A STOCHASTIC INTEGRAL OF OPERATOR-VALUED FUNCTIONS

VOLODYMYR TESKO

To Professor M. L. Gorbachuk on the occasion of his 70th birthday.

Abstract. In this note we define and study a Hilbert space-valued stochastic integral of operator-valued functions with respect to Hilbert space-valued measures. We show that this integral generalizes the classical Itô stochastic integral of adapted processes with respect to normal martingales and the Itô integral in a Fock space.

1. Introduction

Here and subsequently, we fix a real number \( T > 0 \). Let \( \mathcal{H} \) be a complex Hilbert space, \( M \) be a fixed vector from \( \mathcal{H} \) and \([0, T] \ni t \mapsto E_t \) be a resolution of identity in \( \mathcal{H} \). Consider the \( \mathcal{H} \)-valued function \((t) = E_t M \in \mathcal{H}\). In this paper we construct and study an integral

\[
\int_{[0, T]} A(t) dM_t
\]

for a certain class of operator-valued functions \([0, T] \ni t \mapsto A(t)\) whose values are linear operators in the space \( \mathcal{H} \). We define such an integral as an element of the Hilbert space \( \mathcal{H} \) and call it a Hilbert space-valued stochastic integral (or \( \mathcal{H} \)-stochastic integral). By analogy with the classical integration theory we first define integral (1) for a certain class of simple operator-valued functions and then extend this definition to a wider class.

We illustrate our abstract constructions with a few examples. Thus, we show that the classical Itô stochastic integral is a particular case of the \( \mathcal{H} \)-stochastic integral. Namely, let \( \mathcal{H} := L^2(\Omega, \mathcal{A}, P) \) be a space of square integrable functions on a complete probability space \((\Omega, \mathcal{A}, P)\), \( \{\mathcal{A}_t\}_{t \in [0, T]} \) be a filtration satisfying the usual conditions and \( \{N_t\}_{t \in [0, T]} \) be a normal martingale on \((\Omega, \mathcal{A}, P)\) with respect to \( \{\mathcal{A}_t\}_{t \in [0, T]} \), i.e.,

\[
\{N_t\}_{t \in [0, T]} \quad \text{and} \quad \{N_t^2 - t\}_{t \in [0, T]}
\]

are martingales for \( \{\mathcal{A}_t\}_{t \in [0, T]} \). It follows from the properties of martingales that

\[
N_t = E[N_T | \mathcal{A}_t], \quad t \in [0, T],
\]

where \( E[\cdot | \mathcal{A}_t] \) is a conditional expectation with respect to the \( \sigma \)-algebra \( \mathcal{A}_t \). It is well known that \( E[\cdot | \mathcal{A}_t] \) is the orthogonal projector in the space \( L^2(\Omega, \mathcal{A}, P) \) onto its subspace \( L^2(\Omega, \mathcal{A}_t, P) \) and, moreover, the corresponding projector-valued function \( \mathbb{R}_+ \ni t \mapsto E_t := E[\cdot | \mathcal{A}_t] \) is a resolution of identity in \( L^2(\Omega, \mathcal{A}, P) \), see e.g. [13, 3, 4, 12, 7]. In this way the normal martingale \( \{N_t\}_{t \in [0, T]} \) can be interpreted as an abstract martingale, i.e.,

\[
[0, T] \ni t \mapsto N_t = E[N_T | \mathcal{A}_t] = E_t N_T \in \mathcal{H}.
\]

Hence, in the space \( L^2(\Omega, \mathcal{A}, P) \) we can construct the \( \mathcal{H} \)-stochastic integral with respect to the normal martingale \( N_t \). Let \( F \in L^2([0, T] \times \Omega, dt \times P) \) be a square integrable

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In this paper we prove that the \( H \)-stochastic integral of \([0, T] \ni t \mapsto A_F(t) \) whose values are operators \( A_F(t) \) of multiplication by the function \( F(t) = F(t, \cdot) \in L^2(\Omega, \mathcal{A}, P) \) in the space \( L^2(\Omega, \mathcal{A}, P) \),

\[
L^2(\Omega, \mathcal{A}, P) \ni \text{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, \mathcal{A}, P).
\]

In this paper we prove that the \( H \)-stochastic integral of \([0, T] \ni t \mapsto A_F(t) \) coincides with the classical Itô stochastic integral \( \int_{[0, T]} F(t) \, dN_t \) of \( F \). That is,

\[
\int_{[0, T]} A_F(t) \, dN_t = \int_{[0, T]} F(t) \, dN_t.
\]

In the last part of this note we show that the Itô integral in a Fock space is the \( H \)-stochastic integral and establish a connection of such an integral with the classical Itô stochastic integral. The corresponding results are given without proofs (the proofs will be given in a forthcoming publication). Note that the Itô integral in a Fock space is a useful tool in the quantum stochastic calculus, see e.g. [2] for more details.

We remark that in [3, 4] the authors gave a definition of the operator-valued stochastic integral

\[
B := \int_{[0, T]} A(t) \, dE_t
\]

for a family \( \{A(t)\}_{t \in [0, T]} \) of commuting normal operators in \( \mathcal{H} \). Such an integral was defined using a spectral theory of commuting normal operators. It is clear that for a fixed vector \( M \in \text{Dom}(B) \subset \mathcal{H} \) the formula

\[
\int_{[0, T]} A(t) \, dM_t := \left( \int_{[0, T]} A(t) \, dE_t \right) M
\]

can be regarded as a definition of integral (1). In this way we obtain another definition of integral (1) different from the one we have proposed in this paper.

2. THE CONSTRUCTION OF THE \( H \)-STOCHASTIC INTEGRAL

Let \( \mathcal{H} \) be a complex Hilbert space, \( \mathcal{L}(\mathcal{H}) \) be a space of all bounded linear operators in \( \mathcal{H} \), \( M \neq 0 \) be a fixed vector from \( \mathcal{H} \) and

\[
[0, T] \ni t \mapsto E_t \in \mathcal{L}(\mathcal{H})
\]

be a resolution of identity in \( \mathcal{H} \), that is a right-continuous increasing family of orthogonal projections in \( \mathcal{H} \) such that \( E_T = 1 \). Note that the resolution of identity \( E \) can be regarded as a projector-valued measure \( \mathcal{B}([0, T]) \ni \alpha \mapsto E(\alpha) \in \mathcal{L}(\mathcal{H}) \) on the Borel \( \sigma \)-algebra \( \mathcal{B}([0, T]) \). Namely, for any interval \( (s, t) \subset [0, T] \) we set

\[
E((s, t)) := E_t - E_s, \quad E(\{0\}) := E_0, \quad E(\emptyset) := 0,
\]

and extend this definition to all Borel subsets of \([0, T]\), see e.g. [6] for more details.

By definition, the \( \mathcal{H} \)-valued function

\[
[0, T] \ni t \mapsto M_t := E_t M \in \mathcal{H}
\]

is an abstract martingale in the Hilbert space \( \mathcal{H} \).

In this section we give a definition of integral (1) for a certain class of operator-valued functions with respect to the abstract martingale \( M_t \). A construction of such an integral is given step-by-step, beginning with the simplest class of operator-valued functions. Let us introduce the required class of simple functions.

For each point \( t \in [0, T] \), we denote by

\[
\mathcal{H}_M(t) := \text{span}\{M_{s_2} - M_{s_1} \mid (s_1, s_2) \subset (t, T]\} \subset \mathcal{H}
\]

the linear span of the set \( \{M_{s_2} - M_{s_1} \mid (s_1, s_2) \subset (t, T]\} \) in \( \mathcal{H} \) and by

\[
\mathcal{L}_M(t) = \mathcal{L}(\mathcal{H}_M(t) \to \mathcal{H})
\]
the set of all linear operators in \( H \) that continuously act from \( H_M(t) \) to \( H \). The increasing family \( \mathcal{L}_M = \{ \mathcal{L}_M(t) \}_{t \in [0,T]} \) will play here a role of the filtration \( \{ \mathcal{A}_t \}_{t \in [0,T]} \) in the classical martingale theory.

For a fixed \( t \in [0,T] \), a linear operator \( A \) in \( H \) will be called \( \mathcal{L}_M(t) \)-measurable if

(i) \( A \in \mathcal{L}_M(t) \) and, for all \( s \in [t,T] \),

\[
\| A \|_{\mathcal{L}_M(t)} = \| A \|_{\mathcal{L}_M(s)} := \sup \left\{ \frac{\| Ag \|_H}{\| g \|_H} \mid g \in \mathcal{H}_M(s), \ g \neq 0 \right\}.
\]

(ii) \( A \) is partially commuting with the resolution of identity \( E \). More precisely,

\[
AE_s g = E_s Ag, \quad g \in \mathcal{H}_M(t), \quad s \in [t,T].
\]

Such a definition of \( \mathcal{L}_M(t) \)-measurability is motivated by a number of reasons:

- \( \mathcal{L}_M(t) \)-measurability is a natural generalization of the usual \( \mathcal{A}_t \)-measurability in classical stochastic calculus, see Lemma 1 (Section 3) for more details;
- in some sense, \( \mathcal{L}_M(t) \)-measurability (for each \( t \) ) is the minimal restriction on the behavior of a simple operator-valued function \( [0,T] \ni t \mapsto A(t) \) that will allow us to obtain an analogue of the Itô isometry property (see inequality (4) below) and to extend the \( H \)-stochastic integral from a simple class of functions to a wider one.

In what follows, it is convenient for us to call \( \mathcal{L}_M(T) \)-measurable all linear operators in \( H \). Evidently, if a linear operator \( A \) in \( H \) is \( \mathcal{L}_M(t) \)-measurable for some \( t \in [0,T] \) then \( A \) is \( \mathcal{L}_M(s) \)-measurable for all \( s \in [t,T] \).

A family \( \{ A(t) \}_{t \in [0,T]} \) of linear operators in \( H \) will be called a \textit{simple \( \mathcal{L}_M \)-adapted operator-valued function on } \( [0,T] \) if, for each \( t \in [0,T] \), the operator \( A(t) \) is \( \mathcal{L}_M(t) \)-measurable and there exists a partition \( 0 = t_0 < t_1 < \cdots < t_n = T \) of \( [0,T] \) such that

\[
A(t) = \sum_{k=0}^{n-1} A_k \nu_{(t_k,t_{k+1})}(t), \quad t \in [0,T],
\]

where \( \nu_{\alpha} (\cdot) \) is the characteristic function of the Borel set \( \alpha \in \mathcal{B}([0,T]) \).

Let \( S = S(M) \) denote the space of all simple \( \mathcal{L}_M \)-adapted operator-valued functions on \( [0,T] \). For a function \( A \in S \) with representation (2) we define an \( H \)-\textit{stochastic integral} of \( A \) with respect to the abstract martingale \( M_t \) through the formula

\[
\int_{[0,T]} A(t) \ dM_t := \sum_{k=0}^{n-1} A_k (M_{t_{k+1}} - M_{t_k}) \in H.
\]

We can show that this definition does not depend on the choice of representation of the simple function \( A \) in the space \( S \).

In the space \( S \) we introduce a quasinorm by setting

\[
\| A \|_{S_2} := \left( \int_{[0,T]} \| A(t) \|_{\mathcal{L}_M(t)}^2 \ d\mu(t) \right)^{\frac{1}{2}} := \left( \sum_{k=0}^{n-1} \| A_k \|_{\mathcal{L}_M(t_k)}^2 \mu((t_k,t_{k+1}))) \right)^{\frac{1}{2}}
\]

for each \( A \in S \) with representation (2). Here the measure \( \mu \) is defined by the formula

\[
\mathcal{B}([0,T]) \ni \alpha \mapsto \mu(\alpha) := \| M(\alpha) \|_{H}^2 = (E(\alpha)M,M)_H \in \mathbb{R}_+,
\]

where \( M(\alpha) := E(\alpha)M \) for all \( \alpha \in \mathcal{B}([0,T]) \), in particular,

\[
M((t_k,t_{k+1})) := E((t_k,t_{k+1})) M = M_{t_{k+1}} - M_{t_k}, \quad (t_k,t_{k+1}) \subset [0,T].
\]

The following statement is fundamental.

\textbf{Theorem 1.} Let \( A, B \in S \) and \( a, b \in \mathbb{C} \). Then

\[
\int_{[0,T]} \left( aA(t) + bB(t) \right) \ dM_t = a \int_{[0,T]} A(t) \ dM_t + b \int_{[0,T]} B(t) \ dM_t.
\]
and
\[\|\int_{[0,T]} A(t) \, dM_t\|_\mathcal{H}^2 \leq \int_{[0,T]} \|A(t)\|_{\mathcal{L}_M(t)}^2 \, d\mu(t).\]

Proof. The first assertion is trivial.

Let us check inequality (4). Using (i), (ii) and properties of the resolution of identity \(E\), for \(A \in S\) with representation (2), we obtain
\[
\|\int_{[0,T]} A(t) \, dM_t\|_\mathcal{H}^2 = \left( \int_{[0,T]} A(t) \, dM_t, \int_{[0,T]} A(t) \, dM_t \right)_\mathcal{H}
\]
\[
= \sum_{k,m=0}^{n-1} (A_k M(\Delta_k), A_m M(\Delta_m))_\mathcal{H}
\]
\[
= \sum_{k,m=0}^{n-1} (A_k E(\Delta_k) M, A_m E(\Delta_m) M)_\mathcal{H}
\]
\[
= \sum_{k=0}^{n-1} (A_k E(\Delta_k) M, A_k E(\Delta_k) M)_\mathcal{H}
\]
\[
\leq \sum_{k=0}^{n-1} \|A_k\|_{\mathcal{L}_M(t_k)}^2 \|M(\Delta_k)\|_\mathcal{H}^2
\]
\[
= \int_{[0,T]} \|A(t)\|_{\mathcal{L}_M(t)}^2 \, d\mu(t),
\]
where \(\Delta_k := (t_k, t_{k+1})\) for all \(k \in \{0, \ldots, n-1\}\).

Inequality (4) enables us to extend the \(H\)-stochastic integral to operator-valued functions \([0,T] \ni t \mapsto A(t)\) which are not necessarily simple. Namely, denote by \(S_2 = S_2(M)\) a Banach space associated with the quasinorm \(\| \cdot \|_{S_2}\). For its construction, it is first necessary to pass from \(S\) to the factor space \(\hat{S} := S/\{A \in S \mid \|A\|_{S_2} = 0\}\) and then to take the completion of \(\hat{S}\). It is not difficult to see that elements of the space \(S_2\) are equivalence classes of operator-valued functions on \([0,T]\) whose values are linear operators in the space \(\mathcal{H}\).

An operator-valued function \([0,T] \ni t \mapsto A(t)\) will be called \(H\)-stochastic integrable with respect to \(M_t\) if \(A\) belongs to the space \(S_2\). It follows from the definition of the space \(S_2\) that for each \(A \in S_2\) there exists a sequence \((A_n)_{n=0}^{\infty}\) of simple operator-valued functions \(A_n \in S\) such that
\[
\int_{[0,T]} \|A(t) - A_n(t)\|_{\mathcal{L}_M(t)}^2 \, d\mu(t) \to 0 \quad \text{as} \quad n \to \infty.
\]
Due to (4), for such a sequence \((A_n)_{n=0}^{\infty}\), the limit
\[
\lim_{n \to \infty} \int_{[0,T]} A_n(t) \, dM_t
\]
exists in \(\mathcal{H}\) and does not dependent on the choice of the sequence \((A_n)_{n=0}^{\infty} \subset S\) satisfying (5). We denote this limit by
\[
\int_{[0,T]} A(t) \, dM_t := \lim_{n \to \infty} \int_{[0,T]} A_n(t) \, dM_t
\]
and call it the \(H\)-stochastic integral of \(A \in S_2\) with respect to the abstract martingale \(M_t\). It is clear that for all \(A \in S_2\) the assertions of Theorem 1 still hold.
Note one simple property of the integral introduced above. Let $U$ be some unitary operator acting from $H$ onto another complex Hilbert space $K$. Then

$$[0, T] \ni t \mapsto G_t := U M_t \in K$$

is an abstract martingale in the space $K$ because, for any $t \in [0, T],

$$G_t = U M_t = X_t G, \quad X_t := U E_t U^{-1}, \quad G := U M \in K,$$

and $X_t$ is a resolution of identity in the space $K$.

Let an operator-valued function $[0, T] \ni t \mapsto A(t)$ be $H$-stochastic integrable with respect to $M_t$. One can show that the operator-valued function $[0, T] \ni t \mapsto UA(t)U^{-1}$ is $H$-stochastic integrable with respect to $G_t$ and

$$U \left( \int_{[0,T]} A(t) \, dM_t \right) = \int_{[0,T]} UA(t)U^{-1} \, dG_t \in K.$$

3. The Itô stochastic integral as an $H$-stochastic integral

Let $(\Omega, A, P)$ be a complete probability space and $\{A_t\}_{t \in [0, T]}$ be a right continuous filtration. Suppose that the $\sigma$-algebra $A_0$ contains all $P$-null sets of $A$ and $A = A_T$. Moreover, we assume that $A_0$ is trivial, i.e., every set $\alpha \in A_0$ has probability 0 or 1.

Let $N = \{N_t\}_{t \in [0, T]}$ be a normal martingale on $(\Omega, A, P)$ with respect to $\{A_t\}_{t \in [0, T]}$. That is, $N_t \in L^2(\Omega, A_t, P)$ for all $t \in [0, T]$ and

$$E[N_t - N_s | A_s] = 0, \quad E[(N_t - N_s)^2 | A_s] = t - s$$

for all $s, t \in [0, T]$ such that $s < t$. Without loss of generality one can assume that $N_0 = 0$. Note that there are many examples of normal martingales, — the Brownian motion, the compensated Poisson process, the Azéma martingales and others, see for instance [10, 8, 12].

We will denote by $L^2_n([0, T] \times \Omega)$ the set of all functions (equivalence classes), adapted to the filtration $\{A_t\}_{t \in [0, T]}$, from the space

$$L^2([0, T] \times \Omega) := L^2([0, T] \times \Omega, B([0, T]) \times A, dt \times P)$$

where $dt$ is the Lebesgue measure on $B([0, T])$.

Let us show that the Itô stochastic integral $\int_{[0,T]} F(t) \, dN_t$ of $F \in L^2_n([0, T] \times \Omega)$ with respect to the normal martingale $N$ can be considered as an $H$-stochastic integral (see e.g. [15, 16] for the definition and properties of the classical Itô integral). To this end, we set $H := L^2(\Omega, A, P)$ and consider, in this space, the resolution of identity

$$[0, T] \ni t \mapsto E_t := E[\cdot | A_t] \in L(H)$$

generated by the filtration $\{A_t\}_{t \in [0, T]}$. Let $M := N_T \in L^2(\Omega, A, P)$, then the corresponding abstract martingale

$$[0, T] \ni t \mapsto N_t := E_t N_T = E(N_T | A_t) \in H$$

is our normal martingale. Note also that

$$\mu([0, t]) = \|N([0, t])\|_{L^2(\Omega, A, P)}^2 = \|N_t\|_{L^2(\Omega, A, P)}^2 = E[N_T^2] = E[N_T^2 | A_0] = t,$$

i.e., $\mu$ is the Lebesgue measure on $B([0, T])$.

In the context of this section, $L_M(t)$-measurability is equivalent to the usual $A_t$-measurability. More precisely, the following result holds.

**Lemma 1.** Let $t \in [0, T)$. For given $F \in L^2(\Omega, A, P)$ the operator $A_F$ of multiplication by the function $F$ in the space $L^2(\Omega, A, P)$ is $L_N(t)$-measurable if and only if the function $F$ is $A_t$-measurable, i.e., $F = E[F | A_t]$. Moreover, if $F \in L^2(\Omega, A, P)$ is an $A_t$-measurable function then

$$\|A_F\|_{L_N(t)} = \|A_F\|_{L_N(s)} = \|F\|_{L^2(\Omega, A, P)}, \quad s \in [t, T).$$
Proof. Suppose $F \in L^2(\Omega, \mathcal{A}, P)$ is an $\mathcal{A}_t$-measurable function. Let us show that the operator $A_F$ is $\mathcal{L}_N(t)$-measurable.

First, we prove that $A_F \in \mathcal{L}_N(t)$. Taking into account that $F$ is an $\mathcal{A}_t$-measurable function, $\{N_t\}_{t \in [0,T]}$ is the normal martingale and the $\sigma$-algebra $\mathcal{A}_0$ is trivial, for any interval $(s_1, s_2] \subset (t, T)$, we obtain

$$
\|A_F(N_{s_2} - N_{s_1})\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \|F(N_{s_2} - N_{s_1})\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \mathbb{E}[F^2(N_{s_2} - N_{s_1})^2] \\
= \mathbb{E}[F^2(N_{s_2} - N_{s_1})^2|\mathcal{A}_0] = \mathbb{E}[F^2\mathbb{E}[(N_{s_2} - N_{s_1})^2|\mathcal{A}_s]|\mathcal{A}_0] \\
= \mathbb{E}[F^2\mathbb{E}[(N_{s_2} - N_{s_1})^2|\mathcal{A}_s] = \mathbb{E}[F^2](s_2 - s_1) \\
= \mathbb{E}[F^2]\mathbb{E}[(N_{s_2} - N_{s_1})^2] \\
= \|F\|_{L^2(\Omega, \mathcal{A}, P)}^2 \|N_{s_2} - N_{s_1}\|_{L^2(\Omega, \mathcal{A}, P)}^2.
$$

We can similarly show that

$$
\|A_FG\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \|F\|_{L^2(\Omega, \mathcal{A}, P)}^2 \|G\|_{L^2(\Omega, \mathcal{A}, P)}^2
$$

for all $G \in \mathcal{H}_N(t) = \text{span}\{N_{s_2} - N_{s_1} | (s_1, s_2] \subset (t, T]\}$. Hence $A_F \in \mathcal{L}_N(t)$ and, moreover, equality (6) takes place.

Let us check that $A_F$ is partially commuting with $E$, i.e.,

$$A_FE_sG = E_sA_FG, \quad G \in \mathcal{H}_N(t), \quad s \in [t, T].$$

Since $F \in L^2(\Omega, \mathcal{A}, P)$ is an $\mathcal{A}_t$-measurable function and $FG \in L^2(\Omega, \mathcal{A}, P)$, for any $s \in [t, T]$ and any function $G \in \mathcal{H}_N(t)$, we have

$$A_FE_sG = FE_sG = FE[G|\mathcal{A}_s] = \mathbb{E}[FG|\mathcal{A}_s] = E_sA_FG.$$

Thus, the first part of the lemma is proved.

Let us prove the converse statement of the lemma: if for a given $F \in L^2(\Omega, \mathcal{A}, P)$ the operator $A_F$ is $\mathcal{L}_N(t)$-measurable then $F$ is an $\mathcal{A}_t$-measurable function.

Since $A_F$ is an $\mathcal{L}_N(t)$-measurable operator, we see that for any $s \in [t, T]$

$$A_FE_sG = E_sA_FG, \quad G \in \mathcal{H}_N(t),$$

or, equivalently,

$$(7) \quad A_FE[G|\mathcal{A}_s] = \mathbb{E}[A_FG|\mathcal{A}_s], \quad G \in \mathcal{H}_N(t).$$

Let $s \in (t, T)$ and $(s_1, s_2] \subset (t, s]$. We take $G := N_{s_2} - N_{s_1} \in \mathcal{H}_N(t)$. Evidently, $G$ is an $\mathcal{A}_t$-measurable function and

$$A_FE[G|\mathcal{A}_s] = AFG = FG, \quad \mathbb{E}[A_FG|\mathcal{A}_s] = \mathbb{E}[FG|\mathcal{A}_s] = G\mathbb{E}[F|\mathcal{A}_s].$$

Hence, using (7), we obtain

$$FG = G\mathbb{E}[F|\mathcal{A}_s].$$

As a result,

$$F = \mathbb{E}[F|\mathcal{A}_s], \quad s \in (t, T].$$

Since the resolution of identity $[0, T] \ni s \mapsto E_s = \mathbb{E}[\cdot|\mathcal{A}_s] \in \mathcal{L}(\mathcal{H})$ is a right-continuous function, the latter equality still holds for $s = t$, and therefore $F$ is an $\mathcal{A}_t$-measurable function.

As a simple consequence of Lemma 1 we obtain the following result.

**Theorem 2.** Let $F$ belong to $L^2([0, T] \times \Omega)$. The family $\{A_F(t)\}_{t \in [0, T]}$ of the operators $A_F(t)$ of multiplication by $F(t) = F(t, \cdot) \in L^2(\Omega, \mathcal{A}, P)$ in the space $L^2(\Omega, \mathcal{A}, P)$,

$$L^2(\Omega, \mathcal{A}, P) \supset \text{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, \mathcal{A}, P),$$

is $H$-stochastic integrable with respect to the normal martingale $N$ (i.e. belongs to $S_2$) if and only if $F$ belongs to the space $L^2([0, T] \times \Omega)$. 

\[\square\]
The next theorem shows that the Itô stochastic integral with respect to the normal martingale \( N \) can be interpreted as an \( H \)-stochastic integral.

**Theorem 3.** Let \( F \in L^2_a([0, T] \times \Omega) \) and \( \{ A_F(t) \}_{t \in [0, T]} \) be the corresponding family of the operators \( A_F(t) \) of multiplication by \( F(t) \) in the space \( L^2(\Omega, \mathcal{A}, P) \). Then

\[
\int_{[0,T]} A_F(t) \, dN_t = \int_{[0,T]} F(t) \, dN_t.
\]

**Proof.** Taking into account Theorem 2, Lemma 1 and the definitions of the integrals

\[
\int_{[0,T]} A_F(t) \, dN_t \quad \text{and} \quad \int_{[0,T]} F(t) \, dN_t,
\]

it is sufficient to prove Theorem 3 for simple functions \( F \in L^2_a([0, T] \times \Omega) \). But in this case Theorem 3 is obvious. \( \square \)

4. **The Itô integral in a Fock space as an \( H \)-stochastic integral**

Let us recall the definition of the Itô integral in a Fock space, see e.g. \cite{2} for more details. We denote by \( \mathcal{F} \) the symmetric Fock space over the real separable Hilbert space \( L^2([0, T]) := L^2([0, T], dt) \). By definition (see e.g. \cite{5}),

\[
\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{F}_n n!,
\]

where \( \mathcal{F}_0 := \mathbb{C} \) and, for each \( n \in \mathbb{N}, \mathcal{F}_n := (L^2([0, T]))^{\otimes n} \) is an \( n \)-th symmetric tensor power \( \otimes \) of the complex Hilbert space \( L^2([0, T]) \). Thus, the Fock space \( \mathcal{F} \) is the complex Hilbert space of sequences \( f = (f_n)_{n=0}^\infty \) such that \( f_n \in \mathcal{F}_n \) and

\[
\|f\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} \|f_n\|_{\mathcal{F}_n}^2 n! < \infty.
\]

We denote by \( L^2([0, T]; \mathcal{F}) \) the Hilbert space of all \( \mathcal{F} \)-valued functions

\[
[0, T] \ni t \mapsto f(t) \in \mathcal{F}, \quad \|f\|_{L^2([0, T]; \mathcal{F})} := \left( \int_{[0,T]} \|f(t)\|_{\mathcal{F}}^2 \, dt \right)^{\frac{1}{2}} < \infty
\]

with the corresponding scalar product. A function \( f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L^2([0, T]; \mathcal{F}) \) is called **Itô integrable** if, for almost all \( t \in [0, T] \),

\[
f(t) = (f_0(t), x_{[0,t],1} f_1(t), \ldots, x_{[0,t],n} f_n(t), \ldots).
\]

We denote by \( L^2_a([0, T]; \mathcal{F}) \) the set of all Itô integrable functions.

Let \( f \) belong to the space \( L^2_a([0, T]; \mathcal{F}) \) of all **simple Itô integrable functions**. That is, \( f \) belongs to \( L^2_a([0, T]; \mathcal{F}) \) and there exists a partition \( 0 = t_0 < t_1 < \cdots < t_n = T \) of \([0, T]\) such that

\[
f(t) = \sum_{k=0}^{n-1} f(k) \otimes (t_k, t_{k+1}) \in \mathcal{F}
\]

for almost all \( t \in [0, T] \). The **Itô integral** \( \mathbb{I}(f) \) of such a function \( f \) is defined by the formula

\[
\mathbb{I}(f) := \sum_{k=0}^{n-1} f(k) \hat{\otimes} (0, x_{(t_k, t_{k+1})}, 0, 0, \ldots) \in \mathcal{F},
\]

where the symbol \( \hat{\otimes} \) denotes the Wick product in the Fock space \( \mathcal{F} \). Let us recall that for given \( f = (f_n)_{n=0}^\infty \) and \( g = (g_n)_{n=0}^\infty \) from \( \mathcal{F} \) the Wick product \( f \hat{\otimes} g \) is defined by

\[
f \hat{\otimes} g := (\sum_{m=0}^{n} f_m \otimes g_{n-m})_{n=0}^\infty,
\]
provided the latter sequence belongs to the Fock space $\mathcal{F}$.

The Itô integral $\mathbb{I}(f)$ of a simple function $f \in L^2_{a,n}([0,T]; \mathcal{F})$ has the isometry property

$$\|\mathbb{I}(f)\|_{\mathcal{F}}^2 = \int_{[0,T]} \|f(t)\|_{\mathcal{F}}^2 \, dt,$$

see e.g. [2, 1]. Hence, extending the mapping

$$L^2_a([0,T]; \mathcal{F}) \ni L^2_{a,n}([0,T]; \mathcal{F}) \ni f \mapsto \mathbb{I}(f) \in \mathcal{F}$$

by continuity we obtain a definition of the Itô integral $\mathbb{I}(f)$ for each $f \in L^2_a([0,T]; \mathcal{F})$ (we keep the same notation $\mathbb{I}$ for the extension).

Let us show that the Itô integral $\mathbb{I}(f)$ of $f \in L^2_a([0,T]; \mathcal{F})$ can be considered as an $H$-stochastic integral. To do this we set $\mathcal{H} := \mathcal{F}$ and consider in this space the resolution of identity

$$[0,T] \ni t \mapsto \mathcal{H}_t := (f_0, \nu_{[0,t]} f_1, \ldots, \nu_{[0,t]} f_n, \ldots) \in \mathcal{L}(\mathcal{F}), \quad f = (f_n)_{n=0}^\infty \in \mathcal{F}.$$ 

Let $Z := ([0,1,0,0,\ldots]) \in \mathcal{F}$ and

$$[0,T] \ni t \mapsto Z_t := \mathcal{H}_t Z = (0, \nu_{[0,t]} 0, 0, 0, \ldots) \in \mathcal{F}$$

be the corresponding abstract martingale in the Fock space $\mathcal{F}$. Note that now

$$\mu([0,t]) := \|Z_t\|^2_{\mathcal{F}} = \|\nu_{[0,t]} 0\|^2_{\mathcal{F}^2([0,T])} = t, \quad t \in [0,T],$$

i.e., $\mu$ is the Lebesgue measure on $\mathcal{B}([0,T])$.

We have the following analogues of Theorems 2 and 3.

**Theorem 4.** A function $f \in L^2([0,T]; \mathcal{F})$ belongs to the space $L^2_{a,n}([0,T]; \mathcal{F})$ if and only if the corresponding operator-valued function $[0,T] \ni t \mapsto A_f(t)$ whose values are operators $A_f(t)$ of Wick multiplication by $f(t) \in \mathcal{F}$ in the Fock space $\mathcal{F}$,

$$\mathcal{F} \ni \text{Dom}(A_f(t)) \ni g \mapsto A_f(t)g := f(t) \circ g \in \mathcal{F},$$

belongs to the space $S_2$.

**Theorem 5.** Let $f \in L^2_{a,n}([0,T]; \mathcal{F})$ and $\{A_f(t)\}_{t \in [0,T]}$ be the corresponding family of the operators $A_f(t)$ of Wick multiplication by $f(t) \in \mathcal{F}$ in the Fock space $\mathcal{F}$. Then

$$\mathbb{I}(f) = \int_{[0,T]} A_f(t) \, dZ_t.$$ 

Taking into account Theorem 5, in what follows we will denote the Itô integral $\mathbb{I}(f)$ of $f \in L^2_{a,n}([0,T]; \mathcal{F})$ by $\int_{[0,T]} f(t) \, dZ_t$. Note that this integral can be expressed in terms of the Fock space $\mathcal{F}$. Namely, for any $f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L^2_{a,n}([0,T]; \mathcal{F})$, we have

$$\int_{[0,T]} f(t) \, dZ_t = (0, \hat{f}_1, \ldots, \hat{f}_n, \ldots) \in \mathcal{F},$$

where, for each $n \in \mathbb{N}$ and almost all $(t_1, \ldots, t_n) \in [0,T]^n$,

$$\hat{f}_n(t_1, \ldots, t_n) := \frac{1}{n} \sum_{k=1}^n f_{n-1}(t_k; t_1, \ldots, t_k, \ldots, t_n),$$

i.e., $\hat{f}_n$ is the symmetrization of $f_{n-1}(t; t_1, \ldots, t_{n-1})$ with respect to $n$ variables.

5. A CONNECTION BETWEEN THE CLASSICAL ITÔ INTEGRAL AND THE ITÔ INTEGRAL IN THE FOCK SPACE

As before, let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a right continuous filtration $\{\mathcal{A}_t\}_{t \in [0,T]}$, $\mathcal{A}_0$ be the trivial $\sigma$-algebra containing all $P$-null sets of $\mathcal{A}$ and $\mathcal{A} = \mathcal{A}_T$.
Let $N = \{N_t\}_{t \in [0,T]}$ be a normal martingale on $(\Omega, \mathcal{A}, P)$ with respect to $\{\mathcal{A}_t\}_{t \in [0,T]}$, $N_0 = 0$. It is known that the mapping
\[ F \ni f = (f_n)_{n=0}^{\infty} \mapsto I_f := \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega, \mathcal{A}, P) \]
is well-defined and isometric. Here $I_0(f_0) := f_0$ and, for each $n \in \mathbb{N}$,
\[ I_n(f_n) := n! \int_0^T \int_0^{t_n} \cdots (\int_0^{t_1} f_n(t_1, \ldots, t_n) \, dN_{t_1}) \cdots dN_{t_{n-1}} \, dN_{t_n} \]
is an $n$-iterated Itô integral with respect to $N$. We suppose that the normal martingale $N$ has the chaotic representation property (CRP). In other words, we assume that the mapping $I : F \to L^2(\Omega, \mathcal{A}, P)$ is a unitary. Note that
\[ N_t = IZ_t \in L^2(\Omega, \mathcal{A}, P), \quad t \in [0,T], \]
i.e., $N$ is the $I$-image of the abstract martingale $[0, T] \ni t \mapsto Z_t = (0, \xi_{[0,t]}, 0, 0, \ldots) \in F$.

The Brownian motion, the compensated Poisson process and some Azéma martingales are examples of normal martingales which possess the CRP; see e.g. [10, 11].

We note that the spaces $L^2_0([0, T] \times \Omega)$ and $L^2([0, T]; F)$ can be understood as the tensor products $L^2([0, T]) \otimes L^2(\Omega, \mathcal{A}, P)$ and $L^2([0, T]) \otimes F$, respectively. Therefore,
\[ 1 \otimes I : L^2([0, T]; F) \to L^2([0, T] \times \Omega) \]
is a unitary operator.

The next result gives a relationship between the classical Itô integral with respect to the normal martingale with CRP and the Itô integral in the Fock space $F$.

**Theorem 6.** We have
\[ L^2_0([0, T] \times \Omega) = (1 \otimes I) L^2_0([0, T]; F) \]
and, for arbitrary $f \in L^2_0([0, T]; F)$,
\[ I \left( \int_{[0,T]} f(t) \, dZ_t \right) = \int_{[0,T]} I_f(t) \, dN_t. \]

Since $N$ has CRP, for any $F \in L^2_0([0, T] \times \Omega)$ there exists a uniquely defined vector $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L^2_0([0, T]; F)$ such that
\[ F(t) = I_f(t) = \sum_{n=0}^{\infty} I_n(f_n(t)) \]
for almost all $t \in [0, T]$. Hence, using Theorem 6 and equality (8) we obtain
\[ \int_{[0,T]} F(t) \, dN_t = I \left( \int_{[0,T]} f(t) \, dZ_t \right) = \sum_{n=1}^{\infty} I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P). \]

It should be noticed that the right hand side of the latter equality was used by Hitsuda [9] and Skorohod [14] to define an extension of the Itô integral. Namely, a function
\[ F(\cdot) = \sum_{n=0}^{\infty} I_n(f_n(\cdot)) \in L^2([0, T] \times \Omega) \]
is Hitsuda-Skorohod integrable if and only if
\[ \sum_{n=1}^{\infty} I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P) \quad \text{or, equivalently,} \quad \sum_{n=1}^{\infty} \|\hat{f}_n\|_{L^2}^2 n! < \infty. \]

The corresponding **Hitsuda-Skorohod integral** $I_{HS}(F)$ of $F$ is defined by the formula
\[ I_{HS}(F) := \sum_{n=1}^{\infty} I_n(\hat{f}_n). \]
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Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs’ka, Kyiv, 01601, Ukraine
E-mail address: tesko@imath.kiev.ua

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