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THE INVESTIGATION OF BOGOLIUBOV FUNCTIONALS BY OPERATOR METHODS OF MOMENT PROBLEM

YU. M. BEREZANSKY AND V. A. TESKO

To the memory of Natasha Yevdokymova (Berezanska) borned on August 20, 1950 and tragically went away from life on June 19, 2014

ABSTRACT. The article is devoted to a study of Bogoliubov functionals by using methods of the operator spectral theory being applied to the classical power moment problem. Some results, similar to corresponding ones for the moment problem, where obtained for such functionals. In particular, the following question was studied: under what conditions a sequence of nonlinear functionals is a sequence of Bogoliubov functionals.

1. INTRODUCTION

A Bogoliubov functional was introduced by M. M. Bogoliubov [18] to define correlation functions for statistical mechanics systems. Corresponding historical remarks and results can be found in [32]. These functionals have many applications in statistical physics. For applications and properties of such functional, see [28, 33, 23, 24].

The aim of this article is to consider Bogoliubov functionals from the point of view of the classical moment problem. A more detailed account of such a connection will be given in the first subsection of Section 7 when we have all necessary definitions.

It is well known [1, 4] that the classical moment problem is formulated in the following way. For a given a sequence of real numbers $s_0, s_1, \ldots,$ *what are conditions on the sequence so that we can assert that these numbers are power moments of some probability Borel measure σ on $\mathbb{R}$, that is,*

$$(1.1) \qquad s_n = \int_{\mathbb{R}} \lambda^n d\sigma(\lambda), \quad n \in \mathbb{N}_0 := \{0, 1, \ldots\}.$$

The answer is classical, — *the sequence $(s_n)_{n=0}^{\infty}$ must be nonnegative, i.e., for an arbitrary finite sequence $f = (f_n)_{n=0}^{\infty}$ of complex numbers $f_n \in \mathbb{C}$, the following inequality takes place:*

$$(1.2) \qquad \sum_{j,k=0}^{\infty} s_{j+k} f_j \bar{f}_k \geq 0.$$

Bogoliubov functionals are defined as follows. Let X be some Riemannian locally compact manifold and $C_{\text{fin}}^{\infty}(X)$ be the space of all real-valued infinitely differentiable finite functions on X. The Bogoliubov functionals $B(\varphi)$ are defined as the mapping

$$(1.3) \qquad C_{\text{fin}}^{\infty}(X) \ni \varphi \mapsto B(\varphi) = \int_{\Gamma} \prod_{x \in \gamma} (1 + \varphi(x)) \, d\sigma(\gamma),$$

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where σ is a probability measure on the space Γ of all finite and infinite configurations $\gamma = [x_1, x_2, \ldots]$, where $x_n \in X$, $x_i \neq x_j$, $i \neq j$ and "tend to infinity" on X.

We will show that representation (1.3) for $B(\varphi)$ is an analog of representation (1.1) for s_n. The condition on $B(\varphi)$, which gives (1.3), is similar to the nonnegativity condition (1.2). But now, instead of the classical convolution of finite sequences $f = (f_n)_{n=0}^{\infty}$ of complex numbers,

$$(f * g)_n := \sum_{i+j=n} f_i g_j = \sum_{k=0}^{n} f_k g_{n-k},$$

which is connected with condition (1.2), it is necessary to take the Kondratiev–Kuna convolution $\star$, defined in [21, 22].

It is necessary to say that a large part of results of this article was published in [8] and a detailed account of the results in [8] can be found in [13]. But presentation in [8, 13] is complicated, whereas the present article gives a more clear account of these results. Note that the main new result of this work is Theorem 6.6 on the structure of the spectrum of the corresponding family of commuting selfadjoint operators. In the articles [10, 8, 13], an analog of this result was proved under very restrictive conditions on the positive functional that defines the scalar product. Note also that a starting work in the considered direction was the article [10], the articles [35, 19] also play an essential role. The same connection between the Bogoliubov functionals and the moment problem has appeared in an unpublished report of Yu. M. Berezansky at the Ukrainian Mathematical Congress (Kyiv, 2009).

The methods of this work are based on the spectral approach to the moment problem, representations of positive definite kernels, etc., which was initiated by M. G. Krein [26, 27]. A subsequent extension of this approach, using the theory of generalized eigenvector expansion, is given in the works [2, 25, 3, 4, 6, 9, 12, 7, 14, 15] and many others. Note also that in this article we proposed some new approach for constructing measures on the space Γ of configurations (Section 2).

In this article we also give a criterion for a representation of the Bogoliubov functional $C_{\text{fin}}^{\infty}(X) \ni \varphi \mapsto B(\varphi) \in \mathbb{R}$ as in the classical moment problem but in a form more complicated than (1.2). It is also possible to extend such an investigation for real–valued not necessarily smooth functions $\varphi(x)$, $x \in X$. In order to extend the results to complex-valued functions $\varphi(x)$, it is necessary to use corresponding analogues of the complex moment problem [1, 14, 15].

2. Initial spaces. The Ruelle and Kondratiev–Kuna convolutions. Lenard transform

Let X be a connected oriented C^{∞} (non-compact) Riemannian manifold. We denote by $\mathcal{D} := C_{\text{fin}}^{\infty}(X)$ the set of all real-valued infinitely differentiable functions on X with compact support. We will regard $\mathcal{D}$ as a nuclear topological space with the projective limit topology (see, e.g., [16, 9] and Section below). Let $\mathcal{F}_0(\mathcal{D}) := \mathbb{C}$ and $\mathcal{F}_n(\mathcal{D}) := \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$ for all $n \in \mathbb{N}$. Here and below $\widehat{\otimes}$ denotes a symmetric tensor product ($\otimes$ denotes the usual tensor product), the subindex $\mathbb{C}$ denotes the complexification of the real space. Below we always identify in the natural way the space $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$ with the space of all complex-valued symmetric infinitely differentiable functions on X^n with compact supports.

Consider the Fock-type space $\mathcal{F}_{\text{fin}}(\mathcal{D})$. By definition, it is the topological direct sum of the spaces $\mathcal{F}_n(\mathcal{D})$

$$(2.1) \qquad \mathcal{F}_{\text{fin}}(\mathcal{D}) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{D}).$$

This space consists of all finite sequences $(f_n)_{n=0}^\infty$, $f_n \in \mathcal{F}_n(\mathcal{D})$ (when we speak about a finite sequence, we mean a sequence $(f_n)_{n=0}^\infty$ where at most finitely many entries f_n are non-zero). The convergence in this space is equivalent to the uniform finiteness and coordinate-wise convergence. Note that the linear topological space $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ is nuclear, being a topological direct sum of nuclear spaces (see, e. g., [9, 16]).

It will be convenient for us to interpret elements of space (2.1) as functions on a certain set, — the set of finite configurations. Namely, an n-point configuration is, by definition, a (non-ordered) set $\xi_n = [x_1, \ldots, x_n]$ of points $x_1, \ldots, x_n \in X$, $x_k \neq x_j$ if $k \neq j$. The set of all such finite configurations will be denoted by $\Gamma^{(n)} = \Gamma_X^{(n)}$. It is clear that

$$\Gamma^{(n)} = \{\xi \subset X \,|\, |\xi| = n\},$$

where $|\cdot|$ means cardinality of the set. The topology in $\Gamma^{(n)}$ is introduced as the image of topology in the space

$$\widehat{X}^n := \{(x_1, \ldots, x_n) \in X^n \,|\, x_k \neq x_j \text{ if } k \neq j\}$$

($\widehat{X}^n$ inherits the topology of X^n) under the mapping

$$\widehat{X}^n \ni (x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n] = \xi_n \in \Gamma^{(n)}.$$

Thus, a sequence $\xi_n^{(m)} = [x_1^{(m)}, \ldots, x_n^{(m)}]$ convergences to $\xi_n = [x_1, \ldots, x_n]$ as $m \to \infty$ in the topology of $\Gamma^{(n)}$ if and only if $x_1^{(m)} \to x_1, \ldots, x_n^{(m)} \to x_n$ as $m \to \infty$.

Put $\Gamma^{(0)} = \Gamma_X^{(0)} := \{\varnothing\}$. Define the space of (all) finite configuration by the formula

$$(2.2) \qquad \Gamma_0 := \bigsqcup_{n=0}^\infty \Gamma^{(n)} = \{\xi \subset X \,|\, \exists n \in \mathbb{N}_0 \text{ such that } \xi \in \Gamma^{(n)}\}.$$

The topology in Γ_0 is introduced in the following way. A sequence $(\xi^{(m)})_{m=1}^\infty \subset \Gamma_0$ convergences to $\xi \in \Gamma_0$ as $m \to \infty$ if and only if, starting with some m, all $\xi^{(m)}$ belong to some $\Gamma^{(n)}$ and $\xi^{(m)} \to \xi$ as $m \to \infty$ in $\Gamma^{(n)}$.

It is easy to understand that elements of space (2.1) can be treated as functions on the space Γ_0, i.e., one can embed $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ into the space $\mathrm{Fun}(\Gamma_0)$ of all complex-valued functions on Γ_0. Namely, since $f_0 \in \mathbb{C}$ and each f_n, $n \in \mathbb{N}$, is a complex-valued symmetric function on X^n, there is a natural injective mapping

$$(2.3) \qquad \mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \ni f = (f_n)_{n=0}^\infty \mapsto \hat{f}(\cdot) := \sum_{n=0}^\infty \hat{f}_n(\cdot) \in \mathrm{Fun}(\Gamma_0),$$

where $\hat{f}_0(\varnothing) := f_0$ and

$$\Gamma_0 \ni \xi \mapsto \hat{f}_n(\xi) := \begin{cases} f_n(x_1, \ldots, x_n), & \text{if } \xi = [x_1, \ldots, x_n] \in \Gamma^{(n)}, \\ 0, & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$. Using the latter mapping *we can interpret the vectors $f = (f_n)_{n=0}^\infty$ from space $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ as the corresponding functions $\hat{f}(\xi)$ on Γ_0.*

Conversely, if a function $\Gamma_0 \ni \xi \mapsto F(\xi) \in \mathbb{C}$ is such that

$$f_n := F \upharpoonright \Gamma^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}, \quad f_n(x_1, \ldots, x_n) := F([x_1, \ldots, x_n]),$$

and equals zero starting with some n, then we can interpret F as a corresponding vector $(f_n)_{n=0}^\infty$ from the space $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$.

In what follows we will use such interpretations without some additional explanations.

We will often use some subspaces of $\Gamma^{(n)}, \Gamma_0$. Namely, let $Y \subset X$ be some infinite subset of the space X, we topologize it by the relatively topology. We will apply the above constructions of $\Gamma^{(n)}$ and Γ_0 starting from Y instead of X; we will denote then by $\Gamma_Y^{(n)}$ and $\Gamma_{Y,0}$; thus $\Gamma_X^{(n)} = \Gamma^{(n)}$, $\Gamma_{X,0} = \Gamma_0$, $\Gamma_Y^{(n)} \subset \Gamma^{(n)}$, $\Gamma_{Y,0} \subset \Gamma_0$.

We will need also the space Γ of infinite (including finite) configurations over X, i.e., the set of all locally finite subsets of X

$$(2.4) \qquad \Gamma := \Gamma(X) := \big\{ \gamma \subset X \,\big|\, |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset X \big\}.$$

Obviously, each $\gamma \in \Gamma$ consists of distinct points from X, and Γ consists of all different non overlapping configurations γ (subsets of X). *We stress that $\Gamma_0 \subset \Gamma$. The topology in Γ will be introduced later in this Section. This way of introducing the topology also defines a topology on Γ_0 (as the relative topology); this topology on Γ_0 is different from the one considered in (2.2).*

We pass now to the definition of some essential convolutions on the space $\mathrm{Fun}(\Gamma_0)$. For arbitrary functions $f, g \in \mathrm{Fun}(\Gamma_0)$ and $\xi \in \Gamma_0$, the Ruelle convolution $*$ and the Kondratiev–Kuna convolution $\star$ are defined by

$$(2.5) \qquad (f * g)(\xi) = \sum_{\xi' \sqcup \xi'' = \xi} f(\xi') g(\xi'') = \sum_{\xi' \subset \xi} f(\xi') g(\xi \setminus \xi'),$$

$$(2.6) \qquad (f \star g)(\xi) = \sum_{\xi' \cup \xi'' = \xi} f(\xi') g(\xi'') = \sum_{\xi' \sqcup \xi'' \sqcup \xi''' = \xi} f(\xi' \cup \xi'') g(\xi'' \cup \xi''').$$

Recall that $\alpha \sqcup \beta$ means the union of two disjoint sets α and β. All sums in (2.5) and (2.6) are finite, the cases $\xi' = \varnothing$ and $\xi'' = \varnothing$ are included into (2.5) and (2.6). From these definitions we see that the sum for $\star$ has more terms than the sum for $*$. In the next four sections we will investigate the Kondratiev–Kuna convolution and the related to it operators. The Ruelle convolution [34] will be considered in another article.

From (2.6) it immediately follows that if $f, g \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ then $f \star g$ also belongs to $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$. Moreover, the convolution $\star$ (being a convolution on $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$) is commutative, associative, additive and continuous with respect to both variables (see [21, 22, 28, 10, 33]). So, $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ with $\star$ is a commutative topological nuclear algebra $\mathcal{A}$ with the unit

$$\Gamma_0 \ni \xi \mapsto e(\xi) := \begin{cases} 1, & \text{if } \xi = \varnothing, \\ 0, & \text{otherwise}. \end{cases}$$

The convolution $\star$ is closely connected with the Lenard transform K. The corresponding definitions and facts will be given below (for more details, see [29, 30, 31]).

Denote by $\mathrm{Fun}_{\mathrm{bs}}(\Gamma_0)$ the space of all functions in $\mathrm{Fun}(\Gamma_0)$ which are equal to zero on $\Gamma_0 \setminus \Gamma_{\Lambda,0}$ for a compact set $\Lambda \subset X$, i.e., such functions would have bounded support "in the direction of space X". Note that $\mathrm{Fun}_{\mathrm{bs}}(\Gamma_0)$ is a linear set and $\star$ transfers it into an algebra similar to $\mathcal{A}$ which contains the algebra $\mathcal{A}$.

The transform K is defined as a mapping that acts from $\mathrm{Fun}_{\mathrm{bs}}(\Gamma_0)$ to $\mathrm{Fun}(\Gamma)$ by the formula

$$(2.7) \qquad \mathrm{Fun}_{\mathrm{bs}}(\Gamma_0) \ni f \mapsto (Kf)(\gamma) := \sum_{\xi \subset \gamma} f(\xi) =: F(\gamma) \in \mathbb{C},$$

where the summation is taken over all finite subconfigurations of γ. Since the function f in (2.7) belongs to $\mathrm{Fun}_{\mathrm{bs}}(\Gamma_0)$, the sum in (2.7) is finite for every $\gamma \in \Gamma$. Namely, let $f(\xi) = 0$ for $\xi \in \Gamma_0 \setminus \Gamma_{\Lambda,0}$, where $\Lambda \subset X$ is a compact set, i.e., $f(\xi) \neq 0$ only for $\xi = [x_1, \ldots, x_n]$, $n \in \mathbb{N}$, where all x_j belong to Λ. Then for fixed $\gamma = [x_1, x_2, \ldots]$, according to (2.4), only finite summands in (2.7) are nonzero.

Note that $(Kf)(\gamma)$ is a function of $\gamma \in \Gamma$. Different $\gamma \in \Gamma$ are non overlapping if considered as subsets of X, therefore in the sum (2.7) for every fixed $\gamma \in \Gamma$ there exists only one $\xi \in \Gamma_0$ belonging to γ.

The transform K (the Lenard or the key transform) was introduced in the works [21, 22, 28] based on the articles [29, 30, 31].

A connection between K-transform and $\star$-convolution is the following.

Proposition 2.1. *For every $f, g \in \mathrm{Fun}_{\mathrm{bs}}(\Gamma_0)$ and $\gamma \in \Gamma$ we have*

$$(2.8) \qquad (K(f \star g))(\gamma) = (Kf)(\gamma)(Kg)(\gamma).$$

As we have mentioned above, the vectors $f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ can be understood as functions on Γ_0. Of course, $\mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \subset \mathrm{Fun}_{\mathrm{bs}}(\Gamma_0)$ and the transform K is well defined on $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$.

For $f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, $(Kf)(\gamma)$ is a function of all configurations $\gamma \in \Gamma$. In particular, every finite configuration $\eta \in \Gamma^{(n)} \subset \Gamma$, $n \in \mathbb{N}$, and also $\varnothing$ can be its argument.

It turns out that the function $f(\xi)$, $\xi \in \Gamma_0$, can be restored from these values $(Kf)(\eta)$, $\eta \in \Gamma^{(n)}$, $n \in \mathbb{N}_0$. Moreover, let $\Gamma_0 \ni \eta \mapsto F(\eta) \in \mathbb{C}$ be a given function such that $F(\eta)$, $\eta = [y_1, \ldots, y_n]$ is a symmetric infinitely differentiable finite function of point $(y_1, \ldots, y_n) \in X^n$, $n \in \mathbb{N}$; $F(\varnothing) \in \mathbb{C}$. We assert that one can find a function $f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ such that, for every $\eta \in \Gamma_0$, $(Kf)(\eta) = F(\eta)$. So, the inverse transform K^{-1} exists in the just explained sense. The following proposition gives the existence of such f and a formula for it.

Proposition 2.2. *For any above described function $F(\eta)$ on Γ_0, the following formulas hold:*

$$(2.9) \qquad \begin{aligned} (K^{-1}F)(\xi) &= \sum_{\eta \subset \xi} (-1)^{|\xi \setminus \eta|} F(\eta), \quad \xi \in \Gamma_0, \\ (K(K^{-1}F))(\eta) &= F(\eta), \quad \eta \in \Gamma_0. \end{aligned}$$

Thus, it is possible to find $f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ for which

$$(2.10) \qquad (Kf)(\eta) = F(\eta), \quad \eta \in \Gamma_0; \quad f(\xi) = (K^{-1}(Kf))(\xi), \quad \xi \in \Gamma_0.$$

Then inverse operator K^{-1} (2.9) is, in some sense, continuous: the following fact is true.

Lemma 2.3. *Let $\Lambda \subset X$ be a compact set. Then for the function $F(\eta)$ described above, the following estimate holds:*

$$|(K^{-1}F)(\xi)| \leq 2^n \max_{\eta \in \bigsqcup_{j=0}^{n} \Gamma_\Lambda^{(j)}} |F(\eta)|, \quad n \in \mathbb{N}_0,$$

thus

$$(2.11) \qquad \max_{\xi \in \Gamma_\Lambda^{(n)}} |(K^{-1}F)(\xi)| \leq 2^n \max_{\eta \in \bigsqcup_{j=0}^{n} \Gamma_\Lambda^{(j)}} |F(\eta)|, \quad n \in \mathbb{N}.$$

Proof. For $n \in \mathbb{N}$, $\xi \in \Gamma_\Lambda^{(n)}$ we have, according to (2.9), that

$$|(K^{-1}F)(\xi)| = \left| \sum_{\eta \subset \xi} (-1)^{|\xi \setminus \eta|} F(\eta) \right| \leq \max_{\eta \in \bigsqcup_{j=0}^{n} \Gamma_\Lambda^{(j)}} |F(\eta)| \sum_{\eta \subset \xi} (-1)^{|\xi \setminus \eta|} \leq 2^n \max_{\eta \in \bigsqcup_{j=0}^{n} \Gamma_\Lambda^{(j)}} |F(\eta)|.$$

We used that for $\xi \in \Gamma_\Lambda^{(n)}$, each configuration $\eta \subset \xi = [x_1, \ldots, x_n]$ (and also $\eta = \varnothing$) belongs to $\bigsqcup_{j=0}^{n} \Gamma_\Lambda^{(j)}$ and the number of these configurations equals the number of all subsets of the set $\{x_1, \ldots, x_n\}$, i.e., 2^n.

For $n = 0$ the first formula in (2.9) gives $f_0 = F(\varnothing)$.

The second inequality from Lemma 2.3 follows from first inequality. $\qquad \square$

Let us again look at the results of Proposition 2.2 and Lemma 2.3. We have the sequence

$$(2.12) \qquad F = (F_0, F_1(x_1), F_2(x_1, x_2), \ldots, F_m(x_1, \ldots, x_m), 0, 0, \ldots), \quad m \in \mathbb{N}_0,$$

where F_n is a symmetric infinite differentiable finite function of variable $(x_1, \ldots, x_n) \in \hat{X}^n \subset X^n$.

The sequence (2.12), using (2.3), can be understood as a function $F(\xi)$, where $\xi \in \Gamma_0$ and we put $F(\xi) := F_n(x_1, \ldots, x_n)$ for $\xi = [x_1, \ldots, x_n]$; $F(\varnothing) := F_0$. Using (2.12) we see that $F(\xi) = 0$ for all $\xi = [x_1, \ldots, x_n]$ such that $n > m$.

Interchange $\xi = [x_1, \ldots, x_n]$ and $\eta = [y_1, \ldots, y_n]$ in (2.12). Apply the first formula from (2.9) to function $F(\eta)$. As a result, we get the function $f(\xi) = (K^{-1}F)(\xi)$, $\xi \in \Gamma_0$. This function f is equal to zero, when $\xi = [x_1, \ldots, x_n]$, $n > m$. Indeed, from (2.9) it follows that $\eta \subset \xi$ and $F(\eta) = 0$ for $\eta = [y_1, \ldots, y_n]$, $n > m$. *If we apply the operator K to f, we get F*, i.e., the first formula in (2.10) is true. For this it is necessary to note that in the formula (2.7) the sum is taken over $\xi \in \gamma$, where $\gamma \in \Gamma$ are different non overlapping configurations.

In Section 6 we will use, instead of (2.12), the finite vectors

$$(2.13) \qquad F = (F_0, F_1(x_1), F_2(x_1, x_2), \ldots, F_m(x_1, \ldots, x_m)), \quad m \in \mathbb{N}_0.$$

We investigate now, in more details, the set $\Gamma = \Gamma(X)$ (2.4) of all finite and infinite configurations, i.e., the subsets of X of the form (2.4). If X is replaced with its subset $Y \subset X$ we denote the corresponding set (2.4) by $\Gamma(Y)$. It is clear that $\Gamma(Y) \subset \Gamma(X)$. If Λ is compact than $\Gamma(\Lambda)$ consists only of finite configurations.

Since X is a separable locally compact space, there exists a sequence of its compact subspaces Λ_n, $n \in \mathbb{N}$, for which

$$(2.14) \qquad \Lambda_1 \subset \Lambda_2 \subset \ldots \quad \text{and} \quad X = \bigcup_{n=1}^{\infty} \Lambda_n.$$

We have the following decomposition of the space X:

$$(2.15) \qquad \begin{aligned} X &= \Lambda_1 \cup (\Lambda_2 \setminus \Lambda_1) \cup (\Lambda_3 \setminus \Lambda_2) \cup \ldots = K_1 \cup K_2 \cup K_3 \cup \ldots, \\ K_n &:= \Lambda_n \setminus \Lambda_{n-1} \subset \Lambda_n, \quad n \in \mathbb{N} \quad (\Lambda_0 := \varnothing), \end{aligned}$$

where the sets K_1, K_2, K_3, $\ldots$ are pairwise disjoint and have compact closures. Let $\gamma \in \Gamma(X)$ be an arbitrary configuration, i.e., some subset (2.4) of points from X. Then representation (2.15) gives that

$$(2.16) \qquad \begin{aligned} \gamma &= (\gamma \cap \Lambda_1) \cup (\gamma \cap (\Lambda_2 \setminus \Lambda_1)) \cup (\gamma \cap (\Lambda_3 \setminus \Lambda_2)) \cup \ldots \\ &= (\gamma \cap K_1) \cup (\gamma \cap K_2) \cup (\gamma \cap K_3) \cup \ldots \\ &= \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \ldots, \quad \gamma_n := \gamma \cap K_n, \quad n \in \mathbb{N}. \end{aligned}$$

Note that every $\gamma \cap K_n$ is a finite configuration which belongs to $\Gamma(K_n)$. Some of the sets $\gamma \cap K_n$ may be empty.

So, due to (2.16) we have, for every set $\gamma \in \Gamma(X)$, that

$$(2.17) \qquad \gamma \subset \bigcup_{n=1}^{\infty} \Gamma(K_n).$$

A use of (2.17) gives the following important lemma.

Lemma 2.4. *The representations (2.14), (2.15) give that*

$$(2.18) \qquad \Gamma(X) = \bigsqcup_{n=1}^{\infty} \Gamma(K_n),$$

i.e., every $\gamma \in \Gamma(X)$ is a union of its parts γ_n (2.16) from $\Gamma(K_n)$. If $\gamma \in \Gamma(X) \setminus \Gamma_0$, then the disjoint union (2.18) is necessarily infinite.

Proof. Since K_n are disjoint subsets of X, $\Gamma(K_n)$ are also disjoint sets and, in right side of (2.17), we can write $\bigsqcup_{n=1}^{\infty} \Gamma(K_n)$. An arbitrary element of $\bigsqcup_{n=1}^{\infty} \Gamma(K_n)$ (i.e., some subset of X) has the form $(\gamma_1, \gamma_2, \ldots)$, where $\gamma_n \in \Gamma(K_n)$ is some finite configuration (some γ_n may be empty). Points from $\gamma_n \in \Gamma(K_n)$ are different, also different are points from different $\Gamma(K_n)$ and $\Gamma(K_m)$. Therefore $(\gamma_1, \gamma_2, \ldots)$ is some configuration $\gamma = (\gamma_1, \gamma_2, \ldots)$ (see (2.4)).

Thus every configuration γ (2.4) from $\Gamma(X)$ is some subset of $\bigsqcup_{n=1}^{\infty} \Gamma(K_n)$ (see (2.17)) and, conversely, every element from $\bigsqcup_{n=1}^{\infty} \Gamma(K_n)$ is some configuration from $\Gamma(X)$. So, equality (2.18) takes place.

Last assertion of Lemma is evident. $\qquad\qquad\square$

We will construct and investigate some measures on $\Gamma(X)$. We stress that these measures will be constructed using some measure on X. This measure will be given on a fixed copy of X.

The aim of this article is, in particular, to investigate some measures on linear functionals on the space $C_{\mathrm{fin}}(X)$ of finite real-valued continuous functions $X \ni x \mapsto f(x) \in \mathbb{R}$. This space with uniform convergence on compact sets is a linear topological space.

Recall that uniformly finite convergence means the following: $C_{\mathrm{fin}}(X) \ni f_m \mapsto f \in C_{\mathrm{fin}}(X)$, $m \to \infty$, where $f_m \to f$ uniformly and $f_m(x) = 0$ for $x \in X \setminus \Lambda$, where Λ is some compact set.

Introduce the space $(C_{\mathrm{fin}}(X))' =: C'_{\mathrm{fin}}(X)$ of linear continuous functionals l with weak topology. We will investigate the finite non-negative measures on the some sub σ-algebra of Borel σ-algebra $\mathcal{B}(C'_{\mathrm{fin}}(X))$ of the space $C'_{\mathrm{fin}}(X)$.

Let μ be a locally finite non-negative Borel measure on the space X. So, the integral

$$(2.19) \qquad l_\mu(f) := \int_X f(x)\, d\mu(x)$$

exists for every $f \in C_{\mathrm{fin}}(X)$ and it is a linear continuous functional $l_\mu \in C'_{\mathrm{fin}}(X)$. Of course, the correspondence $l_\mu \leftrightarrow \mu$ is one-to-one and we can identify the measure μ and the functional l_μ via the identity (2.19).

Let us interpret finite and infinite configurations as some functionals on the space $C_{\mathrm{fin}}(X)$. For every fixed point $x_0 \in X$ there is the δ-function δ_{x_0}, i.e., a linear continuous functional $l_{\mu_{x_0}}$ generated by the atomic Borel measure μ_{x_0},

$$(2.20) \qquad l_{\mu_{x_0}}(f) = \int_X f(x)\, d\mu_{x_0}(x) = f(x_0) =: \delta_{x_0}(f), \quad f \in C_{\mathrm{fin}}(X).$$

If $\gamma \in \Gamma(X)$, we will interpreted $\gamma = [x_1, x_2, \ldots]$ as the continuous functional

$$(2.21) \qquad l_\gamma : C_{\mathrm{fin}}(X) \to \mathbb{C}, \quad l_\gamma := \sum_{x \in \gamma} \delta_x \in C'_{\mathrm{fin}}(X),$$

i.e.,

$$l_\gamma(f) := \sum_{n=1}^{\infty} f(x_n), \quad f \in C_{\mathrm{fin}}(X)$$

(note that the latter sum is finite). In this way, the space $\Gamma(X)$ is embedded in the space $C'_{\mathrm{fin}}(X)$. The week topology on $\Gamma(X) \subset C'_{\mathrm{fin}}(X)$ is called a vague topology on $\Gamma(X)$. We stress that for every $\gamma = [x_1, x_2, \ldots]$ all $\delta_{x_n} \in C'_{\mathrm{fin}}(X)$.

Let us prove that the space $\Gamma(X)$ belongs to $\mathcal{B}(C'_{\mathrm{fin}}(X))$. We start with some general facts. Let Q be some locally compact metric space of points $q, p, r, \ldots$; $\rho(p, q)$ be the corresponding metric. Let $C_{\mathrm{fin}}(Q)$ be the linear space of all real-valued continuous finite functions on Q with uniformly finite convergence, i.e., $C_{\mathrm{fin}}(Q) \ni f_m \to f \in C_{\mathrm{fin}}(Q)$, $m \to \infty$, if $f_m \to f$ uniformly and all $f_m(q) = 0$ for $q \in Q \setminus F$, where F is some compact set $F \subset Q$. We can introduce the weak convergence in the adjoint space

$(C_{\mathrm{fin}}(Q))' = C'_{\mathrm{fin}}(Q)$ of linear continuous functionals l: $C'_{\mathrm{fin}}(Q) \ni l_m \mapsto l \in C'_{\mathrm{fin}}(Q)$, $m \to \infty$, if $l_m(f) \to l(f)$ for all $f \in C_{\mathrm{fin}}(Q)$.

In the space $C'_{\mathrm{fin}}(Q)$ we can introduce the notion of a δ-function δ_q for $q \in Q$:

$$\delta_q(f) := f(q), \quad f \in C_{\mathrm{fin}}(Q).$$

Lemma 2.5. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be fixed. Consider an arbitrary sequence $(l_m)_{m \in \mathbb{N}}$,*

$$(2.22) \qquad l_m = \sum_{j=1}^{n} \delta_{q_j^{(m)}}, \quad q_1^{(m)}, \ldots, q_n^{(m)} \in Q, \quad \rho(q_j^{(m)}, q_k^{(m)}) \geq \varepsilon, \quad if \quad k \neq j.$$

Let $l_m \to l$, $m \to \infty$, in the weak sense in $C'_{\mathrm{fin}}(Q)$. Then $l = 0$ or

$$(2.23) \qquad\qquad l = \sum_{j=1}^{n} \delta_{p_j},$$

where $p_1, \ldots, p_n$ are some distinct points from Q for which $\rho(p_j, p_k) \geq \varepsilon$ if $k \neq j$.

Proof. We will use the following simple construction. Consider the direct product of Q^n of n copies of the space Q, $Q^n := Q \times \cdots \times Q$ (n times), $(q_1, \ldots, q_n)$ is its points. Note that sequence $Q^n \ni (q_1^{(m)}, \ldots, q_n^{(m)})$, $m \to \infty$, tends to some point $(p_1, \ldots, p_n) \in Q^n$ if and only if $q_j^{(m)} \to p_j$ in the space Q for every $j \in \{1, \ldots, n\}$.

Let F be some compact subset of from Q, $F^n \subset Q^n$ its n-product. Consider the following subspace of F^n:

$$(2.24) \qquad F_\varepsilon^n = F^n \setminus \bigcup_{j,k=1; j \neq k}^{n} \left\{ (q_1, \ldots, q_n) \in F^n \, | \, \rho(q_j, q_k) < \varepsilon \right\}$$

with some $\varepsilon > 0$. Since F^n is compact in Q^n, F_ε^n is also a compact.

Using (2.22) we consider a sequence of points $(q_1^{(m)}, \ldots, q_n^{(m)})$, $m \in \mathbb{N}$, and assume at first that there is a compact set $F \subset Q$ for which all the points $(q_1^{(m)}, \ldots, q_n^{(m)})$, $m \in \mathbb{N}$, belong to F^n. But the last condition in (2.22) means that these points $(q_1^{(m)}, \ldots, q_n^{(m)})$ belong to (2.24). Since F_ε^n is compact in Q^n, this sequence of points $(q_1^{(m)}, \ldots, q_n^{(m)})$ has at least one accumulation point $(p_1, \ldots, p_n) \in F_\varepsilon^n$.

It is easy to prove that such an accumulation point is unique. Indeed, let $(r_1, \ldots, r_n) \in F_\varepsilon^n$ be some other accumulation point. Consider a subsequence $(m')_{m'=1}^{\infty}$ of the sequence $(m)_{m=1}^{\infty}$ for which the points $(q_1^{(m')}, \ldots, q_n^{(m')}) \in F_\varepsilon^n$ tend to $(r_1, \ldots, r_n)$ and a similar subsequence $(m'')_{m''=1}^{\infty}$ of the sequence $(m)_{m=1}^{\infty}$ for which the points $(q_1^{(m'')}, \ldots, q_n^{(m'')}) \in F_\varepsilon^n$ tend to $(p_1, \ldots, p_n)$.

The way of introducing the topology in Q^n is such that above asserted convergence are equivalent to the following convergences: for all $j \in \{1, \ldots, n\}$

$$(2.25) \qquad\qquad q_j^{(m')} \xrightarrow[m' \to \infty]{} r_j, \quad q_j^{(m'')} \xrightarrow[m'' \to \infty]{} p_j.$$

All points $(q_1^{(m)}, \ldots, q_n^{(m)})$ belong to F_ε^n (2.24), therefore for every $j, k \in \{1, \ldots, n\}$, $j \neq k$, $\rho(q_j^{(m')}, q_k^{(m')}) \geq \varepsilon$, $\rho(q_j^{(m'')}, q_k^{(m'')}) \geq \varepsilon$. This gives that $\rho(r_j, r_k) \geq \varepsilon$ and $\rho(p_j, p_k) \geq \varepsilon$ for the above indicated j, k.

Assume that $(r_1, \ldots, r_n) \neq (p_1, \ldots, p_n)$ and let $r_{j_1} \neq p_{j_1}, \ldots, r_{j_h} \neq p_{j_h}$ be all different points from the last two sets (clearly, $h \geq 1$). Consider two open sets $U \subset V$ in Q, where $U \supset \{r_{j_1}, \ldots, r_{j_h}\}$, V has compact closure and $Q \setminus V$ contains all points from the set $\{r_1, \ldots, r_n; p_1, \ldots, p_n\} \setminus \{r_{j_1}, \ldots, r_{j_h}\}$.

Consider a function $f \in C_{\text{fin}}(Q)$ which is equal to 1 on U and to 0 on $Q \setminus V$. From relations (2.25) we conclude that

(2.26)
$$\lim_{m' \to \infty} f(q_j^{(m')}) = f(r_j) = \delta_{r_j}(f) = \begin{cases} 1, & \text{if } j \in \{j_1, \ldots, j_h\}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\lim_{m'' \to \infty} f(q_j^{(m'')}) = f(p_j) = \delta_{p_j}(f) = 0, \quad j \in \{1, \ldots, n\}.$$

The relation (2.22) and (2.26) give

(2.27)
$$\lim_{m' \to \infty} l_{m'}(f) = \lim_{m' \to \infty} \Big(\sum_{j=1}^{n} \delta_{q_j^{(m')}}(f) \Big) = \sum_{j=1}^{n} f(r_j) = h \neq 0,$$

$$\lim_{m'' \to \infty} l_{m''}(f) = \lim_{m'' \to \infty} \Big(\sum_{j=1}^{n} \delta_{q_j^{(m'')}}(f) \Big) = \sum_{j=1}^{n} f(p_j) = 0.$$

By conditions of our lemma, $l_m \to l$, $m \to \infty$ in weak sense, i.e., for every $f \in C_{\text{fin}}(Q)$ $\lim_{m \to \infty} l_m(f)$ exists and is equal to $l(f)$. Relation (2.27) is a contradiction to such existence. So, we have proved that the accumulation point $(p_1, \ldots, p_n)$ is unique.

The existence of a unique accumulation point for the sequence $((q_1^{(m)}, \ldots, q_n^{(m)}))_{m=1}^{\infty}$ of points from Q^n means that this point $(p_1, \ldots, p_n)$ is a limit in the space Q^n of our sequences. But the topology in Q^n implies that $q_j^{(m)} \to p_j$, $m \to \infty$, in Q for every $j = 1, \ldots, n$.

Therefore for every $f \in C_{\text{fin}}(Q)$, according to (2.22) and (2.23), we have

$$l_m(f) = \Big(\sum_{j=1}^{n} \delta_{q_j^{(m)}} \Big)(f) = \sum_{j=1}^{n} \delta_{q_j^{(m)}}(f) = \sum_{j=1}^{n} f(q_j^{(m)})$$

$$\xrightarrow[m \to \infty]{} \sum_{j=1}^{n} f(p_j) = \Big(\sum_{j=1}^{n} \delta_{p_j} \Big)(f) = l(f).$$

Thus, in the case of our first assumption, the lemma is proved. Recall that our assumption is the following: for a given sequence $(q_1^{(m)}, \ldots, q_n^{(m)}) \in Q^n$, there exists some compact set $F \subset Q$ for which all points of this sequence belong to F^n.

Consider another possible situation. So, consider the have previous sequence

$$(q_1^{(m)}, \ldots, q_n^{(m)}) \in Q^n$$

and for every compact set $F \subset Q$ there exist some infinite number of points

$$(q_1^{(m)}, \ldots, q_n^{(m)}) \in Q^n$$

outside of F^n. Thus, we can choose from these points a sequence $((q_1^{(m')}, \ldots, q_n^{(m')}))_{m'=1}^{\infty}$ with the following property: for every compact set $F \subset Q$ all points $(q_1^{(m')}, \ldots, q_n^{(m')})$ lie outside of $F^n \subset Q^n$ starting with some index m (depending on F). If it is impossible to find another part of points $(q_1^{(m)}, \ldots, q_n^{(m)})$ which belong to some compact set $F^n \subset Q_n$, then our points $(q_1^{(m)}, \ldots, q_n^{(m)})$ "tend to infinity", i.e., for every $f \in C_{\text{fin}}(Q)$ and $j \in \{1, \ldots, n\}$ starting from some m $f(q_j^{(m)}) = \delta_{q_j^{(m)}}(f) = 0$. Therefore in this case $l_m \to 0$, $m \to \infty$.

It is possible to have another situation: from this sequence $((q_1^{(m)}, \ldots, q_n^{(m)}))_{m=1}^{\infty}$ we can chose two subsequences such that the first one, $((q_1^{(m')}, \ldots, q_n^{(m')}))_{m'=1}^{\infty}$, has the above mentioned property of tending to infinity (i.e., the corresponding functionals tends to zero) and the second one, $((q_1^{(m'')}, \ldots, q_n^{(m'')}))_{m''=1}^{\infty}$, such that its points belong to $F^n \subset Q^n$, where F is some compact subset of Q. Thus, for this second subsequence we

have the situation considering above, in the first part of the proof of lemma. Therefore, we can assert that $(q_1^{(m'')}, \ldots, q_n^{(m'')})$ tends to some l of form (2.23). But such l is not equal to zero, since it is a limit of the first subsequence. Therefore, the last situation is impossible, that is, the weak limit of (2.22) does not exist. □

Using Lemma 2.5 we can prove the following essential result.

Lemma 2.6. *Let* $\Lambda \subset X$ *be some compact set. Then we have that*

(2.28) $$\Gamma(\Lambda) \in \mathcal{B}(C'_{\mathrm{fin}}(X)).$$

Proof. As we have noted, every configuration from $\Gamma(\Lambda)$ is finite. Denote

$$\Gamma_\Lambda^{(n)} := \big\{ \gamma \in \Gamma(X) \,\big|\, |\gamma| = n, \gamma \subset \Lambda \big\}, \quad n \in \mathbb{N}; \quad \Gamma_\Lambda^{(0)} := \varnothing.$$

It is clear that

(2.29) $$\Gamma(\Lambda) = \bigsqcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}.$$

Let $n \in \mathbb{N}$ be fixed. We will apply Lemma 2.5. Let $Q = \Lambda$, points $p, q, \ldots$ from Q we denote by $x, y, \ldots$; $\delta_x, \delta_y, \ldots$ are the corresponding δ-functions. According to (2.21), a configuration $\gamma = [x_1, \ldots, x_n] \in \Gamma_\Lambda^{(n)}$ will be interpreted as a functional on $C_{\mathrm{fin}}(X)$,

(2.30) $$l_\gamma = \sum_{j=1}^{n} \delta_{x_j} \in C'_{\mathrm{fin}}(X).$$

We will fix some sequence $(\varepsilon_r)_{r=0}^{\infty}$ of positive numbers tending to zero, $\varepsilon_r > 0$, $\varepsilon_r \to 0$, $r \to \infty$. Let $\Gamma_{\varepsilon_r}^{(n)} \subset \Gamma_\Lambda^{(n)}$ be the set of all configurations $\gamma = [x_1, \ldots, x_n]$ for which $\rho(x_j, x_k) \geq \varepsilon_r$ for every $j, k \in \{1, \ldots, n\}$, $j \neq k$. Since, in every configuration γ, all the points $x_1, \ldots, x_n$ are different, we can write

(2.31) $$\Gamma_\Lambda^{(n)} = \bigcup_{r=1}^{\infty} \Gamma_{\varepsilon_r}^{(n)}.$$

Therefore the set $\Gamma_\Lambda^{(n)}$ will be Borel if we prove that every $\Gamma_{\varepsilon_r}^{(n)}$ is a Borel set.

Consider the set $\Gamma_{\varepsilon_r}^{(n)} \cup \{0\}$, where 0 denotes the zero functional from $C'_{\mathrm{fin}}(X)$. From Lemma 2.5 it is easy to conclude that this set is closed in the weak topology.

So, let $(l_m)_{m=1}^{\infty}$ be some sequence from $\Gamma_{\varepsilon_r}^{(n)} \cup \{0\}$ which tends, in the weak sense, to some $l \in C'_{\mathrm{fin}}(X)$. If among the functionals l_m there exists an infinite subsequence consisting of the zero functionals, then $l_m \to 0$, $m \to \infty$, since another part of functionals has form (2.30) and according to Lemma 2.5 tends to a functional of the form (2.23) or to zero. A functional of the form (2.30) is not equal to zero and, by the assumption, the limit $\lim_{m \to \infty} l_m$ exists, therefore in this case limit functional $l = 0$.

The second possibility is the following: among the functionals l_m we have only a finite set of zero functionals, then according to Lemma 2.5 this sequence tends to a functional l of the form (2.30) or to zero.

So, in every case, the limit functional $l \in \Gamma_{\varepsilon_r}^{(n)} \cup \{0\}$, i.e., the latter set is closed in the weak topology. Therefore $\Gamma_{\varepsilon_r}^{(n)}$ is a Borel set. Then according to (2.31) $\Gamma_\Lambda^{(n)}$ is also a Borel set.

According (2.29) $\Gamma(\Lambda)$ is also a Borel set. □

Remark 2.7. From the proof of Lemma 2.6 it follows that for every compact set $\Lambda \subset X$,

(2.32) $$\Gamma_\Lambda^{(n)} \in \mathcal{B}(C'_{\mathrm{fin}}(X)), \quad n \in \mathbb{N}_0.$$

In this place of the article it is necessary to return to the definition of a configuration and to formula (2.4), which are needed for understanding this article.

It is possible to say that if we have some configuration γ, then we have some sequence $\gamma = [x_1, x_2, \ldots]$ of points x_n from X (finite or not), which are necessarily different and "tend to infinity", if γ is infinite. Note that we consider a configuration as a subset γ of points, but not as an independent object. These subsets γ are non overlapping and every point $x \in X$ belongs to some γ.

We will denote by $\Gamma(X)$ the set of all possible configurations, which can be constructed using the space X. It is clear that every "partition or stratification" of the space X into subsets γ ("layers") gives a complete set of configurations $\gamma \in \Gamma(X)$. For example, when we use the trivial stratification of the space X into the set of its points x, then we have the set of all one-points configurations, $X \ni x \mapsto [x]$, which also belongs to $\Gamma(X)$.

Thus, every stratification of the space X into the layers $X \to X_{\mathrm{str}}$ gives some $\Gamma(X_{\mathrm{str}}) =: \Gamma_{\mathrm{str}}(X)$. Conversely, every "complete" set of configurations γ (i.e. every points $x \in X$ must be belongs to some γ) gives a corresponding stratification of the space X.

So, we can say that the operations $X \to X_{\mathrm{str}}$ and $\Gamma \to \Gamma_{\mathrm{str}}$ are in one-to-one correspondence. The full set of configurations of the space X, i.e., the set $\Gamma(X)$, in our point of view is

$$\Gamma(X) = \bigsqcup_{\mathrm{str}} \Gamma_{\mathrm{str}}(X) = \bigsqcup_{\mathrm{str}} \Gamma(X_{\mathrm{str}}), \tag{2.33}$$

where $\bigsqcup_{\mathrm{str}}$ means a disjoint union over all str.

For us it is essential to repeat equality (2.18).

Lemma 2.8. *Suppose we have compact subspaces $\Lambda_n \subset X$, $n \in \mathbb{N}$, for which (2.14) takes place, and subspaces $K_n = \Lambda_n \setminus \Lambda_{n-1}$, $n \in \mathbb{N}$ $(\Lambda_0 = \varnothing)$. Then*

$$\Gamma(X) = \bigsqcup_{n=1}^{\infty} \Gamma(K_n), \tag{2.34}$$

where we understand $\Gamma(X)$ to be in the form (2.33) and $\Gamma(K_n)$ is also in the form (2.33) (with X replaced with K_n).

Proof. The proof is trivial. It is necessary to use Lemma 2.4 and definition (2.33). More precisely, we use equality (2.18) and interpret it according to formulation of Lemma 2.4. Note that every $\Gamma(K_n)$, according to (2.33), consists only of finite configurations. $\square$

Lemma 2.9. *The space $\Gamma(X)$ belongs to $\mathcal{B}(C'_{\mathrm{fin}}(X))$.*

Proof. Lemmas 2.4 and 2.8 give

$$\Gamma(X) = \bigsqcup_{n=1}^{\infty} \Gamma(K_n) \subset \bigcup_{n=1}^{\infty} \Gamma(\Lambda_n), \tag{2.35}$$

since $\Gamma(K_n) \subset \Gamma(\Lambda_n)$.

Using Lemma 2.6 we conclude from (2.35) that the set $\Gamma(X)$ also belongs to $\mathcal{B}(C'_{\mathrm{fin}}(X))$. $\square$

The σ-algebra $\mathcal{B}(C'_{\mathrm{fin}}(X))$ contains sufficiently many sets from $\Gamma(X)$, — the set $\Gamma(X)$ itself, $\Gamma(\Lambda)$ because of Lemma 2.6, where Λ is a compact subset of X (in particular, every point $x \in X$). Note that $\Gamma(K_n) = \Gamma(\Lambda_n) \setminus \Gamma(\Lambda_{n-1})$, $n \in \mathbb{N}$, also belongs to $\mathcal{B}(C'_{\mathrm{fin}}(X))$, since Λ_n is compact. Also, $\Gamma_\Lambda^n \in \mathcal{B}(C'_{\mathrm{fin}}(X))$ (see (2.32)).

The aim of this Section is to study some non-negative finite measures σ on the σ-algebra $\mathcal{B}(C'_{\mathrm{fin}}(X))$. Using Lemma 2.9 we conclude that the measure σ is also defined on $\Gamma(X)$ with vague topology, i.e., on the σ-algebra $\mathcal{B}(\Gamma(X))$ as σ-subalgebra of $\mathcal{B}(C'_{\mathrm{fin}}(X))$.

More precisely, we will investigate σ as a finite measure on $\mathcal{B}(\Gamma(X))$, i.e.,

$$(2.36) \qquad \mathcal{B}(C'_{\mathrm{fin}}(X)) \supset \mathcal{B}(\Gamma(X)) \ni \alpha \mapsto \sigma(\alpha) \geq 0.$$

Theorem 2.10. *If the measure σ (2.36) is nontrivial on $\Gamma(X)$, i.e., $\sigma(\Gamma(X)) > 0$, then there exists $n \in \mathbb{N}$ and some compact set $\Lambda \subset X$ for which*

$$(2.37) \qquad \sigma(\Gamma^{(n)}_{\Lambda}) > 0,$$

where $\Gamma^{(n)}_{\Lambda}$ is a corresponding set of finite configurations (2.29).

Proof. The σ-additivity of the measure (2.36) and identities (2.18), (2.34) give

$$\sigma(\Gamma(X)) = \sigma\left(\bigsqcup_{m=1}^{\infty} \Gamma(K_m)\right) = \sum_{m=1}^{\infty} \sigma(\Gamma(K_m)) > 0,$$

since $\sigma(\Gamma(X)) > 0$. Therefore there exists $m_0 \in \mathbb{N}$ such that $\sigma(\Gamma(K_{m_0})) > 0$. Since $K_{m_0} \subset \Lambda_{m_0}$ we have $\Gamma(K_{m_0}) \subset \Gamma(\Lambda_{m_0})$, therefore $\sigma(\Gamma(\Lambda_{m_0})) > 0$. Using the latter inequality and (2.29) we obtain

$$0 < \sigma(\Gamma(\Lambda_{m_0})) = \sigma\left(\bigsqcup_{n=0}^{\infty} \Gamma^{(n)}_{\Lambda_{m_0}}\right) = \sum_{n=0}^{\infty} \sigma(\Gamma^{(n)}_{\Lambda_{m_0}}).$$

Therefore, we have $\sigma(\Gamma^{(n)}_{\Lambda_{m_0}}) > 0$ at least for one $n \in \mathbb{N}$. We put $\Lambda = \Lambda_{m_0}$. Inequality (2.37) is proved. $\qquad\square$

Some simple consequence of this result is the following: if a nonnegative finite measure σ, entering Theorem 2.10, on $\mathcal{B}(\Gamma(X))$ is such that $\sigma(\Gamma_0) = 0$, then $\sigma = 0$. This poses the following question: for what measure σ Theorem 2.10 is true? For example, for a Poisson measure π, we have $\pi(\Gamma_0) = 0$ and the condition (2.37) cannot be fulfilled. We explain this situation in next article.

3. Hilbert spaces and their riggings

In this section we introduce some Hilbert spaces, their riggings and family of commuting operators connected with the material of Section 2. These objects are necessary for what follows.

At first it is necessary to recall some results concerning weighted Fock spaces constructed similarly to (2.1) (see, e. g., [16, 9, 11]). It is known that $\mathcal{D}$ is the projective limit of real Sobolev spaces $H_\tau = W^{\tau_1}_{2,\mathrm{Re}}(X, \tau_2(x)\,dm(x))$, where $\tau = (\tau_1, \tau_2(x))$, $\tau_1 \in \mathbb{N}_0$, $\tau_2(x) \geq 1$ is a C^∞ weight, m is a Riemannian measure on X. The projective limit (uncountable) is taken over the set T of all such τ. Note that for every $\tau \in T$ there exists $\tau' = (\tau'_1, \tau'_2(x)) \in T$, $\tau'_1 > \tau_1$, $\forall x \in X$ $\tau'_2(x) \geq \tau_2(x)$ (we will write $\tau' > \tau$) such that the embedding $H_{\tau'} \subset H_\tau$ is quasi-nuclear, i.e., the embedding operator is of Hilbert–Schmidt type.

Using the space H_τ we construct the corresponding weighted Fock spaces. So, let $p = (p_n)_{n=0}^{\infty}$, where $\forall n \in \mathbb{N}_0$ $p_n > 0$, is a number weight. Let $\mathcal{F}(H_\tau, p)$ be the weighted Fock space consisting of sequences $f = (f_n)_{n=0}^{\infty}$, $f_n \in H^{\widehat{\otimes} n}_{\tau,c} =: \mathcal{F}_n(H_\tau)$, such that

$$
\begin{aligned}
\|f\|^2_{\mathcal{F}(H_\tau, p)} &= \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{F}_n(H_\tau)} p_n < \infty, \\
(f, g)_{\mathcal{F}(H_\tau, p)} &= \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{F}_n(H_\tau)} p_n.
\end{aligned}
$$
(3.1)

The number weight $p = (p_n)_{n=0}^{\infty}$ such that all $p_n \geq 1$ will be denoted by $p \geq 1$. The weight $(p_n^{-1})_{n=0}^{\infty}$, $p_n > 0$, is denote by p^{-1}.

The space $\mathcal{F}_{\text{fin}}(\mathcal{D})$ (2.1), which is a base space in this article, can be understood as the projective limit of the spaces $\mathcal{F}(H_\tau, p)$. More exactly, we have the following result.

Proposition 3.1. *The space $\mathcal{F}_{\text{fin}}(\mathcal{D})$ is a projective limit of the spaces $\mathcal{F}(H_\tau, p)$, where $\tau \in T$ and $p \geq 1$ are arbitrary,*

$$\mathcal{F}_{\text{fin}}(\mathcal{D}) = \text{pr} \lim_{\tau \in T,\ p \geq 1} \mathcal{F}(H_\tau, p) = \bigcap_{\tau \in T,\ p \geq 1} \mathcal{F}(H_\tau, p).$$

The proof of this result in the case $X = \mathbb{R}^d$, $d \in \mathbb{N}$, can be found in [16], Ch. 14; for an arbitrary Riemannian manifold X, the proof is similar.

For us, the following result will be essential.

Proposition 3.2. *For arbitrary $\mathcal{F}(H_\tau, p)$ there exists $\mathcal{F}(H_{\tau'}, p')$ with $\tau' \geq \tau$, $p' \geq p$ such that the embedding $\mathcal{F}(H_{\tau'}, p') \subset \mathcal{F}(H_\tau, p)$ is quasi-nuclear.*

The proof of this fact can be found in [4, 9, 11].

Since $\mathcal{F}_{\text{fin}}(\mathcal{D})$ is a projective limit of the spaces $\mathcal{F}(H_\tau, p)$, the following systems of open balls,

(3.2) $\qquad \{f \in \mathcal{F}_{\text{fin}}(\mathcal{D}) \,|\, \exists \tau \in T, p \geq 1 : \|f - h\|_{\mathcal{F}(H_\tau, p)} < \varepsilon\}, \quad h \in \mathcal{F}_{\text{fin}}(\mathcal{D}), \quad \varepsilon > 0,$

can be taken as a system of neighborhood of $\mathcal{F}_{\text{fin}}(\mathcal{D})$.

The system (3.2) implies validity of the following two proposition.

Proposition 3.3. *A linear functional l on $\mathcal{F}_{\text{fin}}(\mathcal{D})$ is continuous if and only if there exists a constant $C_1 > 0$ and a space $\mathcal{F}(H_\tau, p)$ such that*

(3.3) $$|l(f)| \leq C_1 \|f\|_{\mathcal{F}(H_\tau, p)}, \quad f \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

Proposition 3.4. *The convolution $\star$ has the following property: for an arbitrary space $\mathcal{F}(H_\tau, p)$ there exists a space $\mathcal{F}(H_{\tau'}, p')$ and a constant C_2 such that*

(3.4) $$\|f \star g\|_{\mathcal{F}(H_\tau, p)} \leq C_2 \|f\|_{\mathcal{F}(H_{\tau'}, p')} \|g\|_{\mathcal{F}(H_{\tau'}, p')}$$

for all $f, g \in \mathcal{F}_{\text{fin}}(\mathcal{D})$.

Note that (3.3) is equivalent to continuity of l and (3.4) is equivalent to continuity of the product $\star$ on the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$ (see [13]).

From Proposition 3.3 it follows that the dual space $(\mathcal{F}_{\text{fin}}(\mathcal{D}))' =: \mathcal{F}'_{\text{fin}}(\mathcal{D})$ is the inductive limit of the space $(\mathcal{F}(H_\tau, p))' =: \mathcal{F}(H_{-\tau}, p^{-1})$. So, we have the following rigging (see, e.g., [9, 16]):

(3.5)
$$\text{ind} \lim_{\tau \in T,\ p \geq 1} \mathcal{F}(H_{-\tau}, p^{-1}) = (\mathcal{F}_{\text{fin}}(\mathcal{D}))' \supset \mathcal{F}(H_{-\tau}, p^{-1}) \supset \mathcal{F}(H)$$
$$\supset \mathcal{F}(H_\tau, p) \supset \mathcal{F}_{\text{fin}}(\mathcal{D}) = \text{pr} \lim_{\tau \in T,\ p \geq 1} \mathcal{F}(H_\tau, p),$$

where $\mathcal{F}(H) := \mathcal{F}(H, p)$ is an ordinary Fock space over $H = L^2_{\text{Re}}(X, dm(x))$ and $p = (1, 1, \ldots)$.

We will usually denote the action of a vector from the negative space on a vector from the positive space of rigging (3.5) with zero space $\mathcal{F}(H)$ by $(\cdot, \cdot)_{\mathcal{F}(H)}$ and also by $\langle \cdot, \cdot \rangle_{\mathcal{F}(H)}$ or $\langle \cdot, \cdot \rangle$. The same is true for other riggings.

We will investigate positive functionals on the nuclear algebra $\mathcal{A} := \mathcal{F}_{\text{fin}}(\mathcal{D})$ of complex-valued finite functions with convolution $\star$ by means of the spectral theory of commuting selfadjoint operators. To this end, it is necessary to introduce at first the corresponding Hilbert space.

Introduce, in the algebra $\mathcal{A}$, a natural involution $\mathcal{A} \ni f = f(\xi) \to \bar{f} := \overline{f(\xi)} \in \mathcal{A}$. It follows from (2.6) that $\bar{f} \star \bar{g} = \overline{f \star g}$ for all $f, g \in \mathcal{A}$. *A continuous linear functional $s \in \mathcal{A}' := \mathcal{F}'_{\text{fin}}(\mathcal{D})$ is called nonnegative if*

$$s(f \star \bar{f}) \geq 0, \quad f \in \mathcal{A}.$$

Any nonnegative functional s generates the following quasi-scalar product on $\mathcal{A}$:

$$(3.6) \qquad\qquad (f, g)_{\mathcal{H}_s} := s(f \star \bar{g}), \quad f, g \in \mathcal{A}.$$

Identifying every $f \in \mathcal{A}$ such that $s(f \star \bar{f}) = 0$ with zero, considering the corresponding classes of $f \in \mathcal{A}$ and completing the space of these classes, we construct a Hilbert space $\mathcal{H}_s$. Let $\{f\}$ be the class containing $f \in \mathcal{A}$, and let $\{\mathcal{A}\}$ be the space of all such classes. Then $\{\mathcal{A}\} \subset \mathcal{H}_s$ and $\{\mathcal{A}\}$ is dense in $\mathcal{H}_s$. The topology in $\{\mathcal{A}\}$ is induced by the topology in $\mathcal{A}$, i.e., by the topology of the space $\mathcal{A} = \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$.

In the space $\mathcal{H}_s$ we will investigate some family of Hermitian operators $A(\varphi)$, where $\varphi \in \mathcal{D} \subset \mathcal{F}_1(\mathcal{D})$ is a real-valued function of point $\xi \in \Gamma^{(1)} \subset \Gamma_0$.

Introduce at first the operation

$$\mathcal{A} \ni f \mapsto \varphi \star f \in \mathcal{A}.$$

This operation is Hermitian in the quasi-scalar product (3.6)

$$(\varphi \star f, \; g)_{\mathcal{H}_s} = s(\varphi \star f \star \bar{g}) = s(f \star \overline{(\varphi \star g)}) = (f, \; \varphi \star g)_{\mathcal{H}_s}, \quad f, g \in \mathcal{A}.$$

Therefore (see, e. g., [4, 9]) this operation can be considered as acting in the set of the corresponding classes: $\{\mathcal{A}\} \ni \{f\} \mapsto \{\varphi \star f\} \in \{\mathcal{A}\}$. So, we have introduced a Hermitian operator $A(\varphi)$ defined densely in $\mathcal{H}_s$:

$$(3.7) \qquad \mathrm{Dom}(A(\varphi)) = \{\mathcal{A}\} \ni \{f\} \mapsto A(\varphi)\{f\} := \{\varphi \star f\} \in \{\mathcal{A}\}, \quad f \in \mathcal{A}.$$

Any two such operators $A(\varphi)$, $A(\psi)$ $(\varphi, \psi \in \mathcal{D})$ commute formally: $A(\varphi)\{\mathcal{A}\} \subset \{\mathcal{A}\} = \mathrm{Dom}(A(\psi))$, $A(\psi)\{\mathcal{A}\} \subset \{\mathcal{A}\} = \mathrm{Dom}(A(\varphi))$, and for every $\{f\} \in \{\mathcal{A}\}$, according to (3.7),

$$A(\varphi)A(\psi)\{f\} = A(\varphi)\{\psi \star f\} = \{\varphi \star \psi \star f\} = \{\psi \star \varphi \star f\} = A(\psi)A(\varphi)\{f\}.$$

Then how to check whether the set of all closures $\tilde{A}(\varphi)$ of $A(\varphi)$ is a family of selfadjoint (strongly) commuting operators?

Now a sufficient condition for this fact is the following (see [9], Ch. 5, Theorem 1.15; or [16], Ch. 13, Theorem 9.3; also [5]): there exists $z \in \mathbb{C} \setminus \mathbb{R}$ such that for each $\varphi, \psi \in \mathcal{D}$ there exists a total set of vectors which are quasi-analytic for the operators $A(\varphi)$, $A(\psi)$, $A(\varphi) \upharpoonright (A(\psi) - z\mathbf{1})\{\mathcal{A}\}$ (Note that some another approach to selfadjointness and commutativity of operators $A(\varphi)$ is contained in [15], Theorem 3 and pp. 11–12. It based is on some general theorems, cited in [15]).

Recall that for an operator A acting on a Hilbert space $\mathcal{H}$, a vector $f \in \mathcal{H}$ is called quasi-analytic if $f \in \bigcap_{n=1}^{\infty} \mathrm{Dom}(A^n)$ and the class $C\{m_n\}$ with $m_n = \|A^n f\|_{\mathcal{H}}$ is quasi-analytic, i.e.,

$$\sum_{n=1}^{\infty} \|A^n f\|_{\mathcal{H}}^{-\frac{1}{n}} = \infty.$$

According to (2.5), $\{\mathcal{A}\} = \bigcap_{n=1}^{\infty} \mathrm{Dom}((A(\varphi))^n)$ for every $\varphi \in \mathcal{D}$. In what follows we demand the following condition to hold.

Condition 3.5 (of selfadjointness). There exists a linear set $M \subset \{\mathcal{A}\}$ such that

 (1) M is invariant w. r. t. every operator $A(\varphi)$ $(\varphi \in \mathcal{D})$;
 (2) M is total in $\mathcal{H}_s$;
 (3) every vector $\{f\} \in M$ is quasi-analytic for every operator $A(\varphi)$ $(\varphi \in \mathcal{D})$, i. e., the class

$$(3.8) \qquad\qquad C\{\|(A(\varphi))^n\{f\}\|_{\mathcal{H}_s}\}$$

 is quasi-analytic.

From 1) – 3) it follows now that $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$, where $\tilde{A}(\varphi)$ denotes the closure of $A(\varphi)$, is a family of selfadjoint commuting operators.

Note that, due to the fact that M is invariant for $A(\psi)$ ($\psi \in \mathcal{D}$), we have $(A(\psi) - z\mathbf{1})M \subset M$ for $\operatorname{Im} z \neq 0$, and thus the condition (3.8) provides the conditions in the above mentioned theorem from [9].

Note that for the functional $s \in \mathcal{F}'_{\text{fin}}(\mathcal{D})$, which is generated by some Borel measure ν on Γ_0, i.e.,

$$(3.9) \qquad s(f) = \int_{\Gamma_0} f(\xi) \, d\nu(\xi) = \sum_{n=0}^{\infty} \int_{\Gamma^{(n)}} f(\xi) \, d\nu(\xi), \quad f \in \mathcal{F}_{\text{fin}}(\mathcal{D}),$$

it is possible to give some sufficient condition which would guaranty Condition 3.5 to hold. Namely (see [13]), construct, for every compact $\Lambda \subset X$ and for every $k \in \mathbb{N}$, the sequence

$$m_n = \left(\sum_{\ell=0}^{2k} \left(\frac{(\ell + 2n)!}{\ell!} \sum_{j=0}^{2n} \nu \left(\Gamma_\Lambda^{(\ell+j)} \right) \right) \right)^{\frac{1}{2}}, \quad n \in \mathbb{N}_0.$$

If the class $C\{m_n\}$ is quasi-analytic for every Λ and k, then Condition 3.5 is fulfilled. The latter formula is complicated. It is possible to give a more simple sufficient estimate which would imply that the above class $C\{m_n\}$ is quasi-analytic and therefore Condition 3.5 is fulfilled. This estimate is the following: for every compact set $\Lambda \subset X$ there exists a constant C_Λ such that

$$(3.10) \qquad \nu(\Gamma_\Lambda^{(n)}) \leq C_\Lambda^n, \quad n \in \mathbb{N}_0.$$

We will apply the generalized eigenvectors expansion for operators $\tilde{A}(\varphi)$. To this end it is necessary to construct a rigging of the space $\mathcal{H}_s$.

The following result takes place.

Lemma 3.6. *There exists weights $\tau \in T$, $p \geq 1$, and a constant $C > 0$ such that*

$$(3.11) \qquad |s(f \star \bar{f})| = \|f\|^2_{\mathcal{H}_s} \leq C\|f\|^2_{\mathcal{F}(H_\tau, p)}, \quad f \in \mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

Proof. Using inequality (3.3) for $l = s$ and (3.4) we immediately get (3.11). $\qquad \square$

We fix now $\tau \in T$, $p \geq 1$ such that inequality (3.11) holds. Consider the space $\mathcal{F}(H_\tau, p)$ which is a completion of $\mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D})$ with respect to the corresponding norm (3.1). Introduce, according to Lemma 3.6, the linear subspace

$$(3.12) \qquad N := \{f \in \mathcal{F}(H_\tau, p) \,|\, s(f \star \bar{f}) = 0\}.$$

Then the space $\mathcal{H}_s$ can be regarded as an orthogonal complement to N in the space $\mathcal{F}(H_\tau, p)$ (see [9], Ch. 5, § 5),

$$(3.13) \qquad \mathcal{F}(H_\tau, p) = \mathcal{H}_s \oplus N, \quad \text{i.e.,} \quad \mathcal{H}_s = \mathcal{F}(H_\tau, p) \ominus N.$$

Namely, there is a one-to-one correspondence between the class $\{f\} \in \mathcal{H}_s$ and the corresponding vector from $\mathcal{H}_s$, $\{f\} \leftrightarrow f \oplus N$. The scalar product of two classes is equal to the scalar product of the corresponding components in $\mathcal{H}_s$ from (3.13) etc.

Construct a quasi-nuclear rigging of the space $\mathcal{H}_s$ (3.1). To do this, we construct a rigging of type appearing in Proposition 3.2

$$(3.14) \qquad \mathcal{F}(H_\tau, p) \supset \mathcal{F}(H_{\tau'}, p'),$$

where the weights τ', p' are so large that the embedding (3.14) is quasi-nuclear. We fix these weights τ', p'.

Applying representation (3.13) to $\mathcal{F}(H_{\tau'}, p')$ from (3.14) we get

$$(3.15) \qquad \mathcal{F}(H_{\tau'}, p') = \mathcal{H}_{s,+} \oplus (N \cap \mathcal{F}(H_{\tau'}, p')),$$

where
$$\mathcal{H}_{s,+} := \mathcal{H}_s \cap \mathcal{F}(H_{\tau'}, p').$$

From the quasi-nuclearity of the embedding (3.14) and (3.13), (3.15) it follows that the embedding $\mathcal{H}_{s,+} \subset \mathcal{H}_s$ is also quasi-nuclear.

As a result, we have a rigging of the base space $\mathcal{H}_s$,

$$(3.16) \qquad\qquad \mathcal{H}_s \supset \mathcal{H}_{s,+} \supset \{\mathcal{A}\} =: D$$

with a quasi-nuclear embedding $\mathcal{H}_{s,+} \subset \mathcal{H}_s$. On the set $\{\mathcal{A}\}$ of classes, every operators $A(\varphi)$, $\varphi \in \mathcal{D}$, is defined. Recall that this set $\{\mathcal{A}\}$ is topologized by means of the topology in $\mathcal{A} = \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, therefore the restriction $A(\varphi) \upharpoonright \{\mathcal{A}\}$ acts continuously from $D = \{\mathcal{A}\}$ into $\mathcal{H}_{s,+}$.

Thus rigging (3.16) is standardly connected with our family of operators $(\bar{A}(\varphi))_{\varphi \in}$ (see [9], Ch. 3, §2; [16], Ch. 15, §2). Denoting by $\mathcal{H}_{s,-}$ the negative space corresponding to the zero space $\mathcal{H}_s$ and the positive $\mathcal{H}_{s,+}$ we get the rigging

$$(3.17) \qquad\qquad \mathcal{H}_{s,-} \supset \mathcal{H}_s \supset \mathcal{H}_{s,+} \supset D.$$

We can construct a generalized eigenvectors expansion for the family $(\bar{A}(\varphi))_{\varphi \in}$ using this rigging. But we will have a certain complication, — the structure of the zero space $\mathcal{H}_s$ is fairly complex. According to (3.16), it is an orthogonal complement to a sufficiently complicated subspace N. Therefore in the case $N \neq 0$, the use of the rigging (3.17) is inconvenient.

At first we consider the case $N = 0$, the case $N \neq 0$ will be considered in another article.

Recall that for arbitrary rigging

$$(3.18) \qquad\qquad H_- \supset H \supset H_+$$

it is easy to prove the existence of bounded operators $I_H : H_- \to H_+$ such that

$$(3.19) \qquad (\alpha, u)_H = (I_H \alpha, u)_{H_+} = (\alpha, I_H^{-1} u)_{H_-}, \quad \alpha \in H_-, \quad u \in H_+$$

(see [4], Ch. 5; [16], Ch. 14).

Lemma 3.7. *Consider the riggings with equal positive spaces*

$$(3.20) \qquad\qquad H_- \supset H \supset H_+, \quad F_- \supset F \supset F_+ = H_+.$$

Then a unitary operator $U : F_- \to H_-$ exists for which

$$(3.21) \qquad\qquad (U\alpha, u)_H = (\alpha, u)_F, \quad \alpha \in F_-, \quad u \in F_+.$$

Proof. It is very simple. We put $U = I_H^{-1} I_F$, where the operators I_H, I_F are connected with the riggings (3.20) by (3.18), (3.19). Then using (3.19) we get

$$(U\alpha, u)_H = (I_H^{-1} I_F \alpha, u)_H = (I_H(I_H^{-1} I_F \alpha), u)_{H_+}$$
$$= (I_F \alpha, u)_{H_+} = (I_F \alpha, u)_{F_+} = (\alpha, u)_F$$

for all $\alpha \in F_-$ and $u \in F_+ = H_+$. $\qquad\qquad\qquad\qquad \square$

So, we will start now with initial weights $\tau = (0, 1) \in T$, $p = (1, 1, \ldots)$, and fix the corresponding weights τ', p'; we denote them by τ^0, p^0. In other words we consider the ordinary Fock space $\mathcal{F}(H)$ constructed from the space $H = L^2_{\mathrm{Re}}(X, dm(x))$. i.e,

$$(3.22) \qquad \mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H), \quad \mathcal{F}_n(H) := H^{\hat{\otimes}n},$$

$$f = (f_n)_{n=0}^{\infty} \in \mathcal{F}(H), \quad \|f\|^2_{\mathcal{F}(H)} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{F}_n(H)} < \infty.$$

The corresponding part of the chain has the form

$$(3.23) \qquad \mathcal{F}(H_{-\tau^0}, (p^0)^{-1}) \supset \mathcal{F}(H) \supset \mathcal{F}(H_{\tau^0}, p^0);$$

the last embedding in (3.23) is quasi-nuclear.

As we have mentioned above, we will consider the case where

$$(3.24) \qquad \{f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \,|\, s(f \star \bar{f}) = 0\} = 0,$$

i.e., a *positive (non-degenerate) case*. This conditions gives that the subspace N (3.12) is equal to zero and therefore according to (3.15) we have $\mathcal{F}(H_{\tau^0}, p^0) = \mathcal{H}_{s,+}$. Thus our main rigging (3.17) now has the form

$$(3.25) \qquad \mathcal{H}_{s,-} \supset \mathcal{H}_s \supset \mathcal{H}_{s,+} \supset D,$$

where $\mathcal{H}_{s,+} = \mathcal{F}(H_{\tau^0}, p^0)$.

Comparing the rigging (3.23), (3.25) with rigging (3.20) we get the following corollary to Lemma 3.7

Corollary 3.8. *In the positive case (3.24), i.e., $N = 0$, there exists a unitary operator $U : \mathcal{H}_{s,-} \to \mathcal{F}(H_{-\tau^0}, (p^0)^{-1})$ such that (3.21) holds.*

This fact will be used bellow in the projection spectral theorem for the family

$$(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$$

of selfadjoint commuting operators.

Remark 3.9. The positivity condition (3.24) it is not essential for developing the theory in Sections 4–7. See the articles [10, 8, 13] and the general theory of eigenfunctions expansion, e.g. [9] Ch. 5, Sect. 5, Subsect. 1; see also [4, 5]. But for applying this spectral theory to Bogoliubov functionals, some changes are needed.

4. Projective spectral theorem for a family of commuting selfadjoint operators linearly depending on a parameter

In this section we recall some general result concerning generalized eigenvector expansion which we will use below. For a detailed account of these results, see [9], Ch. 4, Theorem 1.6, Ch. 3, Theorems 3.1 and 3.2 and also in [4], Ch. 5; [5], Ch. 2; [16], Ch.14; [6, 7].

Let $\mathcal{H}$ be a separable complex Hilbert space and let $A = (A(\varphi))_{\varphi \in \Phi}$ be a family a selfadjoint, strongly commuting operators $A(\varphi)$ in $\mathcal{H}$, Φ be some space of parameters. Let

$$(4.1) \qquad \mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \supset D$$

be a rigging of $\mathcal{H}$ such that $\mathcal{H}_+$ is a Hilbert space topologically and quasi-nuclear embedded into $\mathcal{H}$, $\mathcal{H}_-$ is the dual of $\mathcal{H}_+$ with respect to the zero space $\mathcal{H}$, D is a linear topological space that is topologically embedded into $\mathcal{H}_+$.

We suppose that operator $A(\varphi)$ and chain (4.1) are standardly connected, i.e., $D \subset \mathrm{Dom}(A(\varphi))$ for all $\varphi \in \Psi$ and the restriction $A(\varphi) \restriction D$ acts from D into $\mathcal{H}_+$ continuously.

We will assume that the operators $A(\varphi)$ depend linearly on φ . This mean that the space Φ is real linear topological and we have the following rigging

$$(4.2) \qquad \Phi' \supset H \supset \Phi.$$

Here H is some real Hilbert space, Φ is dense in H and the embedding $\Phi \hookrightarrow H$ is continuous; Φ' is the dual space of Φ. We stress that the spaces in (4.2) are real, the embedding $\Phi \hookrightarrow H$ is not necessarily nuclear.

The main condition is the following: we assume that for every $f \in D$ the mapping

$$(4.3) \qquad \Phi \ni \varphi \mapsto A(\varphi)f \in \mathcal{H}_+$$

is linear and weakly continuous. Of course, the rigging (4.2) may be degenerate: it is possible that $\Phi = H$.

We will consider only the situation where there exists a strong cyclic vector Ω for our family A of operators $A(\varphi)$. Then the join spectrum of this family is simple and the corresponding Fourier transform is a scalar-valued function.

Let us recall that a vector $\Omega \in D$ is called a strong cyclic vector for the family $A = (A(\varphi))_{\varphi \in \Phi}$ if for some $\varphi_1, \ldots, \varphi_p \in \Phi$ and some nonnegative integers $m_1, \ldots, m_p$ $\Omega \in \mathrm{Dom}(A^{m_1}(\varphi_1) \ldots A^{m_p}(\varphi_p))$, the vectors $A^{m_1}(\varphi_1) \ldots A^{m_p}(\varphi_p)\Omega$ belong to D and the set of these vectors, when $m_1, \ldots, m_p \in \mathbb{N}_0$, $p \in \mathbb{N}$, is total in the space $\mathcal{H}_+$ (and, hence, also in $\mathcal{H}$).

The corresponding projection spectral theorem is the following.

Theorem 4.1. *Let $A = (A(\varphi))_{\varphi \in \Phi}$ be a family of strongly commuting selfadjoint operators $A(\varphi)$ in the space $\mathcal{H}$ and all of the above mentioned conditions are fulfilled (the existence of rigging (4.1), standardly connected with each $A(\varphi)$, linearity condition (4.3), and existence of a strong cyclic vector Ω).*

Then, on the space Φ', with the topology of weak convergence, there exists a nonnegative finite Borel measure ρ (a spectral measure) such that for ρ-almost every $\lambda \in \Phi'$ there is a generalized joint eigenvector $\xi(\lambda) \in \mathcal{H}_-$ ($\xi(\lambda) \neq 0$), i.e.,

$$(4.4) \qquad (\xi(\lambda), A(\varphi)f)_{\mathcal{H}} = (\lambda, \varphi)_H (\xi(\lambda), f)_{\mathcal{H}}, \quad \varphi \in \Phi, \quad \lambda \in \Phi', \quad f \in D.$$

The corresponding Fourier transform

$$(4.5) \qquad \mathcal{H} \supset \mathcal{H}_+ \ni f \mapsto (If)(\lambda) := \hat{f}(\lambda) := (\xi(\lambda), f)_{\mathcal{H}} \in L^2(\Phi', d\rho(\lambda))$$

is an isometric operator acting from the space $\mathcal{H}$ into $L^2(\Phi', d\rho(\lambda))$. The extension of I by continuity is a unitary operator between these spaces. The image of any operator $A(\varphi)$ under I is the operator of multiplication by $(\lambda, \varphi)_H$ in the space $L^2(\Phi', d\rho(\lambda))$.

We will use this theorem in the next section. So, the rigging used is the rigging (3.25) (with condition (3.24), i.e. $N = 0$). The family A of operators consists of operators $\bar{A}(\varphi)$ (3.7) in the space $\mathcal{H}_s$, where $\varphi \in \Phi = \mathcal{D}$. The space H is equal to $L^2_{\mathrm{Re}}(X, dm(x))$.

5. SPECTRAL REPRESENTATION

We will apply now the general facts from Section 4 to our situation. At first we prove some lemmas. We consider the following linear operators $A(\varphi)$ defined on the algebra $\mathcal{A}$:

$$(5.1) \qquad \mathcal{A} \ni f \mapsto A(\varphi)f = \varphi \star f \in \mathcal{A},$$

where φ is a real-valued function in $\mathcal{D} \subset \mathcal{F}_1(\mathcal{D}) \subset \mathcal{A}$ (using (5.1), and the operator $A(\varphi)$ has been constructed in Section 3, see (3.7)).

Note that it is sufficient to define $A(\varphi)$ by (5.1) on f that are equal to $\psi^{\otimes n}$, where ψ is an arbitrary complex-valued function from $\mathcal{D}_{\mathbb{C}} := C^{\infty}_{\mathrm{fin}, \mathbb{C}}(X)$ and $n \in \mathbb{N}_0$ is arbitrary. This follows from density of linear combinations of such $\psi^{\otimes n}$ in the space $\mathcal{A} = \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$.

Lemma 5.1. *For the operators (5.1), the following representation takes place:*

$$(5.2) \qquad A(\varphi) = A^+(\varphi) + A^0(\varphi), \quad \varphi \in \mathcal{D},$$

where $A^+(\varphi)$ and $A^0(\varphi)$ are the creation and neutral operators, i.e.,

$$A^+(\varphi)\psi^{\otimes n} = (n+1)\varphi \widehat{\otimes} \psi^{\otimes n}, \quad A^0(\varphi)\psi^{\otimes n} = n(\varphi\psi) \widehat{\otimes} \psi^{\otimes(n-1)}, \quad n \in \mathbb{N}_0.$$

Here $\psi \in \mathcal{D}_c$ and thus $\psi^{\otimes n} \in \mathcal{D}_c^{\widehat{\otimes} n} = \mathcal{F}_n(\mathcal{D})$, $(\varphi\psi)(x) := \varphi(x)\psi(x)$, $\psi^{\otimes(-1)} := 0$.

Proof. Consider at first the main case where $n \in \mathbb{N}$ in (5.2). According to Section 2 we will regard the function φ as the real-valued function $\varphi(\xi) = \varphi([x_1])$ on $\Gamma^{(1)} \subset \Gamma_0$ and $f(x_1, \ldots, x_n) = \psi^{\otimes n}(x_1, \ldots, x_n) = \psi(x_1) \cdots \psi(x_n)$ as the function $f(\xi) = f([x_1, \ldots, x_n])$ on $\Gamma^{(n)} \subset \Gamma_0$. According to (5.2) and (2.6) we have that $\forall \xi = [x_1, \ldots, x_n]$

$$(A(\varphi)f)(\xi) = (\varphi \star f)(\xi) = (f \star \varphi)(\xi) = (f \star \varphi)([x_1, \ldots, x_n])$$

(5.3)
$$= \sum_{\xi' \cup \xi'' = \xi} f(\xi')\varphi(\xi'') = \sum_{\xi' \sqcup \xi'' \sqcup \xi''' = \xi} f(\xi' \cup \xi'')\varphi(\xi'' \cup \xi''').$$

Since φ is a function on $\Gamma^{(1)}$, i.e., it depends on configuration $[x]$ of order 1; ξ'' and ξ''' in (5.3) can only be either $\varnothing$ or $[x]$. Since f depends on $[x_1, \ldots, x_n]$, the variable ξ in (5.3) can be from $\Gamma^{(n)}$ or $\Gamma^{(n+1)}$ only. Therefore, ξ'' can be either $[x]$ or $\varnothing$, respectively, and then, as it is easy to calculate, the last sum in (5.3) is equal to $n(\varphi\psi)\widehat{\otimes}\psi^{\otimes(n-1)}$ in the first case and to $(n+1)\varphi\widehat{\otimes}\psi^{\otimes n}$ in the second one. As a result, we have proved representation (5.3) for $n \in \mathbb{N}$.

In the case $n = 0$ the above calculation gives that the last term is equal to zero. $\qquad\square$

Formulas (5.2) show that the operator $A(\varphi)$ is an operator-valued matrix operation in the Fock space $\mathcal{F}(H)$ (3.22).

Namely, at first we note that in every space $\mathcal{F}_n(H) = H^{\widehat{\otimes}n}$ from (3.22), the set of functions $\psi^{\otimes n}$, where $\psi \in C^\infty_{\mathrm{fin},\mathbb{C}}(X)$ are arbitrary, is total. Therefore the operators $A^+(\varphi)$, $A^0(\varphi)$, for every fixed $\varphi \in \mathcal{D}$, can be extended by linearity and continuity from $\psi^{\otimes n}$ to $f_n \in \mathcal{F}_n(H)$. Formulas (5.2) show that the resulting operators are continuous in the sense of the norm of the spaces $\mathcal{F}_n(H)$. As a result, we have constructed the continuous operators: $\forall n \in \mathbb{N}_0$, $\forall \varphi \in \mathcal{D}$

(5.4)
$$\mathcal{F}_n(H) \ni f_n \mapsto a_n(\varphi)f_n := (n+1)\varphi\widehat{\otimes}f_n \in \mathcal{F}_{n+1}(H),$$
$$\mathcal{F}_n(H) \ni f_n \mapsto b_n(\varphi)f_n \in \mathcal{F}_n(H).$$

Note that, for $f_n = \psi^{\otimes n}$, formula (5.4) has form (5.2). The operators $a_n(\varphi)$, $b_n(\varphi)$ (5.4) are bounded, $b_n(\varphi)$ is selfadjoint.

Now we can say that the operator $A(\varphi)$ from (5.2) in the Fock space $\mathcal{F}(H)$ has the form of an operator Jacobi matrix: $\forall \varphi \in \mathcal{D}$

(5.5)
$$A(\varphi) = A^+(\varphi) + A^0(\varphi) = \begin{pmatrix} b_0(\varphi) & 0 & 0 & 0 & \ldots \\ a_0(\varphi) & b_1(\varphi) & 0 & 0 & \ldots \\ 0 & a_1(\varphi) & b_2(\varphi) & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the matrix $A^+(\varphi)$ consists of $a_n(\varphi)$ and $A^0(\varphi)$ of $b_n(\varphi)$.

Note that this matrix generates, in the Fock space $\mathcal{F}(H)$, a corresponding operator. At first it is defined on finite sequences $f = (f_n)_{n=0}^\infty$ from $\mathcal{F}(H)$, and after this, it is necessary to take its closure. A connection between this operator and the operator $\tilde{A}(\varphi)$ on the space $\mathcal{H}_s$ we will be considered later.

Let us again consider the operator $\bar{A}(\varphi)$, $\varphi \in \mathcal{D} = \Psi$, (3.7) acting on the space $\mathcal{H}_s$ from the rigging (3.25). *We will assume that the condition (3.24) (i.e., $N = 0$) is fulfilled.*

Using the elements of matrix $A(\varphi)$ (5.5), we can rewrite the action of this operator in the following form (see (5.4)): $\forall \varphi \in \mathcal{D}$ and $\forall f = (f_n)_{n=0}^\infty \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$

(5.6)
$$(A(\varphi)f)_n = a_{n-1}(\varphi)f_{n-1} + b_n(\varphi)f_n = n\varphi\widehat{\otimes}f_{n-1} + b_n(\varphi)f_n, \quad n \in \mathbb{N},$$
$$(A(\varphi)f)_0 = b_0(\varphi)f_0.$$

Lemma 5.2. *The vector $\Omega = (1, 0, 0, \ldots) \in \mathcal{D} \subset \mathcal{H}_{s,+}$ is a strong cyclic vector for the family $(\bar{A}(\varphi))_{\varphi \in \mathcal{D}}$.*

Proof. In our case the positive space from rigging (3.25) is the space $\mathcal{H}_{s,+} = \mathcal{F}(\tau_0, p_0)$. Using the formula (5.6)) we conclude that, for $m_1, \ldots, m_p \in \mathbb{N}_0$, $p \in \mathbb{N}$, the vectors $A^{m_1}(\varphi_1) \ldots A^{m_p}(\varphi_p)\Omega$ belong to $\mathcal{F}_{\text{fin}}(\mathcal{D})$. More exactly, we have

$$(5.7) \quad A^{m_1}(\varphi_1) \ldots A^{m_p}(\varphi_p)\Omega \in \bigoplus_{n=0}^{m_1+\cdots+m_p} \mathcal{F}_n(H), \quad \varphi_1, \ldots, \varphi_p \in \mathcal{D}, \quad m_1, \ldots, m_p \in \mathbb{N}_0.$$

Let us stress that in (5.7) every φ_k is an arbitrary function from $\mathcal{D}$.

A vector Ω is a strong cyclic vector for the family $(\bar{A}(\varphi))_{\varphi \in \mathcal{D}}$ if the set (5.7) is total in the space $\mathcal{F}(\tau_0, p_0)$. From the form of the scalar product in the space $\mathcal{F}(\tau_0, p_0)$ it is easy to conclude that from the totality of the set (5.7) in $\bigoplus_{n=0}^{m_1+\cdots+m_p} \mathcal{F}_n(H)$ it follows its totality in the space $\mathcal{F}(\tau_0, p_0)$.

Therefore it is sufficient to prove the totality of (5.7) in the space $\bigoplus_{n=0}^{m_1+\cdots+m_p} \mathcal{F}_n(H)$.

Using the commutativity of the operators $A(\varphi)$, $\varphi \in \mathcal{D}$ we can rewrite the relation (5.7) in the following form: for arbitrary $m \in \mathbb{N}$ the set

$$(5.8) \qquad \{A^{m_1}(\varphi_1) \ldots A^{m_p}(\varphi_p)\Omega \,|\, \varphi_1, \ldots, \varphi_p \in \mathcal{D}\} \subset \bigoplus_{n=0}^{m} \mathcal{F}_n(H) =: F_m.$$

It is necessary to prove that $\forall m \in \mathbb{N}$ the set (5.7) is total in F_m.

For the proof at first we note that the formulas (5.5), (5.6) give

$$(5.9) \qquad a_{n-1}(\varphi) \ldots a_0(\varphi)1 = ((A^+(\varphi))^n \Omega)_n \in \mathcal{F}_n(H), \quad \varphi \in \mathcal{D}, \quad n \in \mathbb{N}_0.$$

From (5.6) we conclude that the expression in (5.9) is equal to $n! \varphi^{\otimes n}$ with arbitrary $\varphi \in \mathcal{D}$ and therefore the set (5.9) is total in the space $\mathcal{F}_n(H)$ for every $n \in \mathbb{N}_0$.

From this totality for $n = 0, \ldots, m$ we conclude that the set

$$(5.10) \qquad \{A^+(\varphi_1) \ldots A^+(\varphi_m)\Omega \,|\, \varphi_1, \ldots, \varphi_m \in \mathcal{D}\} \subset F_m$$

is total in F_m, $m \in \mathbb{N}$.

To prove totality of (5.8) in F_m (and, consequently, to prove the lemma) it is necessary to conclude, from the totality of the set (5.10) in F_m for every $m \in \mathbb{N}$, that the set (5.8) is total. Note that we can rewrite (5.8) in a similar form but with the matrices $A(\varphi)$ instead of the operators $A(\varphi)$. Then it is necessary to prove that for all $m \in \mathbb{N}$ the set

$$(5.11) \qquad \{A(\varphi_1) \ldots A(\varphi_m)\Omega \,|\, \varphi_1, \ldots, \varphi_m \in \mathcal{D}\} \subset F_m$$

is total in F_m.

We will prove this fact by induction. Let the totality of (5.11) in F_m be true for every $m = 1, \ldots, k$ and we will prove that it is true for $m = k+1$.

Let f be a vector from F_{k+1} and $\varepsilon > 0$. Since set (5.10) is total for $m = k+1$, we have that there is a vector g, which is a linear combination of the vectors Ω, $A^+(\varphi_1^{(1)})\Omega, \ldots, A^+(\varphi_{k+1}^{(k+1)}) \ldots A^+(\varphi_1^{(k+1)})\Omega$ with some $\varphi_{k+1}^{(k+1)}, \ldots, \varphi_1^{(k+1)}$ from $\mathcal{D}$ such that $\|f - g\|_{F_{k+1}} < \varepsilon$.

By using the equality $A^+(\varphi) = A(\varphi) - A^0(\varphi)$ (see (5.5)) we construct a corresponding to g vector h by replacing, in its representation, the matrices $A^+(\varphi)$ by $A(\varphi)$. Then we can write that $g = h + r$, where h is a linear combination of Ω,

$$A^+(\varphi_1^{(1)})\Omega, \ldots, A^+(\varphi_{k+1}^{(k+1)}) \ldots A^+(\varphi_1^{(k+1)})\Omega$$

and r is constructed as h but using the matrices $A^0(\varphi)$ instead of $A(\varphi)$. From the way the matrices $A^0(\varphi)$ act on the Fock space $\mathcal{F}(H)$ (see (5.5)) we conclude that vector r belongs to F_k.

By the inductive hypothesis, the vector r can be approximated by a linear combination of vectors from (5.11)) for $m = k$: $\|r - s\|_{F_k} < \varepsilon$. As a result,

$$\|f - (h + s)\|_{F_{k+1}} \leq \|f - (h + r)\|_{F_{k+1}} + \|r - s\|_{F_{k+1}}$$
$$= \|f - (h + r)\|_{F_{k+1}} + \|r - s\|_{F_k} < 2\varepsilon$$

and $h + s$ is a linear combination vectors (5.11) for $m = k + 1$.

From the totality of the set (5.10) in the case $m = 1$ and the identity

$$A(\varphi)\Omega = A(\varphi)(1, 0, 0, \ldots) = (b_0(\varphi)1, a_0(\varphi)1, 0, 0, \ldots)$$

it follows that our assumptions is true for $k = 1$. Therefore (5.11) is total for every $k \in \mathbb{N}$. $\qquad\square$

We now pass to the spectral representation for our family of selfadjoint commuting operators $(\bar{A}(\varphi))_{\varphi \in \mathcal{D}}$ acting on the space $\mathcal{H}_s$ from rigging (3.25). Condition 3.5 and the positivity assumption (3.24) will be assumed in Section 5.

We will apply now the result of Section 4, in particular, Theorem 4.1 to our family of operators $A = (\bar{A}(\varphi))_{\varphi \in \mathcal{D}}$. Now the role of rigging (4.1) is played by rigging (3.25) which is standardly connected with our operators (3.7); the space D is now equal to $\mathcal{F}_{\text{fin}}(\mathcal{D})$ (see (3.16) and condition (3.24)). The space Φ is equal to $\mathcal{D}$, the "eigenvalue" $\lambda \in \Phi' = \mathcal{D}'$ will be denoted by $\omega \in \mathcal{D}'$; $H = L^2(X, dm(x))$. Now equality (4.4) for the generalized eigenvector $\xi(\omega) \in \mathcal{H}_{s,-}$ has the form

$$(5.12) \quad (\xi(\omega), A(\varphi)f)_{\mathcal{H}_s} = \langle \omega, \varphi \rangle (\xi(\omega), f)_{\mathcal{H}_s}, \quad \varphi \in \mathcal{D}, \quad \omega \in \mathcal{D}', \quad f \in D = \mathcal{F}_{\text{fin}}(\mathcal{D});$$

$$\langle \omega, \varphi \rangle = \langle \varphi, \omega \rangle := (\omega, \varphi)_H = (\omega, \varphi)_{L^2(X, dm(x))}.$$

For us it is convenient to pass from the generalized eigenvector $\xi(\omega)$ to its "polynomial" form $P(\omega) \in \mathcal{F}(H_{-\tau^0}, (p^0)^{-1})$, where the latter space is the negative space from rigging (3.23) (compare with [7]). So, using Lemma 3.7 and Corollary 3.8 we can assert that there is a unitary operator $U : \mathcal{H}_{s,-} \to \mathcal{F}(H_{-\tau^0}, (p^0)^{-1})$ for which we have equality of type (3.21),

$$(5.13) \qquad (U\xi(\omega), f)_{\mathcal{F}(H)} = (\xi(\omega), f)_{\mathcal{H}_s}, \quad f \in D = \mathcal{F}_{\text{fin}}(\mathcal{D})$$

(note that now, in equality (3.21), rigging (3.20) are equal to (3.5) and (3.23), respectively).

We put, for all $\omega \in \mathcal{D}'$,

$$(5.14) \qquad U\xi(\omega) =: P(\omega) \in \mathcal{F}(H_{-\tau^0}, (p^0)^{-1}), \quad \text{i.e.,} \quad P(\omega) = (P_n(\omega))_{n=0}^{\infty},$$

in particular, for all $n \in \mathbb{N}_0$,

$$P_n(\omega) \in (\mathcal{D}^{\widehat{\otimes}n})'.$$

Equalities (5.12) and (5.13) give

$$(5.15) \quad (P(\omega), A(\varphi)f)_{\mathcal{F}(H)} = \langle \omega, \varphi \rangle (P(\omega), f)_{\mathcal{F}(H)}, \quad \varphi \in \mathcal{D}, \quad \omega \in \mathcal{D}', \quad f \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

The functions $\mathcal{D}' \ni \omega \mapsto P_n(\omega) \in (\mathcal{D}^{\widehat{\otimes}n})'$, $n \in \mathbb{N}_0$, are similar to polynomials of the first kind. Using these "polynomials" we can write the Fourier transform (4.5) in our case in the form

$$(5.16) \qquad (If)(\omega)\widehat{f}(\omega) = (f, P(\omega))_{\mathcal{F}(H)} = \sum_{n=0}^{\infty}(f_n, P_n(\omega))_{\mathcal{F}_n(H)}$$

for all $\omega \in \mathcal{D}'$ and all $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\text{fin}}(\mathcal{D})$.

So, due to the projective spectral theorem (Theorem 4.1) and (5.16) we can claim the following (main in this Section) result.

Theorem 5.3. *Let Condition 3.5 and assumption (3.24) for the family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ be fulfilled. Then this family generates a Fourier transform I given by*

$$\mathcal{F}_{\text{fin}}(\mathcal{D}) \ni f = (f_n)_{n=0}^{\infty} \mapsto (If)(\omega) =: \widehat{f}(\omega) = (f, P(\omega))_{\mathcal{F}(H)}$$

(5.17)

$$= \sum_{n=0}^{\infty} (f_n, P_n(\omega))_{\mathcal{F}_n(H)} \in L^2(\mathcal{D}', d\rho(\omega)).$$

Here ρ is the spectral measure of the family being a probability Borel measure on the space $\mathcal{D}'$ with weak topology. The closure $\tilde{I}$ by continuity of the operator I is a unitary operator between the spaces $\mathcal{H}_s$ and $L^2(\mathcal{D}', d\rho(\omega))$, it turns each operator $\tilde{A}(\varphi)$ into the operator of multiplication by the function $\langle \omega, \varphi \rangle$.

Proof. It follows from Theorem 4.1 and the considerations given above.   □

Note that this Theorem is also true in the degenerate case (i.e., if (3.24) is not valid). See [10, 8, 13].

6. A study of the Fourier transform and spectral measure. Main technical results

In this section we prove that the space $\mathcal{D}'$ in representation (5.17) can be replaced with the space Γ (2.4) of all infinite configuration, and the Fourier transform I coincides with the Lenard transform K. This result is actually the main theorem of the article. We will assume that assumption (3.24) is fulfilled.

At first we will investigate in more details the "polynomials" $P_n(\omega)$ introduced in (5.14)–(5.16). Return to the operator Jacobi matrices $A(\varphi)$, (5.5), acting in the Fock space $\mathcal{F}(H)$, their elements act by (5.4). We have $a_n(\varphi) : \mathcal{F}_n(H) \to \mathcal{F}_{n+1}(H)$ and $b_n(\varphi) : \mathcal{F}_n(H) \to \mathcal{F}_n(H)$ (it is selfadjoint) are real creation and neutral operators defined in accordance with (5.4); $*$ and $+$ are used respectively for the adjoint operator in $\mathcal{F}(H)$ and for the adjoint operator w. r. t. the zero space of the corresponding chain. It is known that the annihilation operator $a_n^*(\varphi) : \mathcal{F}_{n+1}(H) \to \mathcal{F}_n(H)$ acts as follows:

(6.1) $a_n^*(\varphi)\psi^{\otimes(n+1)} = (n+1)(\varphi, \psi)_H \psi^{\otimes n}.$

From (5.5), (5.6), (6.1) and (5.15) we get

$$\forall \varphi \in \mathcal{D}, \quad \forall f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$$

$$(P(\omega), A(\varphi)f)_{\mathcal{F}(H)} = \sum_{n=0}^{\infty} (P_n(\omega), \ a_{n-1}(\varphi)f_{n-1} + b_n(\varphi)f_n)_{\mathcal{F}_n(H)}$$

(6.2)

$$= \sum_{n=0}^{\infty} ((a_n^*(\varphi))^+ P_{n+1}(\omega) + (b_n(\varphi))^+ P_n(\omega), \ f_n)_{\mathcal{F}_n(H)}$$

$$= \langle \omega, \varphi \rangle \, (P(\omega), \ f)_{\mathcal{F}(H)} = \sum_{n=0}^{\infty} (\langle \omega, \varphi \rangle \, P_n(\omega), \ f_n)_{\mathcal{F}_n(H)}.$$

Here $+$ denotes the conjugation w. r. t. the chain

$$\mathcal{D}' \supset H_{-\tau^0} \supset L^2(X, dm(x)) \supset H_{\tau^0} \supset \mathcal{D}.$$

Since f in (6.2) is arbitrary, we get the following *recurrence relation for $P_n(\omega)$*:

(6.3)

$$\forall \varphi \in \mathcal{D}, \quad \forall \omega \in \mathcal{D}', \quad \forall n \in \mathbb{N}_0$$

$$(a_n^*(\varphi))^+ P_{n+1}(\omega) = \langle \omega, \varphi \rangle \, P_n(\omega) - (b_n(\varphi))^+ P_n(\omega), \quad P_0(\omega) = 1.$$

The equality (6.3) can be regarded as a recurrence formula for the calculating $P_n(\omega)$. But it is possible to write (6.3) in another more simple form.

Namely, since $P_n(\omega) \in (\mathcal{D}^{\widehat{\otimes} n})'$ and thus it is symmetric and real, in order to find $P_n(\omega)$, it is sufficient to know $(P_n(\omega), \varphi^{\otimes n})_{\mathcal{F}(H)}$ for every $\varphi \in \mathcal{D}$. Let us apply (6.2) for $f = (f_n)_{n=0}^{\infty}$, where $f_n = \varphi^{\otimes n}$ and every other $f_m = 0$. Then (6.2) turns into

(6.4)
$$(P_{n+1}(\omega), \ a_n(\varphi)\varphi^{\otimes n})_{\mathcal{F}_{n+1}(H)} + (P_n(\omega), \ b_n(\varphi)\varphi^{\otimes n})_{\mathcal{F}_n(H)}$$
$$= \langle \omega, \varphi \rangle \, (P_n(\omega), \ \varphi^{\otimes n})_{\mathcal{F}_n(H)}.$$

Taking into account the formulas (5.4), (5.2) for $a_n(\varphi)$ and $b_n(\varphi)$ we rewrite (6.4) as follows:

$$\forall \varphi \in \mathcal{D}, \quad \forall n \in \mathbb{N}_0$$

(6.5)
$$(P_{n+1}(\omega), \ \varphi^{\otimes (n+1)})_{\mathcal{F}_{n+1}(H)}$$
$$= \frac{1}{n+1} \left((P_n(\omega) \otimes \omega, \ \varphi^{\otimes(n+1)})_{\mathcal{F}_{n+1}(H)} - (P_n(\omega), \ n\varphi^2 \otimes \varphi^{\otimes(n-1)})_{\mathcal{F}_n(H)} \right),$$
$$P_0(\omega) = 1.$$

As a result, we have *the recurrence formula (6.5) for calculating $P_n(\omega)$*. We see that $P_1(\omega) = \omega$, $P_n(\omega)$ is a "polynomial" of order n with real coefficient but in its expression it is necessary to write $\varphi^{\otimes m}$ instead of the ordinary power of variable.

For calculation of polynomials $P_n(\omega)$ there is also a formula other than (6.5). It is connected with an expansion of some function into a power series. Let us formulate the corresponding result, its complete full proof can be found in the paper [13], Theorem 4.1.

For any $\omega \in \mathcal{D}'$ consider the function

$$e^{\langle \omega, \log(1+\varphi) \rangle},$$

where $\varphi \in \mathcal{D}$ and $\forall x \in X \ \ \varphi(x) > -1$. It is analytic w. r. t. φ in a neighborhood $U(0)$ of 0 from $\mathcal{D}_c$, and thus it can be decomposed into a series w. r. t. tensor powers $\varphi^{\otimes n}$. *It is claimed that the coefficients of this decomposition are just $P_n(\omega)$, i. e.,*

(6.6)
$$e^{\langle \omega, \ \log(1+\varphi) \rangle} = \sum_{n=0}^{\infty} (P_n(\omega), \varphi^{\otimes n})_{\mathcal{F}_n(H)}.$$

After result (6.5) for $P_n(\omega)$ we can pass to investigation of the spectral measure ρ of our family $(A(\varphi))_{\varphi \in \mathcal{D}'}$.

At first we will consider the set $\Gamma(X)$, (2.4), of finite and infinite configurations over X into $\mathcal{D}'$. From the results of Section 2 we will use only the initial definitions (2.2), (2.3) and (2.4). Let $\gamma = [x_1, x_2, \ldots]$ be some infinite configuration, i.e., the points $x_j \in X$ are distinct and, for every compact set $\Lambda \subset X$, only a finite number of these points belongs to Λ. Recall that for every $n \in \mathbb{N}$ the finite configuration $\xi = [x_1, \ldots, x_n]$ belongs to Γ, i.e., $\Gamma^{(n)} \subset \Gamma$.

We will identify γ with a σ-finite Borel measure on X of the kind

$$\mu_\gamma := \sum_{x \in \gamma} \mu_x,$$

where μ_x is a unit measure concentrated at the point x. From the other side, each measure μ_γ generates a linear continuous functionals ω_γ over the space $\mathcal{D}$,

(6.7)
$$\mathcal{D} \ni \varphi \mapsto \omega_\gamma(\varphi) = \int_X \varphi(x) \, d\mu_\gamma(x) = \sum_{x \in \gamma} \varphi(x) \in \mathbb{R}.$$

Because of finiteness of φ and condition (2.4), the mapping (6.7) is indeed a linear continuous functional over $\mathcal{D}$, i. e., $\omega_\gamma \in \mathcal{D}'$. We have

(6.8)
$$\omega_\gamma := \sum_{x \in \gamma} \delta_x, \quad \gamma \in \Gamma,$$

where $\delta_x \in \mathcal{D}'$ denotes the δ-function concentrated at the point $x \in X$. This series is convergent in the weak topology in $\mathcal{D}'$ since the functions $\varphi \in \mathcal{D}$ are finite (moreover, each sum $\langle \omega_\gamma, \varphi \rangle$ is finite).

We will identify $\gamma \in \Gamma$ and $\omega_\gamma \in \mathcal{D}'$ and, as a rule, use the same notations for them, $\gamma = \omega_\gamma$. Therefore, we can rewrite, for example, in the weak topology on $\mathcal{D}'$,

$$(6.9) \qquad \mathcal{D}' \supset \Gamma \ni \gamma = [x_1, x_2, \ldots] = \sum_{n=0}^{\infty} [x_n].$$

So, *identifying γ with ω_γ we get the inclusion $\Gamma(X) \subset \mathcal{D}'$. We will endow $\Gamma(X)$ with the relative topology generated by the weak topology of the space $\mathcal{D}'$.*

Consider the results of Section 2 connected with vague topology and topologization of $\Gamma = \Gamma(X)$ with week topology in $\mathcal{D}'$, regarding Γ as some part of $\mathcal{D}'$.

We start with an almost obvious fact.

Proposition 6.1. *The vague topology on $\Gamma(X)$ is the same as the week topology on $\Gamma(X) \subset \mathcal{D}'$.*

Proof. Let $\gamma^{(m)} = [x_1^{(m)}, x_2^{(m)}, \ldots] \in \Gamma$, $m \in \mathbb{N}$, converge to $\gamma = [x_1, x_2, \ldots] \in \Gamma$ with respect to the vague topology. This means that for every $f \in C_{\mathrm{fin}}(X)$ we have

$$(6.10) \qquad \sum_{j=1}^{\infty} f(x_j^{(m)}) \to \sum_{j=1}^{\infty} f(x_j) \quad \text{as} \quad m \to \infty.$$

But this is the same as (6.10) being valid for every $f \in \mathcal{D}$. $\qquad \square$

We will consider the spectral measure of the family (5.1) of commuting selfadjoint operators $A(\varphi)$, $\varphi \in \mathcal{D}$. This spectral measure is a Borel measure on $\mathcal{D}'$ equipped with weak convergence. This measure on sets from $\Gamma(X) \subset \mathcal{D}'$ is given according to Propositions 6.1 on the σ-algebra $\mathcal{B}(\Gamma(X))$, i.e., *we can use for our spectral measure the results of Section 2, in particular, Theorem 2.10.*

The polynomials $P_n(\omega)$ can be calculated in a simple way in the case $\omega = \gamma \in \Gamma \subset \mathcal{D}'$.

Lemma 6.2. *The following formula holds:*

$$(6.11) \qquad \forall \gamma \in \Gamma \subset \mathcal{D}', \quad \forall n \in \mathbb{N} \quad P_n(\gamma) = \sum_{\xi \subset \gamma, \, |\xi| = n} \widehat{\otimes}_{x \in \xi} \delta_x, \quad P_0(\gamma) = 1.$$

Proof. For $n = 1$ the formula (6.11) is obvious (recall that $P_1(\gamma) = \gamma$). Let us suppose that it is true for $n \in \mathbb{N}$ and prove it for $n + 1$. According to (6.5) we have for all $\varphi \in \mathcal{D}$ that

$$
\begin{aligned}
&(P_{n+1}(\gamma), \, \varphi^{\otimes(n+1)})_{\mathcal{F}_{n+1}(H)} \\
&= \frac{1}{n+1} \left((P_n(\gamma)\widehat{\otimes}\gamma, \, \varphi^{\otimes(n+1)})_{\mathcal{F}_{n+1}(H)} - (P_n(\gamma), \, n\varphi^2\widehat{\otimes}\varphi^{\otimes(n-1)})_{\mathcal{F}_n(H)} \right) \\
&= \frac{1}{n+1} \left((P_n(\gamma), \, \varphi^{\otimes n})_{\mathcal{F}_n(H)} \langle \gamma, \varphi \rangle - (P_n(\gamma), \, n\varphi^2\widehat{\otimes}\varphi^{\otimes(n-1)})_{\mathcal{F}_n(H)} \right) \\
&= \frac{1}{n+1} \left(\left(\sum_{\xi \subset \gamma, \, |\xi| = n} \widehat{\otimes}_{x \in \xi} \delta_x, \, \varphi^{\otimes n} \right)_{\mathcal{F}_n(H)} \langle \gamma, \varphi \rangle \right. \\
&\qquad \left. - \left(\sum_{\xi \subset \gamma, \, |\xi| = n} \widehat{\otimes}_{x \in \xi} \delta_x, \, n\varphi^2\widehat{\otimes}\varphi^{\otimes(n-1)} \right)_{\mathcal{F}_n(H)} \right)
\end{aligned}
$$

$$= \frac{1}{n+1}\left(\left(\sum_{\xi\subset\gamma,\,|\xi|=n}\left(\prod_{x\in\xi}\varphi(x)\right)\right)\left(\sum_{x\in\gamma}\varphi(x)\right)\right.$$

$$\left. - \sum_{\xi\subset\gamma,\,|\xi|=n}\left(\sum_{y\in\xi}\left(\varphi^2(y)\prod_{x\in\xi\setminus\{y\}}\varphi(x)\right)\right)\right)$$

$$= \frac{1}{n+1}\sum_{\xi\subset\gamma,\,|\xi|=n}\left(\left(\prod_{x\in\xi}\varphi(x)\right)\left(\sum_{x\in\gamma}\varphi(x)\right) - \sum_{y\in\xi}\left(\varphi^2(y)\prod_{x\in\xi\setminus\{y\}}\varphi(x)\right)\right)$$

$$= \frac{1}{n+1}\sum_{\xi\subset\gamma,\,|\xi|=n}\left(\sum_{y\in\gamma}\left(\varphi(y)\prod_{x\in\xi}\varphi(x)\right) - \sum_{y\in\xi}\left(\varphi(y)\prod_{x\in\xi}\varphi(x)\right)\right)$$

$$= \frac{1}{n+1}\sum_{\xi\subset\gamma,\,|\xi|=n}\left(\sum_{y\in\gamma\setminus\xi}\left(\varphi(y)\prod_{x\in\xi}\varphi(x)\right)\right) = \sum_{\xi\subset\gamma,\,|\xi|=n+1}\left(\prod_{x\in\xi}\varphi(x)\right)$$

$$= \left(\sum_{\xi\subset\gamma,\,|\xi|=n+1}\widehat{\otimes}_{x\in\xi}\delta_x,\ \varphi^{\otimes(n+1)}\right)_{\mathcal{F}_{n+1}(H)}.$$

Since here $\varphi\in\mathcal{D}$ is arbitrary, we conclude that (6.11) is true for $n+1$, and the lemma is proved by induction. $\qquad\square$

The following fact is important: we can calculate the Fourier transform $(If)(\omega)$ in the point $\omega=\gamma\in\Gamma$ using the simple formula (2.7).

Lemma 6.3. *For the Fourier transform $(If)(\omega)$, $f\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ (see (5.17)) in the point $\omega=\gamma\in\Gamma\subset\mathcal{D}'$, the following identity holds:*

$$(6.12) \qquad\qquad (If)(\gamma) = (Kf)(\gamma), \quad \gamma\in\Gamma.$$

Proof. This proposition is a simple consequence of Lemma 6.2. Indeed, let $f=(f_n)_{n=0}^\infty\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$. According to (5.17), (6.11), and (2.7), we have

$$(If)(\gamma) = \sum_{n=0}^\infty (f_n, P_n(\gamma))_{\mathcal{F}_n(H)} = f_0 + \sum_{n=1}^\infty \left(f_n, \sum_{\xi\subset\gamma,\,|\xi|=n}\widehat{\otimes}_{x\in\xi}\delta_x\right)_{\mathcal{F}_n(H)}$$

$$= f(\varnothing) + \sum_{\xi\subset\gamma,\,|\xi|>0} f(\xi) = (Kf)(\gamma).$$

$\qquad\square$

Note that the relative topology on $\Gamma_0\subset\Gamma$ is not the same as the topology on Γ_0, introduced on page 3.

Let us introduce an important essential notion of a (generalized) character. Namely, let a function $\varphi\in\mathcal{D}$ be given. We construct the following function:

$$(6.13)\qquad \chi_\varphi:\Gamma_0\to\mathbb{R},\quad \xi\mapsto\chi_\varphi(\xi) := \begin{cases} 1, & \text{if } \xi=\varnothing, \\ \prod_{x\in\xi}\varphi(x), & \text{otherwise.}\end{cases}$$

This function will be called a character[1], generated by φ. Of course, $\chi_\varphi\in\mathrm{Fun}_{\mathrm{bs}}(\Gamma_0)$, and therefore, the transform K (2.7) is defined on χ_φ.

Note that it is easy to calculate the action of the transform K on $\chi_\varphi(\xi)$. Namely, for all $\varphi\in\mathcal{D}$, we have

$$(6.14)\qquad (K\chi_\varphi)(\gamma) = \left(K\prod_{x\in\xi}\varphi(x)\right)(\gamma) = \prod_{x\in\gamma}(1+\varphi(x)), \quad \gamma\in\Gamma.$$

---

[1]Such a name makes sense: this follows from an article about the Ruelle convolution, which is under preparation, and from [20, 17]. Usually, the functions (6.13) are called Lebesgue–Poisson exponents.

For every fixed γ this product is finite (see, e.g., [28, 33, 22, 23]).

Introduce a $\mathbb{C}$-linear space of finite linear combinations of $\varphi^{\otimes n}$,

$$(6.15) \qquad \mathcal{F}_{\text{lin}}(\mathcal{D}) := \text{span}\big\{\varphi^{\otimes n} \mid \varphi \in \mathcal{D},\, n \in \mathbb{N}_0\big\}.$$

The topology in the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$ is such that the space $\mathcal{F}_{\text{lin}}(\mathcal{D})$ is dense in $\mathcal{F}_{\text{fin}}(\mathcal{D})$ (see Lemma 6.5 below) and, therefore, it is dense in every space $\mathcal{F}(H_\tau, p)$ (3.1) and $\mathcal{H}_s$. Note that according (6.13), for all $\varphi \in \mathcal{D}$ and $\xi = [x_1, \dots, x_n] \in \Gamma^{(n)} \subset \Gamma$, we have

$$\chi_\varphi(\xi) = \prod_{x \in \xi} \varphi(x) = \varphi(x_1) \dots \varphi(x_n) = \varphi^{\otimes n}(x_1, \dots, x_n).$$

Let us introduce the notion of a subcharacter. By definition, for given $\varphi \in \mathcal{D}$, the function $\chi_{\varphi,\text{sub}} : \Gamma_0 \to \mathbb{R}$ is a subcharacter if $\chi_{\varphi,\text{sub}}(\xi)$ is equal to some character $\chi_\varphi(\xi)$ for $\xi \in \bigsqcup_{n=0}^{k} \Gamma^{(n)}$ and equal to zero for $\xi \in \bigsqcup_{n=k+1}^{\infty} \Gamma^{(n)}$; here $k \in \mathbb{N}_0$ depends on $\chi_{\varphi,\text{sub}}$. Thus, $\chi_{\varphi,\text{sub}} \in \mathcal{F}_{\text{fin}}(\mathcal{D})$. We will also denote such subcharacters by $\chi_{\varphi,\text{sub};k}$; $\chi_{\varphi,\text{sub};0} = e$. In terms of sequences, we can write

$$(6.16) \qquad \chi_{\varphi,\text{sub};k} = (1, \varphi, \varphi^{\otimes 2}, \dots, \varphi^{\otimes k}, 0, 0, \dots), \quad \varphi \in \mathcal{D}, \quad k \in \mathbb{N}_0.$$

Using this notation, for the space $\mathcal{F}_{\text{lin}}(\mathcal{D})$ we can also write

$$(6.17) \qquad \mathcal{F}_{\text{lin}}(\mathcal{D}) := \text{span}\big\{\chi_{\varphi,\text{sub}} \mid \varphi \in \mathcal{D}\big\}.$$

We will consider the question about density of the linear space $\mathcal{F}_{\text{lin}}(\mathcal{D})$ in the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$. To this end, we introduce a linear topological space $C_{\text{fin}}(\Gamma_0)$. It consists of complex-valued functions $\Gamma_0 = \bigsqcup_{n=0}^{\infty} \Gamma^{(n)} \ni \xi \mapsto f(\xi) \in \mathbb{C}$ such that $f \upharpoonright \Gamma^{(n)} = 0$ for all $n > k$ (k depends on f) and $f \upharpoonright \Gamma^{(n)}$ is finite and continuous on $\hat{X}^n \subset X^n$ for all $n \le k$. We endow $C_{\text{fin}}(\Gamma_0)$ with uniform finite topology, $C_{\text{fin}}(\Gamma_0) \ni f^{(m)}(\xi) \to f(\xi) \in C_{\text{fin}}(\Gamma_0)$ if and only if $f^{(m)}$ are uniformly finite with respect to n (i.e., there exists $k \in \mathbb{N}$ such that $f_n^{(m)} = 0$ if $n > k$ and $f_n^{(m)}$ uniformly converge to $f^{(m)}$ as $m \to \infty$ on $X^n \supset \Gamma^{(n)}$).

It is clear that, if above we will use functions $\Gamma_0 \ni \xi \mapsto f(\xi) \in \mathbb{C}$ such that $f \upharpoonright \Gamma^{(n)} \in C_{\text{fin}}^\infty(\Gamma_0)$ and demand, in the definition of the topology, uniform convergence of $f_n^{(m)}$ to $f^{(m)}$ all corresponding derivatives, we get, as a result, the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$.

Let us now look at the question about density.

Lemma 6.4. *The linear space $\mathcal{F}_{\text{lin}}(\mathcal{D})$ is dense in the space $C_{\text{fin}}(\Gamma_0)$.*

Proof. Let us fix some compact $\Lambda \subset X$ and $k \in \mathbb{N}_0$, and consider the space

$$Q_{\Lambda,k} = \bigsqcup_{n=0}^{k} \Lambda^{(n)}, \quad \Lambda^{(0)} = \varnothing,$$

with the topology of the space (2.2) (i.e., $Q_{\Lambda,k} \ni \xi^{(m)} = [x_1^{(m)}, \dots, x_k^{(m)}] \to [x_1, \dots, x_k] \in Q_{\Lambda,k}$, $m \to \infty$, if and only if $x_n^{(m)} \to x_n$, $m \to \infty$, in the space X). In this topology $Q_{\Lambda,k}$ is compact.

Let $f \in \mathcal{F}_{\text{lin}}(\mathcal{D})$, we will consider its restriction $f \upharpoonright Q_{\Lambda,k}$ to the space $Q_{\Lambda,k}$. It is easy to see that a linear span of such restrictions forms an algebra $\mathcal{F}(Q_{\Lambda,k})$ of functions with respect the ordinary addition and multiplication. This follows from (6.17) and the following important remark: *the ordinary product of two arbitrary subcharacters is also a subcharacter*,

$$\chi_{\varphi,\text{sub}}(\xi)\chi_{\psi,\text{sub}}(\xi) = \chi_{\varphi\psi,\text{sub}}(\xi), \quad \varphi, \psi \in \mathcal{D}.$$

This algebra $\mathcal{F}(Q_{\Lambda,k})$ contains all constants and, for arbitrary distinct points $\xi', \xi'' \in Q_{\Lambda,k}$, we can find a subcharacter $\chi_{\varphi,\text{sub}}$ for which $\chi_{\varphi,\text{sub}}(\xi') \ne \chi_{\varphi,\text{sub}}(\xi'')$ (taking a corresponding function φ). According to the Stone theorem (see, e.g., [20]) we can assert that $\mathcal{F}(Q_{\Lambda,k})$ is dense in the space $C(Q_{\Lambda,k})$ of continuous complex-valued functions on $Q_{\Lambda,k}$ with uniform metric.

From this fact it follows that $\mathcal{F}_{\text{lin}}(\mathcal{D})$ is dense in the space $C_{\text{fin}}(\Gamma_0)$ with respect to its topology; indeed, it is necessary to take into account that every function $f \in C_{\text{fin}}(\Gamma_0)$ is equal to zero on the set $\Gamma_0 \setminus Q_{\Lambda,k}$ with a corresponding compact set $\Lambda \subset X$ and $k \in \mathbb{N}_0$. □

Lemma 6.5. *The linear space $\mathcal{F}_{\text{lin}}(\mathcal{D})$ is dense in the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$.*

Proof. Denote by ∂ the first derivative of the function $X \ni x \mapsto \varphi(x) \in \mathbb{C}$ with respect to some given coordinate on X. We will construct, using (6.13), subcharacters $\chi_{\varphi,\text{sub}}$ and $\chi_{\partial\varphi,\text{sub}}$ using the functions $\varphi \in \mathcal{D} = C_{\text{fin}}^{\infty}(X)$ and their derivatives $\partial\varphi$. For a function $f(\xi)$, we will denote by ∂f, where $\xi = [x_1, \ldots, x_n]$, its derivative with respect to a given coordinate in every points x_j.

By repeating the proof of Lemma 6.4 we can see that the algebra $\mathcal{F}^{\partial}(Q_{\Lambda,k})$ constructed as the algebra $\mathcal{F}(Q_{\Lambda,k})$, but using the functions φ and their derivatives $\partial\varphi$, is dense in the space $C^{\partial}(Q_{\Lambda,k})$.

The latter space, by definition, is a space of all functions in $C(Q_{\Lambda,k})$ which have continuous derivatives ∂f and are equipped with the norm

$$(6.18) \qquad \|f\|_{\partial} := \max_{\xi \in Q_{\Lambda,k}} (|f(\xi)| + |\partial f(\xi)|).$$

Using the derivatives ∂ with respect to all coordinates in X and taking the norm, which is equal to sum of norms (6.18), we can conclude that $\mathcal{F}_{\text{lin}}(\mathcal{D})$ is dense in the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$ in sense of space $C_{\text{fin}}^1(\Gamma_0)$ (endowed with the uniform norm with respect to all first derivatives). Extending this procedure for the derivatives of the second, third, etc orders, we finish proving our lemma. □

Now by means of Theorem 2.10 and Lemmas 6.3, 6.2 and 6.5 we can prove the following important form of the spectral Theorem 5.3.

Note at first that it is possible to say about the measure ρ of these sets Γ and $\mathcal{D}' \setminus \Gamma$ since the spectral measure is defined on the Borel sets from $\mathcal{D}'$ with weak topology and we have proved that the set Γ is Borel (Lemma 2.9).

Theorem 6.6. *Let for a family of operators $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ Condition 3.5 is fulfilled and the functional s is positive. Assume also that the corresponding spectral measure ρ of the set Γ is positive,*

$$(6.19) \qquad \rho(\Gamma) > 0.$$

Then the role of the space $\mathcal{D}'$ in Theorem 5.3 is played by the space $\Gamma \subset \mathcal{D}'$ with the topology inducted by the weak topology in $\mathcal{D}'$. The space Γ is of full spectral measure ρ (i.e., $\rho(\mathcal{D}' \setminus \Gamma) = 0$) and the corresponding Fourier transform (5.17) is

$$\mathcal{F}_{\text{fin}}(\mathcal{D}) \ni f = (f_n)_{n=0}^{\infty} \mapsto \hat{f}(\gamma) = (If)(\gamma) = (Kf)(\gamma) = (f, P(\gamma))_{\mathcal{F}(H)}$$

$$(6.20)$$

$$= \sum_{n=0}^{\infty} (f_n, P_n(\gamma))_{\mathcal{F}_n(H)} \in L^2(\Gamma, d\rho(\gamma)),$$

$$\forall \gamma \in \Gamma \subset \mathcal{D}'; \quad \forall n \in \mathbb{N} \quad P_n(\gamma) = \sum_{\xi \subset \gamma, \, |\xi| = n} \hat{\otimes}_{x \in \xi} \delta_x, \quad P_0(\gamma) = 1.$$

The function (6.20) is continuous with respect to the weak topology on $\Gamma \subset \mathcal{D}'$.

The closure $\tilde{I}$ of the operator I by continuity is a unitary operator of $\mathcal{H}_s$ onto

$$L^2(\Gamma, d\rho(\gamma)) = L^2(\mathcal{D}', d\rho(\omega)).$$

Proof. It is necessary to prove that the measure ρ of the set $\mathcal{D}' \setminus \Gamma$ is equal to zero. We assume the contrary. So, we let $\rho(\mathcal{D}' \setminus \Gamma) > 0$ and arrive to a contradiction.

By assumption (6.19) of the theorem, the spectral measure ρ of Γ is also positive. From the identity $\mathcal{D}' = \Gamma \cup (D' \setminus \Gamma)$ we conclude that

$$(6.21) \qquad L^2(\mathcal{D}', d\rho(\omega)) = L^2(\Gamma, d\rho(\omega)) \bigoplus L^2(\mathcal{D}' \setminus \Gamma, d\rho(\omega)) =: L_1^2 \bigoplus L_2^2.$$

Since $\rho(\Gamma) > 0$, $\rho(\mathcal{D}' \setminus \Gamma) > 0$, both subspaces L_1^2 and L_2^2 are not zero spaces. We, as usual, in For an orthogonal sum of Hilbert spaces $H_1 \oplus H_2 = H$, we, as usual, denote the vectors $f_1 \in H_1$ $(f_2 \in H_2)$ by $(f_1, 0) \in H$ $((0, f_2) \in H)$.

Consider the Fourier transform I (5.17) and apply it to a subcharacter $\chi_{\varphi,\mathrm{sub}} \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, where $\varphi \in \mathcal{D}$ and the number $k = m \in \mathbb{N}_0$ are the same as in the definition of a subcharacter (see (6.16)). We will use, in our proof, only subcharacters $\chi_{\varphi,\mathrm{sub}}$ for some fixed $m \in \mathbb{N}_0$.

Since $\chi_{\varphi,\mathrm{sub}} \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, the function $I\chi_{\varphi,\mathrm{sub}}$ belongs to $\mathrm{Ran}\,(I) \subset L^2(\mathcal{D}', d\rho(\omega))$ and, according to Lemma 6.3, we have

$$(6.22) \qquad (I\chi_{\varphi,\mathrm{sub}})(\omega) = \begin{cases} (K\chi_{\varphi,\mathrm{sub}})(\gamma), & \text{if } \omega = \gamma \in \Gamma, \\ F(\omega), & \text{if } \omega \in \mathcal{D}' \setminus \Gamma, \end{cases}$$

where $F(\omega)$ is some function from $L^2(\mathcal{D}' \setminus \Gamma, d\rho(\omega)) \subset L^2(\mathcal{D}', d\rho(\omega))$.

Applying expansion (6.21) to the function (6.22) we get, for ρ-almost all $\omega \in \mathcal{D}'$, that

$$(6.23) \qquad (I\chi_{\varphi,\mathrm{sub}})(\omega) = F_1(\omega) + F_2(\omega), \quad F_1 \in L_1^2, \quad F_2 \in L_2^2,$$

where the function $F_1(\gamma)$ is equal to $(K\chi_{\varphi,\mathrm{sub}})(\gamma)$ for ρ-almost all $\omega = \gamma \in \Gamma$ and zero for $\omega \in \mathcal{D}' \setminus \Gamma$; the function $F_2(\omega)$ is equal to zero for $\omega = \gamma \in \Gamma$ and $F(\omega)$ for $\omega \in \mathcal{D}' \setminus \Gamma$.

Applying the operator $\tilde{I}^{-1}$ to the left-hand side of equality (6.23) gives

$$(6.24) \qquad \tilde{I}^{-1} I\chi_{\varphi,\mathrm{sub}} = \tilde{I}^{-1}\tilde{I}\chi_{\varphi,\mathrm{sub}} = \chi_{\varphi,\mathrm{sub}}.$$

Let us calculate the result of applying the operator $\tilde{I}^{-1}$ to the right-hand side of equality. These calculations will be considerably more difficult. We first recall the discussion in pages 4–5. Denote by $\mathcal{F}_{\mathrm{fin},m}(\mathcal{D})$, $m \in \mathbb{N}_0$, the subspace of the space $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ that consists of vectors $f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ of the form $f = (f_0, \ldots, f_m, 0, 0, \ldots)$. Then the transform $(Kf)(\gamma) = F(\gamma)$ (2.7) is indeed a function of the finite configurations $\gamma = \eta \in \bigsqcup_{n=0}^m \Gamma^{(n)}$.

As F, we can take an arbitrary vector-valued function

$$F(\eta) = (F_0, F_1(y_1), \ldots, F_m(y_1, \ldots, y_m)),$$

where $F_0 \in \mathbb{C}$ and $F_n(y_1, \ldots, y_n)$ are arbitrary symmetric infinitely differentiable finite functions of points $(y_1, \ldots, y_n) \in \hat{X}^n \subset X^n$. See (2.12), (2.13).

The inverse operator K^{-1} on such functions F exists and satisfies the inequality (see (2.11))

$$(6.25) \qquad \max_{\xi \in \Gamma_\Lambda^m} |(K^{-1}F)(\xi)| \le 2^m \max_{\eta \in \bigsqcup_{n=0}^m \Gamma_\Lambda^{(n)}} |F(\eta)|, \quad m \in \mathbb{N}.$$

If we take F to be any infinitely differentiable finite symmetric function $F(\eta)$, then $K^{-1}F$ belongs to $\mathcal{F}_{\mathrm{fin},m}(\mathcal{D})$.

Condition (6.19) can be rewritten in another form, using finite configurations. Namely, consider the sets

$$(6.26) \qquad \Gamma(m, \Lambda) := \bigsqcup_{n=0}^m \Gamma_\Lambda^{(n)} \subset \Gamma_{\Lambda,0} \subset \Gamma, \quad m \in \mathbb{N}_0, \quad \Lambda \subset X \text{ is compact.}$$

We now apply Theorem 2.10, where the general measure is replaced with our spectral measure ρ. Such a replacement is correct. To see this it is necessary to apply Proposition 6.1. So, by this theorem we have inequality (2.37). One can see that there exist $m_0 \in \mathbb{N}$ and a compact set $\Lambda_0 \subset X$ such that

$$(6.27) \qquad \rho(\Gamma(m, \Lambda)) > 0, \quad m \ge m_0, \quad \Lambda \supset \Lambda_0.$$

Let us introduce the following subsets of space $\mathcal{F}_{\text{fin}}(\mathcal{D})$. Let $m \in \mathbb{N}$ and a compact subspace $\Lambda \subset X$ be fixed. Then

$$(6.28) \quad \mathcal{F}_{\text{fin};m,\Lambda}(\mathcal{D}) := \{f = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\text{fin}}(\mathcal{D}) \,|\, \sup f_n \subset \Lambda^n, \, n \leq m \,;\, f_n = 0, \, n > m\}.$$

If $f \in \mathcal{F}_{\text{fin};m,\Lambda}(\mathcal{D})$ then its K-transform has the form (2.12), (2.13), where the functions $F_n(x_1, \ldots, x_n)$ have supports Λ^n for $n \leq m$ and $F_n = 0$ for $n > m$. This fact follows directly from definition (2.7); recall that configurations in (2.7) are non overlapping subsets of X.

Let $f \in \mathcal{F}_{\text{fin};m,\Lambda}(\mathcal{D})$. Then $(Kf)(\gamma)$ is a function of $\gamma = \eta \in \bigsqcup_{n=0}^{m} \Gamma_{\Lambda}^{(n)} = \Gamma(m, \Lambda)$, for another $\gamma \in \Gamma$ it is equal to zero. But $(Kf)(\gamma)$ is equal to $(If)(\gamma)$ (Lemma 6.3) and $\operatorname{Ran} \tilde{I} = L^2(\mathcal{D}', d\rho(\omega))$, therefore $If \in L^2(\Gamma(m, \Lambda), d\rho(\omega))$, and we can write

$$(6.29) \qquad\qquad Kf \in L^2(\Gamma(m, \Lambda), d\rho(\omega)), \quad f \in \mathcal{F}_{\text{fin};m,\Lambda}(\mathcal{D}).$$

In particularly, every subcharacter $\chi_{\varphi,\text{sub};m}$ belongs to $\mathcal{F}_{\text{fin};m,\Lambda}(\mathcal{D})$, therefore the inclusion (6.29) is true also for $\chi_{\varphi,\text{sub};m}$, where Λ is the support of φ. We fix the corresponding $m \in \mathbb{N}$ and a compact set Λ satisfying the condition (6.27).

Similarly to (6.21) we construct the orthogonal decomposition

$$(6.30) \quad \begin{aligned} L^2(\mathcal{D}', d\rho(\omega)) &= L^2(\Gamma(m, \Lambda), d\rho(\omega)) \bigoplus L^2(\mathcal{D}' \setminus \Gamma(m, \Lambda), d\rho(\omega)) \\ &=: L^2_{1;m,\Lambda} \bigoplus L^2_{2;m,\Lambda}. \end{aligned}$$

The idea of the remaining part of the proof of theorem consists in the following: we consider some new Hilbert space $H \supset \mathcal{F}_{\text{fin}}(\mathcal{D})$ ($H \subset \mathcal{H}_s$, $\|\cdot\|_H = \|\cdot\|_{\mathcal{H}_s}$) and construct some isometrical operator U which acts from H into the given space $L^2_{1;m,\Lambda}$. We construct it by using the transform K. Since the function $(K\chi_{\varphi,\text{sub}})(\gamma)$ belongs to $L^2_{1;m,\Lambda}$, one can prove that U^{-1} from this function belongs to the space H and coincides with $\chi_{\varphi,\text{sub}}$. The construction of U is such that $U^{-1} = \tilde{I}^{-1}$ on $(K\chi_{\varphi,\text{sub}})(\gamma)$, therefore $\tilde{I}^{-1}F_1 = \chi_{\varphi,\text{sub}}$.

Lemma 6.7. *The function (6.20) is continuous with respect to the weak topology in $\Gamma \subset \mathcal{D}'$.*

Proof. From (6.20) and (6.11) we see that it is sufficient to prove the following fact: for every $n \in \mathbb{N}$ and fixed $\varphi \in \mathcal{D}$, the function

$$(6.31) \quad \begin{aligned} \Gamma \ni \gamma \mapsto (\varphi^{\otimes n}, P_n(\gamma))_{\mathcal{F}_n(H)} &= (\varphi^{\otimes n}, \sum_{\xi \subset \gamma, \, |\xi| = n} \widehat{\otimes}_{x \in \xi} \delta_x)_{\mathcal{F}_n(H)} \\ &= \sum_{\xi \subset \gamma, \, |\xi| = n} \Big(\prod_{x \in \xi} \varphi(x) \Big) \in \mathbb{R} \end{aligned}$$

is continuous.

It is necessary to consider the configurations γ, for which $|\gamma| = n$. Therefore, it is necessary to prove that the following function is continuous with respect to the weak topology:

$$\Gamma \supset \Gamma^{(n)} \ni \gamma = [y_1, \ldots, y_n] \mapsto \sum_{|\xi| = n} \Big(\prod_{x \in \xi} \varphi(x) \Big) = \sum_{k=1}^{n} \varphi(y_1) \ldots \varphi(y_n) \in \mathbb{R}.$$

Its continuity is evidently, since the weak convergence $\gamma^m = [y_1^m, \ldots, y_n^m] \to \gamma = [y_1, \ldots, y_n]$ means that $\psi(y_k^m) \to \psi(y)$ for every function $\psi \in \mathcal{D}$ and $k = 1, \ldots, n$. $\qquad\square$

Lemma 6.8. *Let H be a normed space (possibly, not complete) and L, M be Banach spaces. Suppose we have an isometric operator $U : H \to L$ (i.e., $\|Uf\|_L = \|f\|_H$, $f \in H$) and a bounded operator $A : M \to H$ such that if for some $\varphi \in M$ $\|A\varphi\|_H = 0$, then $\varphi = 0$. Then for U there exists a bounded inverse operator $U^{-1} : L \to H$.*

Proof. We will denote elements of these spaces by $H = \{f, g, \ldots\}$, $L = \{F, G, \ldots\}$ and $M = \{\varphi, \psi, \ldots\}$. Consider the operator $B = UA : M \to L$. The operator B is bounded and algebraically invertible: if for some $\varphi \in M$ $B\varphi = UA\varphi = 0$, than we have: $0 = \|B\varphi\|_L = \|UA\varphi\|_L = \|A\varphi\|_H$ (U is isometric), therefore $\varphi = 0$.

Operator B acts between complete spaces M and L, therefore using the Banach inverse operator theorem (see, e.g., [16]) we can assert that an inverse bounded operator $B^{-1} : L \to M$ exists. Consider the bounded operator $AB^{-1} : L \to H$, it is an inverse to U, since algebraically $AB^{-1} = AA^{-1}U^{-1} = U^{-1}$. $\qquad\qquad\square$

We continue with the proof of our theorem.

We will apply Lemma 6.8, at the beginning, with the following spaces (see (6.28), (6.30)):

(6.32)
$$H = \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D}) \subset \mathcal{H}_s, \quad L = L^2_{1;m,\Lambda} = L^2(\Gamma(m,\Lambda), d\rho(\omega)), \quad M = C_{\mathrm{sym}}\Big(\bigsqcup_{n=1}^{m} \Lambda^n\Big).$$

The space H with the norm $\|\cdot\|_H = \|\cdot\|_{\mathcal{H}_s}$ is not complete. Note that $m \in \mathbb{N}$ and the compact set $\Lambda \subset X$ are such that the condition (6.27) is fulfilled: M is a complete space of continuous complex-valued functions on the compact set $\bigsqcup_{n=1}^{m} \Lambda^n$ with the usual norm for the space C. These functions must be symmetric on Λ^n for every $n = 2, \ldots, m$.

Construct the isometrical operator $U: H = \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D}) \to L = L^2_{1;m,\Lambda}$ from Lemma 6.8. We put

(6.33)
$$\mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D}) = H \ni f \mapsto (Kf)(\gamma) =: (Uf)(\gamma)$$

and consider this function on $\gamma \in \Gamma(m,\Lambda) \subset \Gamma$. According to Lemma 6.3 and (6.12), (6.29), (6.30) we have for every $f \in H = \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D})$

(6.34)
$$\Gamma(m,\Lambda) \ni \gamma \mapsto (Uf)(\gamma) = (Kf)(\gamma) = (If)(\gamma) = (\tilde{I}f)(\gamma) \in L^2_{1;m,\Lambda}.$$

Of course, the operator $U : H \to L^2_{1;m,\Lambda}$ constructed above is isometric since the norm in H is $\|\cdot\|_{\mathcal{H}_s}$ and the operator $\tilde{I}$ is unitary between the spaces $\mathcal{H}_s$ and $L^2(\mathcal{D}', d\rho(\omega))$.

Construct the bounded operator

(6.35)
$$A : M = C_{\mathrm{sym}}\Big(\bigsqcup_{n=1}^{m} \Lambda^n\Big) \to H = \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D})$$

(in fact, in (6.32) we will use some closed subspace M_0 of the space M).

Consider once more the vector $f \in \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D})$ and the restriction of the function $\Gamma \ni \gamma \mapsto F(\gamma) = (Kf)(\gamma)$ on the set $\Gamma(m,\Lambda) \subset \Gamma$. It is easy to show from (2.7) that this restriction is a continuous function on the compact set $\bigsqcup_{n=1}^{m} \Lambda^n$, symmetric on Λ^n for every $n = 2, \ldots, m$. They make a linear set, and the closure of this set with respect to the norm $\|\cdot\|_M$ will be denoted by M_0.

Using (6.34), (6.29) we have for $f \in \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D})$

$$\Gamma(m,\Lambda) \ni \gamma \mapsto F(\gamma) = (Kf)(\gamma) \in L^2_{1;m,\Lambda},$$
$$\|f\|_{\mathcal{H}_s} = \|Kf\|_{L^2_{1;m,\Lambda}}.$$

Since the function $F(\gamma)$ is continuous on the compact set $\bigsqcup_{n=1}^{m} \Lambda^n$, and $\rho(\mathcal{D}') = 1$, we get

(6.36)
$$\|Kf\|_M = \max_{\eta \in \bigsqcup_{n=1}^{m} \Lambda^n} |(Kf)(\eta)| \geq \|Kf\|_{L^2_{1;m,\Lambda}} = \|f\|_{\mathcal{H}_s}.$$

Using existence of the operator K^{-1} and Lemma 2.3 we conclude from (6.36) that

(6.37)
$$\|K^{-1}F\|_{\mathcal{H}_s} \leq \|F\|_M, \quad F = Kf, \quad f \in \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D}).$$

We are ready to introduce the operator A (6.35). We put

$$(6.38) \qquad AF := K^{-1}F, \quad F(\eta) = (Kf)(\eta), \quad f \in \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D}), \quad \eta \in \bigsqcup_{n=1}^{m} \Lambda^n.$$

From (6.37) we have

$$\|AF\|_{\mathcal{H}_s} \le \|F\|_M, \quad F = Kf, \quad f \in \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D}).$$

Using the restriction mentioned above, this inequality can be rewritten as

$$(6.39) \qquad \|AF\|_H \le \|F\|_{M_0}, \quad F = Kf, \quad f \in \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D}).$$

Here $\| \cdot \|_H = \| \cdot \|_{\mathcal{H}_s}$, $\| \cdot \|_{M_0} = \| \cdot \|_M$.

So, it is necessary to apply Lemma 6.8, where the spaces (6.32) are such that

$$(6.40) \qquad H = \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D}) \subset \mathcal{H}_s, \quad L = L^2_{1;m,\Lambda}, \quad M_0 \subset M = C_{\mathrm{sym}}\left(\bigsqcup_{n=1}^{m} \Lambda^n\right),$$

and the operators are: $U : H \to L$ (6.33), $A : M_0 \to H$ (6.38). This proves that U is isometric, see (6.31). A necessary condition on A is the following: if for $F \in M_0$ we have $\|Af\|_H = 0$, then $F = 0$. This follows from (6.38) and existence of the inverse operator A^{-1}, i.e. K and (6.36).

Thus, by Lemma 6.8 the inverse bounded operator $U^{-1} : L^2_{1;m,\Lambda} \to H = \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D})$ exists. It is clear that $U^{-1} = \tilde{I}^{-1}$ on $L^2_{1;m,\Lambda}$. This property follows from (6.34).

Thus, we will continue our proof starting from formulas (6.30). We have proved that if conditions (6.27) are fulfilled then *the action of the operator $\tilde{I}^{-1}$ on $L^2_{1;m,\Lambda}$ is equal to the action of the operator U^{-1}, i.e., the set $\tilde{I}^{-1}L^2_{1;m,\Lambda}$ is equal to $H = \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D})$. In other words,*

$$(6.41) \qquad \tilde{I}^{-1}L^2_{1;m,\Lambda} \supset H = \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D}), \quad \tilde{I}\mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D}) = \tilde{I}H \subset L^2_{1;m,\Lambda}.$$

Due to the latter it is easy to finish the proof of our theorem.

So we take such $m \in \mathbb{N}$ and a compact set Λ for which the inequality (6.27) is fulfilled. Consider the subcharacter $\chi_{\varphi,\mathrm{sub};m}$ (6.16) and $\varphi \in \mathcal{D}$, for which $\varphi(x) = 0$, $x \in X \setminus \Lambda$, i.e., for every $\varphi \in \mathcal{D}$.

The inclusion $\tilde{I}f \subset L^2_{1;m,\Lambda}$, where $f \in \mathcal{F}_{\mathrm{fin};m,\Lambda}(\mathcal{D})$, means that the Fourier transform $(\tilde{I}f)(\omega)$, $\omega \in \mathcal{D}'$, belongs to subspace $L^2_{1;m,\Lambda}$ of the space $L^2(\mathcal{D}', d\rho(\omega))$, i.e., $(\tilde{I}f)(\omega)$ for ρ-almost all $\omega \in \mathcal{D}' \setminus \Gamma(m,\Lambda)$ is equal to zero. But we can take f to be our $\chi_{\varphi,\mathrm{sub};m}$. Therefore we can assert that the Fourier transform (6.20) $(\tilde{I}\chi_{\varphi,\mathrm{sub};m})(\omega)$ of the subcharacter $\chi_{\varphi,\mathrm{sub};m}$ is equal to zero for $\omega \in \mathcal{D}' \setminus \Gamma(m,\Lambda) \supset \mathcal{D}' \setminus \Gamma$. So, *the function $F_2(\omega)$ from equality (6.23) is equal to zero for ρ-almost all $\omega \in \mathcal{D}'$ in our case $m \ge m_0$ and $\varphi \in \mathcal{D}$.*

The case $m = 0, 1, \ldots, m_0 - 1$ follows from the case which has been proved, since $\chi_{\varphi,\mathrm{sub};m} \in \mathcal{F}_{\mathrm{fin};m_0,\Lambda_0}(\mathcal{D})$. Now we also can assert that $F_2 = 0$.

Thus, using (6.23) and the construction of the function F_2 we can conclude: *for every $\varphi \in \mathcal{D}$ and $m \in \mathbb{N}_0$, the function $(I\chi_{\varphi,\mathrm{sub}})(\omega)$ is equal to $(K\chi_{\varphi,\mathrm{sub}})(\gamma)$ for $\omega = \gamma \in \Gamma$ and equal to zero for ρ-almost all $\omega \in \mathcal{D}' \setminus \Gamma$.*

After this result we can easily get a contradiction. Namely, according to Lemma 6.5 and (6.15), for every vector $f \in \mathcal{H}_s$ there exists a sequence $(f^{(k)})_{k=1}^{\infty}$ of finite linear combinations $f^{(k)}$ of subcharacters $\chi_{\varphi,\mathrm{sub}}$, which tends to f in the space $\mathcal{H}_s$. Since $\tilde{I} : \mathcal{H}_s \to L^2(\mathcal{D}', d\rho(\omega))$ is a unitary operator, we can write: $\tilde{I}f = \lim_{k \to \infty} \tilde{I}f^{(k)}$ in the space $L^2(\mathcal{D}', d\rho(\omega))$ and, therefore, $P_2\tilde{I}f = \lim_{k \to \infty} P_2\tilde{I}f^{(k)}$, where P_2 is a projection in the space (6.21) onto its subspace L^2_2. But for every $k \in \mathbb{N}$ $(P_2f^{(k)})(\omega) = 0$ for ρ-almost all $\omega \in \mathcal{D}' \setminus \Gamma$ since this is true for every $P_2\tilde{I}\chi_{\varphi,\mathrm{sub}}$.

As a result, $(\tilde{I}f)(\omega) = 0$ for ρ-almost all $\omega \in \mathcal{D}' \setminus \Gamma$, i.e., $\tilde{I}f \in L_1^2$. Here f is an arbitrary vector from $\mathcal{H}_s$, therefore the last inclusion is impossible since $\tilde{I}$ is a unitary operator between $\mathcal{H}_s$ and $L^2(\mathcal{D}', d\rho(\omega))$ and $L_2^2 \neq 0$.

Other assertions of the theorem easily follow from Theorem 5.3 and Lemma 6.2. $\qquad\square$

Let us give some simple remarks concerning the fulfilment of condition (6.19) $\rho(\Gamma) > 0$. At first we assert that

$$(6.42) \qquad s(f) = \int_{\mathcal{D}'} (\tilde{I}f)(\omega) d\rho(\omega), \quad f \in \mathcal{H}_s.$$

For the proof of (6.42) we note that from Theorem 5.3 if follows the Parseval equality:

$$(6.43) \qquad (f, g)_{\mathcal{H}_s} = \int_{\mathcal{D}'} (\tilde{I}f)(\omega) \overline{(\tilde{I}g)(\omega)} d\rho(\omega), \quad f, g \in \mathcal{H}_s.$$

Let $g(\xi) = e(\xi)$, $\xi \in \Gamma_0$ (e is the unit of algebra $\mathcal{A}$) in (6.43). Then $(\tilde{I}g)(\omega) = 1$, $\omega \in \mathcal{D}'$ and from (3.6) we get equality (6.42).

Now we will give some conclusions from equality (6.6) and Theorem 5.3. Let $\varphi \in \mathcal{D}$, $\varphi(x) > -1$, $x \in X$, belong to some neighborhood $U(0)$ of 0 in the space $\mathcal{D}$. If this $U(0)$ is small enough then we have expansion (6.6) which we can write in the form

$$(6.44) \qquad \begin{aligned} e^{\langle \omega, \, \log(1+\varphi) \rangle} &= \sum_{n=0}^{\infty} (P_n(\omega), \varphi^{\otimes n})_{\mathcal{F}_n(H)} \\ &= \lim_{m \to \infty} \sum_{n=0}^{m} (P_n(\omega), \varphi^{\otimes n})_{\mathcal{F}_n(H)} \\ &= \lim_{m \to \infty} I\big((1, \varphi, \ldots, \varphi^{\otimes m}, 0, 0, \ldots)\big)(\omega) \end{aligned}$$

for any $\omega \in \mathcal{D}'$ and $\varphi \in U(0)$.

From (6.42) and (6.44), for $\varphi \in U(0)$, we can conclude that

$$(6.45) \qquad \begin{aligned} \int_{\mathcal{D}'} e^{\langle \omega, \, \log(1+\varphi) \rangle} d\rho(\omega) &= \lim_{m \to \infty} \int_{\mathcal{D}'} I\big((1, \varphi, \ldots, \varphi^{\otimes m}, 0, 0, \ldots)\big)(\omega) d\rho(\omega) \\ &= \lim_{m \to \infty} s((1, \varphi, \ldots, \varphi^{\otimes m}, 0, 0, \ldots)). \end{aligned}$$

We give now a simple sufficient condition on the functional s which guarantee (6.19). This condition is necessary in the situation where a "separate" $\gamma_0 \in \Gamma$ exists with positive measure ρ, i.e. $\rho(\gamma_0) > 0$.

Theorem 6.9. *The condition $\rho(\Gamma) > 0$ is fulfilled if the following is satisfied.*

Assume that there are a point $x_0 \in X$ and its neighborhood $U(x_0)$ such that: firstly,

$$(6.46) \qquad \lim_{m \to \infty} s((1, \varphi, \ldots, \varphi^{\otimes m}, 0, 0, \ldots)) = \varepsilon > 0$$

for arbitrary $\varphi \in \mathcal{D}$ which is equal to zero on $X \setminus U(x_0)$ and in some neighborhood of x_0, entering into $U(x_0)$; ε in (6.46) is independent of φ. And secondly, there exists some $\varphi = \varphi_0 \in \mathcal{D}$, which is equal to zero on $X \setminus U(x_0)$ but $\varphi_0(x) = \varepsilon_1 \neq 0$ for all x in some neighborhood of x_0, for which limit (6.46) is different from ε.

Proof. At first we note that according to (5.12) and Theorem 5.3 every δ-function δ_x, $x \in X$, belongs to the set on which the spectral measure of our family of operators $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ is defined. We have $\rho(\delta_x) \geq 0$, $x \in X$. The equality (6.8) gives

$$(6.47) \qquad \rho(\gamma) = \sum_{x \in \gamma} \rho(\delta_x), \quad \gamma \in \Gamma.$$

Therefore, if for some $x_0 \in X$ the measure $\rho(\delta_{x_0}) > 0$, then $\rho([x_0, x_2, x_3, \ldots]) > 0$ and $\rho(\Gamma) > 0$, since $[x_0, x_2, x_3, \ldots] \in \Gamma$.

Assume now that for some point x_0 the conditions (6.46) (first and second situation: 1) and 2)) are fulfilled. Then we prove that $\rho(\delta_{x_0}) > 0$ and our theorem will be proved.

So, in the case 1) the functions $\log(1 + \varphi(x)) \in \mathcal{D}$ and equal to zero on $X \setminus U(x_0)$ and on some neighborhood $V(x_0) \subset U(x_0)$ of x_0; in other points it is arbitrary. The integral

$$(6.48) \qquad \int_{\mathcal{D}'} e^{\langle \omega, \, \log(1+\varphi) \rangle} d\rho(\omega) = \varepsilon$$

from (6.45) is constant only in the case where ρ, on the set of all generalized functions ω with supports in $U(x_0) \setminus x_0$, is equal to zero.

But if in (6.46) we take $\varphi = \varphi_0$ then this integral is not equal to ε (case 2)). Such a situation is possible only in the case where the spectral measure ρ, on the set $\mathcal{D}'(x_0)$ of generalized functions $\omega \in \mathcal{D}'$ the support of which is equal to one point x_0, is not equal to zero.

But every generalized function, the support of which is equal to one point x_0, is equal to a finite linear combination of δ_{x_0} and its derivatives (see, e.g. [36], Ch. 1, § 2). It is easy to see that the set $\mathcal{D}'(x_0)$ in our case necessarily equals to the δ-function δ_{x_0}. Namely, for every derivative $D^\alpha \delta_{x_0}$ we have $\langle D^\alpha \delta_{x_0}, \varphi_0 \rangle = 0$, since φ_0 is some constant in the neighborhood of x_0. Therefore $\rho(\mathcal{D}'(x_0) \setminus \delta_{x_0})$ must be equal to zero and then integral (6.46) is also equal to ε, but its value must be different from ε. Thus $\mathcal{D}'(x_0) = c\delta_{x_0}$ with some constant $c \neq 0$. The spectral measure ρ on $c\delta_{x_0}$ must be positive, since otherwise, integral (6.46) for $\varphi = \varphi_0$ must be equal to ε. $\qquad \square$

Note that this theorem deals only with fulfillment of the condition $\rho(\Gamma) > 0$.

Consider two examples of functional s generated by the measure ν on Γ_0, see (3.9). Now the measure ν must be Borel and σ-finite, i.e., finite on every compact subset from $\Gamma^{(n)}$, $n \in \mathbb{N}_0$.

1. If for every compact $\Lambda \subset X$ the following condition is fulfilled:

$$(6.49) \qquad \sum_{n=0}^{\infty} 2^n \nu(\Gamma_\Lambda^{(n)}) < \infty,$$

then the results of Theorem 6.6 are true and $\rho(\Gamma) > 0$. This statement follows from articles [10, 8, 13].

2. Let τ be some Borel probability measure on X. Consider the functional s of type (3.9), where $\forall n \in \mathbb{N} \, \nu \restriction \Gamma^{(n)} = \tau^{\otimes n}$; $\nu(\Gamma^{(0)}) = 1$. Now the functional s has the form

$$(6.50) \qquad s(f) = \sum_{n=0}^{\infty} \int_{\Gamma^{(n)}} f(\xi) d(\tau^{\otimes n})(\xi), \quad f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D}).$$

For $f = (\varphi^{\otimes n})_{n=0}^{\infty}$, where $\varphi \in \mathcal{D}$, $|\varphi(x)| < 1$, $x \in X$, from (6.50) we get

$$(6.51) \qquad \begin{aligned} \lim_{m \to \infty} s((1, \varphi, \ldots, \varphi^{\otimes m}, 0, 0, \ldots)) &= \sum_{n=0}^{\infty} \left(\int_X \varphi(x) d\tau(x) \right)^n \\ &= \left(1 - \int_X \varphi(x) d\tau(x) \right)^{-1}. \end{aligned}$$

Identity (6.51) permits to formulate the conditions 1), 2) from Theorem 6.9 in the terms of the measure τ.

7. The one-dimensional case

The theory developed in the previous sections has a more simple and clear form in the case of a single selfadjoint operator $A(\varphi) = A$; then an analogue of $\mathcal{D}'$ from Theorem 5.3 is $\mathbb{R}$ (it is convenient to denote the corresponding points of $\mathbb{R}$ by λ instead of ω).

So, we have tree sets: the space X which consists of one point x (i.e., $X = \{x\}$), numbers $\varphi \in \mathbb{R}$ (every such number φ can be understood as the function $\varphi(x) := \varphi$, $x \in X$) and the set of numbers $\xi \in \mathbb{N}_0$.

Then the role of the Fock space $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ is played by the space l_{fin} of all finite sequences $f = (f_0, \ldots, f_n, 0, 0, \ldots)$ of complex numbers $f_n \in \mathbb{C}$ with coordinate-wise convergence. The positive functional s is a functional on l_{fin}.

It is clear that now we consider only one point configurations, therefore we should regard the multiple configurations, i.e., $\Gamma^{(n)}$ to be zero-dimensional and equal to $\{n\}$; therefore $\Gamma_0 = \mathbb{N}_0$. The space Γ of "infinite configurations" is also equal to $\mathbb{N}_0$. Of course, this is a not complete analogy, since we can only construct multiple configurations.

The subcharacters (6.16) are equal to

$$(7.1) \qquad \chi_{\varphi,\mathrm{sub;k}} = (1, \varphi, \varphi^2, \ldots, \varphi^k, 0, 0, \ldots), \quad \varphi \in \mathbb{R}, \quad k \in \mathbb{N}_0.$$

It is easy to show that their linear spans (with different φ) are dense in the space l_{fin} (an analog to Lemma 6.5).

Let us look at the analog of the Kondratiev–Kuna convolution $\star$.

Consider the so-called Newton (binomial) polynomials

$$(7.2) \qquad \mathbb{R} \ni \lambda \mapsto (\lambda)_n := \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda - 1) \cdots (\lambda - n + 1), & \text{if } n \in \mathbb{N}. \end{cases}$$

The generating function of $(\lambda)_n$ has the form

$$(1 + \varphi)^\lambda = e^{\lambda \log(1+\varphi)} = \sum_{n=0}^\infty \frac{\varphi^n}{n!} (\lambda)_n, \quad \varphi > -1.$$

For all $n \in \mathbb{N}_0$ we set

$$(7.3) \qquad P_n(\lambda) := \frac{1}{n!} (\lambda)_n.$$

It is clear that the family $(P_n)_{n=0}^\infty$ is a basis in the space $\mathbb{C}[\lambda]$ of all complex-valued polynomials $F : \mathbb{R} \to \mathbb{C}$, and the mapping

$$(7.4) \qquad I : l_{\mathrm{fin}} \to \mathbb{C}[\lambda], \quad f = (f_n)_{n=0}^\infty \mapsto (If)(\lambda) := \sum_{n=0}^\infty f_n P_n(\lambda)$$

is a bijection between l_{fin} and $\mathbb{C}[\lambda]$. Note that the inverse mapping I^{-1} exists. This mapping is an analog of the Fourier transform in Theorem 6.6.

The restriction of the function If to $\mathbb{N}_0$ gives the mapping (an analog of the K-transform)
$$(7.5)$$
$$K : l_{\mathrm{fin}} \to \mathbb{C}^\infty, \quad f = (f_k)_{k=0}^\infty \mapsto (Kf)(n) := (If)(n) = \sum_{k=0}^n f_k \frac{n!}{k!(n-k)!}, \quad n \in \mathbb{N}_0.$$

An analog of Lemma 6.3 is the evident identity

$$(7.6) \qquad (If)(n) = (Kf)(n), \quad n \in \mathbb{N}_0, \quad f \in l_{\mathrm{fin}}.$$

Define an analog of $\star$-convolution on the space l_{fin} by setting

$$(7.7) \qquad f \star g := I^{-1}(If \cdot Ig), \quad f, g \in l_{\mathrm{fin}}.$$

From the definition it immediately follows that the vector $f \star g = ((f \star g)_k)_{k=0}^\infty$ is uniquely defined by the identity

$$(7.8) \qquad (If)(\lambda)(Ig)(\lambda) = \sum_{n=0}^\infty f_n P_n(\lambda) \sum_{n=0}^\infty g_n P_n(\lambda) = \sum_{n=0}^\infty (f \star g)_n P_n(\lambda), \quad \lambda \in \mathbb{R}.$$

It follows from [35], Theorem 3.2, that

$$(f \star g)_n = \sum_{i+j+k=n} \frac{1}{i!k!j!} f_{i+j} g_{j+k}$$

for all $f = (f_n)_{n=0}^\infty$, $g = (g_n)_{n=0}^\infty \in l_{\text{fin}}$ and $n \in \mathbb{N}_0$.

It is convenient now to pass to the corresponding generalized moment problem.

A sequence $s = (s_n)_{n=0}^\infty$ of complex numbers $s_n \in \mathbb{C}$ we will called a *a generalized moment sequences* if there exists a non-negative Borel measure σ on $\mathbb{R}$ such that

$$(7.9) \qquad s_n = \int_{\mathbb{R}} P_n(\lambda)\, d\sigma(\lambda) = \frac{1}{n!} \int_{\mathbb{R}} (\lambda)_n d\sigma(\lambda), \quad n \in \mathbb{N}_0.$$

A solution of this problem is given by the following theorem.

Theorem 7.1. *A sequence $s = (s_n)_{n=0}^\infty$ is a moment one if and only if s is $\star$-positive (more exactly, non-negative), that is,*

$$(7.10) \qquad s(f \star \bar{f}) = \sum_{n=0}^\infty s_n (f \star \bar{f})_n \geq 0, \quad f \in l_{\text{fin}}.$$

A method of proving this result is similar to the considerations of [7], [35] and is based on the theory of a generalized eigenfunction expansion. In the case of the classical moment problem this method was first proposed in [4], Ch. 8. This method is a simplest variant of arguments which we have used in the proof of Theorem 5.3.

Proof. The necessity of condition (7.10) is trivial. Indeed, using (7.8) we get

$$s(f \star \bar{f}) = \sum_{n=0}^\infty s_n (f \star \bar{f})_n = \int_{\mathbb{R}} \sum_{n=0}^\infty s_n (f \star \bar{f})_n P_n(\lambda) d\sigma(\lambda)$$

$$= \int_{\mathbb{R}} (If)(\lambda)(I\bar{f})(\lambda) d\sigma(\lambda) = \int_{\mathbb{R}} |(If)(\lambda)|^2 d\sigma(\lambda) \geq 0, \quad f \in l_{\text{fin}}.$$

For the proof of the sufficiency of condition (7.10), we will apply the theory of generalized eigenfunction expansion to a certain self-adjoint operator connected to our moment problem.

Let a sequence $s = (s_n)_{n=0}^\infty \in \mathbb{C}^\infty$ be a positive, that is (7.10) holds. Using this sequence and the convolution $\star$ we construct, in a standard way, a Hilbert space H_s. Namely, we define H_s to be the Hilbert space associated with the quasiscalar product

$$(7.11) \qquad (f, g)_{H_s} := s(f \star \bar{g}), \quad f, g \in l_{\text{fin}}.$$

For the construction of H_s, firstly it is necessary to pass from l_{fin} to the factor space $\tilde{l}_{\text{fin}} := l_{\text{fin}}/\{f \in l_{\text{fin}} \,|\, (f, f)_{H_s} = 0\}$ and then to take the completion of $\tilde{l}_{\text{fin}}$. For simplicity we will suppose that $\tilde{l}_{\text{fin}} \equiv l_{\text{fin}}$, i.e., $(f, f)_{H_s} = 0$ if and only if $f = 0$.

Using (7.7) and (7.4) we define the operator

$$(7.12) \qquad J : l_{\text{fin}} \to l_{\text{fin}}, \quad Jf := I^{-1}\mathbb{J}I = \delta_1 \star f, \quad f \in l_{\text{fin}},$$

where $\delta_1 = (0, 1, 0, 0, \ldots)$, I is defined by formula (7.4) and $\mathbb{J}$ is the operator of multiplication by λ in the space $\mathbb{C}[\lambda]$, i.e.,

$$(\mathbb{J}F)(\lambda) := P_1(\lambda)F(\lambda) = \lambda F(\lambda), \quad F \in \mathbb{C}[\lambda].$$

The operator $J : l_{\text{fin}} \to l_{\text{fin}}$ is Hermitian in the Hilbert space H_s

$$(Jf, g)_{H_s} = s(\delta_1 \star f \star \bar{g}) = s(f \star \overline{\delta_1 \star g}) = (f, Jg)_{H_s}, \quad f, g \in l_{\text{fin}},$$

and, moreover, it is real (i.e., $\overline{Jf} = J\bar{f}$) with respect to the involution

$$(7.13) \qquad l_{\text{fin}} \ni f = (f_n)_{n=0}^\infty \mapsto \bar{f} := (\bar{f}_n)_{n=0}^\infty \in l_{\text{fin}},$$

where $\bar{f}_n$ denotes the complex conjugation. Therefore, by a theorem of von Neumann J has self-adjoint extensions.

Denote by A a certain self-adjoint extension of J on H_s. We will apply the projection spectral theorem to this operator. Consider the rigging

$$(7.14) \qquad (l^2(p))'_{H_s} \supset H_s \supset l^2(p) \supset l_{\mathrm{fin}},$$

where $(l^2(p))'_{H_s}$ is the negative space with respect to the positive space $l^2(p)$ and the zero space H_s. The space l_{fin} is provided with uniform finite coordinate-wise convergence, i.e., the sequence $\{f^{(j)}, j \in \mathbb{N}\} \subset l_{\mathrm{fin}}$ converges to $f \in l_{\mathrm{fin}}$ if and only if there exists $N \in \mathbb{N}$ such that $f_n^{(j)} = 0$ for all $n > N$, $j \in \mathbb{N}$ and $f_n^{(j)} \to f_n$ as $j \to \infty$ for all $n \in \mathbb{N}_0$.

It follows, for example, from [7] that there exists a weight $p = (p_n)_{n=0}^{\infty}$, $p_n \geq 1$, such that the embedding $l^2(p) \hookrightarrow H_s$ is well-defined and quasinuclear. In what follows we fix a weight $p = (p_n)_{n=0}^{\infty}$, $p_n \geq 1$, with such a property. It is clear that the operator A is standardly connected with chain (7.14). Let us show that the vector $\Omega = \delta_0 = (1, 0, 0, \ldots) \in l_{\mathrm{fin}}$ is a strong cyclic vector for A.

To this end, it suffices to show that $\mathrm{span}\{A^n\Omega \mid n \in \mathbb{N}_0\} = l_{\mathrm{fin}}$. But this is evidently true, since $I : l_{\mathrm{fin}} \to \mathbb{C}[\lambda]$ is a bijection, $\mathrm{span}\{\lambda^n \mid n \in \mathbb{N}_0\} = \mathbb{C}[\lambda]$ and by (7.12)

$$A^n\Omega = J^n\delta_0 = I^{-1}(\lambda^n), \quad n \in \mathbb{N}_0.$$

So, the operator A satisfies all assumptions of the projection spectral theorem. Let ρ be the corresponding spectral measure of A and $\xi(\lambda) \in (l^2(p))'_{H_s}$ be the generalized eigenvector of A with an eigenvalue $\lambda \in \mathbb{R}$, i.e.,

$$(7.15) \qquad \langle \xi(\lambda), Af \rangle_{H_s} = \lambda \langle \xi(\lambda), f \rangle_{H_s}, \quad f \in l_{\mathrm{fin}}.$$

Then the mapping

$$(7.16) \qquad H_s \supset l_{\mathrm{fin}} \ni f \mapsto (I_Af)(\cdot) := \langle f, \xi(\cdot) \rangle_{H_\tau} \in L^2(\mathbb{R}, d\rho(\lambda))$$

is isometric.

To prove (7.9), it suffices to check that

$$(7.17) \qquad (I_Af)(\lambda) = (If)(\lambda) = \sum_{n=0}^{\infty} f_n P_n(\lambda), \quad f \in l_{\mathrm{fin}},$$

for ρ-almost all $\lambda \in \mathbb{R}$.

Indeed, suppose that (7.17) takes place. Since I_A is an isometric mapping, we have

$$(7.18) \qquad \begin{aligned} (f, g)_{H_s} &= \int_{\mathbb{R}} (I_Af)(\lambda)\overline{(I_Ag)(\lambda)} \, d\rho(\lambda) \\ &= \int_{\mathbb{R}} (If)(\lambda)\overline{(Ig)(\lambda)} \, d\rho(\lambda), \quad f, g \in l_{\mathrm{fin}}. \end{aligned}$$

Therefore, taking into account the identities

$$s_n = s(\delta_n) = s(\delta_n \star \delta_0) = (\delta_n, \delta_0)_{H_s},$$

$$(I\delta_n)(\lambda) = P_n(\lambda), \quad \delta_n := (\underbrace{0, \ldots, 0}_{n \text{ times}}, 1, 0, 0, \ldots),$$

we get

$$s_n = (\delta_n, \delta_0)_{H_s} = \int_{\mathbb{R}} P_n(\lambda) \, d\rho(\lambda), \quad n \in \mathbb{N}_0,$$

i.e., representation (7.9) with measure $\sigma = \rho$.

Let us check (7.17). According to Lemma 3.7, there exists a uniquely determined unitary operator $U : (l^2(p))'_{H_s} \to l^2(p^{-1})$ such that

$$\langle U\eta, g \rangle_{l^2} = \langle \eta, g \rangle_{H_s}, \quad \eta \in (l^2(p))'_{H_s}, \quad g \in l^2(p).$$

Therefore, it suffices to show that the generalized eigenvector $\xi(\lambda)$ has the property

$$(U\xi)(\lambda) = P(\lambda) := (P_n(\lambda))_{n=0}^{\infty}, \quad \lambda \in \mathbb{R},$$

or, equivalently,

$$\langle P(\lambda), Af \rangle_{l^2} = \lambda \langle P(\lambda), f \rangle_{l^2}, \quad \lambda \in \mathbb{R}, \quad f \in l_{\text{fin}}.$$

But the latter equality takes place, since, on the one hand,

$$\lambda \langle P(\lambda), f \rangle_{l^2} = \lambda \sum_{n=0}^{\infty} f_n P_n(\lambda) = \lambda \cdot (If)(\lambda).$$

On the other hand, using (7.8) and taking into account that $Af = \delta_1 \star f$ we get

$$\langle P(\lambda), Af \rangle_{l^2} = \langle P(\lambda), \delta_1 \star f \rangle_{l^2} = \sum_{n=0}^{\infty} (\delta_1 \star f)_n P_n(\lambda)$$
$$= (I\delta_1)(\lambda) \cdot (If)(\lambda) = \lambda \cdot (I_P f)(\lambda).$$

Thus, Theorem 7.1 is proved. $\qquad\square$

Remark 7.2. It can be showed that the polynomials $(\lambda)_n$ obey the recurrence relation

$$\lambda(\lambda)_n = (\lambda)_{n+1} + n(\lambda)_n.$$

Therefore the polynomials $P_n(\lambda)$ (see (7.3)) obey the recurrence relation

$$\lambda P_n(\lambda) = (n+1)P_{n+1}(\lambda) + nP_n(\lambda).$$

Hence the operator $J : l_{\text{fin}} \to l_{\text{fin}}$, $Jf := \delta_1 \star f$, has the following matrix representation:

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 2 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 3 & 0 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \end{pmatrix}.$$

Remark 7.3. It is easy to give an example of a sequence $(s_n)_{n=0}^{\infty}$ that admits representation (7.10) with a measure σ such that $\sigma(\mathbb{N}_0) > 0$ and $\sigma(\mathbb{R} \setminus \mathbb{N}_0) > 0$.

For example, let $\sigma := \mu_\pi + \mu_g$, where μ_π and μ_g be a Poisson and a Gaussian measures on $\mathbb{R}$, respectively. Recall that for any Borel set $\alpha \subset \mathbb{R}$

$$\mu_\pi(\alpha) := \frac{1}{e} \sum_{n \in \mathbb{N}_0} \frac{1}{n!} \kappa_\alpha(n)$$

(κ_α is the indicator function of a set α) and

$$\mu_g(\alpha) := \frac{1}{\sqrt{2\pi}} \int_\alpha e^{-x^2/2} dx.$$

It easy to see that $\sigma(\mathbb{N}_0) > 0$, $\sigma(\mathbb{R} \setminus \mathbb{N}_0) > 0$ and

$$s_n := \int_{\mathbb{R}} P_n(\lambda) \, d\sigma(\lambda) = \frac{1}{n!} \int_{\mathbb{R}} (\lambda)_n d\sigma(\lambda) < \infty, \quad n \in \mathbb{N}_0.$$

Let us now examine Theorem 6.6. Recall that in the one-dimensional case, the set $\Gamma = \mathbb{N}_0$ and $\mathcal{D}' = \mathbb{R}$. Therefore Remark 7.3 allows to proceed as follows. In representation (7.9) of moments s_n we have the spectral measure $\rho = \sigma$ from Theorem 7.1 which, for the one-dimensional case, is in fact the measure ρ from Theorem 6.6. Therefore the conditions $\rho(\mathbb{N}_0) > 0$ and $\rho(\mathbb{R} \setminus \mathbb{N}_0) > 0$ are impossible: according to Theorem 6.6 the condition (6.19) gives that the set $\Gamma = \mathbb{N}_0$ is the set of full spectral measure.

But actually, there is no contradiction; in this case the space X consists only of one point, and, therefore, infinite configurations do not exist. Therefore we must have the results only *of type* of Theorem 6.6.

The place in the proof of Theorem 6.6, which can not be overcame, as far as Theorem 7.1 is concerned, is the following. The subcharacter $\chi_{\varphi,\text{sub}}$ now has the form (7.1), therefore an analog of the set $\Gamma(m, \Lambda)$, (6.26), is $\{0, \dots, m\}$. An analog of the operator K is (7.5), it has an inverse, K^{-1}, on the space l_{fin}. This inverse operator can not be interpreted as the inverse operator $\tilde{I}^{-1} : L^2(\mathbb{R}, d\rho(\lambda)) \to \mathcal{H}_s$, since inequality (6.25) in our situation is absent. Thus, we cannot prove an analog of Theorem 6.6 in the one-dimensional case.

8. Bogoliubov functionals and their representation

We will now introduce Bogoliubov functionals. For their introduction and investigation, it is useful to present some point of view on the classical moment problem. We will use convenient for us notations.

8.1. **Introduction.** We have, as in Section 7, tree sets: the space X, which consists of a single point x, points $\varphi \in \mathbb{R}$ (every such a number φ can be understood as the function $\varphi(x) := \varphi$, $x \in X$) and a set of numbers $\xi \in \mathbb{N}_0$ (it more convenient for us to write ξ instead of n). Consider the function that enters the moment representation

$$(8.1) \qquad\qquad \mathbb{R} \times \mathbb{N}_0 \ni (\varphi, \xi) \mapsto \varphi^\xi =: \chi_\varphi(\xi).$$

The main question in the classical moment problem is to find conditions so that the sequence of numbers $\mathbb{N}_0 \ni \xi \mapsto s_\xi =: s(\xi) \in \mathbb{R}$ can be represent in the form

$$(8.2) \qquad s(\xi) = \int_{\mathbb{R}} \varphi^\xi d\sigma(\varphi) = \int_{\mathbb{R}} \chi_\varphi(\xi)\, d\sigma(\varphi), \quad \xi \in \mathbb{N}_0,$$

where σ is a Borel measure on space $\mathbb{R}$.

For obtaining an answer, we need to introduce, on the sequences of numbers $\mathbb{N}_0 \ni \xi \mapsto f(\xi) \in \mathbb{C}$, the convolution

$$(8.3) \qquad (f * g)(\xi) := \sum_{\xi' + \xi'' = \xi} f(\xi')g(\xi''), \quad \xi \in \mathbb{N}_0.$$

The classical fact is the following: the representation (8.2) takes place if and only if $s(\xi)$ generates, on finite sequences $f(\xi)$, the functional $s(f) := \sum_{\xi=0}^{\infty} s(\xi)f(\xi)$, which is non-negative with respect to $*$, i.e.,

$$(8.4) \qquad s(f * \bar{f}) = \sum_{\xi \in \mathbb{N}_0} s(\xi)(f * \bar{f})(\xi) \geq 0$$

for arbitrary such f.

Here, it is fundamental that functions (8.1) possess the following property with respect to the convolution $*$: $\forall \varphi \in \mathbb{R}$

$$(8.5) \qquad (\chi_\varphi * \chi_\varphi)(\xi) = (\xi + 1)\chi_\varphi(\xi), \quad \xi \in \mathbb{N}_0.$$

Note that it is easy to rewrite condition (8.4) in the following form: for an arbitrary finite sequence $(f_n)_{n=0}^{\infty}$, $f_n \in \mathbb{C}$, we have

$$\sum_{j,k=0}^{\infty} s(j + k) f_j \bar{f}_k \geq 0.$$

Note that representation (8.2) can be obtained from the projection spectral theorem for a single selfadjoint operator (the shift operator connected with (8.3)) acting on the Hilbert space constructed by means of convolution (8.3) and positive functional (8.4) (see, e.g., [7, 1, 4]).

The introduction of Bogoliubov functionals and the obtaining for them a representation of type (8.4) is similar but considerably more difficult. The main essential difference is the absence of a semigroup structure and that, instead of a single operator, we need to

consider now a family of commuting selfadjoint operators indexed by real-valued functions $\varphi \in C_{\text{fin}}^{\infty}(X) = \mathcal{D}$.

So, we now the simple sets $X = \{x\}$, $\mathbb{R} \ni \varphi$, $\mathbb{N}_0 \ni \xi$ have to be replaced with the following four sets (see (2.2), (2.4)):

$$X \ni x, \quad C_{\text{fin}}^{\infty}(X) = \mathcal{D} \ni \varphi, \quad \Gamma_0 \ni \xi; \quad \Gamma \ni \gamma.$$

Instead of functions (8.1), we have now the functions (see (6.13))

$$(8.6) \qquad \mathcal{D} \times \Gamma \ni (\varphi, \gamma) \mapsto (K\chi_{\varphi})(\gamma) \in \mathbb{R}, \quad \chi_{\varphi}(\xi) = \prod_{x \in \xi} \varphi(x), \quad \chi_{\varphi}(\varnothing) = 1,$$

where the operator K is defined by (2.7), (6.14).

The Bogoliubov functional is some nonlinear functional $\mathcal{D} \ni \varphi \mapsto B(\varphi) \in \mathbb{R}$ admitting a representation of type (8.2)

$$(8.7) \qquad B(\varphi) = \int_{\Gamma} (K\chi_{\varphi})(\gamma) d\sigma(\gamma), \quad \varphi \in \mathcal{D},$$

where σ is some Borel measure on the space Γ.

It is easy to see that representation (8.7) is, in some sense, similar to (8.2). Namely, functions (8.1) are of the "character" type (8.5) but, in moment problem, we consider the product $\varphi^{\xi} = (\varphi(x))^{\xi}$, $x \in X = \{x\}$, only in one fixed point x (thus we have a situation of "multiple configuration" type, see [29, 30, 31, 10, 8, 13]). In representation (8.2) we integrate the characters φ^{ξ} on "functions" φ. *In representation (8.7) we have invertible operator K (instead of the unit operator in (8.2)) that maps functions on Γ_0 onto functions on Γ and integrate over "points" γ. Thus the presence of K changes the representation of type (8.2).*

The condition on the function $B(\varphi)$, which gives representation (8.7), can be obtained similarly to the classical moment problem.

Now, instead of the classical convolution (8.3), it is necessary to use Kondratiev–Kuna convolution $\star$ on the functions $\Gamma_0 \ni \xi \mapsto f(\xi) \in \mathbb{C}$ (see (2.6)),

$$(8.8) \qquad (f \star g)(\xi) = \sum_{\xi' \cup \xi'' = \xi} f(\xi')g(\xi''), \quad \xi \in \Gamma_0.$$

Formally, (8.8) is similar to (8.3) but, instead of the simple semigroup $\mathbb{N}_0$ in (8.3), we have the complicated space Γ_0 in (8.8) (without a group operation).

To understand the way of solving the problem of representation (8.7), it is convenient to pass at first, from representation (8.2) for the moment problem, to some a little more general problem. Namely, if we have (8.2) then we can write, for an arbitrary finite sequence $f = (f(\xi))_{\xi \in \mathbb{N}_0}$, the representation

$$(8.9) \qquad s(f) := \sum_{\xi=0}^{\infty} s(\xi)f(\xi) = \int_{\mathbb{R}} P_f(\varphi) \, d\sigma(\varphi), \quad P_f(\varphi) = \sum_{\xi=0}^{\infty} f(\xi)\varphi^{\xi}.$$

Of course, for the moment problem representations (8.2) and (8.9) are tautological. But for Bogoliubov functionals the connection between (8.7) and an analog of (8.9) is not so trivial.

So, similarly to the moment problem, we introduce at first, on finite "sequences" $f=(f(\xi))_{\xi \in \Gamma_0}$ (i.e., on functions $\Gamma_0 \ni \xi \mapsto f(\xi) \in \mathbb{C}$), a positive functional of type (8.4)

$$f \mapsto s(f), \quad s(f \star \bar{f}) \geq 0.$$

Then we will prove (see Subsection 7.2) that there is a representation of type (8.9) for such a functional s. It has the form

$$(8.10) \qquad s(f) = \int_{\Gamma} (Kf)(\gamma) \, d\sigma(\gamma) = \int_{\Gamma} P_f(\gamma) \, d\sigma(\gamma), \quad P_f(\gamma) := (Kf)(\gamma).$$

The pair (8.10), (8.7) and the pair (8.9), (8.2) are similar. The connection in the last pair is trivial, but the connection between (8.10) and (8.7) is more complicated: it is necessary to find $s(f)$ using the knowledge of the "moments" $B(\varphi)$, where $\varphi \in \mathcal{D}$, and conversely. In the next subsection we will find at first representation (8.10).

8.2. **Exact calculation.** By definition, the Bogoliubov functional (corresponding to a measure σ) is nonlinear function

$$(8.11) \qquad C^{\infty}_{\text{fin}}(X) = \mathcal{D} \ni \varphi \mapsto B(\varphi) := \int_{\Gamma} \prod_{x \in \gamma} (1 + \varphi(x)) \, d\sigma(\gamma) \in \mathbb{R},$$

where σ is a probability Borel measure on the space Γ (with the topology of weak convergence in $\mathcal{D}'$, see (6.8) and (6.7)).

For us it is essential to rewrite definition (8.11) in a form connected to the one given in the previous section.

We have introduced the notion of a character χ_{φ} by formula (6.13); characters satisfy equality (6.14). Therefore definition (8.11) can be rewritten as

$$(8.12) \qquad \mathcal{D} \ni \varphi \mapsto B(\varphi) = \int_{\Gamma} (K\chi_{\varphi})(\gamma) \, d\sigma(\gamma) \in \mathbb{R}.$$

It is important to note the following. The knowledge of the function $\varphi(x)$, $x \in X$, from $\mathcal{D}$ and the corresponding character χ_{φ}, $\varphi \in \mathcal{D}$, is equivalent: if we known φ then, according to (6.13), we known χ_{φ}. Conversely, if we known $\Gamma_0 \ni \xi \mapsto \chi_{\varphi}$ then we known $\chi_{\varphi}([x_1]) = \varphi(x_1)$, $x_1 \in X$. As a result, instead of Bogoliubov functional (8.11), (8.12) we can investigate the mapping

$$(8.13) \qquad \chi_{\varphi} \mapsto b(\chi_{\varphi}) := \int_{\Gamma} (K\chi_{\varphi})(\gamma) \, d\sigma(\gamma) = B(\varphi) \in \mathbb{R}, \quad \varphi \in \mathcal{D}.$$

Investigations of them are equivalent.

To study (8.12), (8.13), we will use the results of Section 6, in particular, the Theorem 6.6. We assume that *the measure σ in (8.11) is the spectral measure ρ* of our family operators $(\bar{A}(\varphi))_{\varphi \in \mathcal{D}}$. Condition 3.5 and assumption (3.24) of positivity are assumed to be fulfilled.

We follow the proof of some results of type (6.42). So, the Fourier transform I (6.20) (after taking the closure) is a unitary operator between $\mathcal{H}_s$ and $L^2(\Gamma, d\rho(\gamma))$. Therefore we have the corresponding Parseval equality:

$$(8.14) \qquad \begin{aligned} s(f \star \bar{g}) &= (f, g)_{\mathcal{H}_s} = (If, Ig)_{L^2(\Gamma, d\rho(\gamma))} = (Kf, Kg)_{L^2(\Gamma, d\rho(\gamma))} \\ &= \int_{\Gamma} (Kf)(\gamma) \overline{(Kg)(\gamma)} \, d\rho(\gamma), \quad f, g \in \mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D}). \end{aligned}$$

Put $g = e$ in (8.14), where e is the unity of algebra $\mathcal{A}$, i.e., $e(\xi) = 1$ if $\xi = \varnothing$ and $e(\xi) = 0$ if $\xi \neq \varnothing$ (or $e = (1, 0, 0, \dots)$). According to (6.20) we get

$$(8.15) \qquad (Ke)(\gamma) = (e, P(\gamma))_{\mathcal{F}(H)} = \sum_{n=0}^{\infty} (e_n, P_n(\gamma))_{\mathcal{F}_n(H)} = P_0(\gamma) = 1, \quad \gamma \in \Gamma.$$

Formulas (8.15) and (8.14) give the *essential equality*, an expression for the functional s in terms of the spectral measure and the K-transform,

$$(8.16) \qquad s(f) = \int_{\Gamma} (Kf)(\gamma) \, d\rho(\gamma), \quad f \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

For the vector f (i.e., the function of $\xi \in \Gamma_0$) in (8.16), we can take the subcharacter $\chi_{\varphi, \text{sub}}$. Then (8.16) and (8.13) give

$$(8.17) \qquad s(\chi_{\varphi, \text{sub}}) = \int_{\Gamma} (K\chi_{\varphi, \text{sub}})(\gamma) \, d\rho(\gamma), \quad \varphi \in \mathcal{D}.$$

Note that integral (8.17) is similar to representation (8.13) of the Bogoliubov functional $B(\varphi)$. In connection with this, consider the subcharacter of the following form:

$$(8.18) \qquad \mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \ni \chi_{\varphi,\mathrm{sub}} = (1, \varphi, \ldots, \varphi^{\otimes m}, 0, 0, \ldots) =: \chi_{\varphi,\mathrm{sub};m}, \qquad \varphi \in \mathcal{D}, \quad m \in \mathbb{N}_0.$$

For each $\gamma \in \Gamma$ we have (see (6.14) and [21, 22]) that

$$(8.19) \qquad (K\chi_\varphi)(\gamma) = \lim_{m \to \infty} (K\chi_{\varphi,\mathrm{sub};m})(\gamma).$$

Let the measure σ in (8.13) be equal to our spectral measure ρ. Then under some additional conditions from (8.18), it follows that

$$(8.20) \qquad \int_\Gamma (K\chi_\varphi)(\gamma) d\rho(\gamma) = \lim_{m \to \infty} \int_\Gamma (K\chi_{\varphi,\mathrm{sub};m})(\gamma)\, d\rho(\gamma), \quad \varphi \in \mathcal{D},$$

and representation (8.17) for $\chi_{\varphi,\mathrm{sub}}$ gives representation (8.13) for the Bogoliubov functional $B(\varphi)$.

Now it is convenient to introduce the notion of Bogoliubov subfunctional. Namely, *we will say that a nonlinear functional $B_{\mathrm{sub}} : \mathcal{D} \to \mathbb{R}$ is Bogoliubov subfunctional if it has the form*

$$(8.21) \qquad B_{\mathrm{sub}}(\varphi) := \int_\Gamma (K\chi_{\varphi,\mathrm{sub}})(\gamma)\, d\sigma(\gamma), \quad \varphi \in \mathcal{D},$$

with some probability measure σ.

We can formulate now the following general result.

Theorem 8.1. *Consider the space $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, the Kondratiev–Kuna convolution $\star$ and a linear continuous functional s on $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ which is positive with respect to $\star$,*

$$s(f \star \bar{f}) > 0, \quad f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D}), \quad f \neq 0.$$

Using s, we introduce a Hilbert space $\mathcal{H}_s$, starting with the scalar product $(f,g)_{\mathcal{H}_s} = s(f \star \bar{g}), f, g \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$. Let $(A(\varphi))_{\varphi \in \mathcal{D}}$ be a family of commuting Hermitian operators $A(\varphi)f = \varphi \star f$, $f \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, in the space $\mathcal{H}_s$.

We assume that Condition 3.5 is fulfilled and, therefore, their closures $\tilde{A}(\varphi)$ form a family of commuting selfadjoint operators in $\mathcal{H}_s$. We assume also that condition (6.19) of Theorem 6.6 is satisfied, and therefore the corresponding Fourier transform has form (6.20).

As a result we have the representation

$$(8.22) \qquad s(\chi_{\varphi,\mathrm{sub}}) = \int_\Gamma (K\chi_{\varphi,\mathrm{sub}})(\gamma)\, d\rho(\gamma) = B_{\mathrm{sub}}(\varphi), \quad \varphi \in \mathcal{D},$$

where $\chi_{\varphi,\mathrm{sub}}(\xi), \xi \in \mathbb{N}_0$, is a subcharacter connected with (6.13), K the Lenard transform, ρ a spectral measure of the family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$, and $B_{\mathrm{sub}}(\varphi)$ the corresponding Bogoliubov subfunctional.

Additionally assume that $\forall \varphi \in \mathcal{D}$ the character χ_φ belongs to the space $\mathcal{H}_s$ as a limit of $\chi_{\varphi,\mathrm{sub};m}$ in $\mathcal{H}_s$ (8.18) for $m \to \infty$. Then the Bogoliubov functional corresponding to measure ρ has the representation

$$(8.23) \qquad B(\varphi) = \int_\Gamma (K\chi_\varphi)(\gamma)\, d\rho(\gamma) = \lim_{m \to \infty} B_{\mathrm{sub};m}(\varphi), \quad \varphi \in \mathcal{D},$$

where the subfunctionals $B_{\mathrm{sub};m}(\varphi)$ are given by (8.21) with $\chi_{\varphi,\mathrm{sub};m}$ as the integrand.

Proof. All statements of this theorem, save for equality (8.23), follow directly from the results of Section 6. Consider equality (8.23).

It is only necessary to explain why, in our case, we can pass to limit in (8.20). In our case, $K\chi_{\varphi,\mathrm{sub};m} = \tilde{I}\chi_{\varphi,\mathrm{sub};m}$ and $\tilde{I} : \mathcal{H}_s \to L^2(\Gamma, d\rho(\omega))$ is a unitary operator. Since $\chi_{\varphi,\mathrm{sub};m} \to \chi_\varphi$ in $\mathcal{H}_s$ by assumption, taking the limit in (8.19) can be carried out in the

space $L^2(\Gamma, d\rho(\omega))$. The spectral measure ρ is finite, therefore we can assert that (8.20) takes place. $\qquad\square$

It is useful to give some example of situation, when formula (8.23) takes place.

1. Assume that the functional s of type (3.10) satisfies, instead of condition (6.49), the following stronger condition: for every compact $\Lambda \subset X$ and every $C_\Lambda > 0$ we have

$$(8.24) \qquad \sum_{n=0}^{\infty} C_\Lambda^n \nu(\Gamma_\Lambda^{(n)}) < \infty.$$

Then (8.23) takes place. This statement easily follows from estimates of type (6.50), (6.51).

2. Note that the Lebesgue-Poisson measure ν on Γ_0 satisfies condition (8.23).

We will pass now to another main result of this Section. Let us firstly explain, using the language of the classical moment problem, the situation which we have now.

We developed some spectral theory and prove that integrals (8.2) are moments of a spectral measure. But the main result of the moment problem is: under what condition on a given sequence $(s(n))_{n=0}^{\infty}$ the representation (8.2) takes place. This measure σ depends, of course, on the values $s(n)$.

Now we have a similar characterization problem for Bogoliubov functionals: under what conditions on a given nonlinear functional $\mathcal{D} \ni \varphi \mapsto B(\varphi) \in \mathbb{R}$ we can present it in the form (8.12) with some measure σ? This measure σ depends on the scalar product in $\mathcal{H}_s$, i.e., on the functional s (the convolution $\star$ is fixed). In what way we can find s from given $B(\varphi)$?

We will investigate this problem but, at first, it is necessary to understand in what way we can construct the corresponding Bogoliubov subfunctional $B_{\mathrm{sub};m}(\varphi)$ from given $B(\varphi)$, since the former functional is an analog of moments $s(n)$.

It is easy to prove the following statement.

Lemma 8.2. *Let conditions of Theorem 6.6 be fulfilled and $\chi_\varphi \in \mathcal{H}_s$ for all $\varphi \in \mathcal{D}$. Then, for the Bogoliubov functional and a corresponding spectral measure ρ, we have*

$$(8.25) \qquad B(\varphi) = \sum_{n=0}^{\infty} \int_\Gamma \left(P_n(\gamma), \varphi^{\otimes n}\right)_{\mathcal{F}_n(H)} d\rho(\gamma), \quad \varphi \in \mathcal{D}, \quad \varphi(x) > -1, \quad x \in X.$$

Proof. Using Theorem 8.1 (equality (8.23)), Lemma 6.5, and 6.4 we get

$$B(\varphi) = \int_\Gamma (K\chi_\varphi)(\gamma)\, d\rho(\gamma) = \lim_{m\to\infty} B_{\mathrm{sub},m}(\varphi)$$

$$= \lim_{m\to\infty} \int_\Gamma (K(1, \varphi, \ldots, \varphi^{\otimes m}, 0, 0, \ldots))(\gamma)\, d\rho(\gamma)$$

$$= \lim_{m\to\infty} \int_\Gamma \sum_{n=0}^{m} \left(P_n(\gamma), \varphi^{\otimes n}\right)_{\mathcal{F}_n(H)} d\rho(\gamma) = \sum_{n=0}^{\infty} \int_\Gamma \left(P_n(\gamma), \varphi^{\otimes n}\right)_{\mathcal{F}_n(H)} d\rho(\gamma).$$

$\qquad\square$

Similarly, for Bogoliubov subfunctional, we have

$$(8.26) \qquad B_{\mathrm{sub},m}(\varphi) = \sum_{n=0}^{m} \int_\Gamma \left(P_n(\gamma), \varphi^{\otimes n}\right)_{\mathcal{F}_n(H)} d\rho(\gamma), \quad \varphi \in \mathcal{D}.$$

Let $\varphi \in \mathcal{D}$ be fixed. Consider the function $X \ni x \mapsto t\varphi(x) \in \mathbb{R}$, where $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ is sufficiently small and fixed. Then this function satisfies the conditions for which

the representations (8.25) takes place. Therefore we have

$$
\begin{aligned}
B(t,\varphi) :=& B(t\varphi) = \sum_{n=0}^{\infty} \int_{\Gamma} \left(P_n(\gamma), (t\varphi)^{\otimes n} \right)_{\mathcal{F}_n(H)} d\rho(\gamma) \\
=& \sum_{n=0}^{\infty} t^n \int_{\Gamma} \left(P_n(\gamma), \varphi^{\otimes n} \right)_{\mathcal{F}_n(H)} d\rho(\gamma), \quad \varphi \in \mathcal{D}.
\end{aligned}
$$
(8.27)

Assume that the function $(-\varepsilon,\varepsilon) \ni t \mapsto B(t\varphi) \in \mathbb{R}$ is infinitely differentiable, then using series (8.27) we get

$$
D_t^k B(t,\varphi) = \sum_{n=k}^{\infty} k! t^{n-k} \int_{\Gamma} \left(P_n(\gamma), \varphi^{\otimes n} \right)_{\mathcal{F}_n(H)} d\rho(\gamma), \quad k \in \mathbb{N}_0.
$$

Therefore for all $\varphi \in \mathcal{D}$ we have

$$
(D_t^k B(t,\varphi))(0,\varphi) = k! \int_{\Gamma} \left(P_k(\gamma), \varphi^{\otimes k} \right)_{\mathcal{F}_k(H)} d\rho(\gamma), \quad k \in \mathbb{N}_0.
$$
(8.28)

From (8.26) and (8.28) we conclude that

$$
B_{\mathrm{sub},m}(\varphi) = \sum_{n=0}^{m} \frac{1}{n!} (D_t^k B(t,\varphi))(0,\varphi), \quad \varphi \in \mathcal{D}, \quad m \in \mathbb{N}_0.
$$
(8.29)

We have proved the following result.

Theorem 8.3. *Let conditions of Theorem 6.6 be fulfilled and $\chi_\varphi \in \mathcal{H}_s$ for all $\varphi \in \mathcal{D}$. Consider the Bogoliubov functional $B(\varphi)$, $\varphi \in \mathcal{D}$ corresponding to a spectral measure ρ. Assume that, for every fixed $\varphi \in \mathcal{D}$ and some $\varepsilon > 0$, the function $(-\varepsilon,\varepsilon) \ni t \mapsto B(t\varphi) \in \mathbb{R}$ is infinitely differentiable. Then Bogoliubov subfunctionals $B_{\mathrm{sub},m}(\varphi)$ are reconstructed by formula (8.29).*

Consider some additional properties of Bogoliubov functional and subfunctional connected with functional positivity. As above, we consider these objects constructed from a spectral measure ρ using Theorem 8.1.

In Theorem 8.1 we have assumed that the every character χ_φ, $\varphi \in \mathcal{D}$, belongs to the space $\mathcal{H}_s$ since they are limits of $\chi_{\varphi,\mathrm{sub},m}$ in $\mathcal{H}_s$ (8.18) as $m \to \infty$. We will assume now that the set of all characters χ_φ, $\varphi \in \mathcal{D}$, is total in $\mathcal{H}_s$.

The definitions (8.12), (8.21) and relation (8.16) give the following expressions for the functional s on characters and subcharacters:

$$
s(\chi_\varphi) = B(\varphi), \quad s(\chi_{\varphi,\mathrm{sub};m}) = B_{\mathrm{sub};m}(\varphi), \quad \varphi \in \mathcal{D}, \quad m \in \mathbb{N}_0.
$$
(8.30)

Below we will use (8.30) and the following result.

Lemma 8.4. *For arbitrary $\varphi, \psi \in \mathcal{D}$ the following formula takes place:*

$$
(\chi_\varphi \star \chi_\psi)(\xi) = \chi_{\varphi+\psi+\varphi\psi}(\xi), \quad \xi \in \Gamma_0.
$$
(8.31)

Proof. Apply the operator K to the left-hand side of (8.31) and use (2.8) and (6.14) to get

$$
\begin{aligned}
(K(\chi_\varphi \star \chi_\psi))(\gamma) =& (K\chi_\varphi)(\gamma)(K\chi_\psi)(\gamma) = \left(\prod_{x\in\gamma}(1+\varphi(x)) \right)\left(\prod_{x\in\gamma}(1+\psi(x)) \right) \\
=& \prod_{x\in\gamma}(1 + \varphi(x) + \psi(x) + \varphi(x)\psi(x)), \quad \gamma \in \Gamma.
\end{aligned}
$$
(8.32)

Application of K to the right-hand side of (8.31) and equality (6.14) gives

$$
(K\chi_{\varphi+\psi+\varphi\psi})(\gamma) = \prod_{x\in\gamma}(1 + \varphi(x) + \psi(x) + \varphi(x)\psi(x)), \quad \gamma \in \Gamma.
$$
(8.33)

Identities (8.32), (8.33) and invertibility of the operator K (see Proposition 2.2) gives (8.31). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Note that equality (8.31) is well-known, see e.g. [28], [33, p. 127].

Consider an arbitrary finite sequence $(c_j)_{j=0}^{\infty}$ of complex numbers and a sequence $(\varphi_j)_{j=0}^{\infty}$, $\varphi_j \in \mathcal{D}$. The vectors $f = \sum_{j=0}^{\infty} c_j \chi_{\varphi_j}$ belong to $\mathcal{H}_s$ and their set is dense in $\mathcal{H}_s$. Positivity of the functional s gives

$$(8.34) \qquad s(f \star \bar{f}) = \sum_{j,k=0}^{\infty} c_j \bar{c}_k s(\chi_{\varphi_j} \star \chi_{\varphi_k}) > 0, \quad f \neq 0.$$

Note also that using the limiting procedure from (8.16) we get

$$(8.35) \qquad s(f) = \int_{\Gamma} (\tilde{I}f)(\gamma) d\rho(\gamma), \quad f \in \mathcal{H}_s.$$

Using (8.34) and (8.31) we obtain

$$(8.36) \qquad s(f \star \bar{f}) = \sum_{j,k=0}^{\infty} c_j \bar{c}_k s(\chi_{\varphi_j} \star \chi_{\varphi_k}) = \sum_{j,k=0}^{\infty} c_j \bar{c}_k s(\chi_{\varphi_j + \varphi_k + \varphi_j \varphi_k}) > 0, \quad f \neq 0.$$

This equality can be rewritten in terms of Bogoliubov functionals. Namely, using (8.35), the fact that the operators $\bar{I}$ and K coincide on the characters, and (8.12) we can write

$$(8.37) \qquad \sum_{j,k=0}^{\infty} c_j \bar{c}_k B(\varphi_j + \varphi_k + \varphi_j \varphi_k) = s(f \star \bar{f}) \geq 0$$

for arbitrary $f = \sum_{j=0}^{\infty} c_j \chi_{\varphi_j} \neq 0$, where $(c_j)_{j=0}^{\infty}$ is a finite sequence of complex numbers and $\varphi_j \in \mathcal{D}$.

So, *the Bogoliubov functionals $B(\varphi)$, $\varphi \in \mathcal{D}$, satisfy the following condition of positivity: for an arbitrary finite sequence $(c_j)_{j=0}^{\infty}$, $c_j \in \mathbb{C}$, and a sequence $(\varphi_j)_{j=0}^{\infty}$, $\varphi_j \in \mathcal{D}$, we have inequality (8.37).*

A similar construction can be repeated for Bogoliubov subfunctionals: it is necessary replace χ_{φ} with $\chi_{\varphi,\mathrm{sub}}$ in (8.34) and (8.36) and use formula (8.17) instead of (8.35). But now an inequality of type (8.37) does not hold, since the formula of type (8.31) has a more complicated form.

We can now give the following conclusion from the results connected with Theorem 8.3 and positivity.

Lemma 8.5. *Consider the situation of Theorem 8.1 in the case where, instead of a formula for the functional s, we only know the Bogoliubov functional $B(\varphi)$, $\varphi \in \mathcal{D}$. We assume that the conditions of totality of the set of characters in $\mathcal{H}_s$ and positivity (8.37) are fulfilled.*

Then we can find Bogoliubov subfunctionals $B_{\mathrm{sub}}(\varphi)$, $\varphi \in \mathcal{D}$, by formula (8.29) and calculate the functional s on linear combinations of characters and subcharacters by formula (8.30).

We can formulate now the following result.

Theorem 8.6. *Let $\mathcal{D} \ni \varphi \mapsto B(\varphi) \in \mathbb{R}$ be continuous nonlinear functional such that*

(1) *For every $\varphi \in \mathcal{D}$ the function $\mathbb{R} \ni t \mapsto B(t, \varphi) := B(t\varphi) \in \mathbb{R}$ is infinitely differentiable;*

(2) *The functional*

$$s : \mathcal{F}_{\mathrm{lin}} \to \mathbb{R}_+, \quad s(\varphi^{\otimes n}) := \frac{1}{n!}(D_t^n B(t, \varphi))(0, \varphi), \quad \varphi \in \mathcal{D}, \quad n \in \mathbb{N}_0,$$

is positive, i.e., $s(f \star \bar{f}) > 0$, $f \neq 0$.

Construct the corresponding space $\mathcal{H}_s$. Additionally we will assume that s satisfies the following three following properties:

(3) *The corresponding operators $A(\varphi)$ satisfy Condition 3.5;*

(4) *Every character χ_φ, $\varphi \in \mathcal{D}$, belongs to $\mathcal{H}_s$, i.e., the sequence $(\chi_{\varphi,\mathrm{sub};n})_{n=0}^\infty$ is fundamental in the space $\mathcal{H}_s$;*

(5) *The spectral measure ρ of the family $(\tilde{A}(\varphi))_{\varphi\in\mathcal{D}}$ is positive on Γ: $\rho(\Gamma) > 0$.*

Then $B(\varphi)$ is a Bogoliubov functional corresponding to the spectral measure ρ.

Proof. Using given objects, including the functional s, according to the results of Sections 2–6 we construct a family of selfadjoint commuting operators $(\tilde{A}(\varphi))_{\varphi\in\mathcal{D}}$, acting on the space $\mathcal{H}_s$. Using Theorem 8.3 we can conclude that the corresponding Bogoliubov subfunctionals and functionals exist and, for them, we have the representations (8.22) and (8.23). Lemma 8.5 shows that for the found subfunctionals we get the formulas such as (8.22), i.e., an a priori given functional $B(\varphi)$ is equal to the constructed Bogoliubov functional. □

It is necessary to make some remarks to this theorem.

Remark 8.7. As a result, we have introduced Bogoliubov functionals, using their properties, as moments in the classical moment problem. In particular, we have imposed on them the positivity condition in the form (8.37). But it seams that, unlike the moment theory, fulfillment of only inequality (8.37) is not sufficient for positivity of the functional s and, therefore, for a complete characterization of Bogoliubov functionals.

Remark 8.8. It remains a problem to give a more constructive form of the above conditions 3) – 5), to formulate them in terms of $B(\varphi)$ (similar to the moment problem).

References

1. N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner, New York, 1965. (Russian edition: Fizmatgiz, Moscow, 1961)

2. Yu. M. Berezanskii, *Generalizations of Bochner's theorem to expansions according to eigenfunctions of partial differential operators*, Dokl. Acad. Nauk SSSR **110** (1956), no. 6, 893–896. (Russian)

3. Yu. M. Berezansky, *Generalizations of multidimensional theorem of Bochner*, Dokl. Acad. Nauk SSSR **135** (1961), no. 5, 1011–1014. (Russian)

4. Yu. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)

5. Yu. M. Berezanskiĭ, *Selfadjoint Operators in Spaces of Functions of Infinitely Many Variables*, Amer. Math. Soc., Providence, RI, 1986. (Russian edition: Naukova Dumka, Kiev, 1978)

6. Yu. M. Berezansky, *Commutative Jacobi fields in Fock space*, Integr. Equ. Oper. Theory **30** (1998), 163–190.

7. Yu. M. Berezansky, *Some generalizations of the classical moment problem*, Integr. Equ. Oper. Theory **44** (2002), 255–289.

8. Yu. M. Berezansky, *The generalized moment problem associated with correlation measures*, Funct. Anal. Appl. **37** (2003), no. 4, 311–315.

9. Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis*, Vols. 1, 2, Kluwer Academic Publishers, Dordrecht—Boston—London, 1995. (Russian edition: Naukova Dumka, Kiev, 1988)

10. Yu. M. Berezansky, Yu. G. Kondratiev, T. Kuna, and E. Lytvynov, *On a spectral representation for correlation measures in configuration space analysis*, Methods Funct. Anal. Topology **5** (1999), no. 4, 87–100.

11. Yu. M. Berezansky, V. O. Livinsky, and E. W. Lytvynov, *A generalization of Gaussian white noise analysis*, Methods Funct. Anal. Topology **1** (1995), no. 1, 28–55.

12. Yu. M. Berezansky and D. A. Mierzejewski, *The construction of the chaotic representation for the Gamma field*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6** (2003), no. 1, 33–56.

13. Yu. M. Berezansky and D. A. Mierzejewski, *The investigation of a generalized moment problem associated with correlation measures*, Methods Funct. Anal. Topology **13** (2007), no. 2, 124–151.

14. Yu. M. Berezansky and M. E. Dudkin, *On the complex moment problem*, Math. Nachr. **280** (2007), no. 1–2, 60–73.

15. Yu. M. Berezansky and M. E. Dudkin, *The complex moment problem and direct and inverse spectal problems for the block Jacobi type bounded normal matrices*, Methods Funct. Anal. Topology **12** (2006), no. 1, 1–31.

16. Yu. M. Berezansky, Z. G. Sheftel, G. F. Us, *Functional Analysis*, Vols. 1, 2, Birkhäuser Verlag, Basel–Boston–Berlin, 1996; 3rd ed., Institute of Mathematics NAS of Ukraine, Kyiv, 2010. (Russian edition: Vyshcha Shkola, Kiev, 1990)

17. Yu. M. Berezans'kyi and V. A. Tesko, *Spaces of test and generalized functions related to generalized translation operators*, Ukrain. Mat. Zh. **55** (2003), no. 12, 1587–1657. (Ukrainian); English transl. Ukrainian Math. J. **55** (2003), no. 12, 1907–1979.

18. N. N. Bogolyubov, *Problems of Dynamical Theory in Statistical Physics*, Gostekhizdat, Moscow, 1946. (Russian); English transl. Studies in Statistical Mechanics (J. de Boez and G. E. Uhlenbeck, eds.), Vol. 1, North-Holland, Amsterdam, 1962, pp. 1–118.

19. D. Finkelshtein, *On convolutions on configuration spaces. I. Spaces of finite configurations*, Ukrain. Mat. Zh. **64** (2012), no. 11, 1547–1567. (Ukrainian); English transl. Ukrainian Math. J. **64** (2013), no. 11, 1919–1944.

20. I. Gelfand, D. Raikov, G. Shilov, *Commutative Normed Rings*, Chelsea Publishing Company, 1964. (Russian edition: Fizmatgiz, Moscow, 1960)

21. Yu. G. Kondratiev and T. Kuna, *Harmonic Analysis on Configuration Space,* I*: General Theory*, SFB 256 Preprint no. 626, University of Bonn, Bonn, 1999.

22. Yu. G. Kondratiev and T. Kuna, *Harmonic analysis on configuration space,* I*: General theory*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5** (2002), no. 2, 201–233.

23. Yu. G. Kondratiev, T. Kuna and M. J. Oliveira, *On the relation between Poissonian white noise analysis and harmonic analysis on configuration spaces*, J. Funct. Anal. **213** (2004), no. 1, 1–30.

24. Yu. G. Kondratiev, T. Kuna, and M. J. Oliveira, *Holomorphic Bogoliubov functionals for interacting particle systems in continuum*, J. Funct. Anal. **238** (2006), no. 2, 375–404.

25. A. G. Kostyuchenko and B. S. Mityagin, *Positive-definite functionals on the nuclear spaces*, Trudy Moskov. Mat. Obshch. **9** (1960), 283–316. (Russian); English transl. Amer. Math. Soc. Transl. Ser. 2 **93** (1970), 1–44.

26. M. G. Krein, *On a general method of decomposition of positive definite kernels on elementary products*, Dokl. Akad. Nauk SSSR **53** (1946), no. 1, 3–6. (Russian)

27. M. G. Krein, *On Hermitian operators with directed functionals*, Akad. Nauk Ukrain. RSR, Zbirnyk Prac' Inst. Mat. (1948), no. 10, 83–106. (Ukrainian)

28. T. Kuna, *Studies in configuration space analysis and applications*, Bonner Mathematische Schriften 324, University of Bonn, PhD Thesis, Bonn, 1999.

29. A. Lenard, *Correlation functions and the uniqueness of the state in classical statistical mechanics*, Comm. Math. Phys. **30** (1972), 35–44.

30. A. Lenard, *States of classical statistical mechanical systems of infinitely many particles,* I, Arch. Rational Mech. Anal. **59** (1975), 219–239.

31. A. Lenard, *States of classical statistical mechanical systems of infinitely many particles,* II, Arch. Rational Mech. Anal. **59** (1975), 241–256.

32. G. I. Nazin, *Method of the generating functional*, J. Soviet Math. **31** (1985), 2859–2886.

33. M. J. Oliveira, *Configuration space analysis and Poissonian white noise analysis*, University of Lisbon, PhD Thesis, Lisbon, 2002.

34. D. Ruelle, *Cluster property of the correlations functions of classical gases*, Rev. Modern Phys. **36** (1964), 580–584.

35. Volodymyr Tesko, *One generalization of the classical moment problem*, Methods Funct. Anal. Topology **17** (2011), no. 4, 356-380.

36. V. S. Vladimirov, *Generalized Functions in Mathematical Physics*, Nauka, Moscow, 1976. (Russian)

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