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The Principal W-Algebra of $\mathfrak{psl}_{2|2}$

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Abstract. We study the structure and representation theory of the principal W-algebra $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ of $\mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{2|2})$. The defining operator product expansions are computed, as is the Zhu algebra, and these results are used to classify irreducible highest-weight modules. In particular, for $\mathsf{k} = \pm\frac{1}{2}$, $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ is not simple and the corresponding simple quotient is the symplectic fermion vertex algebra. We use this fact, along with inverse Hamiltonian reduction, to study relaxed highest-weight and logarithmic modules for the small $N = 4$ superconformal algebra at central charges -9 and -3.

Key words: vertex-operator algebras; conformal field theory; representation theory

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1 Introduction

The simple Lie superalgebra $\mathfrak{psl}_{2|2}$ is somewhat unusual among the basic classical examples in that it has odd roots of multiplicity 2, which may be regarded as being both positive and negative. Nevertheless, it has received an enormous amount of attention due to its applications in physics. In particular, the "baby" version of the AdS/CFT correspondence on $\mathsf{AdS}_3 \times \mathsf{S}^3$ supersymmetrises to a sigma model on the Lie supergroup $\mathsf{PSU}_{1,1|2}$ [12], leading to interest in Wess–Zumino–Witten models with $\widehat{\mathfrak{psl}}_{2|2}$ symmetry (see, for example, [22, 31, 33, 34, 37, 53]).

Intertwined with this story is that of the (small) $N = 4$ superconformal algebra. Introduced initially to describe spacetime supersymmetries of heterotic strings, its unitary representations were first studied in [24], motivated by applications to string theory on hyperKähler manifolds. More recently, there has been a resurgence in interest in $N = 4$ representation theory because of its role in such things as Mathieu moonshine and its variants [23]. However, this representation theory is still poorly understood in general. Classifications are only known for the unitary highest-weight modules [39, 40] and the irreducibles in the case of central charge $\mathsf{c} = -9$ [1]. The $\mathsf{c} = -9$ algebra is also the $p = 2$ member of the family of extensions $\mathcal{V}^{(p)}$ of $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ at $\mathsf{k} = -2 + \frac{1}{p}$, introduced in [1] and studied further in [5].

In a different direction, recent developments in the 3d/2d (and 4d/2d) correspondence has renewed interest in theories with $N = 4$ supersymmetry. In [14], it was argued that generic 3-dimensional gauge theories with an appropriate boundary condition support vertex-operator algebras on the boundary. The motivation here was that the data of the vertex-operator algebra, for example, its fusion algebra, could be used to investigate a related topological quantum field theory (analogous to the well-known story connecting Wess–Zumino–Witten models and 3d Chern–Simons theories).

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A rich source of 3-dimensional gauge theories are the quiver gauge theories [21], in which the field content is determined by specifying a quiver. These theories have an associated geometry described by the corresponding quiver varieties of Nakajima [47, 48]. One example, analysed in [10], is associated with the Jordan quiver for which the associated quiver variety is the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C})$ of n-points in the complex plane. The authors use chiral quantisation to construct a vertex-operator algebra for each n that has $\mathrm{Hilb}^n(\mathbb{C})$ as its associated variety. They also showed that for $n = 2$, their vertex-operator algebra is the small $N = 4$ superconformal algebra at $\mathsf{c} = -9$ (tensored with some additional free-field vertex algebras). As such, a detailed understanding of $N = 4$ representation theory, its modular data and fusion rules, will shed light on the associated topological quantum field theory.

With this goal in mind, our interest is the relationship between the vertex superalgebras constructed from $\mathfrak{psl}_{2|2}$ and $N = 4$. In the formalism developed by Kac, Roan and Wakimoto, these are related by quantum Hamiltonian reduction: the small $N = 4$ superconformal algebra is a minimal reduction of the affine vertex superalgebra constructed from $\mathfrak{psl}_{2|2}$ [41, 42]. But there is also a principal reduction which appears to have received little to no attention in the literature. This paper then amounts to the first steps towards understanding this reduction and its representation theory.

Given the recent interest in $N = 4$ representation theory, we aim to connect this to the representation theory of the principal reduction. Happily, the formalism of inverse quantum Hamiltonian reduction is designed for this purpose. Originally introduced by Semikhatov [51] for the universal Virasoro and $\mathfrak{sl}_2$ vertex algebras, inverse reduction was extended to the simple quotients by Adamović in [2]. The latter also studied certain restriction functors that connect the representation theories, specifically sending irreducible highest-weight Virasoro modules to (spectral flows of) relaxed highest-weight $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules. Subsequent work generalised this to higher ranks and developed methods to prove that these $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-modules were generically irreducible [7] and that every irreducible relaxed $\mathsf{L}_\mathsf{k}(\mathfrak{sl}_2)$-module could be obtained in this fashion [8]. Other recent works addressing inverse reduction include [3, 4, 15, 25, 26, 27, 28, 29, 30].

Inverse reduction therefore suggests a general path to understanding the representation theory of a nonrational simple affine vertex-operator algebra or W-algebra (associated to a simple Lie superalgebra $\mathfrak{g}$ at level k). One starts with the representation theory of the "exceptional" W-algebra of level k [11] and then uses inverse reduction à la Adamović to reconstruct the representation theory of each successive "less reduced" W-algebra. In this way, one ascends through the poset of level-k W-algebras corresponding to $\mathfrak{g}$ and k (parametrised by the even nilpotent orbits of $\mathfrak{g}$), iteratively building up their representation theories.

We remark that when $\mathfrak{g}$ is a Lie algebra and k is (co)admissible, the exceptional W-algebra is rational [11] and so its representation theory is well understood. This need not be the case if k is nonadmissible, see [8, Section 5.2], or $\mathfrak{g}$ is not a Lie algebra (or $\mathfrak{osp}_{1|2n}$), see [15]. In the case of interest here ($\mathfrak{g} = \mathfrak{psl}_{2|2}$), we identify the "exceptional" W-algebra for $\mathsf{k} = \pm\frac{1}{2}$ as the symplectic fermions vertex-operator superalgebra [44]. Inverse reduction then allows us to investigate the representation theory of the small $N = 4$ superconformal algebra at central charge $\mathsf{c} = -9$ (and also at $\mathsf{c} = -3$).

1.1 Outline and results

We begin in Section 2 by recalling some basic facts about the Lie superalgebra $\mathfrak{psl}_{2|2}$. We then compute the principal W-algebra $\mathsf{W}^\mathsf{k}_{\mathrm{pr}}$ from the universal affine vertex-operator superalgebra $\mathsf{V}^\mathsf{k}(\mathfrak{psl}_{2|2})$ via quantum Hamiltonian reduction. The operator product expansions that define $\mathsf{W}^\mathsf{k}_{\mathrm{pr}}$ are given in Theorem 2.1 and we record that $\mathsf{k} = \pm\frac{1}{2}$ is a collapsing level with simple quotient isomorphic to the symplectic fermions vertex-operator superalgebra. We also determine the levels for which $\mathsf{W}^\mathsf{k}_{\mathrm{pr}}$ is not simple (see Theorem 2.2).

In Section 3, we study the lower-bounded weight modules of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ by computing the Zhu algebra $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}\big]$ and determining the irreducible weight $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}\big]$-modules. Every irreducible lower-bounded $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$-module is shown to be highest-weight and the dimensions of their top spaces are determined. In particular, the top spaces are always finite-dimensional. This confirms that the representation theory of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ is not too dissimilar to those of the principal W-algebras with $\mathfrak{g}$ a simple Lie algebra. Given that the symplectic fermions vertex-operator superalgebra is "log-rational", meaning C_2-cofinite but not rational, it is reasonable to ask if the simple quotient of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ is also log-rational (when the latter is not simple). We show that this question is interesting by analysing two further examples.

Section 4 then applies inverse Hamiltonian reduction to $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$. We first realise the minimal W-algebra $\mathsf{W}^{\mathsf{k}}_{\mathrm{min}}$ of $\mathfrak{psl}_{2|2}$ inside $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}} \otimes \Pi$, where Π is the usual bosonisation of symplectic bosons. The argument determining when this realisation descends to the simple quotients $\mathsf{W}_{\mathsf{k}}^{\mathrm{min}}$ and $\mathsf{W}_{\mathsf{k}}^{\mathrm{pr}}$ is then sketched. Granting this, Section 4.2 restricts to the case $\mathsf{k} = \pm\frac{1}{2}$ for which inverse reduction translates the well-known representation theory of the symplectic fermions vertex-operator superalgebra into that of $\mathsf{W}^{\mathrm{min}}_{\pm 1/2}$. We thereby recover the classification of irreducible highest-weight modules [1] and extend it to irreducible relaxed highest-weight modules (in both the Neveu–Schwarz and Ramond sectors). The (super)characters of the latter are easily computed, though we leave the potentially subtle analysis of their modularity to future work.

In anticipation of this modularity study, we turn to the degenerations and spectral flows of the relaxed families in Section 4.3. Interestingly, each member of these families decomposes as a direct sum of two generically irreducible relaxed highest-weight modules. We also exhibit the structures of the reducible examples using Loewy diagrams. Here, we observe a qualitative difference between the $\mathsf{k} = -\frac{1}{2}$ and $\mathsf{k} = +\frac{1}{2}$ cases: unlike the former, one of the reducible relaxed modules of the latter has composition factors whose minimal conformal weights are strictly greater than that of the top space. We conclude in Section 4.4 by applying inverse reduction to the logarithmic module of symplectic fermions, thereby obtaining an uncountably infinite number of logarithmic $\mathsf{W}^{\mathrm{min}}_{\pm 1/2}$-modules. The structures of these modules are completely elucidated, even at degeneration points.

2 Computing the principal W-algebra

In this section, we introduce the principal W-algebras associated to the simple complex Lie superalgebra $\mathfrak{psl}_{2|2}$. For this, we follow the seminal work of Kac, Roan and Wakimoto [41] to identify generators and compute their defining operator product expansions explicitly.

2.1 The Lie superalgebra $\mathfrak{psl}_{2|2}$

Recall that $\mathfrak{sl}_{2|2}$ is the Lie superalgebra whose elements are the endomorphisms of the $\mathbb{Z}_2$-graded vector superspace $\mathbb{C}^{2|2}$ with zero supertrace. The quotient of this superalgebra by the ideal spanned by the identity matrix is denoted by $\mathfrak{psl}_{2|2}$. Let E_{ij}, $i,j = 1,\dots,4$, denote the 4×4 matrix whose (i,j)-th entry is 1 and whose other entries are 0. We fix a basis of $\mathfrak{psl}_{2|2}$ as follows, indicating parity and understanding that each E_{ij} is implicitly replaced by its image in the quotient

$$\begin{aligned}
&\left.\begin{aligned} &E^1 = E_{12}, \quad H^1 = E_{11} - E_{22}, \quad F^1 = E_{21}, \\ &E^2 = E_{34}, \quad H^2 = E_{33} - E_{44}, \quad F^2 = E_{43} \end{aligned}\right. \qquad \text{(even)}, \\
&\left.\begin{aligned} &e^{++} = +E_{14}, \quad e^{+-} = -E_{13}, \quad e^{-+} = +E_{24}, \quad e^{--} = -E_{23}, \\ &f^{++} = -E_{32}, \quad f^{+-} = -E_{42}, \quad f^{-+} = +E_{31}, \quad f^{--} = +E_{41} \end{aligned}\right. \qquad \text{(odd)}.
\end{aligned} \tag{2.1}$$

Here are some useful facts about $\mathfrak{psl}_{2|2}$ and the basis that we have chosen:

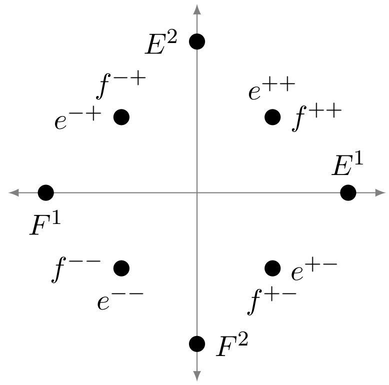


Figure 1. The roots of the simple Lie superalgebra $\mathfrak{psl}_{2|2}$, labelled by their root vectors (the odd roots have multiplicity 2). The horizontal axis indicates the eigenvalue under the adjoint action of H^1 while the vertical axis records it for H^2.

- The even subalgebra is isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Both $\{E^1, H^1, F^1\}$ and $\{E^2, H^2, F^2\}$ are $\mathfrak{sl}_2$-bases.
- The four odd elements labelled with an e span a copy of the tensor product of two fundamental $\mathfrak{sl}_2$-modules. The signs in (2.1) ensure that the actions of the E^i and F^i, $i = 1, 2$, just exchange the i-th $\pm$ index. For example, $F^1 e^{++} = e^{-+}$ and $E^2 e^{+-} = e^{++}$.
- The previous property also holds for the odd elements labelled with an f.
- The odd roots have multiplicity 2. The corresponding root vectors are nilpotent: $\left[e^{\pm\pm}, e^{\pm\pm}\right] = \left[f^{\pm\pm}, f^{\pm\pm}\right] = 0$. (Here and throughout, $[\cdot,\cdot]$ denotes a Lie superbracket, modelled on the anticommutator if both elements are odd and on the commutator if at least one element is even.)
- In fact, the odd elements labelled by an e all anticommute with one another. The same is true for the odd elements labelled by an f.
- The nonzero anticommutators are as follows:

$$\begin{aligned}
&\left[e^{++}, f^{+-}\right] = -E^1, && \left[e^{--}, f^{++}\right] = -\tfrac{1}{2}\left(H^1 - H^2\right), && \left[e^{--}, f^{+-}\right] = +F^2, \\
&\left[e^{+-}, f^{++}\right] = +E^1, && \left[e^{+-}, f^{-+}\right] = -\tfrac{1}{2}\left(H^1 + H^2\right), && \left[e^{+-}, f^{--}\right] = -F^2, \\
&\left[e^{-+}, f^{++}\right] = -E^2, && \left[e^{-+}, f^{+-}\right] = +\tfrac{1}{2}\left(H^1 + H^2\right), && \left[e^{-+}, f^{--}\right] = +F^1, \\
&\left[e^{++}, f^{-+}\right] = +E^2, && \left[e^{++}, f^{--}\right] = +\tfrac{1}{2}\left(H^1 - H^2\right), && \left[e^{--}, f^{-+}\right] = -F^1.
\end{aligned}$$

- The Killing form is identically zero, but the supertrace form in the defining representation (2.1) is supersymmetric and nondegenerate. The nonzero entries of this supertrace form are as follows:

$$\begin{aligned}
&\left\langle E^1, F^1\right\rangle = +1, && \left\langle H^1, H^1\right\rangle = +2, && \left\langle E^2, F^2\right\rangle = -1, && \left\langle H^2, H^2\right\rangle = -2, \\
&\left\langle e^{++}, f^{--}\right\rangle = +1, && \left\langle e^{+-}, f^{-+}\right\rangle = -1, && \left\langle e^{-+}, f^{+-}\right\rangle = -1, \\
&\left\langle e^{--}, f^{++}\right\rangle = +1.
\end{aligned} \tag{2.2}$$

We picture the root system of $\mathfrak{psl}_{2|2}$ in Figure 1, indicating by position the eigenvalues with respect to the basis $\{H^1, H^2\}$ of the Cartan subalgebra.

The Weyl group of $\mathfrak{psl}_{2|2}$ is, by definition, that of the even subalgebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, hence it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. However, the automorphism group of the root system is clearly isomorphic to D_4. The additional automorphisms are generated by the reflection that fixes the odd root of weight $(1,1)$. This automorphism corresponds to swapping the two copies of $\mathfrak{sl}_2$. It moreover lifts to an order-2 automorphism ω of $\mathfrak{psl}_{2|2}$ defined by

$$\begin{aligned} &E^1 \longleftrightarrow E^2, && H^1 \longleftrightarrow H^2, && F^1 \longleftrightarrow F^2, \\ &e^{++} \longleftrightarrow f^{++}, && e^{+-} \longleftrightarrow f^{-+}, && e^{-+} \longleftrightarrow f^{+-}, && e^{--} \longleftrightarrow f^{--}. \end{aligned}$$

Interestingly, this automorphism does not leave the supertrace form invariant, but instead negates it

$$\big\langle \omega(A), \omega(B) \big\rangle = -\langle A, B \rangle, \qquad A, B \in \mathfrak{psl}_{2|2}. \tag{2.3}$$

2.2 The principal reduction

The (untwisted) affine Kac–Moody superalgebra $\widehat{\mathfrak{psl}}_{2|2}$ is spanned by elements A_n and K, for $A \in \mathfrak{psl}_{2|2}$ and $n \in \mathbb{Z}$, where K is even and A_n inherits its parity from that of A. As usual, K is central and the Lie superbrackets for the A_n are given by

$$[A_m, B_n] = [A, B]_{m+n} + m\langle A, B \rangle \delta_{m+n,0} K, \qquad A, B \in \mathfrak{psl}_{2|2}, \quad m, n \in \mathbb{Z}.$$

Because of (2.3), the automorphism ω of $\mathfrak{psl}_{2|2}$ lifts to an automorphism of $\widehat{\mathfrak{psl}}_{2|2}$ by defining

$$\widehat{\omega}(A_n) = \omega(A)_n \qquad \text{and} \qquad \widehat{\omega}(K) = -K, \qquad A \in \mathfrak{psl}_{2|2}, \quad n \in \mathbb{Z}.$$

The universal level-k affine vertex superalgebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{2|2})$ is defined, as always, to be strongly and freely generated by the fields $A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}$, $A \in \mathfrak{psl}_{2|2}$, subject to the operator product expansions

$$A(z)B(w) \sim \frac{\langle A, B \rangle \mathsf{k} \mathbb{1}}{(z-w)^2} + \frac{[A, B](z)}{z-w}, \qquad A, B \in \mathfrak{psl}_{2|2}.$$

Here, $\mathbb{1}$ denotes the identity field and the A_n furnish a representation of $\widehat{\mathfrak{psl}}_{2|2}$ on $\mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{2|2})$ in which K acts as multiplication by the level $\mathsf{k} \in \mathbb{C}$. We note that the automorphism $\widehat{\omega}$ of $\widehat{\mathfrak{psl}}_{2|2}$ again lifts, but this time to an *isomorphism* between $\mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{2|2})$ and $\mathsf{V}^{-\mathsf{k}}(\mathfrak{psl}_{2|2})$. It follows, in particular, that the representation theory of this vertex superalgebra (and any of its quotients) is independent of the sign of the level.

Because the supertrace form equation (2.2) is nondegenerate, the Sugawara construction equips $\mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{2|2})$ with the structure of a vertex-operator superalgebra, when the level is noncritical. The dual Coxeter number of $\mathfrak{psl}_{2|2}$ is 0 because the Killing form vanishes, so for all $\mathsf{k} \neq 0$, the energy-momentum tensor takes the form

$$\begin{aligned} T = \tfrac{1}{2\mathsf{k}} \big(&\tfrac{1}{2} {:}H^1 H^1{:} + {:}E^1 F^1{:} + {:}F^1 E^1{:} - \tfrac{1}{2} {:}H^2 H^2{:} - {:}E^2 F^2{:} - {:}F^2 E^2{:} \\ &- \big({:}e^{++} f^{--}{:} - {:}f^{--} e^{++}{:} - {:}e^{+-} f^{-+}{:} + {:}f^{-+} e^{+-}{:} - {:}e^{-+} f^{+-}{:} \\ &+ {:}f^{+-} e^{-+}{:} + {:}e^{--} f^{++}{:} - {:}f^{++} e^{--}{:} \big) \big). \end{aligned}$$

The central charge is the superdimension of $\mathfrak{psl}_{2|2}$, namely -2.

As there are two nilpotent orbits in $\mathfrak{sl}_2$, there are four even orbits in $\mathfrak{psl}_{2|2}$. Along with the zero orbit, there are two minimal orbits, for which F^1 and F^2 are representatives, and the principal one, for which $F^1 + F^2$ is a representative. It is well known that quantum Hamiltonian reduction $\mathsf{W}^{\mathsf{k}}_{\mathrm{min}}$ with respect to one of the minimal nilpotents is isomorphic to the universal (small) $N = 4$ superconformal vertex-operator superalgebra of central charge $-6(\mathsf{k}+1)$, see [41, Remark 4.1] and [42, Section 8.4]. Applying $\widehat{\omega}$, we see that the other minimal reduction is isomorphic to $\mathsf{W}^{-\mathsf{k}}_{\mathrm{min}}$ with central charge $6(\mathsf{k}-1)$. We leave further consideration of these minimal reductions to Section 4.

Our interest is in the principal reduction $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$, which does not seem to have received as much attention. We observe that $\{E^1+E^2, H^1+H^2, F^1+F^2\}$ is an $\mathfrak{sl}_2$-triple and that the adjoint action of H^1+H^2 gives a good even grading on $\mathfrak{psl}_{2|2}$

$$\mathfrak{psl}_{2|2} = \mathfrak{psl}_{2|2}^{(-2)} \oplus \mathfrak{psl}_{2|2}^{(0)} \oplus \mathfrak{psl}_{2|2}^{(+2)}, \qquad \begin{aligned} \mathfrak{psl}_{2|2}^{(+2)} &= \mathrm{span}\{E^1, E^2 \mid e^{++}, f^{++}\}, \\ \mathfrak{psl}_{2|2}^{(0)} &= \mathrm{span}\{H^1, H^2 \mid e^{+-}, e^{-+}, f^{+-}, f^{-+}\}, \\ \mathfrak{psl}_{2|2}^{(-2)} &= \mathrm{span}\{F^1, F^2 \mid e^{--}, f^{--}\}. \end{aligned}$$

By [42, Theorem 4.1], $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ is then strongly and freely generated, up to some operator product expansions that must be computed, by two even fields of conformal weight 2 and four odd fields of conformal weights 1, 1, 2 and 2.

Following the recipe in [42], we associate to each even basis element of $\mathfrak{psl}_{2|2}^{(+2)}$ a fermionic ghost vertex superalgebra FG^i, $i=1,2$, and to each odd basis element a bosonic ghost vertex algebra BG^j, $j=e,f$. These are strongly and freely generated by fields satisfying

$$b^i(z)c^i(w) \sim \frac{\mathbb{1}}{z-w} \qquad \text{and} \qquad \beta^j(z)\gamma^j(w) \sim -\frac{\mathbb{1}}{z-w},$$

respectively, with all other operator product expansions of the generators being regular. We next form the graded complex $\mathsf{C} = \mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{2|2}) \otimes \mathsf{FG}^1 \otimes \mathsf{FG}^2 \otimes \mathsf{BG}^e \otimes \mathsf{BG}^f$, in which the b^i and β^j have grade -1, the c^i and γ^j have grade $+1$, and all other generators (and vacuum states) have grade 0. This becomes a differential graded complex upon taking the differential to be the zero mode of

$$D = (E^1+\mathbb{1})c^1 + (E^2-\mathbb{1})c^2 - e^{++}\gamma^e - f^{++}\gamma^f.$$

D_0 is odd, has grade $+1$ and is easily checked to square to zero. As a vertex superalgebra, $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ is the grade-0 cohomology of C with respect to D_0.

To give $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$, $\mathsf{k} \neq 0$, the structure of a vertex-operator superalgebra, we choose an energy-momentum tensor L on C. This needs to make D homogeneous of conformal weight 1, so we take

$$L = T + \tfrac{1}{2}(\partial H^1 + \partial H^2) + {:}\partial b^1 c^1{:} + {:}\partial b^2 c^2{:} + {:}\partial\beta^e\gamma^e{:} + {:}\partial\beta^f\gamma^f{:}.$$

It is easy to check that C is now a vertex-operator superalgebra of central charge -2 and that $D_0 L = 0$. Moreover, L defines an energy-momentum tensor on $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$, as desired, by [41, Theorem 2.2].

The next step is to construct the "parenthetical building blocks" of the generators of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$. These are obtained by correcting some of the generators of $\mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{2|2})$ by adding bilinear terms in the ghost fields, see [42, equation (2.4)]. We need to compute them for all elements in $\mathfrak{psl}_{2|2}^{(0)}$, the results for our basis being

$$\begin{aligned} H^{(1)} &= H^1 + 2{:}b^1c^1{:} + {:}\beta^e\gamma^e{:} + {:}\beta^f\gamma^f{:}, & H^{(2)} &= H^2 + 2{:}b^2c^2{:} + {:}\beta^e\gamma^e{:} + {:}\beta^f\gamma^f{:}, \\ e^{(+-)} &= e^{+-} - {:}\beta^e c^2{:} + {:}b^1\gamma^f{:}, & e^{(-+)} &= e^{-+} - {:}\beta^e c^1{:} - {:}b^2\gamma^f{:}, \\ f^{(+-)} &= f^{+-} - {:}\beta^f c^2{:} - {:}b^1\gamma^e{:}, & f^{(-+)} &= f^{-+} - {:}\beta^f c^1{:} + {:}b^2\gamma^e{:}. \end{aligned} \tag{2.4}$$

The generating fields of conformal weight 1 in $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ are then (the images in cohomology of) the linear combinations of the parenthetical building blocks that commute with F [42, Theorem 4.1]. In this way, we find two odd generators $\chi^e = e^{(+-)} - e^{(-+)}$ and $\chi^f = f^{(+-)} - f^{(-+)}$. It is easy to check that these are D_0-closed.

To determine the weight-2 generating fields of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$, we must find D_0-closed corrections to the elements of $\mathfrak{psl}^{(-2)}_{2|2}$ (all of which commute with F). This time, [42] does not give a precise recipe, only guaranteeing that the corrections belong to the subalgebra generated by the parenthetical building blocks (2.4). Finding the corrections by brute force is quite tractable in this case (we used OPEdefs [52]). The resulting four generators are

$$\begin{aligned}
B^1 &= F^1 - \frac{1}{4}{:}H^{(1)}H^{(1)}{:} - \frac{\mathsf{k}}{2}\partial H^{(1)} - {:}e^{(-+)}f^{(-+)}{:} \\
&\quad - \frac{(2\mathsf{k}-1)(3\mathsf{k}+1)}{4\mathsf{k}}{:}\big(e^{(+-)} - e^{(-+)}\big)\big(f^{(+-)} - f^{(-+)}\big){:}, \\
B^2 &= F^2 + \frac{1}{4}{:}H^{(2)}H^{(2)}{:} - \frac{\mathsf{k}}{2}\partial H^{(2)} - {:}e^{(+-)}f^{(+-)}{:} \\
&\quad + \frac{(2\mathsf{k}+1)(3\mathsf{k}-1)}{4\mathsf{k}}{:}\big(e^{(+-)} - e^{(-+)}\big)\big(f^{(+-)} - f^{(-+)}\big){:}, \\
\psi^e &= e^{--} - \frac{1}{2}\big({:}H^{(1)}e^{(+-)}{:} - {:}H^{(2)}e^{(-+)}{:}\big) - \frac{2\mathsf{k}-1}{4}\partial e^{(+-)} - \frac{2\mathsf{k}+1}{4}\partial e^{(-+)}, \\
\psi^f &= f^{--} - \frac{1}{2}\big({:}H^{(1)}f^{(+-)}{:} - {:}H^{(2)}f^{(-+)}{:}\big) - \frac{2\mathsf{k}-1}{4}\partial f^{(+-)} - \frac{2\mathsf{k}+1}{4}\partial f^{(-+)},
\end{aligned}$$

where unique solutions were obtained by requiring in addition that the generators be primary with respect to L. We record that (up to D_0-exact terms)

$$L = -\frac{1}{\mathsf{k}}\big(B^1 + B^2\big) - \frac{1}{2\mathsf{k}}{:}\chi^e\chi^f{:}.$$

2.3 Operator product expansions

Having explicitly computed the generating fields of the universal level-k principal W-algebra $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$, we now turn to its operator product expansions. By [42, Theorem 4.1 and Remark 4.2], $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ admits a Poincaré–Birkhoff–Witt basis in the modes of the generating fields. It follows that $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ is strongly and freely generated by B^1, B^2, χ^e, χ^f, ψ^e and ψ^f, subject only to their operator product expansions. In principle, these relations might only close modulo D_0-exact fields. Happily, this is not the case.

The resulting operator product expansions are nevertheless somewhat messy. To bring out some of the structural features, we first present those of the generating fields of conformal weight 1

$$\chi^e(z)\chi^e(w) \sim 0, \qquad \chi^e(z)\chi^f(w) \sim \frac{2\mathsf{k}\mathbb{1}}{(z-w)^2}, \qquad \chi^f(z)\chi^f(w) \sim 0.$$

The weight-1 fields of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ thus generate a subalgebra isomorphic to a pair of symplectic fermions [44]. We may even remove the apparent level dependence by defining renormalised symplectic fermions thusly $\chi = \frac{1}{\sqrt{\mathsf{k}}}\chi^e$ and $\bar{\chi} = \frac{1}{\sqrt{\mathsf{k}}}\chi^f$.

To choose a better basis of weight-2 generating fields, we require that they transform "cleanly" under the action of the symplectic fermion subalgebra. More precisely, the space of weight-2 generators forms a representation of the zero modes of the symplectic fermions vertex-operator superalgebra $\mathsf{SF} = \mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{1|1})$ (these are isomorphic for all $\mathsf{k} \neq 0$), so we can look for a basis that respects the structure of this representation. Since the zero modes of the generators define a copy of $\mathfrak{psl}_{1|1}$, a Lie superalgebra with relatively few representations, it is in fact rather easy to identify ours. Indeed, if we define

$$S = -\frac{1}{\mathsf{k}}\big(B^1 + B^2\big), \qquad H = \tfrac{1}{2}\big(B^2 - B^1\big) - \tfrac{1}{2}S, \qquad \psi = \frac{1}{\sqrt{\mathsf{k}}}\psi^e \qquad \text{and} \qquad \bar{\psi} = \frac{1}{\sqrt{\mathsf{k}}}\psi^f,$$

it is straightforward to verify (again, using OPEdefs) that

$$\chi_0 H = \psi, \qquad \bar{\chi}_0 H = \bar{\psi}, \qquad \chi_0 \bar{\psi} = +S, \qquad \bar{\chi}_0 \psi = -S$$
$$\text{and} \qquad \chi_0 \psi = \bar{\chi}_0 \bar{\psi} = \chi_0 S = \bar{\chi}_0 S = 0.$$

The space spanned by the weight-2 generators is thus a copy of the unique projective module of $\mathfrak{psl}_{1|1}$, which is itself isomorphic to $\mathsf{U}(\mathfrak{psl}_{1|1})$.

Note that the energy-momentum tensor of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ is now given by $L = S - \frac{1}{2}{:}\chi\bar{\chi}{:}$. Because of this, we may replace S by L in our set of strong generators. We record the defining operator product expansions of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ in the following Theorem 2.1.

Theorem 2.1. *The principal quantum Hamiltonian reduction $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ of $\mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{2|2})$ is strongly and freely generated by two even elements, L and H, and four odd elements, χ, $\bar{\chi}$, ψ and $\bar{\psi}$. Here, L is a conformal vector of central charge -2 and the generators H, χ, $\bar{\chi}$, ψ and $\bar{\psi}$ are Virasoro primaries with respective conformal weights 2, 1, 1, 2 and 2. The remaining operator product expansions are as follows:*

$$\begin{aligned} &\chi(z)\chi(w) \sim \bar{\chi}(z)\bar{\chi}(w) \sim 0, \qquad \chi(z)\bar{\psi}(w) \sim +\frac{S(w)}{z-w}, \\ &\chi(z)\psi(w) \sim \bar{\chi}(z)\bar{\psi}(w) \sim 0, \qquad \bar{\chi}(z)\psi(w) \sim -\frac{S(w)}{z-w}, \qquad \chi(z)\bar{\chi}(w) \sim \frac{2\mathbb{1}}{(z-w)^2}, \end{aligned} \tag{2.5a}$$

$$\begin{aligned} &\chi(z)H(w) \sim -3\big(\mathsf{k}^2-\tfrac{1}{4}\big)\frac{\chi(w)}{(z-w)^2} + \frac{\psi(w)}{z-w}, \qquad \psi(z)\psi(w) \sim \frac{{:}\chi\psi{:}(w)}{z-w}, \\ &\bar{\chi}(z)H(w) \sim -3\big(\mathsf{k}^2-\tfrac{1}{4}\big)\frac{\bar{\chi}(w)}{(z-w)^2} + \frac{\bar{\psi}(w)}{z-w}, \qquad \bar{\psi}(z)\bar{\psi}(w) \sim \frac{{:}\bar{\chi}\bar{\psi}{:}(w)}{z-w}, \end{aligned} \tag{2.5b}$$

$$\begin{aligned} H(z)\psi(w) \sim{}& \tfrac{3}{2}\big(\mathsf{k}^2-\tfrac{1}{4}\big)\left[\frac{\chi(w)}{(z-w)^3} + \frac{\frac{1}{2}\partial\chi(w)}{(z-w)^2} + \frac{\frac{1}{6}\partial^2\chi(w)}{z-w}\right] \\ &+ \frac{\frac{1}{4}\partial\psi(w) - {:}\chi\big(H + \frac{3\mathsf{k}^2}{2}S\big){:}(w)}{z-w}, \\ H(z)\bar{\psi}(w) \sim{}& \tfrac{3}{2}\big(\mathsf{k}^2-\tfrac{1}{4}\big)\left[\frac{\bar{\chi}(w)}{(z-w)^3} + \frac{\frac{1}{2}\partial\bar{\chi}(w)}{(z-w)^2} + \frac{\frac{1}{6}\partial^2\bar{\chi}(w)}{z-w}\right] \\ &+ \frac{\frac{1}{4}\partial\bar{\psi}(w) - {:}\bar{\chi}\big(H + \frac{3\mathsf{k}^2}{2}S\big){:}(w)}{z-w}, \end{aligned} \tag{2.5c}$$

$$\begin{aligned} H(z)H(w) \sim{}& -3(3\mathsf{k}^2-1)\big(\mathsf{k}^2-\tfrac{1}{4}\big)\left[\frac{\mathbb{1}}{(z-w)^4} + \frac{{:}\chi\bar{\chi}{:}(w)}{(z-w)^2} + \frac{\frac{1}{2}\partial{:}\chi\bar{\chi}{:}(w)}{z-w}\right] \\ &+ \frac{7\mathsf{k}^2-1}{2}\left[\frac{S(w)}{(z-w)^2} + \frac{\frac{1}{2}\partial S(w)}{z-w}\right], \end{aligned} \tag{2.5d}$$

$$\begin{aligned} \psi(z)\bar{\psi}(w) \sim{}& 3\big(\mathsf{k}^2-\tfrac{1}{4}\big)\left[\frac{\mathbb{1}}{(z-w)^4} + \frac{{:}\chi\bar{\chi}{:}(w)}{(z-w)^2} + \frac{\frac{1}{2}\partial{:}\chi\bar{\chi}{:}(w)}{z-w}\right] \\ &- \left[\frac{2H(w) + \frac{1}{2}S(w)}{(z-w)^2} + \frac{\partial H(w) + \frac{1}{4}\partial S(w) - \frac{1}{2}{:}\chi\bar{\psi}{:}(w) - \frac{1}{2}{:}\bar{\chi}\psi{:}(w)}{z-w}\right]. \end{aligned} \tag{2.5e}$$

In Theorem 2.1, we chose to present the defining operator product expansions (2.5) so that the right-hand sides involve S rather than L, in order to respect the $\mathfrak{psl}_{1|1}$-symmetry. For completeness, we also give the operator product expansions in which the left-hand sides involve S

$$\chi(z)S(w) \sim \bar{\chi}(z)S(w) \sim 0, \qquad S(z)S(w) \sim \frac{2S(w)}{(z-w)^2} + \frac{\partial S(w)}{z-w}, \tag{2.6a}$$

$$
\begin{aligned}
S(z)\psi(w) &\sim \frac{2\psi(w)}{(z-w)^2} + \frac{\partial\psi(w) - \frac{1}{2}{:}\chi S{:}(w)}{z-w}, \\
S(z)\bar{\psi}(w) &\sim \frac{2\bar{\psi}(w)}{(z-w)^2} + \frac{\partial\bar{\psi}(w) - \frac{1}{2}{:}\bar{\chi} S{:}(w)}{z-w},
\end{aligned} \tag{2.6b}
$$

$$
\begin{aligned}
S(z)H(w) \sim &- 3\big(\mathsf{k}^2 - \tfrac{1}{4}\big)\left[\frac{\mathbb{1}}{(z-w)^4} + \frac{{:}\chi\bar{\chi}{:}(w)}{(z-w)^2} + \frac{\frac{1}{2}\partial{:}\chi\bar{\chi}{:}(w)}{z-w}\right] \\
&+ \frac{2H(w) + \frac{1}{2}S(w)}{(z-w)^2} + \frac{\partial H(w) + \frac{1}{2}{:}\chi\bar{\psi}{:}(w) - \frac{1}{2}{:}\bar{\chi}\psi{:}(w)}{z-w}.
\end{aligned} \tag{2.6c}
$$

We make a few additional comments about $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$:

- The operator product expansions (2.5) only depend on k^2, not k, so that $\mathsf{W}^{\mathsf{k}}_{\text{pr}} \cong \mathsf{W}^{-\mathsf{k}}_{\text{pr}}$. This is an obvious consequence of the isomorphism $\mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{2|2}) \cong \mathsf{V}^{-\mathsf{k}}(\mathfrak{psl}_{2|2})$, noted in Section 2.2, and the fact that the principal nilpotent $F^1 + F^2 \in \mathfrak{psl}_{2|2}$ is invariant under ω.
- S is an energy-momentum tensor of central charge 0. It commutes with the symplectic fermions subalgebra generated by χ and $\bar{\chi}$. It would be interesting to identify the commutant of this subalgebra in $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$.
- When $\mathsf{k}^2 = \frac{1}{4}$ (hence $\mathsf{k} = \pm\frac{1}{2}$), the level is "collapsing" [6], meaning that the simple quotient $\mathsf{W}^{\text{pr}}_{\mathsf{k}}$ of $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ is strongly generated by just its weight-1 fields. To see this, note that H, S, ψ and $\bar{\psi}$ generate an ideal of $\mathsf{W}^{\pm 1/2}_{\text{pr}}$ and that the quotient is the symplectic fermions subalgebra (which is simple): $\mathsf{W}^{\text{pr}}_{\pm 1/2} \cong \mathsf{SF}$.
- $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ admits a horizontal grading in which χ and ψ have grade $+1$, $\bar{\chi}$ and $\bar{\psi}$ have grade -1, and H, S and L have grade 0. Its graded character $\big(\text{ch}^+\big)$ and supercharacter (ch^-) are thus

$$
\text{ch}^{\pm}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big](\mathsf{y};\mathsf{q}) = \mathsf{q}^{1/12}\prod_{i=1}^{\infty}\frac{\big(1\pm\mathsf{y}\mathsf{q}^{i}\big)\big(1\pm\mathsf{y}^{-1}\mathsf{q}^{i}\big)\big(1\pm\mathsf{y}\mathsf{q}^{i+1}\big)\big(1\pm\mathsf{y}^{-1}\mathsf{q}^{i+1}\big)}{\big(1-\mathsf{q}^{i+1}\big)^2}. \tag{2.7}
$$

- Based on numerical investigations with the analogue of the Shapovalov form, we conjecture that when $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ is not simple, then the singular vector of minimal conformal weight has grade $+1$ and multiplicity 1. The conformal weight is 2 for $\mathsf{k} = \pm\frac{1}{2}$, 4 for $\mathsf{k} = \pm\frac{1}{3}, \pm\frac{3}{2}$, and 6 for $\mathsf{k} = \pm\frac{1}{4}, \pm\frac{2}{3}, \pm\frac{5}{2}$.

The simplicity of $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ is obviously of interest. The following result establishes this for all but a few levels.

Theorem 2.2. *The vertex-operator superalgebra* $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ *is simple if* $\mathsf{k} \notin \mathbb{Q}$ *or* $\mathsf{k} \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}$. *It is not simple if* $\mathsf{k} \in \mathbb{Q} \setminus \mathbb{Z}$.

Proof. We make use of the Semikhatov realisation $\mathsf{W}^{\mathsf{k}}_{\text{min}} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{\text{pr}} \otimes \Pi$ of Theorem 4.1 and (a spectral-flow twist of) an Adamović functor (4.3), see Section 4.1 below. They connect the (universal) principal and minimal W-algebras of $\mathfrak{psl}_{2|2}$ via inverse quantum Hamiltonian reduction.

- Building on [35, 38], Gorelik and Kac prove in [36, Corollary 1.1 (iv)] that $\mathsf{W}^{\mathsf{k}}_{\text{min}}$ is simple if $\mathsf{k} \notin \mathbb{Q}$ or $\mathsf{k} \in \mathbb{Z}_{\geqslant 3}$ and that it is not simple if $\mathsf{k} \in \mathbb{Q} \setminus \mathbb{Z}$.
- Assume that $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ has a nonzero proper submodule I. Then, applying an Adamović functor shows that $\mathsf{I} \otimes \Pi$ is a nonzero proper $\mathsf{W}^{\mathsf{k}}_{\text{min}}$-submodule of $\mathsf{W}^{\mathsf{k}}_{\text{pr}} \otimes \Pi$. The argument of [8, Lemma 3.14] now proves that for any $v \in \mathsf{I}$, we have $v \otimes \mathrm{e}^{2nc} \in \mathsf{W}^{\mathsf{k}}_{\text{min}}$ for all sufficiently large $n \in \mathbb{Z}$. $\mathsf{I} \otimes \Pi$ is thus a nonzero proper submodule of $\mathsf{W}^{\mathsf{k}}_{\text{min}}$. Contrapositively, $\mathsf{k} \notin \mathbb{Q}$ or $\mathsf{k} \in \mathbb{Z}_{\geqslant 3}$ implies that $\mathsf{W}^{\mathsf{k}}_{\text{min}}$ is simple, hence so are $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ and $\mathsf{W}^{-\mathsf{k}}_{\text{pr}}$.

- Next, assume that $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ is simple *and* that $\mathsf{k} \notin \mathbb{Z}_{\leqslant 0}$. Then, $\mathsf{W}^{\mathsf{k}}_{\text{pr}} \otimes \Pi$ is (up to a spectral flow) an almost-irreducible $\mathsf{W}^{\mathsf{k}}_{\text{min}}$-module containing $\mathsf{W}^{\mathsf{k}}_{\text{min}}$. But, this forces $\mathsf{W}^{\mathsf{k}}_{\text{min}}$ to be simple because the restriction on k rules out a singular vector of the form $\mathbb{1} \otimes \mathrm{e}^{2nc}$, $n \in \mathbb{Z}_{>0}$. Contrapositively, $\mathsf{k} \in \mathbb{Q} \setminus \mathbb{Z}$ implies that $\mathsf{W}^{\mathsf{k}}_{\text{min}}$ is not simple, hence that $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ is not simple either. ■

We thank Drazen Adamović for pointing out to us the work [36] leading to the proof of Theorem 2.2. The simplicity of $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ is not settled for $\mathsf{k} = \pm 1, \pm 2$ ($\mathsf{k} = 0$ is critical), but we expect a positive answer in these cases. We also thank an anonymous reviewer for sketching an alternative, and extremely interesting, approach to proving this result when $\mathsf{k} \in \mathbb{Q} \setminus \mathbb{Z}$.

3 Irreducible $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$-modules

We next study the representation theory of $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$. Being a principal W-algebra, we expect that its irreducible modules are all ordinary, meaning that the L_0-eigenspaces are all finite-dimensional. This is proven under the additional hypothesis that the irreducible is lower bounded. We also conjecture that every irreducible $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$-module is lower bounded.

3.1 The principal Zhu algebra

The Zhu algebra $\mathsf{Zhu}[\mathsf{V}]$ of a vertex-operator superalgebra V is the unital associative superalgebra of zero modes of the fields, restricted to act on states that are annihilated by all positive modes. Introduced formally in [43, 54], $\mathsf{Zhu}[\mathsf{V}]$ may be realised as a quotient of V with an associative product given by

$$A_\circ B_\circ = \sum_{j=0}^{\Delta_A} \binom{\Delta_A}{j} (A_{-\Delta_A+j} B)_\circ, \qquad A, B \in \mathsf{V}. \tag{3.1}$$

Here, $A_\circ$ denotes the image of A in $\mathsf{Zhu}[\mathsf{V}]$ and Δ_A is the conformal weight of A (which is assumed to be homogeneous). It inherits its parity from that of A. The quotient operation will not be important for us here, except to note that it includes the identification

$$(\partial A)_\circ = -\Delta_A A_\circ, \qquad A \in \mathsf{V}. \tag{3.2}$$

It is now easy to use (3.1) and (3.2) to deduce relations in $\mathsf{Zhu}[\mathsf{W}^{\mathsf{k}}_{\text{pr}}]$ involving the images of the strong generators

$$\begin{gathered} (\chi_\circ)^2 = (\bar{\chi}_\circ)^2 = 0, \qquad [\chi_\circ, S_\circ] = [\bar{\chi}_\circ, S_\circ] = 0, \qquad [\chi_\circ, H_\circ] = \psi_\circ, \qquad [\bar{\chi}_\circ, H_\circ] = \bar{\psi}_\circ, \\ [\chi_\circ, \bar{\chi}_\circ] = 0, \qquad [\chi_\circ, \psi_\circ] = [\bar{\chi}_\circ, \bar{\psi}_\circ] = 0, \qquad [\chi_\circ, \bar{\psi}_\circ] = +S_\circ, \qquad [\bar{\chi}_\circ, \psi_\circ] = -S_\circ, \\ (\psi_\circ)^2 = \tfrac{1}{2}\chi_\circ\psi_\circ, \qquad (\bar{\psi}_\circ)^2 = \tfrac{1}{2}\bar{\chi}_\circ\bar{\psi}_\circ, \qquad [\psi_\circ, S_\circ] = \tfrac{1}{2}\chi_\circ S_\circ, \qquad [\bar{\psi}_\circ, S_\circ] = \tfrac{1}{2}\bar{\chi}_\circ S_\circ, \\ [\psi_\circ, \bar{\psi}_\circ] = \tfrac{1}{2}(\chi_\circ\bar{\psi}_\circ + \bar{\chi}_\circ\psi_\circ), \qquad [\psi_\circ, H_\circ] = \chi_\circ\left(H_\circ + \frac{3\mathsf{k}^2}{2} L_\circ + \tfrac{1}{4}(\mathsf{k}^2 - \tfrac{1}{4})\mathbb{1}_\circ\right) - \tfrac{1}{2}\psi_\circ, \\ [S_\circ, H_\circ] = \tfrac{1}{2}(\chi_\circ\bar{\psi}_\circ + \psi_\circ\bar{\chi}_\circ), \qquad [\bar{\psi}_\circ, H_\circ] = \bar{\chi}_\circ\left(H_\circ + \frac{3\mathsf{k}^2}{2} L_\circ + \tfrac{1}{4}(\mathsf{k}^2 - \tfrac{1}{4})\mathbb{1}_\circ\right) - \tfrac{1}{2}\bar{\psi}_\circ. \end{gathered} \tag{3.3}$$

We shall not do so here, but one can prove that the images of the strong generators, along with the relations (3.3), comprise a presentation of $\mathsf{Zhu}[\mathsf{W}^{\mathsf{k}}_{\text{pr}}]$. Instead, we record some useful observations:

- As usual, $L_\circ = S_\circ + \frac{1}{2}\bar{\chi}_\circ\chi_\circ$ is central.
- The Zhu-images of the odd weight-2 fields are not nilpotent of order 2, as one might expect, but of order 3: $(\psi_\circ)^3 = \frac{1}{2}\chi_\circ(\psi_\circ)^2 = \frac{1}{4}(\chi_\circ)^2\psi_\circ = 0$.

- The horizontal grading of $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$ descends to a grading of $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$ in which $\chi_\circ$ and $\psi_\circ$ have grade $+1$, $\bar{\chi}_\circ$ and $\bar{\psi}_\circ$ have grade -1, and $H_\circ$, $L_\circ$ and $S_\circ$ have grade 0.
- A straightforward consequence of the relations (3.3) is that monomials in the generators of $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$ may be consistently ordered so that the odd elements with bars (no bars) are placed to the left (right) of the even elements. For example, the juxtaposition $\psi_\circ H_\circ$ may be so reordered by writing

$$\begin{aligned}\psi_\circ H_\circ &= H_\circ\psi_\circ + \chi_\circ\left(H_\circ + \frac{3\mathsf{k}^2}{2}L_\circ + \tfrac{1}{4}\big(\mathsf{k}^2 - \tfrac{1}{4}\big)\mathbb{1}_\circ\right) - \tfrac{1}{2}\psi_\circ \\ &= \big(H_\circ + \tfrac{1}{2}\mathbb{1}_\circ\big)\psi_\circ + \left(H_\circ + \frac{3\mathsf{k}^2}{2}L_\circ + \tfrac{1}{4}\big(\mathsf{k}^2 - \tfrac{1}{4}\big)\mathbb{1}_\circ\right)\chi_\circ.\end{aligned}$$

  This Poincaré–Birkhoff–Witt-style ordering suggests that $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$ admits a well behaved highest-weight theory.

3.2 Modules of the Zhu algebra

Let us define a weight vector in a $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$-module to be a simultaneous eigenvector of $H_\circ$ and $L_\circ$. As usual, weight vectors always exist in (nonzero) finite-dimensional modules. We will call a weight vector a highest-weight vector if it is annihilated by both $\chi_\circ$ and $\psi_\circ$. Obviously, a highest-weight vector is also an eigenvector of $S_\circ$ (and the eigenvalue matches that of $L_\circ$).

Lemma 3.1. *If a* $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$*-module has a weight vector, then it has a highest-weight vector.*

Proof. Let v denote the hypothesised weight vector, so that $L_\circ v = \Delta v$ and $H_\circ v = hv$, for some $\Delta, h \in \mathbb{C}$. Then,

- If $\chi_\circ v = \psi_\circ v = 0$, then v is already highest-weight.
- If $\psi_\circ v = 0$ but $\chi_\circ v \neq 0$, then $\chi_\circ v$ is weight because $L_\circ$ is central and $H_\circ\chi_\circ v = \chi_\circ H_\circ v - \psi_\circ v = h\chi_\circ v$. But, $\chi_\circ\chi_\circ v = 0$ and $\psi_\circ\chi_\circ v = -\chi_\circ\psi_\circ v = 0$, so $\chi_\circ v$ is highest-weight.
- If $\chi_\circ v = 0$ but $\psi_\circ v \neq 0$, then $\psi_\circ v$ is weight because

$$H_\circ\psi_\circ v = \psi_\circ H_\circ v + \tfrac{1}{2}\psi_\circ v - \chi_\circ H_\circ v - \frac{3\mathsf{k}^2}{2}\chi_\circ L_\circ v - \tfrac{1}{4}\big(\mathsf{k}^2 - \tfrac{1}{4}\big)\chi_\circ v = \big(h + \tfrac{1}{2}\big)v.$$

  But, $\chi_\circ\psi_\circ v = -\psi_\circ\chi_\circ v = 0$ and $\psi_\circ\psi_\circ v = \frac{1}{2}\chi_\circ\psi_\circ v = 0$, so $\psi_\circ v$ is highest-weight.
- If $\chi_\circ v \neq 0$ and $\psi_\circ v \neq 0$ but $\chi_\circ\psi_\circ v = 0$, then

$$H_\circ\chi_\circ v = h\chi_\circ v - \psi_\circ v, \qquad H_\circ\psi_\circ v = \big(h + \tfrac{1}{2}\big)\psi_\circ v - \left(h + \frac{3\mathsf{k}^2}{2}\Delta + \tfrac{1}{4}\big(\mathsf{k}^2 - \tfrac{1}{4}\big)\right)\chi_\circ v.$$

  Some linear combination of $\chi_\circ v$ and $\psi_\circ v$ is therefore a weight vector. Since $\chi_\circ\psi_\circ v = 0$, this linear combination is annihilated by both $\chi_\circ$ and $\psi_\circ$, hence it is a highest-weight vector.
- Finally, if $\chi_\circ\psi_\circ v \neq 0$ (so $\chi_\circ v \neq 0$ and $\psi_\circ v \neq 0$), then $H_\circ\chi_\circ\psi_\circ v = h\chi_\circ\psi_\circ v$ and $\chi_\circ\chi_\circ\psi_\circ v = \psi_\circ\chi_\circ\psi_\circ v = 0$, so $\chi_\circ\psi_\circ v$ is the desired highest-weight vector. ■

Inspection of (3.3) demonstrates that $H_\circ$, $L_\circ$, $\chi_\circ$ and $\psi_\circ$ generate a unital subalgebra of $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$. We may therefore construct Verma modules in the usual way and realise all highest-weight modules, these being modules generated by a single highest-weight vector, as quotients. Let $\mathcal{V}^{h,\Delta}_\circ$ denote the Verma module generated by a highest-weight vector of $H_\circ$-eigenvalue h and $L_\circ$-eigenvalue Δ.

Proposition 3.2.

- $\mathcal{V}_\circ^{h,\Delta}$ *is* 4*-dimensional, for all* $h,\Delta\in\mathbb{C}$.
- $\mathcal{V}_\circ^{h,\Delta}$ *is irreducible unless* $\Delta=0$.
- *The irreducible quotient of* $\mathcal{V}_\circ^{h,0}$ *is* 1*-dimensional, for all* $h\in\mathbb{C}$.

Proof. Let v be the generating highest-weight vector of $\mathcal{V}_\circ^{h,\Delta}$. Because we can order monomials so that the "negative" generators $\bar{\chi}_\circ$ and $\bar{\psi}_\circ$ act last, $\mathcal{V}_\circ^{h,\Delta}$ is spanned by the monomials obtained by acting on v with these negative generators. It is easy to check that these monomials are v, $\bar{\chi}_\circ v$, $\bar{\psi}_\circ v$ and $\bar{\chi}_\circ\bar{\psi}_\circ v$, hence that $\mathcal{V}_\circ^{h,\Delta}$ is 4-dimensional.

To investigate irreducibility, consider first the submodule of $\mathcal{V}_\circ^{h,\Delta}$ generated by the linear combination $w=a\bar{\chi}_\circ v+b\bar{\psi}_\circ v$, where $a,b\in\mathbb{C}$ are not both 0. Using (3.3), we find that $\chi_\circ w=\Delta bv$ and $\psi_\circ w=\Delta\big(-a+\frac{1}{2}b\big)v$. When $\Delta\neq 0$, it follows that this submodule contains v, for any $(a,b)\neq(0,0)$, and is therefore $\mathcal{V}_\circ^{h,\Delta}$ itself. Similarly, $\Delta\neq 0$ implies that the submodule generated by $\bar{\chi}_\circ\bar{\psi}_\circ v$ contains $\chi_\circ\bar{\chi}_\circ\bar{\psi}_\circ v=-\Delta\bar{\chi}_\circ v$ and thus v, hence is also $\mathcal{V}_\circ^{h,\Delta}$. This proves the irreducibility of Verma modules when $\Delta\neq 0$.

When $\Delta=0$, however, these calculations show that every nonzero linear combination of $\bar{\chi}_\circ v$ and $\bar{\psi}_\circ v$ is a highest-weight vector, hence generates a proper submodule. The sum of these proper submodules obviously contains $\bar{\chi}_\circ\bar{\psi}_\circ v$, so we conclude that the irreducible quotient of $\mathcal{V}_\circ^{h,0}$ is spanned by (the image of) v. ■

An irreducible $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$-module with a weight vector is thus a highest-weight module and so is either 4- or 1-dimensional. There are nevertheless reducible but indecomposable $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$-modules, also possessing weight vectors, of dimensions greater than 4. We shall not describe these here, but note instead that the action of $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$ on Verma modules is not as nice as we might have hoped. In particular, the action of $S_\circ$ on $\mathcal{V}_\circ^{h,\Delta}$, $\Delta\neq 0$, is nondiagonalisable

$$\begin{aligned} S_\circ v&=\Delta v, & S_\circ\bar{\chi}_\circ\bar{\psi}_\circ v&=\Delta\bar{\chi}_\circ\bar{\psi}_\circ v,\\ S_\circ\bar{\chi}_\circ v&=\Delta\bar{\chi}_\circ v, & S_\circ\bar{\psi}_\circ v&=-\frac{1}{2}\Delta\bar{\chi}_\circ v+\Delta\bar{\psi}_\circ v. \end{aligned}$$

The action of $H_\circ$ is even more opaque

$$\begin{aligned} H_\circ v&=hv, & H_\circ\bar{\chi}_\circ\bar{\psi}_\circ v&=h\bar{\chi}_\circ\bar{\psi}_\circ v,\\ H_\circ\bar{\chi}_\circ v&=h\bar{\chi}_\circ v-\bar{\psi}_\circ v, & H_\circ\bar{\psi}_\circ v&=-\bigg(h+\frac{3\mathsf{k}^2}{2}\Delta+\tfrac{1}{4}\big(\mathsf{k}^2-\tfrac{1}{4}\big)\bigg)\bar{\chi}_\circ v+\big(h+\tfrac{1}{2}\big)\bar{\psi}_\circ v. \end{aligned}$$

The eigenvalues of $H_\circ$ on the subspace spanned by $\bar{\chi}_\circ v$ and $\bar{\psi}_\circ v$ are thus

$$h+\tfrac{1}{4}\pm\sqrt{h+\tfrac{1}{4}(6\Delta+1)\mathsf{k}^2}.$$

This action is therefore also nondiagonalisable when $h=-\frac{1}{4}(6\Delta+1)\mathsf{k}^2$.

This classifies irreducible $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$-modules with a weight vector. Could there exist an irreducible $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$-module without a weight vector? To tackle this question, recall that because $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$ has countable dimension, its irreducible modules do too. Dixmier's lemma therefore shows that $L_\circ$ always acts as a multiple of the identity on any irreducible.

Choose $v\neq 0$ in an irreducible $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$-module. By acting with $\chi_\circ$ and $\psi_\circ$, we may assume without loss of generality that v is annihilated by both $\chi_\circ$ and $\psi_\circ$. If v is a weight vector, then our irreducible is a highest-weight module by Lemma 3.1. So suppose that v is not weight. As $H_\circ v$ is then nonzero, the submodule it generates must still contain v, by irreducibility. Because we can order monomials in $\mathsf{Zhu}\big[\mathsf{W}^{\mathsf{k}}_{\text{pr}}\big]$, this submodule is spanned by monomials of the form $(\bar{\psi}_\circ)^\ell(\bar{\chi}_\circ)^m H_\circ^n v$, with $n \geqslant 1$. However, our irreducible admits a grading inherited from that of $\mathsf{Zhu}[\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}]$ by setting the grade of v to 0. To obtain v, we may thus restrict to monomials with $\ell = m = 0$, concluding that $v = p(H_\circ)v$ for some polynomial p with zero constant term. But now, $H_\circ$ preserves the finite-dimensional subspace $\{H_\circ^n v \mid n = 0, 1, \dots, \deg(p) - 1\}$ and so this subspace contains a weight vector. We therefore conclude that every irreducible has a weight vector.

Proposition 3.3. *Every irreducible* $\mathsf{Zhu}[\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}]$*-module is highest weight.*

3.3 Irreducible lower-bounded $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$- and $\mathsf{W}^{\mathrm{pr}}_{\mathsf{k}}$-modules

Recall that a finitely generated module over a vertex-operator superalgebra is lower bounded if its conformal weights have a minimal real part. The subspace realising this minimum is called the top space, and it is naturally a module for the Zhu algebra. We define a weight vector in a $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$-module to be an eigenvector of H_0 that is also a generalised eigenvector of L_0, its weight then being the pair (h, Δ) of eigenvalues of H_0 and L_0 (respectively). A weight vector is a highest-weight vector if it is also annihilated by χ_0, ψ_0 and all the modes of $\widehat{\mathfrak{psl}}_{2|2}$ with positive indices. The previous Proposition 3.3 now has the following consequence for the representation theory of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$.

Theorem 3.4. *Every irreducible lower-bounded* $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$*-module is highest weight, so is determined up to isomorphism by its highest weight* (h, Δ)*. Moreover, the top space of the irreducible is* 1*-dimensional, if* $\Delta = 0$*, and is otherwise* 4*-dimensional.*

The obvious next step is to try to classify the irreducible lower-bounded $\mathsf{W}^{\mathrm{pr}}_{\mathsf{k}}$-modules in terms of their highest weights. For general k, this is clearly beyond the scope of the paper. However, it is interesting to investigate whether the number of irreducibles is finite. When $\mathsf{k} = \pm\frac{1}{2}$, there is only one irreducible as $\mathsf{W}^{\mathrm{pr}}_{\pm 1/2} \cong \mathsf{SF}$. We record the results of our investigations for the two next-most-accessible levels: $\mathsf{k} = \pm\frac{1}{3}$ and $\mathsf{k} = \pm\frac{3}{2}$:

- There is a singular vector in $\mathsf{W}^{\pm 1/3}_{\mathrm{pr}}$ of charge $+1$ and conformal weight 4 (see Section 2.3). It thus has two linearly independent descendants of charge 0 obtained by acting with $\bar{\chi}_0$ and $\bar{\psi}_0$, respectively. Their zero modes act as scalars on a highest-weight vector of weight (h, Δ) and this action must vanish if the vector is to generate a $\mathsf{W}^{\mathrm{pr}}_{\pm 1/3}$-module. Explicit calculation now gives two equations in two unknowns and a finite number of solutions
$$\begin{aligned} &\Delta(24h - 5\Delta + 5) = 0 \qquad \text{and} \qquad 576h^2 - 48h - 120h\Delta - 49\Delta - 63\Delta^2 = 0 \\ &\qquad \Rightarrow \quad (h, \Delta) = (0,0), \left(\tfrac{1}{12}, 0\right), \left(-\tfrac{5}{9}, -\tfrac{5}{3}\right), \left(-\tfrac{5}{36}, \tfrac{1}{3}\right). \end{aligned}$$
$\mathsf{W}^{\mathrm{pr}}_{\pm 1/3}$ therefore has finitely many lower-bounded irreducibles. We conjecture that it is a log-rational vertex-operator superalgebra.
- A similar singular vector in $\mathsf{W}^{\pm 3/2}_{\mathrm{pr}}$ likewise leads to two equations in two unknowns. Our expectation that we again have finitely many solutions is however dashed as explicit calculation gives
$$\begin{aligned} &\Delta(4h + 5\Delta + 2) = 0 \qquad \text{and} \qquad h(4h + 5\Delta + 2) = 0 \\ &\qquad \Rightarrow \quad (h, \Delta) = (0,0), \left(h, -\tfrac{2}{5}(2h+1)\right), \qquad h \in \mathbb{C}. \end{aligned}$$
While this does not prove that $\mathsf{W}^{\mathrm{pr}}_{\pm 3/2}$ has infinitely many irreducibles, we conjecture that it does and so is not log-rational.

4 Inverse reduction and the $N = 4$ superconformal algebra

We conclude by explaining how the paradigm of inverse quantum Hamiltonian reduction can be used to transfer representation-theoretic information between W-algebras. For this, we explicitly construct a so-called Semikhatov realisation of the small $N = 4$ superconformal vertex-operator superalgebra $\mathsf{W}^{\mathsf{k}}_{\mathrm{min}}$ from $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ and sketch how to determine when it descends to $\mathsf{W}_{\mathsf{k}}^{\mathrm{min}}$ and $\mathsf{W}_{\mathsf{k}}^{\mathrm{pr}}$. As the representation theory of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ and $\mathsf{W}_{\mathsf{k}}^{\mathrm{pr}}$ is still mysterious for general levels k, we then restrict to $\mathsf{k} = \pm\frac{1}{2}$. The corresponding $N = 4$ central charges are -3 and -9, both of which have been previously considered in the literature. In particular, inverse reduction allows us to easily recover the $\mathsf{c} = -9$ results of [1, 5]. We take the opportunity to complement these known results with similar ones for $\mathsf{c} = -3$ and, for both central charges, detailed structures for certain nonsemisimple $\mathsf{W}_{\pm 1/2}^{\mathrm{min}}$-modules, some of which are logarithmic.

4.1 Inverse quantum Hamiltonian reduction

As previously mentioned, the quantum Hamiltonian reduction $\mathsf{W}^{\mathsf{k}}_{\mathrm{min}}$ of $\mathsf{V}^{\mathsf{k}}(\mathfrak{psl}_{2|2})$ with respect to a minimal nilpotent element is isomorphic to the small $N = 4$ superconformal vertex-operator superalgebra. In the framework of inverse quantum Hamiltonian reduction [2, 51], we expect a Semikhatov realisation of $\mathsf{W}^{\mathsf{k}}_{\mathrm{min}}$ in terms of $\mathsf{W}^{\mathsf{k}}_{\mathrm{pr}}$ and some free field algebra F, $\mathsf{W}^{\mathsf{k}}_{\mathrm{min}} \hookrightarrow \mathsf{W}^{\mathsf{k}}_{\mathrm{pr}} \otimes \mathsf{F}$. This embedding should moreover descend to an embedding involving the simple quotients $\mathsf{W}_{\mathsf{k}}^{\mathrm{min}}$ and $\mathsf{W}_{\mathsf{k}}^{\mathrm{pr}}$, at least for most k, and thereby provide nontrivial insights into the representation theory of $\mathsf{W}^{\mathsf{k}}_{\mathrm{min}}$ and $\mathsf{W}_{\mathsf{k}}^{\mathrm{min}}$.

To construct such a Semikhatov realisation, we will utilise an explicit presentation of $\mathsf{W}^{\mathsf{k}}_{\mathrm{min}}$, following [41, 42]. This has the following features:

- A strong generating set consists of even elements J^+, J^0, J^- and T, as well as odd elements G^+, G^-, $\bar{G}^+$ and $\bar{G}^-$.
- T is a conformal vector for $\mathsf{W}^{\mathsf{k}}_{\mathrm{min}}$ of central charge $-6(\mathsf{k}+1)$ and the corresponding conformal weights of the strong generators labelled by J are 1, while those of the strong generators labelled by G are $\frac{3}{2}$.
- All the strongly generating fields except $T(z)$ are primary with respect to this conformal structure.
- The zero modes of J^+, J^0 and J^- form an $\mathfrak{sl}_2$-triple and the vertex subalgebra they generate is isomorphic to the affine vertex algebra $\mathsf{V}^{-\mathsf{k}-1}(\mathfrak{sl}_2)$.
- G^+ and G^- span a fundamental module of $\mathfrak{sl}_2$, as do $\bar{G}^+$ and $\bar{G}^-$. Here, the labels $+$ and $-$ correspond to the eigenvalue of J^0. We normalise these $\mathfrak{sl}_2$-actions so that $J^{\pm}_0 G^{\mp} = G^{\pm}$ and $J^{\pm}_0 \bar{G}^{\mp} = \bar{G}^{\pm}$.

Given this information, it only remains to specify the operator product expansions involving $G^{\pm}$ and $\bar{G}^{\pm}$, the nonregular of which are given by

$$G^{\pm}(z)\bar{G}^{\pm}(w) \sim \pm\frac{2J^{\pm}(w)}{(z-w)^2} \pm \frac{\partial J^{\pm}(w)}{z-w},$$
$$G^{\pm}(z)\bar{G}^{\mp}(w) \sim \pm\frac{2(\mathsf{k}+1)\mathbb{1}}{(z-w)^3} - \frac{J^0(w)}{(z-w)^2} - \frac{\frac{1}{2}\partial J^0(w) \pm T(w)}{z-w}.$$

We remark that if one requires that the action of the even generating fields is invariant under a 2π-rotation about the origin, while that of the odd fields need only be invariant under a 4π-rotation, then the representation theories of $\mathsf{W}^{\mathsf{k}}_{\mathrm{min}}$ and $\mathsf{W}_{\mathsf{k}}^{\mathrm{min}}$ decompose into two sectors: the Neveu–Schwarz sector in which all odd fields are invariant and the Ramond sector in which all odd fields are negated under a 2π-rotation. These two sectors are moreover equivalent as categories because there exist spectral flow maps σ^ℓ, parametrised for $\ell \in \frac{1}{2}\mathbb{Z}$ by ℓJ^0 (as in [46]). At the level of the modes of the generating fields, σ^ℓ adds $\mp\ell$ to the indices of the $G^\pm$- and $\bar{G}^\pm$-modes, while it adds $\mp 2\ell$ to those of the $J^\pm$-modes. Otherwise, we have

$$\sigma^\ell\big(J^0_n\big) = J^0_n + 2(\mathsf{k}+1)\ell\delta_{n,0}\mathbb{1} \qquad \text{and} \qquad \sigma^\ell(T_n) = T_n - \ell J^0_n - (\mathsf{k}+1)\ell^2\delta_{n,0}\mathbb{1}. \tag{4.1}$$

There is also the conjugation automorphism w of the mode algebra of $\mathsf{W}^{\mathsf{k}}_{\text{min}}$. It preserves the mode indices and T, while acting as the Weyl reflection of $\mathfrak{sl}_2$ on J^0 and $J^\pm$. The action on the odd generators is given by $\mathsf{w}\big(G^\pm\big) = G^\mp$ and $\mathsf{w}\big(\bar{G}^\pm\big) = -\bar{G}^\mp$.

Let F be the Friedan–Martinec–Shenker bosonisation Π of the bosonic ghosts vertex algebra, also known as the half-lattice vertex algebra. This has strong generators c, d and e^{mc}, for $m \in \mathbb{Z}$, whose nonregular operator product expansions are [32]

$$c(z)d(w) \sim \frac{2\mathbb{1}}{(z-w)^2}, \qquad d(z)\mathrm{e}^{mc}(w) \sim \frac{2m\mathrm{e}^{mc}(w)}{z-w}.$$

We also introduce the following convenient alternative basis for the Heisenberg elements in Π $a = +\frac{1}{2}(\mathsf{k}+1)c + \frac{1}{4}d$, $b = -\frac{1}{2}(\mathsf{k}+1)c + \frac{1}{4}d$. It is easy to check that $a(z)b(w)$ is regular. Finally, we equip Π with the conformal vector $t = \frac{1}{2}{:}cd{:} - \partial a$. It corresponds to central charge $-2(3\mathsf{k}+2)$. The conformal weights of c, d and e^{mc} are 1, 1 and $\frac{1}{2}m$, respectively.

Given this setup, it is easy to verify (for example, using OPEdefs) the following result.

Theorem 4.1. *For $\mathsf{k} \neq 0$, there is a homomorphism $\Phi^{\mathsf{k}} \colon \mathsf{W}^{\mathsf{k}}_{\text{min}} \to \mathsf{W}^{\mathsf{k}}_{\text{pr}} \otimes \Pi$ of vertex-operator algebras, determined by*

$$\begin{aligned}
&\Phi^{\mathsf{k}}\big(J^+\big) = \mathrm{e}^{2c}, \qquad \Phi^{\mathsf{k}}\big(J^0\big) = 2b, \qquad \Phi^{\mathsf{k}}(T) = L + t, \qquad \Phi^{\mathsf{k}}\big(G^+\big) = {:}\chi\mathrm{e}^{c}{:},\\
&\Phi^{\mathsf{k}}\big(\bar{G}^+\big) = {:}\bar{\chi}\mathrm{e}^{c}{:}, \qquad \Phi^{\mathsf{k}}\big(G^-\big) = -{:}\big(\psi + \chi a - \tfrac{1}{2}\big(\mathsf{k}+\tfrac{1}{2}\big)\partial\chi\big)\mathrm{e}^{-c}{:},\\
&\Phi^{\mathsf{k}}\big(\bar{G}^-\big) = -{:}\big(\bar{\psi} + \bar{\chi}a - \tfrac{1}{2}\big(\mathsf{k}+\tfrac{1}{2}\big)\partial\bar{\chi}\big)\mathrm{e}^{-c}{:},\\
&\Phi^{\mathsf{k}}\big(J^-\big) = {:}\big(H - \tfrac{1}{2}(\mathsf{k}-1)S - aa - \tfrac{1}{2}\big(\mathsf{k}+\tfrac{1}{2}\big)(3\mathsf{k}-1)\chi\bar{\chi} + \mathsf{k}\partial a\big)\mathrm{e}^{-2c}{:}.
\end{aligned} \tag{4.2}$$

To convert this into a Semikhatov realisation of $\mathsf{W}^{\mathsf{k}}_{\text{min}}$, we need to demonstrate that Φ^{k} is injective. This follows from a combinatorial argument based on the "triangularity" of Φ^{k}, see [7] for an example of such an argument. Here, triangularity refers to the fact that the strong generators of $\mathsf{W}^{\mathsf{k}}_{\text{min}}$ and $\mathsf{W}^{\mathsf{k}}_{\text{pr}} \otimes \Pi$ may be ordered so that the image of the n-th generator of $\mathsf{W}^{\mathsf{k}}_{\text{min}}$ is expressed in terms of the m-th generators, with $m \leqslant n$. (For Π, we only need to consider d and e^c for this purpose, see [7].) We omit the details.

It remains to obtain a Semikhatov realisation of $\mathsf{W}_{\mathsf{k}}^{\text{min}}$. As in [7], this follows from a remarkable property of inverse quantum Hamiltonian reduction. First, recall [13] that Π admits a family of irreducible lower-bounded modules $\Pi_{[\lambda]} = \Pi\mathrm{e}^{-a+\lambda c}$, $[\lambda] \in \mathbb{C}/\mathbb{Z}$. The top space of $\Pi_{[\lambda]}$ is spanned by the $\mathrm{e}^{-a+\mu c}$ with $\mu \in [\lambda]$ and it has conformal weight $-\frac{1}{4}(\mathsf{k}+1)$. Note that because e^c has conformal weight $\frac{1}{2}$, the $\Pi_{[\lambda]}$ are actually $\mathbb{Z}_2$-twisted Π-modules.

With these (twisted) Π-modules, we introduce the Adamović functors

$$\mathrm{Ad}_{[\lambda]} = \mathrm{Res}^{\mathsf{W}^{\mathsf{k}}_{\text{pr}} \otimes \Pi}_{\mathsf{W}^{\mathsf{k}}_{\text{min}}}(- \otimes \Pi_{[\lambda]}), \qquad [\lambda] \in \mathbb{C}/\mathbb{Z}. \tag{4.3}$$

The remarkable property is now that if $\mathcal{M}$ is a simple $\mathsf{W}^{\mathsf{k}}_{\text{pr}}$-module, then $\mathrm{Ad}_{[\lambda]}(\mathcal{M})$ is almost irreducible. This means that it is lower bounded, generated by its top space, and is such that every nonzero submodule has nonzero intersection with the top space. We will also omit the details that establish this property, referring again to [7] for a similar example.

This property of Adamović functors has many important applications. One such is the precise condition under which Φ^{k} descends to a Semikhatov realisation

$$\Phi_{\mathsf{k}}\colon\ \mathsf{W}_{\mathsf{k}}^{\min} \hookrightarrow \mathsf{W}_{\mathsf{k}}^{\mathrm{pr}} \otimes \Pi \tag{4.4}$$

of the simple quotients. The idea is as follows [7]. Since $\mathsf{W}_{\mathsf{k}}^{\mathrm{pr}}$ is irreducible, $\mathrm{Ad}_{[\lambda]}\big(\mathsf{W}_{\mathsf{k}}^{\mathrm{pr}}\big)$ is an almost-irreducible $\mathsf{W}^{\mathsf{k}}_{\min}$-module, hence any of its nonzero submodules intersect its top space nontrivially. But, for a suitable choice of $[\lambda]$, this module is a spectral flow of $\mathsf{W}_{\mathsf{k}}^{\mathrm{pr}} \otimes \Pi$. It follows that the image of the maximal submodule of $\mathsf{W}^{\mathsf{k}}_{\min}$ is either 0 or it has nonzero intersection with the "spectrally flown top space" $\mathrm{span}\{(J^+)^n|0\rangle \mid n \in \mathbb{Z}\}$ of $\mathsf{W}_{\mathsf{k}}^{\mathrm{pr}} \otimes \Pi$. As $J^+ \in \mathsf{V}^{-\mathsf{k}-1}(\mathfrak{sl}_2)$, the latter is impossible unless $\mathsf{k} \in \mathbb{Z}_{\leqslant -1}$.

Theorem 4.2. *For $\mathsf{k} \notin \mathbb{Z}_{\leqslant 0}$, the embedding of (4.2) induces the simple Semikhatov realisation (4.4).*

4.2 Constructing weight $\mathsf{W}^{\min}_{\pm 1/2}$-modules

Recall that the simple principal W-algebra $\mathsf{W}_{\mathsf{k}}^{\mathrm{pr}}$ collapses to the symplectic fermions vertex-operator superalgebra SF when the level is $\mathsf{k} = \pm\frac{1}{2}$. The corresponding simple minimal W-algebras $\mathsf{W}^{\min}_{1/2}$ and $\mathsf{W}^{\min}_{-1/2}$ have central charges -9 and -3, respectively. For convenience, we specialise the Semikhatov realisation of Theorems 4.1 and 4.2 to these levels by setting H, S, ψ and $\bar{\psi}$ to 0.

Corollary 4.3. *For $\mathsf{k} = \pm\frac{1}{2}$, there is a Semikhatov realisation $\Phi_{\mathsf{k}}\colon \mathsf{W}_{\mathsf{k}}^{\min} \hookrightarrow \mathsf{SF} \otimes \Pi$ given by*

$$\begin{aligned}
&\Phi_{\mathsf{k}}\big(J^+\big) = \mathrm{e}^{2c}, && \Phi_{\mathsf{k}}\big(J^-\big) = -{:}(aa - \mathsf{k}\partial a)\mathrm{e}^{-2c}{:} - \tfrac{1}{2}\big(\mathsf{k}+\tfrac{1}{2}\big)(3\mathsf{k}-1){:}\chi\bar{\chi}{:}\mathrm{e}^{-2c},\\
&\Phi_{\mathsf{k}}\big(J^0\big) = 2b, && \Phi_{\mathsf{k}}\big(T\big) = \tfrac{1}{2}{:}cd{:} - \partial a - \tfrac{1}{2}{:}\chi\bar{\chi}{:},\\
&\Phi_{\mathsf{k}}\big(G^+\big) = \chi\mathrm{e}^{c}, && \Phi_{\mathsf{k}}\big(G^-\big) = \tfrac{1}{2}\big(\mathsf{k}+\tfrac{1}{2}\big)\partial\chi\mathrm{e}^{-c} - \chi{:}a\mathrm{e}^{-c}{:},\\
&\Phi_{\mathsf{k}}\big(\bar{G}^+\big) = \bar{\chi}\mathrm{e}^{c}, && \Phi_{\mathsf{k}}\big(\bar{G}^-\big) = \tfrac{1}{2}\big(\mathsf{k}+\tfrac{1}{2}\big)\partial\bar{\chi}\mathrm{e}^{-c} - \bar{\chi}{:}a\mathrm{e}^{-c}{:}.
\end{aligned} \tag{4.5}$$

These free-field realisations of the $\mathsf{c} = -9$ and $\mathsf{c} = -3$ (small) $N = 4$ superconformal algebra are closely related to other realisations that have appeared in the literature.

- For $\mathsf{W}^{\min}_{1/2}$ ($\mathsf{c} = -9$), [1, Proposition 5.1 and Theorem 6.1] replaces SF and Π with the $\mathsf{c} = -11$ fermionic ghost vertex-operator superalgebra FG and the $\mathsf{c} = 2$ bosonic ghost vertex-operator algebra BG, respectively. We remark that $\mathsf{SF} \subset \mathsf{FG}$ and $\Pi \supset \mathsf{BG}$ as vertex superalgebras (these are not conformal embeddings).
- In [5, Proposition 1], $\mathsf{W}^{\min}_{1/2}$ ($\mathsf{c} = -9$) is shown to embed into a $\mathbb{Z}_2$-orbifold of $\mathsf{SF} \otimes \Pi^{1/2}$, where $\Pi^{1/2}$ is a simple-current extension of Π. This is very close to Corollary 4.3, but is very different in flavour: their realisation uses screening operators and does not identify SF as a W-algebra.
- On the other hand, [16, Theorem 4.14] realises $\mathsf{W}^{\min}_{-1/2}$ ($\mathsf{c} = -3$) as a simple-current extension of the tensor product of the $\mathsf{c} = -2$ triplet vertex-operator algebra $\mathsf{W}(2)$ and the $\mathsf{c} = -1$ affine vertex-operator algebra $\mathsf{L}_{-1/2}(\mathfrak{sl}_2)$. This is also close to Corollary 4.3: $\mathsf{W}(2) \subset \mathsf{SF}$ and $\mathsf{L}_{-1/2}(\mathfrak{sl}_2) \subset \mathsf{BG} \subset \Pi$ as vertex-operator superalgebras (these are conformal embeddings).

Recall that SF has a single irreducible untwisted (hence Neveu–Schwarz) module $\mathcal{S}^{\mathrm{NS}} \cong \mathsf{SF}$ and a single irreducible $\mathbb{Z}_2$-twisted (hence Ramond) module $\mathcal{S}^{\mathrm{R}}$. The highest-weight vectors of $\mathcal{S}^{\mathrm{NS}}$ and $\mathcal{S}^{\mathrm{R}}$, which we denote by $|\mathrm{NS}\rangle$ and $|\mathrm{R}\rangle$, have conformal weights 0 and $-\frac{1}{8}$, respectively. The Adamović functors thus give two families of almost-irreducible (twisted) $\mathsf{W}^{\min}_{\pm 1/2}$-modules, parametrised by $[\lambda] \in \mathbb{C}/\mathbb{Z}$

$$\mathcal{N}^{\mathrm{R}}_{[\lambda]} = \mathrm{Ad}_{[\lambda]}\big(\mathcal{S}^{\mathrm{NS}}\big) \qquad \text{and} \qquad \mathcal{N}^{\mathrm{NS}}_{[\lambda]} = \mathrm{Ad}_{[\lambda]}\big(\mathcal{S}^{\mathrm{R}}\big). \tag{4.6}$$

Note that because $\Pi_{[\lambda]}$ is $\mathbb{Z}_2$-twisted, the $\mathcal{N}^{\mathrm{R}}_{[\lambda]}$ are too – they belong to the Ramond sector of $\mathsf{W}^{\min}_{\pm1/2}$. The $\mathcal{N}^{\mathrm{NS}}_{[\lambda]}$ are untwisted, hence Neveu–Schwarz.

Recall the characters and supercharacters of the irreducible SF-modules

$$\mathrm{ch}^-\big[\mathcal{S}^{\mathrm{NS}}\big](\mathsf{y};\mathsf{q}) = \mathsf{q}^{1/12}\prod_{n=1}^{\infty}(1-\mathsf{y}\mathsf{q}^n)(1-\mathsf{y}^{-1}\mathsf{q}^n) = \frac{1}{\mathsf{y}^{1/2}-\mathsf{y}^{-1/2}}\frac{\mathrm{i}\vartheta_1(\mathsf{y};\mathsf{q})}{\eta(\mathsf{q})},$$
$$\mathrm{ch}^+\big[\mathcal{S}^{\mathrm{NS}}\big](\mathsf{y};\mathsf{q}) = \mathsf{q}^{1/12}\prod_{n=1}^{\infty}(1+\mathsf{y}\mathsf{q}^n)(1+\mathsf{y}^{-1}\mathsf{q}^n) = \frac{1}{\mathsf{y}^{1/2}+\mathsf{y}^{-1/2}}\frac{\vartheta_2(\mathsf{y};\mathsf{q})}{\eta(\mathsf{q})},$$
$$\mathrm{ch}^+\big[\mathcal{S}^{\mathrm{R}}\big](\mathsf{y};\mathsf{q}) = \mathsf{q}^{-1/24}\prod_{n=1}^{\infty}\big(1+\mathsf{y}\mathsf{q}^{n-1/2}\big)\big(1+\mathsf{y}^{-1}\mathsf{q}^{n-1/2}\big) = \frac{\vartheta_3(\mathsf{y};\mathsf{q})}{\eta(\mathsf{q})},$$
$$\mathrm{ch}^-\big[\mathcal{S}^{\mathrm{R}}\big](\mathsf{y};\mathsf{q}) = \mathsf{q}^{-1/24}\prod_{n=1}^{\infty}\big(1-\mathsf{y}\mathsf{q}^{n-1/2}\big)\big(1-\mathsf{y}^{-1}\mathsf{q}^{n-1/2}\big) = \frac{\vartheta_4(\mathsf{y};\mathsf{q})}{\eta(\mathsf{q})}.$$

Here, we have included the horizontal grading as in (2.7). For the lower-bounded Π-modules $\Pi_{[\lambda]}$, we define characters as follows

$$\mathrm{ch}^+\big[\Pi_{[\lambda]}\big](\mathsf{z};\mathsf{q}) = \mathrm{tr}_{\Pi_{[\lambda]}}\,\mathsf{z}^{2b_0}\mathsf{q}^{t_0+(3\mathsf{k}+2)/12} = \frac{\mathsf{q}^{-(\mathsf{k}+1)/4+(3\mathsf{k}+2)/12}}{\prod_{n=1}^{\infty}(1-\mathsf{q}^n)^2}\sum_{n\in\mathbb{Z}}\mathsf{z}^{\lambda+n} = \mathsf{z}^{\lambda}\frac{\delta(\mathsf{z})}{\eta(\mathsf{q})^2}.$$

Here, $\delta(\mathsf{z})$ is the formal series (or distribution) $\sum_{n\in\mathbb{Z}}\mathsf{z}^n$.

Because Corollary 4.3 identifies J^0 and $2b$, we may define graded (super)characters of $\mathsf{W}^{\min}_{\pm1/2}$-modules that keep track of the horizontal grading (y), the J^0_0-eigenvalue (z) and the conformal weight (q). For the images (4.6) under the Adamović functors, these (super)characters have the simple forms

$$\mathrm{ch}^-\big[\mathcal{N}^{\mathrm{R}}_{[\lambda]}\big](\mathsf{y},\mathsf{z};\mathsf{q}) = \frac{\mathsf{z}^{\lambda}}{\mathsf{y}^{1/2}-\mathsf{y}^{-1/2}}\frac{\mathrm{i}\vartheta_1(\mathsf{y};\mathsf{q})\delta(\mathsf{z})}{\eta(\mathsf{q})^3}, \qquad \mathrm{ch}^+\big[\mathcal{N}^{\mathrm{NS}}_{[\lambda]}\big](\mathsf{y},\mathsf{z};\mathsf{q}) = \mathsf{z}^{\lambda}\frac{\vartheta_3(\mathsf{y};\mathsf{q})\delta(\mathsf{z})}{\eta(\mathsf{q})^3},$$
$$\mathrm{ch}^+\big[\mathcal{N}^{\mathrm{R}}_{[\lambda]}\big](\mathsf{y},\mathsf{z};\mathsf{q}) = \frac{\mathsf{z}^{\lambda}}{\mathsf{y}^{1/2}+\mathsf{y}^{-1/2}}\frac{\vartheta_2(\mathsf{y};\mathsf{q})\delta(\mathsf{z})}{\eta(\mathsf{q})^3},$$
$$\mathrm{ch}^-\big[\mathcal{N}^{\mathrm{NS}}_{[\lambda]}\big](\mathsf{y},\mathsf{z};\mathsf{q}) = \mathsf{z}^{\lambda}\frac{\vartheta_4(\mathsf{y};\mathsf{q})\delta(\mathsf{z})}{\eta(\mathsf{q})^3}. \tag{4.7}$$

Given these results, we may also use (4.1) to determine the (super)characters of the spectral flows of these modules.

4.3 Degenerations and spectral flows

Consider now the almost-irreducible $\mathsf{W}^{\min}_{\pm1/2}$-modules $\mathcal{N}^{\mathrm{R}}_{[\lambda]}$. These belong to the Ramond sector and so are completely determined by the action of the zero modes, including $G^{\pm}_0$ and $\bar{G}^{\pm}_0$, on their top spaces [20]. A basis for this top space is given by the vectors

$$|\mu\rangle^{\mathrm{R}} = |\mathrm{NS}\rangle\otimes\mathrm{e}^{-a+\mu c}, \qquad \mu\in[\lambda]. \tag{4.8}$$

Using Corollary 4.3, we compute the action of the generators' zero modes on these top space vectors. The even modes' actions are given by

$$J^+_0|\mu\rangle^{\mathrm{R}} = |\mu+2\rangle^{\mathrm{R}}, \qquad J^0_0|\mu\rangle^{\mathrm{R}} = \mu|\mu\rangle^{\mathrm{R}}, \qquad T_0|\mu\rangle^{\mathrm{R}} = -\tfrac{1}{4}(\mathsf{k}+1)|\mu\rangle^{\mathrm{R}},$$
$$J^-_0|\mu\rangle^{\mathrm{R}} = -\tfrac{1}{4}(\mu-\mathsf{k}-1)(\mu+\mathsf{k}-1)|\mu-2\rangle^{\mathrm{R}}, \qquad \mu\in[\lambda], \tag{4.9}$$

while the odd modes all act as 0. The top space of $\mathcal{N}^{\mathrm{R}}_{[\lambda]}$ thus decomposes as a direct sum of two (generically irreducible) modules for the zero-mode algebra (the Ramond-twisted Zhu algebra) of $\mathsf{W}^{\min}_{\pm 1/2}$. We may therefore write

$$\mathcal{N}^{\mathrm{R}}_{[\lambda]} = \mathcal{M}^{\mathrm{R}}_{[\![\lambda]\!]} \oplus \mathcal{M}^{\mathrm{R}}_{[\![\lambda+1]\!]}, \qquad [\lambda] = \lambda + \mathbb{Z} \in \mathbb{C}/\mathbb{Z},$$

where $[\![\lambda]\!] = \lambda + 2\mathbb{Z} \in \mathbb{C}/2\mathbb{Z}$ and $\mathcal{M}^{\mathrm{R}}_{[\![\lambda]\!]}$ denotes the submodule of $\mathcal{N}^{\mathrm{R}}_{[\lambda]}$ generated by the $|\mu\rangle^{\mathrm{R}}$ with $\mu \in [\![\lambda]\!]$.

Note that the top spaces of the $\mathcal{M}^{\mathrm{R}}_{[\![\lambda]\!]}$, $[\![\lambda]\!] \in \mathbb{C}/2\mathbb{Z}$, may be viewed as dense modules for the even zero-mode subalgebra of $\mathsf{W}^{\min}_{\pm 1/2}$ (which is isomorphic to $\mathfrak{gl}_2$). Because the $\mathcal{M}^{\mathrm{R}}_{[\![\lambda]\!]}$ are almost irreducible, see, for example, [27, Proposition 4.8], it follows immediately from (4.9) that $\mathcal{M}^{\mathrm{R}}_{[\![\lambda]\!]}$ is reducible if and only if there exists a conjugate highest-weight vector $|\mu\rangle^{\mathrm{R}}$ with $\mu \in [\![\lambda]\!]$. Moreover, such a vector only exists when $\mu = 1 \pm \mathsf{k} = \frac{1}{2}, \frac{3}{2}$. We thus say that the family $\{\mathcal{M}^{\mathrm{R}}_{[\![\lambda]\!]} \mid [\![\lambda]\!] \in \mathbb{C}/2\mathbb{Z}\}$ degenerates at $[\![\lambda]\!] = [\![\pm\frac{1}{2}]\!]$.

Let $\mathcal{L}^{\mathrm{R}}_{\mu}$ denote the irreducible highest-weight $\mathsf{W}^{\min}_{\pm 1/2}$-module, in the Ramond sector, whose highest-weight vector has J^0_0-eigenvalue μ and T_0-eigenvalue $-\frac{1}{4}(\mathsf{k}+1)$. Recalling the conjugation automorphism w from Section 4.1, the reducible cases are then characterised by the following nonsplit short exact sequences

$$\begin{aligned} &0 \longrightarrow \mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-1/2}\big) \longrightarrow \mathcal{M}^{\mathrm{R}}_{[\![1/2]\!]} \longrightarrow \mathcal{L}^{\mathrm{R}}_{-3/2} \longrightarrow 0, \\ &0 \longrightarrow \mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-3/2}\big) \longrightarrow \mathcal{M}^{\mathrm{R}}_{[\![-1/2]\!]} \longrightarrow \mathcal{L}^{\mathrm{R}}_{-1/2} \longrightarrow 0. \end{aligned} \tag{4.10}$$

This result deserves a few comments.

- The submodules in (4.10) are irreducible because any nonzero proper submodule would also be one of $\mathcal{M}^{\mathrm{R}}_{[\![\pm 1/2]\!]}$, hence would intersect the top space.
- The quotients are not obviously irreducible, although this would follow easily if one has a classification of highest-weight modules, such as [1, Proposition 6.2] for $\mathsf{k} = \frac{1}{2}$. Otherwise, the irreducibility may be established using the methods developed in [45, Proposition 4.13], which also demonstrate exactness.
- To our knowledge, this is the first time the images of Adamović functors have produced modules that are generically completely reducible rather than generically irreducible. In this case, this may be explained by an observation in [5], whereby $\mathsf{W}^{\min}_{\pm 1/2}$ in fact embeds into the even subalgebra of $\mathsf{SF} \otimes \Pi$ with respect to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. The $\mathbb{Z}_2$ action on SF is the usual one for superalgebras. The $\mathbb{Z}_2$ action on Π is determined by the action on the lattice generators e^{mc}, namely, by whether $m \in \mathbb{Z}$ is even or odd. It is clear from inspecting Corollary 4.3 that even generators of $\mathsf{W}^{\min}_{\pm 1/2}$ are mapped into the $(0,0)$ subspace, and odd generators of $\mathsf{W}^{\min}_{\pm 1/2}$ are mapped into the $(1,1)$ subspace, with respect to this action.
- Finally, the (super)character of $\mathcal{M}^{\mathrm{R}}_{[\![\lambda]\!]}$ may be obtained from that of $\mathcal{N}^{\mathrm{R}}_{[\lambda]}$, given in (4.7), by decomposing into $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-eigenspaces (where we assign $|\lambda\rangle^{\mathrm{R}}$ the grade $(0,0)$). More precisely, that of $\mathcal{M}^{\mathrm{R}}_{[\![\lambda]\!]}$ is the sum (difference) of the characters of the $(0,0)$- and $(1,1)$-eigenspaces.

One can repeat the above analysis in the Neveu–Schwarz sector. Now, the top space of $\mathcal{N}^{\mathrm{NS}}_{[\lambda]}$, $[\lambda] \in \mathbb{C}/\mathbb{Z}$, has the following basis vectors

$$|\mu\rangle^{\mathrm{NS}} = |\mathrm{R}\rangle \otimes \mathrm{e}^{-a+\mu c}, \qquad \mu \in [\lambda].$$

The odd generators have no zero modes, and the even zero modes' actions are given explicitly by

$$\begin{aligned} &J^+_0 |\mu\rangle^{\mathrm{NS}} = |\mu+2\rangle^{\mathrm{NS}}, \qquad J^0_0 |\mu\rangle^{\mathrm{NS}} = \mu |\mu\rangle^{\mathrm{NS}}, \qquad T_0 |\mu\rangle^{\mathrm{NS}} = -\tfrac{1}{4}\big(\mathsf{k} + \tfrac{3}{2}\big) |\mu\rangle^{\mathrm{NS}}, \\ &J^-_0 |\mu\rangle^{\mathrm{NS}} = -\big(\tfrac{1}{4}(\mu - \mathsf{k} - 1)(\mu + \mathsf{k} - 1) + \tfrac{1}{8}\big(\mathsf{k} + \tfrac{1}{2}\big)(3\mathsf{k} - 1)\big) |\mu - 2\rangle^{\mathrm{NS}}, \qquad \mu \in [\lambda]. \end{aligned}$$

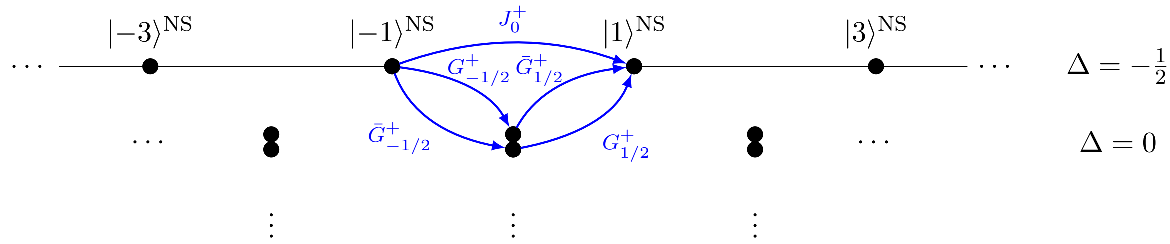


Figure 2. An illustration of the structure of the $\mathsf{W}^{\min}_{1/2}$-module $\mathcal{M}^{\mathrm{NS}}_{[\![1]\!]}$. The dots represent weight vectors, with the J^0_0-eigenvalue increasing from left to right and the T_0-eigenvalue Δ from top to bottom.

We therefore again get a direct sum decomposition

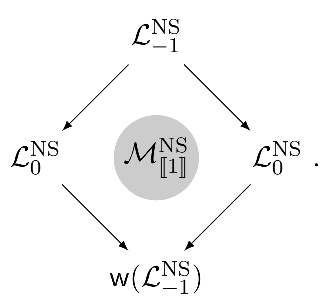
$$\mathcal{N}^{\mathrm{NS}}_{[\lambda]} = \mathcal{M}^{\mathrm{NS}}_{[\![\lambda]\!]} \oplus \mathcal{M}^{\mathrm{NS}}_{[\![\lambda+1]\!]},$$

where $\mathcal{M}^{\mathrm{NS}}_{[\![\mu]\!]}$ denotes the irreducible highest-weight $\mathsf{W}^{\min}_{\pm 1/2}$-module, in the Neveu–Schwarz sector, whose highest-weight vector has J^0_0-eigenvalue μ and T_0-eigenvalue $-\frac{1}{4}\big(\mathsf{k}+\frac{3}{2}\big)$.

When $\mathsf{k} = -\frac{1}{2}$, the Neveu–Schwarz analysis is almost identical to the Ramond case. The family $\big\{\mathcal{M}^{\mathrm{NS}}_{[\![\lambda]\!]} \mid [\![\lambda]\!] \in \mathbb{C}/2\mathbb{Z}\big\}$ degenerates at $[\![\lambda]\!] = [\![\pm\frac{1}{2}]\!]$ and there we have the following nonsplit short exact sequences:

$$\begin{aligned} &0 \longrightarrow \mathsf{w}\big(\mathcal{L}^{\mathrm{NS}}_{-1/2}\big) \longrightarrow \mathcal{M}^{\mathrm{NS}}_{[\![1/2]\!]} \longrightarrow \mathcal{L}^{\mathrm{NS}}_{-3/2} \longrightarrow 0, \\ &0 \longrightarrow \mathsf{w}\big(\mathcal{L}^{\mathrm{NS}}_{-3/2}\big) \longrightarrow \mathcal{M}^{\mathrm{NS}}_{[\![-1/2]\!]} \longrightarrow \mathcal{L}^{\mathrm{NS}}_{-1/2} \longrightarrow 0. \end{aligned}$$

The analysis for $\mathsf{k} = \frac{1}{2}$ is significantly more interesting. In this case, $J^-_0\,|\mu\rangle^{\mathrm{NS}} = 0$ has only one solution: $\mu = 1$. The family $\big\{\mathcal{M}^{\mathrm{NS}}_{[\![\lambda]\!]} \mid [\![\lambda]\!] \in \mathbb{C}/2\mathbb{Z}\big\}$ thus only degenerates at $[\![\lambda]\!] = [\![1]\!]$. But, $\mathcal{M}^{\mathrm{NS}}_{[\![1]\!]}$ has more than two composition factors! Indeed, its Loewy diagram is



This structure is also indicated schematically in Figure 2.

We make a few further comments on the structure of $\mathcal{M}^{\mathrm{NS}}_{[\![1]\!]}$.

- This module is not logarithmic: T_0 acts semisimply (its head and socle are not isomorphic).
- It is well known that an almost-irreducible module may have composition factors that are not detected by the top space, see [45, Section 4.5]. For simple affine vertex algebras (and W-algebras), this appears to be a feature of the representation theory at nonadmissible levels. Given that the level of the affine vertex subalgebra generated by J^0 and $J^\pm$ is admissible for $\mathsf{k} = -\frac{1}{2}$, but nonadmissible for $\mathsf{k} = \frac{1}{2}$, this qualitative structural difference should perhaps not be surprising.
- It is nevertheless quite satisfying to use (4.5) to explicitly verify that $G^-_{-1/2}$ and $\bar{G}^-_{-1/2}$ annihilate $|1\rangle^{\mathrm{NS}}$, while $G^-_{1/2}$ and $\bar{G}^-_{1/2}$ annihilate both $G^+_{-1/2}|{-1}\rangle^{\mathrm{NS}}$ and $\bar{G}^+_{-1/2}|{-1}\rangle^{\mathrm{NS}}$.

- The fact that $\mathcal{M}^{\mathrm{NS}}_{[\![1]\!]}$ does not have more than four composition factors uses the classification [1, Proposition 6.2] of irreducible highest-weight $\mathsf{W}^{\min}_{1/2}$-modules (there are only two: $\mathcal{L}^{\mathrm{NS}}_0$ and $\mathcal{L}^{\mathrm{NS}}_{-1}$). The inverse-reduction technology developed in [8] can also be used to prove this classification result, though we shall not do so here.
- As noted above, $\mathsf{W}^{\min}_{1/2}$ is a vertex subalgebra of the tensor product of a $\mathsf{c} = -11$ fermionic ghost system and a $\mathsf{c} = 2$ bosonic ghost system. In our language, [1, Proposition 6.3] identifies this tensor product as a nonsplit extension of $\mathcal{L}^{\mathrm{NS}}_{-1}$ by $\mathcal{L}^{\mathrm{NS}}_0$, hence as a length-2 quotient of $\mathcal{M}^{\mathrm{NS}}_{[\![1]\!]}$.
- The quotient $\mathcal{M}^{\mathrm{NS}}_{[\![1]\!]}/\mathsf{w}(\mathcal{L}^{\mathrm{NS}}_{-1})$ is a highest-weight $\mathsf{W}^{\min}_{1/2}$-module with two linearly independent singular vectors that share the same J^0_0- and T^0-eigenvalues. Both generate a copy of the vacuum module $\mathcal{L}^{\mathrm{NS}}_0$.

We conclude by reporting the action of spectral flow on the irreducible highest-weight $\mathsf{W}^{\min}_{\pm 1/2}$-modules. For $\mathsf{k} = -\frac{1}{2}$, the spectral flow orbits feature ordinary modules that do not appear as composition factors of the almost-irreducible modules constructed by inverse reduction. These follow from explicit computation using (4.1)

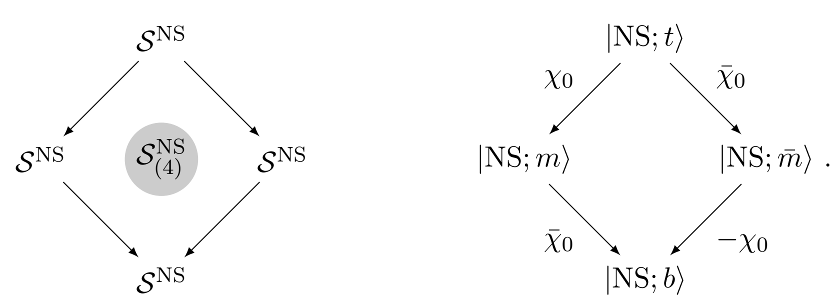
$$\begin{aligned}
&\cdots \xrightarrow{\sigma^{1/2}} \mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-1/2}\big) \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{NS}}_0 \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{R}}_{-1/2} \xrightarrow{\sigma^{1/2}} \cdots, \\
&\cdots \xrightarrow{\sigma^{1/2}} \mathsf{w}\big(\mathcal{L}^{\mathrm{NS}}_{-1/2}\big) \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{R}}_0 \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{NS}}_{-1/2} \xrightarrow{\sigma^{1/2}} \cdots, \\
&\cdots \xrightarrow{\sigma^{1/2}} \mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-3/2}\big) \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{NS}}_1 \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{R}}_{-3/2} \xrightarrow{\sigma^{1/2}} \cdots, \\
&\cdots \xrightarrow{\sigma^{1/2}} \mathsf{w}\big(\mathcal{L}^{\mathrm{NS}}_{-3/2}\big) \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{R}}_1 \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{NS}}_{-3/2} \xrightarrow{\sigma^{1/2}} \cdots.
\end{aligned}$$

Here, the dots indicate irreducible modules that are not lower bounded. We also note the similarity to the spectral flow orbits of $\mathsf{L}_{-1/2}(\mathfrak{sl}_2)$, see [49, Figure 3]. Contrarily, for $\mathsf{k} = \frac{1}{2}$, the orbits contain no new lower-bounded modules

$$\begin{aligned}
&\cdots \xrightarrow{\sigma^{1/2}} \mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-3/2}\big) \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{NS}}_0 \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{R}}_{-3/2} \xrightarrow{\sigma^{1/2}} \cdots, \\
&\cdots \xrightarrow{\sigma^{1/2}} \mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-1/2}\big) \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{NS}}_{-1} \xrightarrow{\sigma^{1/2}} \cdots, \qquad \cdots \xrightarrow{\sigma^{1/2}} \mathsf{w}\big(\mathcal{L}^{\mathrm{NS}}_{-1}\big) \xrightarrow{\sigma^{1/2}} \mathcal{L}^{\mathrm{R}}_{-1/2} \xrightarrow{\sigma^{1/2}} \cdots.
\end{aligned}$$

4.4 Logarithmic $\mathsf{W}^{\min}_{\pm 1/2}$-modules

Recall that the Ramond sector of the symplectic fermions vertex-operator superalgebra SF is semisimple, but the Neveu–Schwarz sector is not. In particular, it contains a logarithmic module $\mathcal{S}^{\mathrm{NS}}_{(4)}$. This is induced from the projective module of $\mathfrak{psl}_{2|2}$ and is thus an indecomposable sum of four copies of the vacuum module. We present its Loewy diagram and a basis for its top space



(4.11)

The logarithmic nature of $\mathcal{S}^{\mathrm{NS}}_{(4)}$ is evident from the action of the Virasoro zero mode: it maps $|\mathrm{NS}; t\rangle$ to $\frac{1}{2}|\mathrm{NS}; b\rangle$.

We can apply Adamović functors to such nonsemisimple SF-modules. To warm up, we will first consider a structurally simpler example: the length-2 submodule of $\mathcal{S}^{\mathrm{NS}}_{(4)}$ generated

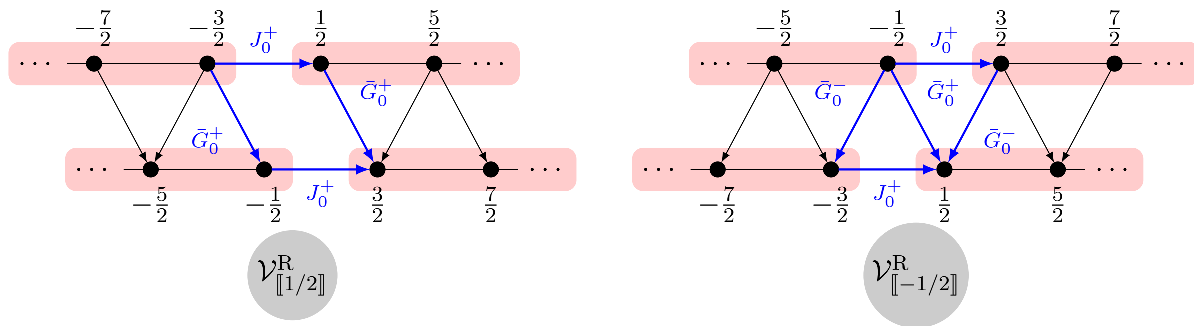


Figure 3. An illustration of the structure of the top spaces of the $\mathsf{W}^{\min}_{\pm 1/2}$-modules $\mathcal{V}^{\mathrm{R}}_{[\![1/2]\!]}$ (left) and $\mathcal{V}^{\mathrm{R}}_{[\![-1/2]\!]}$ (right). The dots represent weight vectors, with the J^0_0-eigenvalue increasing from left to right. The top row corresponds to vectors with label "m" and the bottom row has label "b". The shading suggests the composition factors.

by $|\mathrm{NS}; m\rangle$. This SF-module, which we shall denote by $\mathcal{S}^{\mathrm{NS}}_{(2)}$, has top space $\operatorname{span}\{|\mathrm{NS}; m\rangle, |\mathrm{NS}; b\rangle\}$. It is moreover a self-extension of the vacuum module that coincides with a Verma module of $\widehat{\mathfrak{psl}}_{2|2}$. We shall denote its images under the Adamović functors by $\mathcal{W}^{\mathrm{R}}_{[\lambda]}$, $[\lambda] \in \mathbb{C}/\mathbb{Z}$. These are lower-bounded Ramond-twisted $\mathsf{W}^{\min}_{\pm 1/2}$-modules.

The top space of $\mathcal{W}^{\mathrm{R}}_{[\lambda]}$ has basis $\{|\mu; m\rangle^{\mathrm{R}}, |\mu; b\rangle^{\mathrm{R}} \mid \mu \in [\lambda]\}$, where we follow (4.8) but add the label "m" or "b" appropriately. Using Corollary 4.3, it is easy to determine the action of the zero-mode algebra of $\mathsf{W}^{\min}_{\pm 1/2}$ on this basis. Indeed, the even zero modes act as in (4.9) (preserving labels), and almost all of the odd zero modes act as zero. The exceptions are those involving $\bar{\chi}_0$

$$\bar{G}^+_0 |\mu; m\rangle^{\mathrm{R}} = |\mu+1; b\rangle^{\mathrm{R}} \qquad \text{and} \qquad \bar{G}^-_0 |\mu; m\rangle^{\mathrm{R}} = -\tfrac{1}{2}\big(\mu - \tfrac{1}{2}\big)|\mu-1; b\rangle^{\mathrm{R}}.$$

Because of this, we still have a direct sum decomposition $\mathcal{W}^{\mathrm{R}}_{[\lambda]} = \mathcal{V}^{\mathrm{R}}_{[\![\lambda]\!]} \oplus \mathcal{V}^{\mathrm{R}}_{[\![\lambda+1]\!]}$, $[\lambda] \in \mathbb{C}/\mathbb{Z}$, where $\mathcal{V}^{\mathrm{R}}_{[\![\lambda]\!]}$ is spanned by the $|\mu; m\rangle^{\mathrm{R}}$ and $|\mu+1; b\rangle^{\mathrm{R}}$ with $\mu \in [\![\lambda]\!]$. However, the $\mathcal{V}^{\mathrm{R}}_{[\![\lambda]\!]}$ are nonsemisimple and (generically) have two composition factors

$$0 \longrightarrow \mathcal{M}^{\mathrm{R}}_{[\![\lambda+1]\!]} \longrightarrow \mathcal{V}^{\mathrm{R}}_{[\![\lambda]\!]} \longrightarrow \mathcal{M}^{\mathrm{R}}_{[\![\lambda]\!]} \longrightarrow 0, \qquad [\![\lambda]\!] \neq \big[\!\big[\pm\tfrac{1}{2}\big]\!\big]. \tag{4.12}$$

When $[\![\lambda]\!] = \big[\!\big[\pm\frac{1}{2}\big]\!\big]$, there is a further degeneration corresponding to the fact that $\mathcal{M}^{\mathrm{R}}_{[\![\pm 1/2]\!]}$ is reducible. In this case, J^-_0 annihilates $|\mu; \bullet\rangle^{\mathrm{R}}$, for $\mu = \frac{1}{2}, \frac{3}{2}$ and $\bullet = m, b$, and $\bar{G}^-_0$ annihilates $|\frac{1}{2}; m\rangle^{\mathrm{R}}$. The resulting Loewy diagrams are then

$$\begin{array}{ccc} & \mathcal{L}^{\mathrm{R}}_{-3/2} & \\ \swarrow & & \searrow \\ \mathcal{L}^{\mathrm{R}}_{-1/2} & \mathcal{V}^{\mathrm{R}}_{[\![1/2]\!]} & \mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-1/2}\big) \\ \searrow & & \swarrow \\ & \mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-3/2}\big) & \end{array} \qquad \begin{array}{ccc} & \mathcal{L}^{\mathrm{R}}_{-1/2} & \\ \swarrow & & \searrow \\ \mathcal{L}^{\mathrm{R}}_{-3/2} & \mathcal{V}^{\mathrm{R}}_{[\![-1/2]\!]} & \mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-3/2}\big) \\ \searrow & & \swarrow \\ & \mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-1/2}\big) & \end{array}, \tag{4.13}$$

see also Figure 3. Of course, neither module is logarithmic.

With this analysis complete, we turn to the image of the logarithmic module $\mathcal{S}^{\mathrm{NS}}_{(4)}$ under the Adamović functor, which we denote by $\mathcal{Q}^{\mathrm{R}}_{[\lambda]}$, $[\lambda] \in \mathbb{C}/\mathbb{Z}$. The top space has basis $\{|\mu; t\rangle^{\mathrm{R}}, |\mu; m\rangle^{\mathrm{R}}, |\mu; \bar{m}\rangle^{\mathrm{R}}, |\mu; b\rangle^{\mathrm{R}} \mid \mu \in [\lambda]\}$, where again the second label follows (4.11). On this top space, the even zero modes J^+_0 and J^0_0 act as they did on $\mathcal{W}^{\mathrm{R}}_{[\lambda]}$ (preserving labels). The remaining even zero modes act, for $\bullet = t, m, \bar{m}, b$, as follows

$$J^-_0 |\mu; \bullet\rangle^{\mathrm{R}} = -\tfrac{1}{4}(\mu - \mathsf{k} - 1)(\mu + \mathsf{k} - 1)|\mu - 2; \bullet\rangle^{\mathrm{R}} + \tfrac{1}{2}\big(\mathsf{k} + \tfrac{1}{2}\big)(3\mathsf{k} - 1)|\mu - 2; b\rangle^{\mathrm{R}} \delta_{\bullet, t},$$

$$T_0|\mu;\bullet\rangle^{\mathrm{R}} = -\tfrac{1}{4}(\mathsf{k}+1)|\mu;\bullet\rangle^{\mathrm{R}} + \tfrac{1}{2}|\mu;b\rangle^{\mathrm{R}}\delta_{\bullet,t}. \tag{4.14}$$

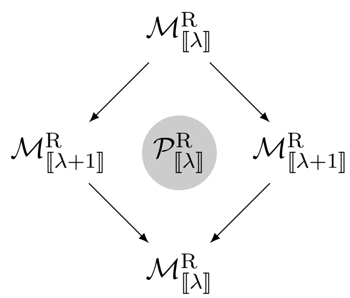
The action of the odd zero modes on the top space basis vectors is again zero, except for

$$\begin{aligned}
&G_0^+|\mu;t\rangle^{\mathrm{R}} = |\mu+1;m\rangle^{\mathrm{R}}, && G_0^+|\mu;\bar{m}\rangle^{\mathrm{R}} = -|\mu+1;b\rangle^{\mathrm{R}},\\
&\bar{G}_0^+|\mu;t\rangle^{\mathrm{R}} = |\mu+1;\bar{m}\rangle^{\mathrm{R}}, && \bar{G}_0^+|\mu;m\rangle^{\mathrm{R}} = +|\mu+1;b\rangle^{\mathrm{R}},\\
&G_0^-|\mu;t\rangle^{\mathrm{R}} = -\tfrac{1}{2}\big(\mu-\tfrac{1}{2}\big)|\mu-1;m\rangle^{\mathrm{R}}, && G_0^-|\mu;\bar{m}\rangle^{\mathrm{R}} = +\tfrac{1}{2}\big(\mu-\tfrac{1}{2}\big)|\mu-1;b\rangle^{\mathrm{R}},\\
&\bar{G}_0^-|\mu;t\rangle^{\mathrm{R}} = -\tfrac{1}{2}\big(\mu-\tfrac{1}{2}\big)|\mu-1;\bar{m}\rangle^{\mathrm{R}}, && \bar{G}_0^-|\mu;m\rangle^{\mathrm{R}} = -\tfrac{1}{2}\big(\mu-\tfrac{1}{2}\big)|\mu-1;b\rangle^{\mathrm{R}}.
\end{aligned}$$

It is clear that we once again have a direct sum decomposition

$$\mathcal{Q}^{\mathrm{R}}_{[\lambda]} = \mathcal{P}^{\mathrm{R}}_{[\![\lambda]\!]} \oplus \mathcal{P}^{\mathrm{R}}_{[\![\lambda+1]\!]}, \qquad [\lambda] \in \mathbb{C}/\mathbb{Z},$$

where $\mathcal{P}^{\mathrm{R}}_{[\![\lambda]\!]}$ is spanned by the $|\mu;t\rangle^{\mathrm{R}}$, $|\mu+1;m\rangle^{\mathrm{R}}$, $|\mu+1;\bar{m}\rangle^{\mathrm{R}}$ and $|\mu;b\rangle^{\mathrm{R}}$, with $\mu \in [\![\lambda]\!]$. Each of these submodules is logarithmic and their Loewy diagrams have the generic form



, $\qquad [\![\lambda]\!] \neq \big[\!\big[\pm\tfrac{1}{2}\big]\!\big]$.

It is clear from (4.14) that these modules are logarithmic.

Consider the degeneration that occurs when $[\![\lambda]\!] = \big[\!\big[\pm\frac{1}{2}\big]\!\big]$. We will suppose first that $\mathsf{k} = -\frac{1}{2}$. Then, the qualitative difference from the generic action given above is that J_0^- acts as zero, when $\mu = \frac{1}{2}$ or $\frac{3}{2}$, while G_0^- and $\bar{G}_0^-$ act as zero, when $\mu = \frac{1}{2}$. The resulting structures are depicted, along with the corresponding Loewy diagrams, in Figure 4.

For $\mathsf{k} = \frac{1}{2}$, the only difference is that J_0^- no longer acts as zero when $\mu = \frac{1}{2}$ or $\frac{3}{2}$. Instead, we have

$$J_0^-|\mu;\bullet\rangle^{\mathrm{R}} = \tfrac{1}{4}|\mu-2;b\rangle^{\mathrm{R}}\delta_{\bullet,t}.$$

This modifies the structural depictions and Loewy diagrams of Figure 4 by adding one arrow to each. Specifically, we should add an arrow in each depiction from the top dot labelled by $\frac{1}{2}$ $\big(\frac{3}{2}\big)$ to the bottom dot labelled by $-\frac{3}{2}$ $\big(-\frac{1}{2}\big)$ and an arrow in each Loewy diagram from the second-row factor $\mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-1/2}\big)$ $\big(\mathsf{w}\big(\mathcal{L}^{\mathrm{R}}_{-3/2}\big)\big)$ to the third-row factor $\mathcal{L}^{\mathrm{R}}_{-3/2}$ $\big(\mathcal{L}^{\mathrm{R}}_{-1/2}\big)$.

We finish with three remarks.

- The structures of the $\mathcal{P}^{\mathrm{R}}_{[\![\lambda]\!]}$, for both $\mathsf{k} = \frac{1}{2}$ and $-\frac{1}{2}$, may be partially characterised in terms of short exact sequences involving the $\mathcal{V}^{\mathrm{R}}_{[\![\lambda]\!]}$ of (4.12) and (4.13)

$$0 \longrightarrow \mathcal{V}^{\mathrm{R}}_{[\![\lambda+1]\!]} \longrightarrow \mathcal{P}^{\mathrm{R}}_{[\![\lambda]\!]} \longrightarrow \mathcal{V}^{\mathrm{R}}_{[\![\lambda]\!]} \longrightarrow 0.$$

  This unsurprising characterisation is nevertheless not as informative as the Loewy diagrams of Figure 4.

- Given experience with other "logarithmic" vertex-operator superalgebras, see [18] for example, one might be surprised by the existence of the family of logarithmic $\mathsf{W}^{\min}_{\pm1/2}$-modules $\big\{\mathcal{P}^{\mathrm{R}}_{[\![\lambda]\!]} \mid [\![\lambda]\!] \in \mathbb{C}/2\mathbb{Z}\big\}$. Indeed, the standard module formalism of [18, 50] suggests that the

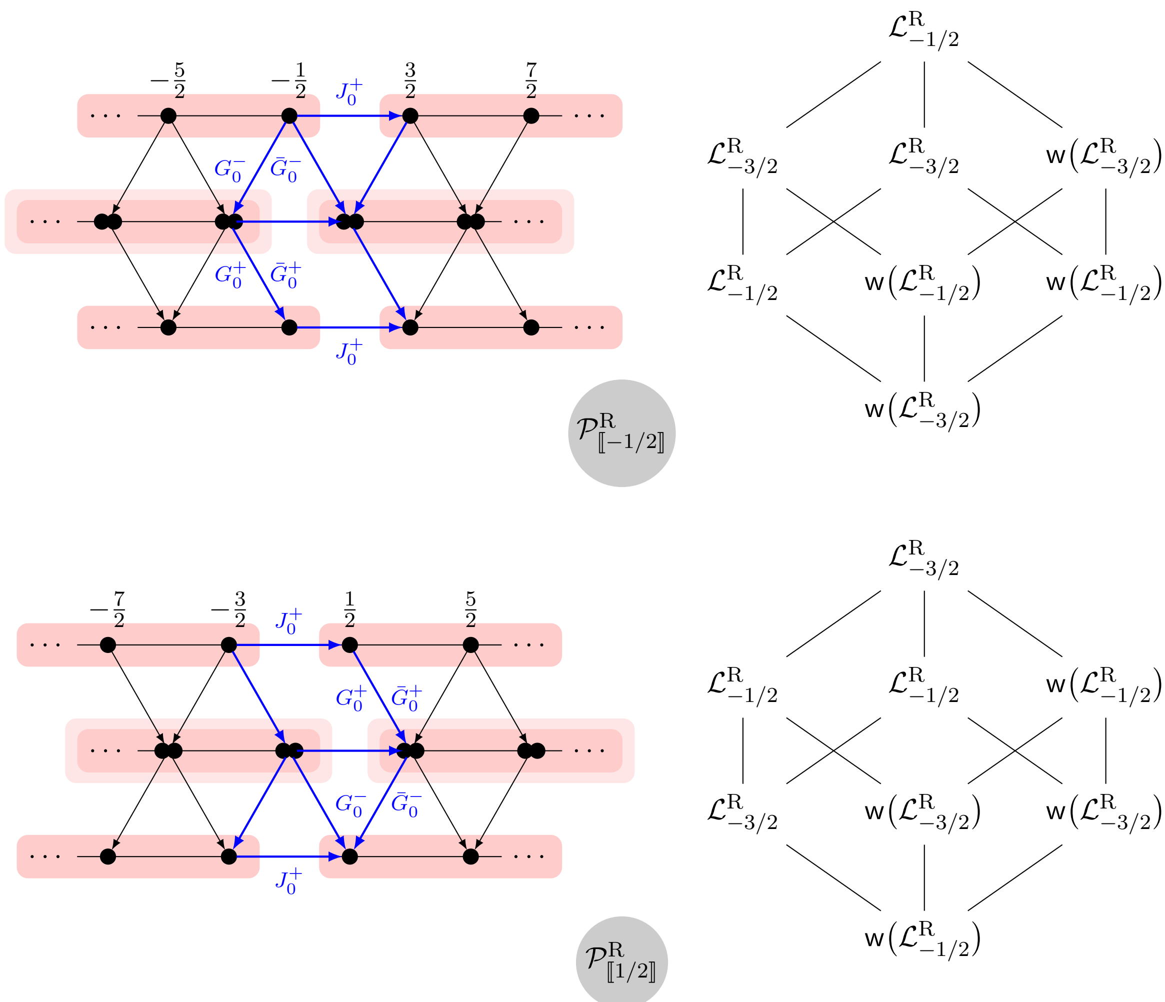


Figure 4. Loewy diagrams and top-space structures of the modules $\mathcal{P}^{\mathrm{R}}_{[\![\pm 1/2]\!]}$ when $\mathsf{k} = -\frac{1}{2}$. The top row of the structures corresponds to vectors with label "t", the middle row to those with labels "m" and "$\bar{m}$", and the bottom row to those with labels "b". The shading again suggests the composition factors.

"typical" members of the family of "standard modules" should be both irreducible and projective, hence logarithmic modules should be (in this sense) rare. In the case at hand, the ubiquity of logarithmic modules is a simple consequence of the fact that $\mathsf{W}^{\mathrm{pr}}_{\pm 1/2}$ is not rational and, in fact, admits a logarithmic module (see [8, Section 4.2] for another example of this phenomenon). Examples like these indicate that the simple "typical/atypical" distinction of the standard module formalism needs generalising.

- A natural question is whether the $\mathcal{P}^{\mathrm{R}}_{[\![\lambda]\!]}$ are projective in a suitable category of $\mathsf{W}^{\min}_{\pm 1/2}$-modules. Experience with projectives for $\mathsf{L}_{\mathsf{k}}(\mathfrak{sl}_2)$ [9, 17] and (conjecturally) $\mathsf{L}_{-3/2}(\mathfrak{sl}_3)$ [19] suggests that the answer is no because the $\mathcal{P}^{\mathrm{R}}_{[\![\lambda]\!]}$ are lower bounded. It would be extremely interesting to investigate the construction of logarithmic $\mathsf{W}^{\min}_{\pm 1/2}$-modules that do not have a lower-bounded spectral flow image. We expect that these modules will feature Jordan blocks of rank greater than 2 in the action of T_0.

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