_r+m+2\langle k\rangle+N-2)f\big)(x).\end{aligned}\tag{2.7}$$

The third equation follows from (2.6) and (2.7). ∎

Proposition 2.7. *Let k be a multiplicity function and $a\in\mathbb{C}\setminus\{0\}$.*

(1) *The differential-difference operators $\mathbb{H}_{k,a}$, $\mathbb{E}^+_{k,a}$ and $\mathbb{E}^-_{k,a}$ form an $\mathfrak{sl}_2$-triple. That is,*

$$\big[\mathbb{H}_{k,a},\mathbb{E}^+_{k,a}\big]=2\mathbb{E}^+_{k,a},\qquad \big[\mathbb{H}_{k,a},\mathbb{E}^-_{k,a}\big]=-2\mathbb{E}^-_{k,a},\qquad \big[\mathbb{E}^+_{k,a},\mathbb{E}^-_{k,a}\big]=\mathbb{H}_{k,a}.$$

(2) *For any $m\in\mathbb{N}$, the differential operators $\mathbb{H}^{(m)}_{k,a}$, $\mathbb{E}^{+\,(m)}_{k,a}$ and $\mathbb{E}^{-\,(m)}_{k,a}$ form an $\mathfrak{sl}_2$-triple. That is,*

$$\big[\mathbb{H}^{(m)}_{k,a},\mathbb{E}^{+\,(m)}_{k,a}\big]=2\mathbb{E}^+_{k,a},\qquad \big[\mathbb{H}^{(m)}_{k,a},\mathbb{E}^{-\,(m)}_{k,a}\big]=-2\mathbb{E}^-_{k,a},\qquad \big[\mathbb{E}^{+\,(m)}_{k,a},\mathbb{E}^{-\,(m)}_{k,a}\big]=\mathbb{H}_{k,a}.$$

Proof. (1) It is [4, Theorem 3.2]. (2) It follows from (1) and Theorem 2.6. ∎

3 Contraction of the $\mathfrak{sl}_2$-triple as $a\to 0$

3.1 The commutative Lie algebras $\mathfrak{g}_{k,0}$ and $\mathfrak{g}^{\mathrm{rad}}_{k,0}$

For a multiplicity function k and $a\in\mathbb{C}\setminus\{0\}$, we write

$$\begin{aligned}\mathfrak{g}_{k,a}&=\mathrm{span}_{\mathbb{R}}\big\{\mathbb{H}_{k,a},\mathbb{E}^+_{k,a},\mathbb{E}^-_{k,a}\big\}=\mathrm{span}_{\mathbb{R}}\big\{a\mathbb{H}_{k,a},a\mathbb{E}^+_{k,a},a\mathbb{E}^-_{k,a}\big\}\\&=\mathrm{span}_{\mathbb{R}}\big\{2E_x+2\langle k\rangle+a+N-2,\mathrm{i}|x|^a,\mathrm{i}|x|^{2-a}\Delta_k\big\}.\end{aligned}$$

Putting $a=0$ in the above equation, we define

$$\mathfrak{g}_{k,0}=\mathrm{span}_{\mathbb{R}}\big\{2E_x+2\langle k\rangle+N-2,\mathrm{i},\mathrm{i}|x|^2\Delta_k\big\}.\tag{3.1}$$

Similarly, we write

$$\begin{aligned}\mathfrak{g}^{(m)}_{k,a}&=\mathrm{span}_{\mathbb{R}}\big\{\mathbb{H}^{(m)}_{k,a},\mathbb{E}^{+\,(m)}_{k,a},\mathbb{E}^{-\,(m)}_{k,a}\big\}=\mathrm{span}_{\mathbb{R}}\big\{a\mathbb{H}^{(m)}_{k,a},a\mathbb{E}^{+\,(m)}_{k,a},a\mathbb{E}^{-\,(m)}_{k,a}\big\}\\&=\mathrm{span}_{\mathbb{R}}\big\{2E_r+2\langle k\rangle+a+N-2,\mathrm{i}r^a,\mathrm{i}r^{-a}(E_r-m)(E_r+m+2\langle k\rangle+N-2)\big\}.\end{aligned}$$

and define

$$\begin{aligned}\mathfrak{g}^{\mathrm{rad}}_{k,0}&=\mathrm{span}_{\mathbb{R}}\{2E_r+2\langle k\rangle+N-2,\mathrm{i},\mathrm{i}(E_r-m)(E_r+m+2\langle k\rangle+N-2)\}\\&=\mathrm{span}_{\mathbb{R}}\big\{2E_r+2\langle k\rangle+N-2,\mathrm{i},\mathrm{i}\big(E_r^2+(2\langle k\rangle+N-2)E_r-m(m+2\langle k\rangle+N-2)\big)\big\}\\&=\mathrm{span}_{\mathbb{R}}\big\{2E_r+2\langle k\rangle+N-2,\mathrm{i},\mathrm{i}\big(E_r^2+(2\langle k\rangle+N-2)E_r\big)\big\}.\end{aligned}\tag{3.2}$$

Note that the right-hand side of the above definition does not depend on m, which justifies the notation $\mathfrak{g}^{\mathrm{rad}}_{k,0}$.

Proposition 3.1. *Let k be a multiplicity function. For $p\in\mathcal{H}^m_k\big(\mathbb{S}^{N-1}\big)$ and $f\in C^\infty(\mathbb{R}_{>0})$, we have*

$$\begin{aligned}&(2E_x+2\langle k\rangle+N-2)(p\otimes f)=p\otimes(2E_r+2\langle k\rangle+N-2)f,\\&\mathrm{i}(p\otimes f)=p\otimes\mathrm{i}f,\qquad \mathrm{i}|x|^2\Delta_k(p\otimes f)=p\otimes\mathrm{i}(E_r-m)(E_r+m+2\langle k\rangle+N-2)f,\end{aligned}$$

where $p\otimes f$ denotes the function $r\omega\mapsto p(\omega)f(r)$ on $\mathbb{R}^N\setminus\{0\}$.

Proof. The proof goes along the same lines as that of Theorem 2.6. ∎

Proposition 3.2. *Let k be a multiplicity function.*

(1) *The space $\mathfrak{g}_{k,0}$ of differential-difference operators on $\mathbb{R}^N \setminus \{0\}$ is a three-dimensional commutative Lie algebra.*
(2) *The space $\mathfrak{g}_{k,0}^{\mathrm{rad}}$ of differential operators on $\mathbb{R}_{>0}$ is a three-dimensional commutative Lie algebra.*

Proof. (1) By Theorem 2.7, we have

$$\big[a\mathbb{H}_{k,a}, a\mathbb{E}^+_{k,a}\big] = 2a\cdot a\mathbb{E}^+_{k,a}, \qquad \big[a\mathbb{H}_{k,a}, a\mathbb{E}^-_{k,a}\big] = -2a\cdot a\mathbb{E}^-_{k,a}, \qquad \big[a\mathbb{E}^+_{k,a}, a\mathbb{E}^-_{k,a}\big] = a\cdot a\mathbb{H}_{k,a}.$$

By taking the limit as $a \to 0$, we have

$$[2E_x + 2\langle k\rangle + N - 2, \mathrm{i}] = 0, \qquad \big[2E_x + 2\langle k\rangle + N - 2, \mathrm{i}|x|^2\Delta_k\big] = 0, \qquad \big[\mathrm{i}, \mathrm{i}|x|^2\Delta_k\big] = 0.$$

(It can also be shown by a direct computation.)

(2) It follows from (1) and Theorem 3.1. ∎

3.2 Joint spectral decomposition for $\mathfrak{g}_{k,0}$ and $\mathfrak{g}_{k,0}^{\mathrm{rad}}$

In the following, we consider a *non-negative* multiplicity function k. In the next two theorems, we use the unitary operator

$$U_{N,k}\colon\ L^2\big(\mathbb{R}_{>0}, r^{2\langle k\rangle+N-3}\mathrm{d}r\big) \to L^2(\mathbb{R}, \mathrm{d}s),$$
$$U_{N,k}f(s) = \mathrm{e}^{\frac{2\langle k\rangle+N-2}{2}s} f(\mathrm{e}^s), \qquad U_{N,k}^{-1}g(r) = r^{-\frac{2\langle k\rangle+N-2}{2}} g(\log r)$$

and the (classical) Fourier transform

$$\mathscr{F}\colon\ L^2(\mathbb{R}, \mathrm{d}s) \to L^2(\mathbb{R}, \mathrm{d}\sigma),$$
$$\mathscr{F}g(\sigma) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(s)\mathrm{e}^{-\mathrm{i}\sigma s}\,\mathrm{d}s, \qquad \mathscr{F}^{-1}h(s) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(s)\mathrm{e}^{\mathrm{i}\sigma s}\,\mathrm{d}s.$$

We recall some terminology related to operators on a Hilbert space. A densely defined operator T on a Hilbert space is called *self-adjoint* (resp. *skew-adjoint*) if its adjoint T^* is equal to T (resp. $\mathrm{i}T$), that is, these have the same domain and coincide on it. A closable operator T on a Hilbert space is called *essentially self-adjoint* (resp. *essentially skew-adjoint*) if its closure $\overline{T}$ is self-adjoint (resp. skew-adjoint).

Theorem 3.3. *Let k be a non-negative multiplicity function. Every differential operator in $\mathfrak{g}_{k,0}^{\mathrm{rad}}$ (see (3.2) for the definition) defined on the domain $C_{\mathrm{c}}^\infty(\mathbb{R}_{>0})$ is an essentially skew-adjoint operator on $L^2\big(\mathbb{R}_{>0}, r^{2\langle k\rangle+N-3}\mathrm{d}r\big)$. Moreover, via the unitary operator*

$$\mathscr{F}\circ U_{N,k}\colon\ L^2\big(\mathbb{R}_{>0}, r^{2\langle k\rangle+N-3}\mathrm{d}r\big) \to L^2(\mathbb{R}, \mathrm{d}\sigma),$$

the closures of

$$(2E_r + 2\langle k\rangle + N - 2)|_{C_{\mathrm{c}}^\infty(\mathbb{R}_{>0})}, \qquad \mathrm{i}\,\mathrm{id}_{C_{\mathrm{c}}^\infty(\mathbb{R}_{>0})},$$
$$\mathrm{i}(E_r - m)(E_r + m + 2\langle k\rangle + N - 2)|_{C_{\mathrm{c}}^\infty(\mathbb{R}_{>0})}$$

correspond to the multiplication operators

$$2\mathrm{i}\sigma, \qquad \mathrm{i}, \qquad -\mathrm{i}\left(\sigma^2 + \left(m + \frac{2\langle k\rangle + N - 2}{2}\right)^2\right),$$

respectively.

Proof. Via the unitary operator $U_{N,k}\colon L^2\big(\mathbb{R}_{>0}, r^{2\langle k\rangle+N-3}\mathrm{d}r\big) \to L^2(\mathbb{R}, \mathrm{d}s)$, the operator

$$\left(E_r + \frac{2\langle k\rangle + N - 2}{2}\right)\bigg|_{C_c^\infty(\mathbb{R})}$$

corresponds to $\frac{\mathrm{d}}{\mathrm{d}s}|_{C_c^\infty(\mathbb{R})}$. As is well-known, for any complex polynomial $P(\sigma)$ such that $P(\mathrm{i}\sigma)$ is real-valued, $P\big(\frac{\mathrm{d}}{\mathrm{d}s}\big)|_{C_c^\infty(\mathbb{R})}$ is an essentially self-adjoint operator on $L^2(\mathbb{R}, \mathrm{d}s)$ and its closure corresponds to the multiplication operator by the function $P(\mathrm{i}\sigma)$ via the Fourier transform $\mathscr{F}\colon L^2(\mathbb{R}, \mathrm{d}s) \to L^2(\mathbb{R}, \mathrm{d}\sigma)$. Since $2E_r + 2\langle k\rangle + N - 2$, i and $\mathrm{i}(E_r - m)(E_r + m + 2\langle k\rangle + N - 2)$ can be expressed as i times such polynomials of $E_r + \frac{2\langle k\rangle+N-2}{2}$, now the assertion follows. ■

We recall that the L^2-theory of the $\mathfrak{sl}_2$-triple $\big(\mathbb{H}_{k,a}, \mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}\big)$ was considered on the Hilbert space $L^2\big(\mathbb{R}^N, w_{k,a}(x)\mathrm{d}x\big)$, where the weight function $w_{k,a}\colon \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ (see [4, equation (1.2)], $\vartheta_{k,a}$ in their notation) is defined by

$$w_{k,a}(x) = |x|^{a-2} \prod_{\alpha\in\mathscr{R}} |\langle \alpha, x\rangle|^{k_\alpha} = |x|^{a-2} \prod_{\alpha\in\mathscr{R}^+} |\langle \alpha, x\rangle|^{2k_\alpha}. \tag{3.3}$$

By the polar decomposition $w_{k,a}(x)\mathrm{d}x = w_k(\omega)\mathrm{d}\omega \otimes r^{2\langle k\rangle+a+N-3}\mathrm{d}r$ (here, w_k is as defined in Theorem 2.1) and Theorem 2.1, we have the orthogonal decomposition

$$\begin{aligned} L^2\big(\mathbb{R}^N, w_{k,a}(x)\mathrm{d}x\big) &= L^2\big(\mathbb{S}^{N-1}, w_k(\omega)\mathrm{d}\omega\big) \mathbin{\widehat{\otimes}} L^2\big(\mathbb{R}_{>0}, r^{2\langle k\rangle+a+N-3}\mathrm{d}r\big) \\ &= \sum_{m\in\mathbb{N}}^{\oplus} \mathcal{H}^m_k\big(\mathbb{S}^{N-1}\big) \otimes L^2\big(\mathbb{R}_{>0}, r^{2\langle k\rangle+a+N-3}\mathrm{d}r\big). \end{aligned} \tag{3.4}$$

Note that $w_{k,a}$ is well-defined and the above orthogonal decomposition holds even for $a = 0$.

We now state the main result of this paper.

Theorem 3.4. *Let k be a non-negative multiplicity function. Every differential-difference operator in $\mathfrak{g}_{k,0}$ (see (3.1) for the definition) defined on the domain*

$$\mathcal{D} = \mathcal{H}\big(\mathbb{S}^{N-1}\big) \otimes C_c^\infty(\mathbb{R}_{>0}) = \operatorname{span}_{\mathbb{C}}\big\{p \otimes f \mid p \in \mathcal{H}\big(\mathbb{S}^{N-1}\big),\ f \in C_c^\infty(\mathbb{R}_{>0})\big\},$$

is an essentially skew-adjoint operator on $L^2\big(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\big)$. Moreover, via the unitary operator

$$\mathrm{id}_{L^2(\mathbb{S}^{N-1}, w_k(\omega)\mathrm{d}\omega)} \mathbin{\widehat{\otimes}} (\mathscr{F}\circ U_{N,k})\colon\ L^2\big(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\big) \to L^2\big(\mathbb{S}^{N-1}, w_k(\omega)\mathrm{d}\omega\big) \mathbin{\widehat{\otimes}} L^2(\mathbb{R}, \mathrm{d}\sigma),$$

the closures of

$$(2E_x + 2\langle k\rangle + N - 2)|_{\mathcal{D}}, \qquad \mathrm{i}\,\mathrm{id}_{\mathcal{D}}, \qquad \big(\mathrm{i}|x|^2\Delta_k\big)|_{\mathcal{D}}$$

correspond to the multiplication operators

$$\mathrm{id}_{L^2(\mathbb{S}^{N-1}, w_k(\omega)\mathrm{d}\omega)} \mathbin{\widehat{\otimes}} 2\mathrm{i}\sigma, \qquad \mathrm{i}, \qquad \sum_{m\in\mathbb{N}}^{\oplus} \mathrm{id}_{\mathcal{H}^m_k(\mathbb{S}^{N-1})} \otimes \left(-\mathrm{i}\left(\sigma^2 + \left(m + \frac{2\langle k\rangle + N - 2}{2}\right)^2\right)\right),$$

respectively.

Proof. It follows from Theorems 3.1 and 3.3, and the orthogonal decomposition (3.4). ■

Remark 3.5. Since the unitary operator $\mathscr{F}\circ U_{N,k}\colon L^2(\mathbb{R}_{>0}, r^{2\langle k\rangle+N-3}\mathrm{d}r) \to L^2(\mathbb{R}, \mathrm{d}\sigma)$ "maps" $\frac{1}{\sqrt{2\pi}} r^{-\frac{2\langle k\rangle+N-2}{2}+\mathrm{i}\sigma}$ to the Dirac distribution δ_σ, we have the direct integral decomposition

$$L^2\big(\mathbb{R}_{>0}, r^{2\langle k\rangle+N-3}\mathrm{d}r\big) = \int_{\mathbb{R}}^{\oplus} \mathbb{C}\frac{1}{\sqrt{2\pi}} r^{-\frac{2\langle k\rangle+N-2}{2}+\mathrm{i}\sigma}\, \mathrm{d}\sigma$$

and may write Theorem 3.3 as

$$2E_r + 2\langle k\rangle + N - 2 = \int_{\mathbb{R}}^{\oplus} 2\mathrm{i}\sigma\,\mathrm{d}\sigma, \qquad \mathrm{i}\,\mathrm{id} = \int_{\mathbb{R}}^{\oplus} \mathrm{i}\,\mathrm{d}\sigma,$$

$$\mathrm{i}(E_r - m)(E_r + m + 2\langle k\rangle + N - 2) = \int_{\mathbb{R}}^{\oplus} \left(-\mathrm{i}\left(\sigma^2 + \left(m + \frac{2\langle k\rangle + N - 2}{2}\right)^2\right)\right) \mathrm{d}\sigma.$$

Similarly, we have the direct integral decomposition

$$L^2\big(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\big) = \sum_{m\in\mathbb{N}}^{\oplus} \int_{\mathbb{R}}^{\oplus} \mathcal{H}_k^m\big(\mathbb{S}^{N-1}\big) \otimes \mathbb{C}\frac{1}{\sqrt{2\pi}} r^{-\frac{2\langle k\rangle + N - 2}{2} + \mathrm{i}\sigma}\,\mathrm{d}\sigma$$

and may write Theorem 3.4 as

$$2E_x + 2\langle k\rangle + N - 2 = \sum_{m\in\mathbb{N}}^{\oplus} \int_{\mathbb{R}}^{\oplus} 2\mathrm{i}\sigma\,\mathrm{d}\sigma, \qquad \mathrm{i}\,\mathrm{id} = \sum_{m\in\mathbb{N}}^{\oplus} \int_{\mathbb{R}}^{\oplus} \mathrm{i}\,\mathrm{d}\sigma,$$

$$\mathrm{i}|x|^2\Delta_k = \sum_{m\in\mathbb{N}}^{\oplus} \int_{\mathbb{R}}^{\oplus} \left(-\mathrm{i}\left(\sigma^2 + \left(m + \frac{2\langle k\rangle + N - 2}{2}\right)^2\right)\right) \mathrm{d}\sigma.$$

Corollary 3.6. *Let k be a non-negative multiplicity function. The (possibly unbounded) normal operator*

$$\exp\left(\frac{z_1}{\mathrm{i}}(2E_x + 2\langle k\rangle + N - 2) + z_2 + z_3|x|^2\Delta_k\right)$$

on $L^2\big(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\big)$ is well-defined for $z_1, z_2, z_3 \in \mathbb{C}$. In particular, the action of the differential-difference operators in $\mathfrak{g}_{k,0}$ lifts to a unique unitary representation of $\mathbb{R}^3$ on $L^2\big(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\big)$, which is given by

$$(t_1, t_2, t_3) \mapsto \exp\big(t_1(2E_x + 2\langle k\rangle + N - 2) + \mathrm{i}t_2 + \mathrm{i}t_3|x|^2\Delta_k\big).$$

Proof. Since $\mathfrak{g}_{k,0}$ admits joint spectral decomposition (see Theorem 3.4), the (possibly unbounded) normal operator

$$\phi\left(\frac{1}{\mathrm{i}}(2E_x + 2\langle k\rangle + N - 2), 1, |x|^2\Delta_k\right)$$

on $L^2\big(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\big)$ is defined for any Borel measurable function $\phi\colon \mathbb{C}^3 \to \mathbb{C}$ by means of the functional calculus. The former assertion is shown by setting $\phi(w_1, w_2, w_3) = \exp(z_1w_1 + z_2w_2 + z_3w_3)$. The latter assertion is a consequence of Stone's theorem. ■

For the operators in Theorem 3.6, the part involving z_2 contributes only as a scalar multiple of the identity. The subsequent two subsections are devoted to the analysis of the parts involving z_1 and z_3.

3.3 The unitary group with infinitesimal generator $2E_x + 2\langle k\rangle + N - 2$

In this subsection, we consider the unitary group with infinitesimal generator $2E_x + 2\langle k\rangle + N - 2$, regarded as a skew-adjoint operator on $L^2\big(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\big)$ based on Theorem 3.4.

Proposition 3.7. *Let k be a non-negative multiplicity function. For $z \in \mathbb{C}$, the (possibly unbounded) normal operator*

$$\exp\left(\frac{z}{\mathrm{i}}(2E_x + 2\langle k\rangle + N - 2)\right)$$

on $L^2\big(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\big)$ is bounded if and only if $\operatorname{Re} z = 0$, and in this case, this operator is unitary.

Proof. By Theorem 3.4, the operator in the assertion corresponds to the multiplication operator $\mathrm{id}_{L^2(\mathbb{S}^{N-1},w_k(\omega)\mathrm{d}\omega)}\mathbin{\widehat{\otimes}}\mathrm{e}^{2z\sigma}$ on $L^2(\mathbb{S}^{N-1},w_k(\omega)\mathrm{d}\omega)\mathbin{\widehat{\otimes}}L^2(\mathbb{R},\mathrm{d}\sigma)$ via the unitary operator $\mathrm{id}_{L^2(\mathbb{S}^{N-1},w_k(\omega)\mathrm{d}\omega)}\mathbin{\widehat{\otimes}}(\mathscr{F}\circ U_{N,k})$. Now the assertion follows since the function $\sigma\mapsto\mathrm{e}^{2z\sigma}$ on $\mathbb{R}$ is bounded if and only if $\operatorname{Re}z=0$, and in this case, $|\mathrm{e}^{2z\sigma}|=1$. ∎

Theorem 3.8. *Let k be a non-negative multiplicity function. For $z=\mathrm{i}t$ with $t\in\mathbb{R}$, the unitary operator on $L^2(\mathbb{R}^N,w_{k,0}(x)\mathrm{d}x)$ in Theorem 3.7 is given by*

$$\exp(t(2E_x+2\langle k\rangle+N-2))F(x)=\mathrm{e}^{(2\langle k\rangle+N-2)t}F\big(\mathrm{e}^{2t}x\big).$$

Proof. We continue our discussion following the proof of Theorem 3.7. The multiplication operator $\mathrm{e}^{2\mathrm{i}t\sigma}$ on $L^2(\mathbb{R},\mathrm{d}\sigma)$ corresponds to the translation operator $g\mapsto g((-)+2t)$ on $L^2(\mathbb{R},\mathrm{d}s)$ via $\mathscr{F}^{-1}$, which in turn corresponds to the scaling operator $f\mapsto\mathrm{e}^{(2\langle k\rangle+N-2)t}f\big(\mathrm{e}^{2t}(-)\big)$ on $L^2\big(\mathbb{R}_{>0},r^{2\langle k\rangle+N-3}\mathrm{d}r\big)$ via $U_{N,k}^{-1}$. Hence, the assertion holds. ∎

3.4 The operator semigroup with infinitesimal generator $|x|^2\Delta_k$

In this subsection, we consider the operator semigroup with infinitesimal generator $|x|^2\Delta_k$, regarded as a self-adjoint operator on $L^2(\mathbb{R}^N,w_{k,0}(x)\mathrm{d}x)$ based on Theorem 3.4.

Proposition 3.9. *Let k be a non-negative multiplicity function. For $z\in\mathbb{C}$, the (possibly unbounded) normal operator $\exp\big(z|x|^2\Delta_k\big)$ on $L^2(\mathbb{R}^N,w_{k,0}(x)\mathrm{d}x)$ is bounded if and only if $\operatorname{Re}z\geq0$, and unitary if and only if $\operatorname{Re}z=0$.*

Proof. By Theorem 3.4, the operator in the assertion corresponds to the multiplication operator

$$\sum_{m\in\mathbb{N}}^{\oplus}\mathrm{id}_{\mathcal{H}_k^m(\mathbb{S}^{N-1})}\otimes\mathrm{e}^{-z(\sigma^2+(m+\frac{2\langle k\rangle+N-2}{2})^2)}$$

on $L^2(\mathbb{S}^{N-1},w_k(\omega)\mathrm{d}\omega)\mathbin{\widehat{\otimes}}L^2(\mathbb{R},\mathrm{d}\sigma)$ via the unitary operator $\mathrm{id}_{L^2(\mathbb{S}^{N-1},w_k(\omega)\mathrm{d}\omega)}\mathbin{\widehat{\otimes}}(\mathscr{F}\circ U_{N,k})$. Now the assertion follows since the function

$$\sigma\mapsto\mathrm{e}^{-z(\sigma^2+(m+\frac{2\langle k\rangle+N-2}{2})^2)}$$

on $\mathbb{R}$ is bounded if and only if $\operatorname{Re}z\geq0$, and has modulus one if and only if $\operatorname{Re}z=0$. ∎

We consider the integral kernel formula for the operator semigroup $\big(\exp\big(z|x|^2\Delta_k\big)\big)_{\operatorname{Re}z\geq0}$. For this purpose, we first focus on the radial part $(E_r-m)(E_r+m+2\langle k\rangle+N-2)$ of $|x|^2\Delta_k$ (see Theorem 3.1).

Fact 3.10 ([25, Section IX.7, Example 3])**.** *The operator semigroup $\big(\exp\big(z\big(\frac{\mathrm{d}}{\mathrm{d}s}\big)^2\big)\big)_{\operatorname{Re}z\geq0}$ on $L^2(\mathbb{R},\mathrm{d}s)$ admits the integral kernel formula*

$$\exp\left(z\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^2\right)g(s)=\frac{1}{\sqrt{4\pi z}}\int_{\mathbb{R}}\exp\left(-\frac{(s-s')^2}{4z}\right)g(s')\,\mathrm{d}s' \tag{3.5}$$

in the following sense. Here, we take the branch of $\sqrt{z}$ such that $\sqrt{z}>0$ when $z>0$.

(1) *For $z\in\mathbb{C}$ with $\operatorname{Re}z>0$ and $g\in L^2(\mathbb{R},\mathrm{d}s')$, the integrand in the right-hand side of (3.5) is integrable for all $s\in\mathbb{R}$, and this integral as a function of s gives $\exp\big(z\big(\frac{\mathrm{d}}{\mathrm{d}s}\big)^2\big)g$.*

(2) *For $z\in\mathbb{C}$ with $\operatorname{Re}z=0$ and $z\neq0$ and $g\in\big(L^1\cap L^2\big)(\mathbb{R},\mathrm{d}s')$, the integrand in the right-hand side of (3.5) is integrable for all $s\in\mathbb{R}$, and this integral as a function of s gives $\exp\big(z\big(\frac{\mathrm{d}}{\mathrm{d}s}\big)^2\big)g$.*

Theorem 3.11. *Let k be a non-negative multiplicity function and $m \in \mathbb{N}$. The operator semigroup $(\exp(z(E_r-m)(E_r+m+2\langle k\rangle+N-2)))_{\operatorname{Re} z\geq 0}$ on $L^2\big(\mathbb{R}_{>0}, r^{2\langle k\rangle+N-3}\mathrm{d}r\big)$ admits the integral kernel formula*

$$\exp(z(E_r-m)(E_r+m+2\langle k\rangle+N-2))f(r) = \int_{\mathbb{R}_{>0}} K_k^{(m)}\big(r,r';z\big)f\big(r'\big)r'^{2\langle k\rangle+N-3}\,\mathrm{d}r', \tag{3.6}$$

where

$$K_k^{(m)}(r,r';z) = \frac{1}{\sqrt{4\pi z}}\exp\left(-z\left(m+\frac{2\langle k\rangle+N-2}{2}\right)^2\right)\exp\left(-\frac{(\log r-\log r')^2}{4z}\right)\big(rr'\big)^{-\frac{2\langle k\rangle+N-2}{2}}$$

for $r, r' \in \mathbb{R}_{>0}$, in the following sense. Here, we take the branch of $\sqrt{z}$ such that $\sqrt{z} > 0$ when $z > 0$.

1. *For $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and $f \in L^2\big(\mathbb{R}_{>0}, r'^{2\langle k\rangle+N-3}\mathrm{d}r'\big)$, the integrand in the right-hand side of (3.6) is integrable for all $r \in \mathbb{R}_{>0}$, and this integral as a function of r gives $\exp(z(E_r-m)(E_r+m+2\langle k\rangle+N-2))f$.*
2. *For $z \in \mathbb{C}$ with $\operatorname{Re} z = 0$ and $z \neq 0$ and $f \in \big(L^1\cap L^2\big)\big(\mathbb{R}_{>0}, r'^{2\langle k\rangle+N-3}\mathrm{d}r'\big)$, the integrand in the right-hand side of (3.6) is integrable for all $r \in \mathbb{R}_{>0}$, and this integral as a function of r gives $\exp(z(E_r-m)(E_r+m+2\langle k\rangle+N-2))f$.*

Proof. As follows from the proof of Theorem 3.3, $(E_r-m)(E_r+m+2\langle k\rangle+N-2)$ corresponds to $\big(\frac{\mathrm{d}}{\mathrm{d}s}\big)^2 - \big(m+\frac{2\langle k\rangle+N-2}{2}\big)^2$ via the unitary operator $U_{N,k}\colon L^2(\mathbb{R}_{>0}, r^{2\langle k\rangle+N-3}\mathrm{d}r) \to L^2(\mathbb{R}, \mathrm{d}s)$. Now the assertion follows from Theorem 3.10. ■

We then combine this result with the spherical part. Recall that $P_k^{(m)}$ denotes the Poisson kernel of the space of k-spherical harmonics of degree m (see (2.1) for the definition), and that C_m^ν (resp. $\widetilde{C}_m^\nu$) denotes the Gegenbauer polynomial (resp. the renormalized Gegenbauer polynomial) (see (2.3) and (2.4) for the definitions).

Lemma 3.12. *Fix a non-negative multiplicity function k. Then, the uniform norm of the Poisson kernel $P_k^{(m)}$ satisfies*

$$\sup_{\omega,\omega'\in\mathbb{S}^{N-1}} \big|P_k^{(m)}(\omega,\omega')\big| = O\big(m^{2\langle k\rangle+N-2}\big), \qquad m\to\infty.$$

Proof. In the case $N = 1$, we have $P_k^{(m)} = 0$ for $m \geq 2$ (see Theorem 2.5), so that the assertion holds trivially. We now consider the case $N \geq 2$. By Theorems 2.3 and 2.2, we have

$$P_k^{(m)}(\omega,\omega') = V_k\big(\widetilde{C}_m^{\frac{2\langle k\rangle+N-2}{2}}(\langle -,\omega'\rangle)\big)(\omega) = \int_{\mathbb{R}^N} \widetilde{C}_m^{\frac{2\langle k\rangle+N-2}{2}}(\langle \xi,\omega'\rangle)\,\mathrm{d}\mu_{k,\omega}(\xi),$$

where $\mu_{k,x}$ is a probability Borel measure on $\mathbb{R}^N$ whose support is contained in the unit ball $\big\{\xi\in\mathbb{R}^N \mid |\xi|\leq 1\big\}$. Hence, we have

$$\big|P_k^{(m)}(\omega,\omega')\big| \leq \int_{\mathbb{R}^N}\big|\widetilde{C}_m^{\frac{2\langle k\rangle+N-2}{2}}(\langle \xi,\omega'\rangle)\big|\,\mathrm{d}\mu_{k,\omega}(\xi) = \sup_{t\in[-1,1]}\big|\widetilde{C}_m^{\frac{2\langle k\rangle+N-2}{2}}(t)\big|.$$

Note that $\frac{2\langle k\rangle+N-2}{2} \geq 0$ since $N \geq 2$. For $\nu \in \mathbb{R}_{>0}$, it is known (see [1, p. 302]) that

$$\sup_{t\in[-1,1]}|C_m^\nu(t)| = C_m^\nu(1) = \frac{\Gamma(m+2\nu)}{m!\Gamma(2\nu)},$$

which implies

$$\sup_{t\in[-1,1]}\left|\widetilde{C}_m^{\nu}(t)\right| = \frac{m+\nu}{\nu}\frac{\Gamma(m+2\nu)}{m!\Gamma(2\nu)} = O\left(m^{2\nu}\right).$$

This also holds for $\nu=0$ since $\widetilde{C}_m^0$ is defined by the limit formula (2.5). The assertion follows by applying this estimate to the case $\nu=\frac{2\langle k\rangle+N-2}{2}$. ■

Recall from (2.2) that $\mathrm{vol}_k\left(\mathbb{S}^{N-1}\right)$ denotes the volume of the sphere $\mathbb{S}^{N-1}$ with respect to the measure $w_k(\omega)\mathrm{d}\omega$.

Theorem 3.13. *Let k be a non-negative multiplicity function. The operator semigroup*

$$\left(\exp\left(z|x|^2\Delta_k\right)\right)_{\mathrm{Re}\,z\geq 0}$$

on $L^2\left(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x\right)$ admits the integral kernel formula

$$\exp\left(z|x|^2\Delta_k\right)F(x) = \int_{\mathbb{R}^N} K_k\left(x,x';z\right)F\left(x'\right)w_{k,0}\left(x'\right)\mathrm{d}x', \tag{3.7}$$

where

$$K_k(r\omega, r'\omega'; z) = \frac{1}{\sqrt{4\pi z}}\exp\left(-\frac{(\log r-\log r')^2}{4z}\right)\left(rr'\right)^{-\frac{2\langle k\rangle+N-2}{2}} \times \frac{1}{\mathrm{vol}_k\left(\mathbb{S}^{N-1}\right)}\sum_{m=0}^{\infty}\exp\left(-z\left(m+\frac{2\langle k\rangle+N-2}{2}\right)^2\right)P_k^{(m)}(\omega,\omega')$$

for $r,r'\in\mathbb{R}_{>0}$ and $\omega,\omega'\in\mathbb{S}^{N-1}$, in the following sense. Here, we take the branch of $\sqrt{z}$ such that $\sqrt{z}>0$ when $z>0$.

For $z\in\mathbb{C}$ with $\mathrm{Re}\,z>0$ and $F\in L^2\left(\mathbb{R}^N, w_{k,0}\left(x'\right)\mathrm{d}x'\right)$, the integrand in the right-hand side of (3.7) is integrable for all $x\in\mathbb{R}^N\backslash\{0\}$, and this integral as a function of x gives $\exp\left(z|x|^2\Delta_k\right)F$.

Remark 3.14. When $k=0$, we have

$$\mathrm{vol}_0\left(\mathbb{S}^{N-1}\right) = \mathrm{vol}\left(\mathbb{S}^{N-1}\right) = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \qquad \text{with} \quad P_0^{(m)}(\omega,\omega') = \widetilde{C}_m^{\frac{N-2}{2}}(\langle\omega,\omega'\rangle)$$

(see Theorem 2.4), so that the integral kernel formula in Theorem 3.13 reduces to

$$K_0(r\omega, r'\omega'; z) = \frac{1}{\sqrt{4\pi z}}\exp\left(-\frac{\left(\log r-\log r'\right)^2}{4z}\right)\left(rr'\right)^{-\frac{N-2}{2}} \times \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\sum_{m=0}^{\infty}\exp\left(-z\left(m+\frac{N-2}{2}\right)^2\right)\widetilde{C}_m^{\frac{N-2}{2}}(\langle\omega,\omega'\rangle).$$

Proof of Theorem 3.13. Fix $r\in\mathbb{R}_{>0}$ and $\omega\in\mathbb{S}^{N-1}$. Then, the function

$$r'\mapsto \exp\left(-\frac{(\log r-\log r')^2}{4z}\right)\left(rr'\right)^{-\frac{2\langle k\rangle+N-2}{2}}$$

is square-integrable with respect to the measure $r'^{2\langle k\rangle+N-3}\mathrm{d}r'$ and the infinite series

$$\sum_{m=0}^{\infty}\exp\left(-z\left(m+\frac{2\langle k\rangle+N-2}{2}\right)^2\right)P_k^{(m)}(\omega,\omega')$$

absolutely converges with respect to the uniform norm for $\omega' \in \mathbb{S}^{N-1}$ by Theorem 3.12, so that the equation

$$\begin{aligned} K_k(r\omega, r'\omega'; z) &= \frac{1}{\sqrt{4\pi z}} \exp\left(-\frac{(\log r - \log r')^2}{4z}\right)(rr')^{-\frac{2\langle k\rangle + N-2}{2}} \\ &\quad \times \frac{1}{\mathrm{vol}_k(\mathbb{S}^{N-1})} \sum_{m=0}^{\infty} \exp\left(-z\left(m + \frac{2\langle k\rangle + N - 2}{2}\right)^2\right) P_k^{(m)}(\omega, \omega') \\ &= \frac{1}{\mathrm{vol}_k(\mathbb{S}^{N-1})} \sum_{m=0}^{\infty} P_k^{(m)}(\omega,\omega') K_k^{(m)}(r, r'; z) \end{aligned}$$

(here, $K_k^{(m)}(r, r'; z)$ is as defined in Theorem 3.11) holds with respect to the topology of $L^2(\mathbb{R}_{>0} \times \mathbb{S}^{N-1}, r'^{2\langle k\rangle + N-3} w_k(\omega') \mathrm{d}r' \mathrm{d}\omega') \cong L^2(\mathbb{R}^N, w_{k,0}(x')\mathrm{d}x')$.

By the result of the previous paragraph, for $F \in L^2(\mathbb{R}^N, w_{k,0}(x')\mathrm{d}x')$ and $x = r\omega \in \mathbb{R}^N \setminus \{0\}$, the function $x' \mapsto K_k(x, x'; z)F(x')$ is integrable with respect to the measure $w_{k,0}(x')\mathrm{d}x'$ and

$$\begin{aligned} \int_{\mathbb{R}^N} K_k(x, x'; z) F(x') w_{k,0}(x')\,\mathrm{d}x' = \frac{1}{\mathrm{vol}_k(\mathbb{S}^{N-1})} \sum_{m=0}^{\infty} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}_{>0}} P_k^{(m)}(\omega, \omega') \\ \times K_k^{(m)}(r, r'; z) F(r'\omega') r'^{2\langle k\rangle + N-3} w_k(\omega')\,\mathrm{d}r'\mathrm{d}\omega'. \end{aligned}$$

If $F = p \otimes f$ with $p \in \mathcal{H}_k^l(\mathbb{S}^{N-1})$ and $f \in L^2(\mathbb{R}_{>0}, r'^{2\langle k\rangle+N-3}\mathrm{d}r')$, by (2.1) and Theorem 3.11 (1), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} K_k(x, x'; z)(p \otimes f)(x') w_{k,0}(x')\,\mathrm{d}x' \\ &\quad = \frac{1}{\mathrm{vol}_k(\mathbb{S}^{N-1})} \sum_{m=0}^{\infty} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}_{>0}} P_k^{(m)}(\omega,\omega') K_k^{(m)}(r, r'; z) p(\omega') f(r') \\ &\quad \times r'^{2\langle k\rangle+N-3} w_k(\omega')\,\mathrm{d}r'\mathrm{d}\omega' \\ &\quad = \sum_{m=0}^{\infty} \left(\frac{1}{\mathrm{vol}_k(\mathbb{S}^{N-1})} \int_{\mathbb{S}^{N-1}} P_k^{(m)}(\omega,\omega') p(\omega') w_k(\omega')\,\mathrm{d}\omega' \right) \\ &\quad \times \left(\int_{\mathbb{R}_{>0}} K_k^{(m)}(r, r'; z) f(r') r'^{2\langle k\rangle+N-3}\,\mathrm{d}r' \right) \\ &\quad = p(\omega) \left(\int_{\mathbb{R}_{>0}} K_k^{(l)}(r, r'; z) f(r') r'^{2\langle k\rangle+N-3}\,\mathrm{d}r' \right) \\ &\quad = p(\omega) \exp(z(E_r - l)(E_r + l + 2\langle k\rangle + N - 2)) f(r) \\ &\quad = \exp(z|x|^2\Delta_k)(p \otimes f)(x). \end{aligned}$$

Hence, (3.7) holds in this case.

Let $F \in L^2(\mathbb{R}^N, w_{k,0}(x')\mathrm{d}x')$ and take a sequence $(F_j)_{j\in\mathbb{N}}$ in

$$\mathcal{H}(\mathbb{S}^{N-1}) \otimes L^2(\mathbb{R}_{>0}, r'^{2\langle k\rangle+N-3}\mathrm{d}r')$$

such that $F_j \to F$ in $L^2(\mathbb{R}^N, w_{k,0}(x')\mathrm{d}x')$. Then, since $\exp(z|x|^2\Delta_k)$ is a bounded operator on $L^2(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x)$ (see Theorem 3.9), we have

$$\exp(z|x|^2\Delta_k)F_j \to \exp(z|x|^2\Delta_k)F \qquad \text{in} \quad L^2(\mathbb{R}^N, w_{k,0}(x)\mathrm{d}x).$$

On the other hand, for each $x \in \mathbb{R}^N \setminus \{0\}$, the function $x' \mapsto K_k(x, x'; z)$ is square-integrable with respect to the measure $w_{k,0}(x')\mathrm{d}x'$, so we have

$$\int_{\mathbb{R}^N} K_k(x, x'; z) F_j(x') w_{k,0}(x')\,\mathrm{d}x' \to \int_{\mathbb{R}^N} K_k(x, x'; z) F(x') w_{k,0}(x')\,\mathrm{d}x'.$$

By the result of the previous paragraph, (3.7) holds for each F_j. By taking the limit as $j \to \infty$, we conclude that (3.7) also holds for F. ∎

Remark 3.15. We recall the definition of the (k,a)-generalized Laguerre semigroup

$$(\mathscr{I}_{k,a}(z))_{\operatorname{Re} z \geq 0}$$

from [4, equation (1.3)]:

$$\mathscr{I}_{k,a}(z) = \exp\Big(\frac{z}{\mathrm{i}}\big(\mathbb{E}^-_{k,a} - \mathbb{E}^+_{k,a}\big)\Big) = \exp\Big(\frac{z}{a}\big(|x|^{2-a}\Delta_k - |x|^a\big)\Big),$$

and the definition of the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ from [4, equation (5.2)]:

$$\begin{aligned}\mathscr{F}_{k,a} &= \mathrm{e}^{\frac{\mathrm{i}\pi}{2}\frac{2\langle k\rangle + a + N - 2}{a}} \mathscr{I}_{k,a}\left(\frac{\mathrm{i}\pi}{2}\right) = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}\frac{2\langle k\rangle + a + N - 2}{a}} \exp\Big(\frac{\pi}{2}\big(\mathbb{E}^-_{k,a} - \mathbb{E}^+_{k,a}\big)\Big) \\ &= \mathrm{e}^{\frac{\mathrm{i}\pi}{2}\frac{2\langle k\rangle + a + N - 2}{a}} \exp\left(\frac{\mathrm{i}\pi}{2a}\big(|x|^{2-a}\Delta_k - |x|^a\big)\right).\end{aligned}$$

These are not well-defined for $a = 0$. However, considering the "renormalized" (k,a)-generalized Laguerre semigroup $\mathscr{I}_{k,a}(az) = \exp\big(z\big(|x|^{2-a}\Delta_k - |x|^a\big)\big)$ and putting $a = 0$, we get the operator $\exp\big(z\big(|x|^2\Delta_k - 1\big)\big) = \mathrm{e}^{-z}\exp\big(z|x|^2\Delta_k\big)$. By Theorem 3.13, for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, the integral kernel of this operator is the function $(x, x') \mapsto \mathrm{e}^{-z} K_k\big(x, x'; z\big)$.

4 Closed-form expressions for the integral kernels in low-dimensional cases

In this section, we give a closed-form expression for the integral kernel $(x, x') \mapsto K_k\big(x, x'; z\big)$ of the operator $\exp\big(z|x|^2\Delta_k\big)$ ($\operatorname{Re} z > 0$), obtained in Theorem 3.13, in the low-dimensional cases $N = 1, 2$ and 4. In the cases $N = 2$ and 4, we assume that $k = 0$, and show that the integral kernel can be expressed in terms of the theta function.

Proposition 4.1. *We consider the case $N = 1$. The reduced root system $\mathscr{R}$ is taken to be $\{\alpha, -\alpha\}$ with $\alpha \in \mathbb{R}_{>0}$, and the non-negative multiplicity function k is identified with $k_\alpha = k_{-\alpha} \in \mathbb{R}_{\geq 0}$. In this case, for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, we have*

$$\begin{aligned}K_k\big(x, x'; z\big) = {}& \frac{1}{2\alpha^{2k}\sqrt{4\pi z}} \exp\left(-\frac{(\log|x| - \log|x'|)^2}{4z}\right)|xx'|^{-k+\frac{1}{2}} \\ &\times \big(\mathrm{e}^{-(k-\frac{1}{2})^2 z} + \mathrm{e}^{-(k+\frac{1}{2})^2 z}\operatorname{sgn}(xx')\big)\end{aligned}$$

for $x, x' \in \mathbb{R}\setminus\{0\}$. In particular, when $k = 0$, we have

$$K_0\big(x, x'; z\big) = \frac{\mathrm{e}^{-\frac{z}{4}}}{2\sqrt{4\pi z}} \exp\left(-\frac{(\log|x| - \log|x'|)^2}{4z}\right)|xx'|^{\frac{1}{2}}(1 + \operatorname{sgn}(xx'))$$

for $x, x' \in \mathbb{R}\setminus\{0\}$.

Proof. It follows from $\operatorname{vol}_k\big(\mathbb{S}^0\big) = 2\alpha^{2k}$ and Theorem 2.5. ∎

In the following, we consider only the case $k = 0$. We recall the definition of the theta function:

$$\vartheta(v, \tau) = \sum_{m=-\infty}^{\infty} \exp\big(\mathrm{i}\pi\tau m^2 + 2\mathrm{i}\pi m v\big) = 1 + 2\sum_{m=1}^{\infty} \exp\big(\mathrm{i}\pi\tau m^2\big)\cos 2\pi m v.$$

Proposition 4.2. *In the case $N = 2$, for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, we have*

$$K_0(r\omega, r'\omega'; z) = \frac{1}{2\pi\sqrt{4\pi z}} \exp\left(-\frac{(\log r - \log r')^2}{4z}\right) \vartheta\left(\frac{1}{2\pi} \arccos\langle \omega, \omega' \rangle, \frac{\mathrm{i}}{\pi} z\right)$$

for $r, r' \in \mathbb{R}_{>0}$ and $\omega, \omega' \in \mathbb{S}^1$. Equivalently, we have

$$K_0\left(r\mathrm{e}^{\mathrm{i}\phi}, r'\mathrm{e}^{\mathrm{i}\phi'}; z\right) = \frac{1}{2\pi\sqrt{4\pi z}} \exp\left(-\frac{\left(\log r - \log r'\right)^2}{4z}\right) \vartheta\left(\frac{1}{2\pi}\left(\phi - \phi'\right), \frac{\mathrm{i}}{\pi} z\right)$$

for $r, r' \in \mathbb{R}_{>0}$ and $\phi, \phi' \in \mathbb{R}$.

Proof. Since

$$\widetilde{C}^0_m(t) = \begin{cases} 1, & m = 0, \\ 2T_m(t), & m \geq 1, \end{cases}$$

where T_m denotes the Chebyshev polynomial of the first kind, which is characterized by the formula and $T_m(\cos\theta) = \cos m\theta$, we have

$$\begin{aligned} \sum_{m=0}^{\infty} \exp\left(-zm^2\right) \widetilde{C}^0_m(\cos\theta) &= 1 + 2\sum_{m=1}^{\infty} \exp\left(-zm^2\right) T_m(\cos\theta) \\ &= 1 + 2\sum_{m=1}^{\infty} \exp\left(-zm^2\right) \cos m\theta = \vartheta\left(\frac{1}{2\pi}\theta, \frac{\mathrm{i}}{\pi} z\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} K_0(r\omega, r'\omega'; z) &= \frac{1}{\sqrt{4\pi z}} \exp\left(-\frac{\left(\log r - \log r'\right)^2}{4z}\right) \times \frac{1}{2\pi} \sum_{m=0}^{\infty} \exp\left(-zm^2\right) \widetilde{C}^0_m\left(\langle \omega, \omega' \rangle\right) \\ &= \frac{1}{2\pi\sqrt{4\pi z}} \exp\left(-\frac{\left(\log r - \log r'\right)^2}{4z}\right) \vartheta\left(\frac{1}{2\pi} \arccos\langle \omega, \omega' \rangle, \frac{\mathrm{i}}{\pi} z\right). \end{aligned}$$

■

Proposition 4.3. *In the case $N = 4$, for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, we have*

$$\begin{aligned} K_0(r\omega, r'\omega'; z) = &-\frac{1}{8\pi^3\sqrt{4\pi z}} \exp\left(-\frac{(\log r - \log r')^2}{4z}\right) \left(rr'\right)^{-1} \\ &\times \left(1 - \langle \omega, \omega' \rangle^2\right)^{-\frac{1}{2}} \frac{\partial \vartheta}{\partial v}\left(\frac{1}{2\pi} \arccos\langle \omega, \omega' \rangle, \frac{\mathrm{i}}{\pi} z\right) \end{aligned}$$

for $r, r' \in \mathbb{R}_{>0}$ and $\omega, \omega' \in \mathbb{S}^3$. Here, we take the branch of $\arccos\langle \omega, \omega' \rangle$ such that $\arccos\langle \omega, \omega' \rangle \in [0, \pi]$.

Proof. Since $\widetilde{C}^1_m(t) = (m+1)C^1_m(t) = (m+1)U_m(t)$, where U_m denotes the Chebyshev polynomial of the second kind, which is characterized by the formula $U_m(\cos\theta) = (\sin(m+1)\theta)/(\sin\theta)$, we have

$$\begin{aligned} \sum_{m=0}^{\infty} \exp\left(-z(m+1)^2\right) \widetilde{C}^1_m(\cos\theta) &= \sum_{m=0}^{\infty} \exp\left(-z(m+1)^2\right) \cdot (m+1) \frac{\sin(m+1)\theta}{\sin\theta} \\ &= \frac{1}{\sin\theta} \sum_{m=1}^{\infty} \exp\left(-zm^2\right) \cdot m \sin m\theta \\ &= -\frac{1}{4\pi \sin\theta} \frac{\partial \vartheta}{\partial v}\left(\frac{1}{2\pi}\theta, \frac{\mathrm{i}}{\pi} z\right). \end{aligned}$$

Hence, we have

$$\begin{aligned}
&K_0(r\omega, r'\omega'; z)\\
&\quad= \frac{1}{\sqrt{4\pi z}} \exp\left(-\frac{\left(\log r - \log r'\right)^2}{4z}\right) \left(rr'\right)^{-1} \times \frac{1}{2\pi^2} \sum_{m=0}^{\infty} \exp(-z(m+1)^2) \widetilde{C}_m^1\left(\langle \omega, \omega' \rangle\right)\\
&\quad= -\frac{1}{8\pi^3\sqrt{4\pi z}} \exp\left(-\frac{\left(\log r - \log r'\right)^2}{4z}\right) \left(rr'\right)^{-1}\\
&\qquad\times \left(1 - \langle \omega, \omega' \rangle^2\right)^{-\frac{1}{2}} \frac{\partial \vartheta}{\partial v}\left(\frac{1}{2\pi}\arccos\langle \omega, \omega' \rangle, \frac{\mathrm{i}}{\pi} z\right). \qquad \blacksquare
\end{aligned}$$

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