THE INTEGRATION OF OPERATOR–VALUED FUNCTIONS WITH RESPECT TO VECTOR–VALUED MEASURES

VOLODYMYR TESKO

Abstract. We investigate the H-stochastic integral introduced in [24]. It is known that this integral generalizes the classical Itô stochastic integral and the Itô integral on a Fock space. In the present paper we construct and study an extension of the H-stochastic integral which will generalize the Hitsuda–Skorokhod integral.

1. Introduction

It is well known that the Itô integral of adapted square integrable functions plays a fundamental role in the classical stochastic calculus. In the case of scalar–valued integrands this integral has a very simple interpretation in the framework of functional analysis. Namely, let $L^2(\Omega, \mathcal{A}, P)$ be a space of square integrable functions on a complete probability space $(\Omega, \mathcal{A}, P)$, $\{\mathcal{A}_t\}_{t\in[0,T]}$ be a filtration of σ-algebras satisfying the usual conditions and $\{M_t\}_{t\in[0,T]}$ be a square integrable martingale on $(\Omega, \mathcal{A}, P)$ with respect to $\{\mathcal{A}_t\}_{t\in[0,T]}$ (here $T > 0$ is fixed). Then the Itô integral $\int_{[0,T]} F(t)\, dM_t$ of a scalar–valued (non-random) function $F\colon [0, T] \to \mathbb{C}$ can be interpreted as an ordinary spectral integral $\int_{[0,T]} F(t)\, dE_t$ applied to M_T, i.e.,

$$\int_{[0,T]} F(t)\, dM_t = \left(\int_{[0,T]} F(t)\, dE_t \right) M_T,$$

where $E_t := \mathbb{E}[\,\cdot\,|\mathcal{A}_t]$ denotes a conditional expectation with respect to the σ-algebra $\mathcal{A}_t$. It can be shown that E_t is an orthogonal projection in the space $L^2(\Omega, \mathcal{A}, P)$ onto its subspace $L^2(\Omega, \mathcal{A}_t, P)$ and, moreover, the corresponding projection–valued function $[0, T] \ni t \mapsto E_t$ is a resolution of identity in $L^2(\Omega, \mathcal{A}, P)$. Note that $M_t = E_t M_T$ for all $t \in [0, T]$ since M is a martingale.

At the same time, it is considerably more difficult to establish a relation between the Itô and spectral integrals for general adapted $L^2(\Omega, \mathcal{A}, P)$-valued integrands, since the main problem is to find an explicit expression for the corresponding spectral integral (see, for example, [6] and reference therein). In this context a natural problem arises, — *to give a suitable definition of "spectral integral" which will generalize the Itô stochastic integral.*

The starting point for solving this problem is the observation that an $L^2(\Omega, \mathcal{A}, P)$-valued function $[0, T] \ni t \mapsto F(t) \in L^2(\Omega, \mathcal{A}, P)$ (integrand) can be naturally viewed as an operator–valued function $[0, T] \ni t \mapsto A_F(t)$ whose values are operators $A_F(t)$ of multiplication by the function $F(t) = F(t, \cdot)$ in the space $L^2(\Omega, \mathcal{A}, P)$. This suggests the consideration of the above mentioned problem in a more comprehensive sense. *Namely, the problem is to give a definition of integral*

$$\int_{[0,T]} A(t)\, dM_t, \quad M_t := E_t M_T, \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 46G99, 47B15, 60H05.

Key words and phrases. Hitsuda–Skorohod integral; resolution of identity; Fock space.
The work is partially supported by the DAAD, Project A/11/05321.

which will generalize the Itô stochastic integral. Here $[0,T] \ni t \mapsto E_t$ is a resolution of identity in a Hilbert space $\mathcal{H}$, $[0,T] \ni t \mapsto A(t)$ is an operator–valued function whose values are linear operators in $\mathcal{H}$ and $M_T \in \mathcal{H}$.

In the papers [24], [25] integral (1.1) was constructed and studied for a certain class of operator–valued functions. This integral was defined as an element of the Hilbert space $\mathcal{H}$ and called a *Hilbert space–valued stochastic integral* (*H-stochastic integral* for short). It was shown that integral (1.1) generalizes the classical Itô stochastic integral with respect to normal martingales and the Itô integral on a Fock space.

On the other hand, the concept of the Hitsuda–Skorokhod integral is of great importance in the stochastic calculus [12], [23]. This integral is one of the central objects of study in the Gaussian analysis, since, on the one hand, it is a natural generalization of the Itô integral for nonadapted integrands and, on the other hand, it is adjoint to the stochastic derivative (also called Malliavin derivative). It should be stressed that these properties of the Hitsuda–Skorokhod integral are a starting point in developing of the anticipating stochastic calculus (see for example Nualart's book [21]). Moreover, we note that the Hitsuda–Skorokhod integral, defined as a "creation operator" on a Fock space, is an extension of the Itô integral not only in the Gaussian case but in the case of any normal martingale with the so-called Chaos Representation Property (see for instance [19] and review [17]).

Since both the Hitsuda–Skorokhod integral and the H-stochastic integral are generalization of the classical Itô integral, it is natural "to relate" these two objects. To this end, in the present article we introduce and study a natural extension of H-stochastic integral (1.1) which will generalize the Hitsuda–Skorokhod integral. In our construction the Berezansky–Gelfand–Kostyuchenko theorem (about the differentiability of an operator–valued measure, see e.g. [5], [11]) will play a crucial role. Roughly speaking, applying this theorem to the resolution of identity E we obtain the representation

$$E(\alpha) = \int_\alpha P(t)\, d\mu(t), \quad \alpha \in \mathcal{B}([0,T]),$$

where $[0,T] \ni t \mapsto P(t)$ is an operator–valued function (an operator–valued density), μ is the so-called spectral measure of the resolution of identity E, $\mathcal{B}([0,T])$ is a Borel σ-algebra on $[0,T]$. Using this integral representation of E, we define an extension of H-stochastic integral (1.1) as a Bochner one of the form

$$\int_{[0,T]} A(t)P(t)M_T\, d\mu(t) \tag{1.2}$$

and show that (1.2) generalizes the Hitsuda–Skorokhod integral.

It should be noticed that (1.2) gives us a new understanding of the Hitsuda–Skorokhod integral. More exactly, let $\int_{[0,T]} F(t)\, dN_t$ be an Itô integral of a "fine" function F with respect to a normal martingale N (by definition the process $N = \{N_t\}_{t\in[0,T]}$ is a normal martingale if $\{N_t\}_{t\in[0,T]}$ and $\{N_t^2 - t\}_{t\in[0,T]}$ are both martingales). Applying the Berezansky–Gelfand–Kostyuchenko theorem to the resolution of identity $E_t := \mathbb{E}[\,\cdot\,|\mathcal{A}_t]$ and replacing $N_t = E_tN_T$ by $\int_{[0,t]} P(s)N_T\, ds$, the Itô integral can be rewriten (at least formally) as

$$\int_{[0,T]} F(t)\, dN_t = \int_{[0,T]} F(t)P(t)N_T\, dt.$$

Since $F(t)P(t)N_T = F(t)\lozenge\langle\cdot,\delta_t\rangle$ (here δ_t is the Dirac delta function and $\lozenge$ is the Wick product), we see that

$$\int_{[0,T]} F(t)\, dN_t = \int_{[0,T]} F(t)\lozenge\langle\cdot,\delta_t\rangle\, dt,$$

where the latter integral is the Hitsuda–Skorokhod integral.

The paper is organized in the following manner. In the forthcoming section we recall the definition and properties of the H-stochastic integral. In Section 3 we present some of the standard facts about the Hitsuda–Skorokhod integral on a Fock space. Finally, in Section 4 we introduce and study an extension of the H-stochastic integral. This extension we define as an ordinary Bochner integral and show that it generalizes the Hitsuda–Skorokhod integral.

2. H-stochastic integral

In this section we recall a definition of the H-stochastic integral and describe the connection of this integral with both the classical Itô integral and the Itô integral on a Fock space. For the convenience of the reader we repeat the relevant material from [24, 25] without proofs, thus making our exposition self-contained.

2.1. **Definition and properties of the H-stochastic integral.** Here and subsequently, we fix $T > 0$. Let $\mathcal{H}$ be a complex Hilbert space with the inner product $(\cdot\,,\cdot)_{\mathcal{H}}$ and the norm $\|\cdot\|_{\mathcal{H}}$, $\mathcal{L}(\mathcal{H})$ be a space of all bounded linear operators in $\mathcal{H}$.

Suppose that $E\colon [0,T] \to \mathcal{L}(\mathcal{H})$ is a right-continuous resolution of identity in $\mathcal{H}$, i.e., $\{E_t\}_{t\in[0,T]}$ is a right-continuous and increasing family of orthogonal projections in $\mathcal{H}$ and $E_T = 1$. Sometimes it will be convenient for us to regard E as a projection–valued measure

$$E : \mathcal{B}([0,T]) \to \mathcal{L}(\mathcal{H}), \quad \alpha \mapsto E(\alpha),$$

on the Borel σ-algebra $\mathcal{B}([0,T])$. To do this, we set

$$E((s,t]) := E_t - E_s, \quad E(\{0\}) := E_0, \quad E(\varnothing) := 0,$$

for any interval $(s,t] \subset [0,T]$ and then extend this definition to all Borel subsets of $[0,T]$.

A slight generalization of the notion of square integrable martingale is the following.

Definition 2.1. A function $M\colon [0,T] \to \mathcal{H},\ t \mapsto M_t$, is called an $\mathcal{H}$-valued martingale with respect to E if $M_t = E_t M_T$ for all $t \in [0,T]$.

It is clear that an ordinary square integrable martingale $M\colon [0,T] \to L^2(\Omega,\mathcal{A},P)$ with respect to the filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$ is an $L^2(\Omega,\mathcal{A},P)$-valued martingale with respect to the resolution of identity $E_t := \mathbb{E}[\,\cdot\,|\mathcal{A}_t]$.

In the sequel we will regard the martingale M as a $\mathcal{H}$-valued measure

$$M : \mathcal{B}([0,T]) \to \mathcal{H}, \quad \alpha \mapsto M(\alpha) := E(\alpha)M_T.$$

Using this martingale we construct a nonnegative Borel measure

$$\mu : \mathcal{B}([0,T]) \to \mathbb{R}_+, \quad \alpha \mapsto \mu(\alpha) := \|M(\alpha)\|^2_{\mathcal{H}}.$$

Let us recall the definition of the *H-stochastic integral*

$$\int_{[0,T]} A(t)\, dM_t \in \mathcal{H}$$

of operator–valued functions $[0,T] \ni t \mapsto A(t)$ whose values are linear operators in the space $\mathcal{H}$. The construction of this integral is given step-by-step, starting from the simplest class of operator–valued functions. Namely, for $t \in [0,T]$, denote by

$$\mathcal{H}_M(t) := \operatorname{span}\{M((s_1,s_2])\,|\,(s_1,s_2] \subset (t,T]\} \subset \mathcal{H}$$

the linear span of the set $\{M((s_1,s_2])\,|\,(s_1,s_2] \subset (t,T]\}$, and

$$\mathcal{L}_M(t) = \mathcal{L}(\mathcal{H}_M(t),\mathcal{H})$$

will represent the set of linear operators in $\mathcal{H}$ that continuously act from $\mathcal{H}_M(t)$ to $\mathcal{H}$. Note that the set $\mathcal{L}_M(t)$ consists of all linear (bounded or non-bounded) operators $A\colon \mathcal{H} \to \mathcal{H}$ such that

$$\|A\|_{\mathcal{L}_M(t)} := \sup\left\{ \frac{\|Ag\|_{\mathcal{H}}}{\|g\|_{\mathcal{H}}} \,\middle|\, g \in \mathcal{H}_M(t),\, g \neq 0 \right\} < \infty.$$

Definition 2.2. A family $\{A(t)\}_{t\in[0,T]}$ of linear operators in $\mathcal{H}$ is called an *M-adapted* operator–valued function if, for every $t \in [0,T)$,

(i) $A(t) \in \mathcal{L}_M(t)$ and $\|A(t)\|_{\mathcal{L}_M(t)} = \|A(t)\|_{\mathcal{L}_M(s)}$ for all $s \in [t,T)$.

(ii) $A(t)$ is partially commuting with the resolution of identity E. More precisely,

$$A(t)E_s g = E_s A(t) g, \quad g \in \mathcal{H}_M(t), \quad s \in [t,T].$$

An M-adapted operator–valued function $[0,T] \ni t \mapsto A(t)$ is called simple if there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that

$$A(t) = \sum_{k=0}^{n-1} A_k \mathbb{1}_{(t_k,t_{k+1}]}(t), \quad t \in [0,T], \tag{2.3}$$

where $\mathbb{1}_\alpha(\cdot)$ denotes the characteristic function of a Borel set $\alpha \in \mathcal{B}([0,T])$. Denote by $S = S(M)$ the space of all simple M-adapted operator–valued functions on $[0,T]$. For each $A \in S$ of kind (2.3), we introduce a seminorm

$$\|A\|_{S_2} := \left(\int_{[0,T]} \|A(t)\|^2_{\mathcal{L}_M(t)}\, d\mu(t)\right)^{\frac{1}{2}} := \left(\sum_{k=0}^{n-1} \|A_k\|^2_{\mathcal{L}_M(t_k)}\, \mu((t_k,t_{k+1}])\right)^{\frac{1}{2}}. \tag{2.4}$$

Since A is the M-adapted function, we conclude that, for each $t \in [0,T]$,

$$\|A(t)\|_{\mathcal{L}_M(t)} = \|A(t)\|_{\mathcal{L}_M(s)}, \quad s \in [t,T].$$

Due to the latter equality and finite additivity of the measure μ, definition (2.4) is correct, i.e., it does not depend on the choice of representation A in S.

According to [24], [25], for $A \in S$ of kind (2.3), the *H-stochastic integral* with respect to M is defined as an element of $\mathcal{H}$ given by

$$\int_{[0,T]} A(t)\, dM_t := \sum_{k=0}^{n-1} A_k M((t_k,t_{k+1}])$$

and has the property

$$\Big\| \int_{[0,T]} A(t)\, dM_t \Big\|^2_{\mathcal{H}} \le \int_{[0,T]} \|A(t)\|^2_{\mathcal{L}_M(t)}\, d\mu(t). \tag{2.5}$$

Inequality (2.5) allows us to extend the H-stochastic integral to operator–valued functions $[0,T] \ni t \mapsto A(t)$ which are not necessarily simple. Namely, denote by $S_2 = S_2(M)$ *a Banach space associated with the seminorm* $\|\cdot\|_{S_2}$. For its construction, at first it is necessary to pass from S to the factor space $\dot{S} := S/\{A \in S \,|\, \|A\|_{S_2} = 0\}$ and then to take the completion of $\dot{S}$. It is not difficult to understand that the elements of the space S_2 are equivalence classes of operator–valued functions on $[0,T]$ whose values are linear operators in the space $\mathcal{H}$. In what follows we will not distinguish between the equivalence class and operator–valued function from this class.

Definition 2.3. An operator–valued function $[0,T] \ni t \mapsto A(t)$ is said to be *H-stochastic integrable* with respect to M if A belongs to the space S_2, i.e. there exists a sequence $(A_n)_{n=0}^\infty$ of simple operator–valued functions $A_n \in S$ such that

$$\int_{[0,T]} \|A(t) - A_n(t)\|^2_{\mathcal{L}_M(t)}\, d\mu(t) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.6}$$

An H*-stochastic integral of* $A \in S_2$ *with respect to* M is defined by

$$\int_{[0,T]} A(t)\, dM_t := \lim_{n\to\infty} \int_{[0,T]} A_n(t)\, dM_t \in \mathcal{H},$$

where $(A_n)_{n=0}^{\infty} \subset S$ is any sequence satisfying (2.6).

Note that due to inequality (2.5) the latter limit exists in $\mathcal{H}$ and does not depend on the choice of the sequence $(A_n)_{n=0}^{\infty} \subset S$ satisfying (2.6).

2.2. **The Itô stochastic integral as the H-stochastic one.** We start from the definition of the classical *Itô stochastic integral*, see books [18], [15], [20], [22] for details. We will consider only normal martingales in our presentation of stochastic integration. This family of martingales includes Brownian motion, the compensated Poisson process and the Azéma martingales as particular cases, see e.g. [13], [10], [20], [19], [3].

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space endowed with a right continuous filtration $\mathbf{A} := \{\mathcal{A}_t\}_{t\in[0,T]}$, i.e., with a family $\{\mathcal{A}_t\}_{t\in[0,T]}$ of σ-algebras $\mathcal{A}_t \subset \mathcal{A}$ such that $\mathcal{A}_s \subset \mathcal{A}_t$ if $s \le t$ and $\mathcal{A}_t = \bigcap_{s>t} \mathcal{A}_s$ for all $t \in [0,T]$. Furthermore, we assume that $\mathcal{A}_0$ contains all the P-null sets of $\mathcal{A}$, $\mathcal{A} = \mathcal{A}_T$ and $\mathcal{A}_0$ is trivial (i.e., every $\alpha \in \mathcal{A}_0$ has probability 0 or 1).

By definition, a process $N = \{N_t\}_{t\in[0,T]}$ is a *normal martingale* on $(\Omega, \mathcal{A}, P)$ with respect to $\{\mathcal{A}_t\}_{t\in[0,T]}$ if $\{N_t\}_{t\in[0,T]}$ and $\{N_t^2 - t\}_{t\in[0,T]}$ are martingales with respect to $\{\mathcal{A}_t\}_{t\in[0,T]}$. In other words, N is a *normal martingale* if $N_t \in L^2(\Omega, \mathcal{A}_t, P)$ for all $t \in [0,T]$ and

$$\mathbb{E}[N_t - N_s | \mathcal{A}_s] = 0, \quad \mathbb{E}[(N_t - N_s)^2 | \mathcal{A}_s] = t - s$$

for all $s, t \in [0,T]$ such that $s \le t$ (in what follows, without loss of generality we will assume that $N_0 = 0$). Recall that a conditional expectation $\mathbb{E}[\,\cdot\,|\mathcal{A}_t]$ is the orthogonal projection in the space $L^2(\Omega, \mathcal{A}, P)$ onto its subspace $L^2(\Omega, \mathcal{A}_t, P)$ and, moreover, the corresponding projection–valued function

$$E : [0,T] \to \mathcal{L}(\mathcal{H}), \quad t \mapsto E_t := \mathbb{E}[\,\cdot\,|\mathcal{A}_t],$$

is a resolution of identity in $L^2(\Omega, \mathcal{A}, P)$.

Let us introduce the space of functions for which the Itô integral is defined. We will denote by $L^2([0,T]\times\Omega)$ the space of all $\mathcal{B}([0,T]) \times \mathcal{A}$-measurable functions (classes of functions) $F\colon [0,T]\times\Omega \to \mathbb{C}$ such that

$$\int_{[0,T]} \int_{\Omega} |F(t,\omega)|^2\, dP(\omega)\, dt = \int_{[0,T]} \|F(t)\|^2_{L^2(\Omega,\mathcal{A},P)}\, dt < \infty,$$

and $L^2_a([0,T]\times\Omega)$ will present the subspace of $\mathbf{A}$-adapted functions. Recall that a function $F \in L^2([0,T]\times\Omega)$ is called $\mathbf{A}$-adapted if $F(t,\cdot)$ is $\mathcal{A}_t$-measurable for almost all $t \in [0,T]$, i.e., $F(t,\cdot) = \mathbb{E}[F(t,\cdot)|\mathcal{A}_t]$ for almost all $t \in [0,T]$.

Assume that $F(t) = F(t,\omega)$ is a simple function from the space $L^2_a([0,T]\times\Omega)$, i.e., there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that

$$F(\cdot) = \sum_{k=0}^{n-1} F_k \mathbb{1}_{(t_k, t_{k+1}]}(\cdot) \in L^2_a([0,T]\times\Omega).$$

The *Itô integral* of such function F with respect to N is defined by

$$\int_{[0,T]} F(t)\, dN_t := \sum_{k=0}^{n-1} F_k (N_{t_{k+1}} - N_{t_k}) \in L^2(\Omega, \mathcal{A}, P) \tag{2.7}$$

and satisfies the Itô isometry property

$$\Big\| \int_{[0,T]} F(t)\, dN_t \Big\|^2_{L^2(\Omega,\mathcal{A},P)} = \int_{[0,T]} \|F(t)\|^2_{L^2(\Omega,\mathcal{A},P)}\, dt.$$

Since the set $L^2_{a,s}([0,T]\times\Omega)$ of all simple functions from $L^2_a([0,T]\times\Omega)$ is dense in the space $L^2_a([0,T]\times\Omega)$, extending the mapping

$$L^2_a([0,T]\times\Omega)\supset L^2_{a,s}([0,T]\times\Omega)\ni F\mapsto\int_{[0,T]}F(t)\,dN_t\in L^2(\Omega,\mathcal{A},P)$$

by continuity, we obtain a definition of the Itô integral on $L^2_a([0,T]\times\Omega)$. In what follows, we keep the same notation $\int_{[0,T]}F(t)\,dN_t$ for the extension.

From [24] (Theorems 2 and 3) it follows that the Itô stochastic integral with respect to the normal martingale N can be interpreted as the H-stochastic integral. Namely, we set $\mathcal{H}:=L^2(\Omega,\mathcal{A},P)$ and consider in this space a resolution of identity

$$E:[0,T]\to\mathcal{L}(\mathcal{H}),\quad t\mapsto E_t:=\mathbb{E}[\,\cdot\,|\mathcal{A}_t],$$

generated by the filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$. It is easy to see that the normal martingale N is the $L^2(\Omega,\mathcal{A},P)$-valued martingale with respect to E and

$$\mu([0,t])=\|N_t\|^2_{L^2(\Omega,\mathcal{A},P)}=\mathbb{E}[N_t^2]=\mathbb{E}[N_t^2\,|\,\mathcal{A}_0]=t$$

is the ordinary Lebesgue measure on $[0,T]$.

Theorem 2.1. *For a given function $F\in L^2([0,T]\times\Omega)$ the family $\{A_F(t)\}_{t\in[0,T]}$ of the operators $A_F(t)$ of multiplication by $F(t)\in L^2(\Omega,\mathcal{A},P)$ in the space $L^2(\Omega,\mathcal{A},P)$,*

$$L^2(\Omega,\mathcal{A},P)\supset\mathrm{Dom}(A_F(t))\ni G\mapsto A_F(t)G:=F(t)G\in L^2(\Omega,\mathcal{A},P),$$
$$\mathrm{Dom}(A_F(t)):=\{G\in L^2(\Omega,\mathcal{A},P)\,|\,F(t)G\in L^2(\Omega,\mathcal{A},P)\},$$

is H-stochastic integrable with respect to the normal martingale N (i.e. A_F belongs to $S_2=S_2(N)$) if and only if F belongs to the space $L^2_a([0,T]\times\Omega)$.

Moreover, if $F\in L^2_a([0,T]\times\Omega)$ then

$$\int_{[0,T]}A_F(t)\,dN_t=\int_{[0,T]}F(t)\,dN_t.$$

2.3. **The Itô integral on a Fock space as the H-stochastic integral.** Let us denote by $L^2([0,T]):=L^2([0,T],dt)$ a complex L^2-space with respect to the Lebesgue measure $dt=dm(t)$. The corresponding symmetric Fock space is defined as

$$\mathcal{F}:=\mathbb{C}\oplus\bigoplus_{n=1}^{\infty}L^2([0,T])^{\odot n}n!,$$

where $\odot$ stands for the symmetric tensor product ($\otimes$ is the ordinary tensor product). Thus, $\mathcal{F}$ is a complex Hilbert space of sequences $f=(f_n)_{n=0}^\infty$ such that each f_n belongs to $L^2([0,T])^{\odot n}$ and

$$\|f\|^2_{\mathcal{F}}=|f_0|^2+\sum_{n=1}^{\infty}\|f_n\|^2_{L^2([0,T])^{\odot n}}n!<\infty.$$

In what follows, we always identify in the natural way the space $L^2([0,T])^{\odot n}$ with the space $L^2_{\mathrm{sym}}([0,T]^n)$ of all complex–valued symmetric functions from $L^2([0,T]^n)$. It is easy to see that

$$\begin{aligned}\|f_n\|^2_{L^2([0,T])^{\odot n}}&=\int_{[0,T]^n}|f_n(t_1,\dots,t_n)|^2\,dt_1\dots dt_n\\&=n!\int\limits_0^T\int\limits_0^{t_n}\cdots\Big(\int\limits_0^{t_2}|f_n(t_1,\dots,t_n)|^2\,dt_1\Big)\dots dt_{n-1}\,dt_n\end{aligned}$$

for all $f_n\in L^2([0,T])^{\odot n}\cong L^2_{\mathrm{sym}}([0,T]^n)$.

Denote by $\Diamond$ the Wick product in the Fock space $\mathcal{F}$. For given $f = (f_n)_{n=0}^{\infty}$ and $g = (g_n)_{n=0}^{\infty}$ from $\mathcal{F}$ the Wick product $f \Diamond g$ is defined by

$$f \Diamond g := \Big(\sum_{m=0}^{n} f_m \odot g_{n-m} \Big)_{n=0}^{\infty},$$

provided the latter sequence belongs to the Fock space $\mathcal{F}$. Let

$$L^2([0,T];\mathcal{F}) := L^2([0,T], dt; \mathcal{F})$$

be the Hilbert space of $\mathcal{F}$-valued functions

$$f\colon [0,T] \to \mathcal{F}, \quad \|f\|^2_{L^2([0,T];\mathcal{F})} := \int_{[0,T]} \|f(t)\|^2_{\mathcal{F}}\, dt < \infty,$$

with the corresponding scalar product.

Now we are ready to recall the definition of the Itô integral on the Fock space proposed in [4] (see also [2]). First of all a function $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L^2([0,T];\mathcal{F})$ is said to be *Itô integrable* if, for almost all $t \in [0,T]$,

$$f(t) = (f_0(t), 1\!\!1_{[0,t]} f_1(t), \ldots, 1\!\!1_{[0,t]^n} f_n(t), \ldots).$$

The set of all Itô integrable functions will be denoted by $L^2_a([0,T];\mathcal{F})$.

Observe that each component $f_n(t_1,\ldots,t_n;t)$ of a function

$$f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L^2([0,T];\mathcal{F})$$

belongs to the space $L^2_{\mathrm{sym}}([0,T]^n) \otimes L^2([0,T])$. It means that f_n belongs to $L^2([0,T]^{n+1})$ and it is a symmetric function in the first n variables, i.e., for m-almost all $t \in [0,T]$ and for $m^{\otimes n}$-almost all $(t_1,\ldots,t_n) \in [0,T]^n$,

$$f_n(t_1,\ldots,t_n;t) = \frac{1}{n!} \sum_{\sigma} f_n(t_{\sigma(1)},\ldots,t_{\sigma(n)};t),$$

where σ running over all permutations of $\{1,\ldots,n\}$, m is the Lebesgue measure.

Denote by $L^2_{a,s}([0,T];\mathcal{F})$ the space of all simple Itô integrable functions. By definition, $f \in L^2_{a,s}([0,T];\mathcal{F})$ if and only if there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that

$$f(\cdot) = \sum_{k=0}^{n-1} f_{(k)} 1\!\!1_{(t_k,t_{k+1}]}(\cdot) \in L^2_a([0,T];\mathcal{F})$$

It can be shown that the set $L^2_{a,s}([0,T];\mathcal{F})$ is dense in the space $L^2_a([0,T];\mathcal{F})$.

The *Itô integral* $\mathbb{I}(f)$ of a simple function $f \in L^2_{a,s}([0,T];\mathcal{F})$ is defined by

$$\mathbb{I}(f) := \sum_{k=0}^{n-1} f_{(k)} \Diamond (0, 1\!\!1_{(t_k,t_{k+1}]}, 0, 0, \ldots) \in \mathcal{F} \tag{2.8}$$

and satisfies the following isometry property (see e.g. [2, 1])

$$\big\|\mathbb{I}(f)\big\|^2_{\mathcal{F}} = \int_{[0,T]} \big\|f(t)\big\|^2_{\mathcal{F}}\, dt.$$

The latter property allows us to extend the mapping

$$L^2_a([0,T];\mathcal{F}) \supset L^2_{a,s}([0,T];\mathcal{F}) \ni f \mapsto \mathbb{I}(f) \in \mathcal{F}$$

by continuity and obtain a definition of the Itô integral $\mathbb{I}(f)$ for each $f \in L^2_a([0,T];\mathcal{F})$ (we keep the same notation $\mathbb{I}$ for the extension).

From [25] (Theorem 5.1) it follows that the Itô integral $\mathbb{I}(f)$ of $f \in L_a^2([0,T];\mathcal{F})$ can be considered as the H-stochastic integral. Namely, we set $\mathcal{H} := \mathcal{F}$ and consider in this space a resolution of identity

$$\mathrm{Exp}\,1\!\!1\colon [0,T] \to \mathcal{L}(\mathcal{F}), \quad t \mapsto \mathrm{Exp}\,1\!\!1_{[0,t]} := \mathrm{id} \oplus \bigoplus_{n=1}^{\infty} 1\!\!1_{[0,t]^n},$$

generated by a resolution of identity

$$[0,T] \ni t \mapsto 1\!\!1_{[0,t]} g \in L^2([0,T]), \quad g \in L^2([0,T]).$$

Thus, we have

$$\mathrm{Exp}\,1\!\!1_{[0,t]} f := (f_0, 1\!\!1_{[0,t]} f_1, \ldots, 1\!\!1_{[0,t]^n} f_n, \ldots) \in \mathcal{F}, \quad f = (f_n)_{n=0}^{\infty} \in \mathcal{F}.$$

It is clear that

$$Z\colon [0,T] \to \mathcal{F}, \quad t \mapsto Z_t := (0, 1\!\!1_{[0,t]}, 0, 0, \ldots),$$

is the $\mathcal{F}$-valued martingale with respect to $\mathrm{Exp}\,1\!\!1_{[0,t]}$ and

$$\mu([0,t]) := \|Z_t\|_{\mathcal{F}}^2 = \|1\!\!1_{[0,t]}\|_{L^2([0,T])}^2 = m([0,t]) = t$$

is the Lebesgue measure on $[0,T]$.

Theorem 2.2. *A function $f \in L^2([0,T];\mathcal{F})$ belongs to the space $L_a^2([0,T];\mathcal{F})$ if and only if the corresponding operator–valued function $[0,T] \ni t \mapsto A_f(t)$ whose values are operators $A_f(t)$ of Wick multiplication by $f(t) \in \mathcal{F}$ in the Fock space $\mathcal{F}$,*

$$\mathcal{F} \supset \mathrm{Dom}(A_f(t)) \ni g \mapsto A_f(t)g := f(t)\Diamond g \in \mathcal{F},$$
$$\mathrm{Dom}(A_f(t)) := \{g \in \mathcal{F} \mid f(t)\Diamond g \in \mathcal{F}\},$$

belongs to the space $S_2 = S_2(Z)$.

Moreover, if $f \in L_a^2([0,T];\mathcal{F})$ then

$$\mathbb{I}(f) = \int_{[0,T]} A_f(t)\, dZ_t.$$

Taking into account Theorem 2.2 it is natural to denote the Itô integral $\mathbb{I}(f)$ of $f \in L_a^2([0,T];\mathcal{F})$ by $\int_{[0,T]} f(t)\, dZ_t$. Note that such integral can be expressed in terms of the Fock space $\mathcal{F}$ structure, see e.g. [25] (Theorem 5.2). Namely, for any $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L_a^2([0,T];\mathcal{F})$ we have

$$\int_{[0,T]} f(t)\, dZ_t = (0, \hat{f}_1, \ldots, \hat{f}_n, \ldots) \in \mathcal{F}, \tag{2.9}$$

where $\hat{f}_n(t_1, \ldots, t_n)$ denotes the symmetrization of $f_{n-1}(t_1, \ldots, t_{n-1}; t)$ with respect to n variables. Since $f_{n-1}(t_1, \ldots, t_{n-1}; t)$ is symmetric in the first $n-1$ variables, we have

$$\hat{f}_n(t_1, \ldots, t_n) := \frac{1}{n} \sum_{k=1}^{n} f_{n-1}(t_1, \ldots, \cancel{t_k}, \ldots, t_n; t_k).$$

2.4. **Relationship between the classical Itô integral and the Itô integral on the Fock space $\mathcal{F}$.** Without going into details, let us give a brief exposition of probabilistic interpretations of the Fock space $\mathcal{F}$, see e.g. [20], [14] for more details. As before, let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a right continuous filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$, $\mathcal{A}_0$ be the trivial σ-algebra containing all the P-null sets of $\mathcal{A}$ and $\mathcal{A} = \mathcal{A}_T$. Suppose that $N = \{N_t\}_{t\in[0,T]}$, $N_0 = 0$, is a *normal martingale* on $(\Omega, \mathcal{A}, P)$ with respect to $\{\mathcal{A}_t\}_{t\in[0,T]}$.

Let $I_{N,0}(f_0) := f_0 \in \mathbb{C}$ and, for each $n \in \mathbb{N}$,

$$I_{N,n}(f_n) := n! \int_{\Delta^n} f_n(t_1, \dots, t_n)\, dN_{t_1} \dots dN_{t_n},$$
$$\Delta^n := \{(t_1, \dots, t_n) \in [0,T] \,|\, t_1 < \dots < t_n\}, \quad f_n \in L^2_{\mathrm{sym}}([0,T]^n),$$

be an n-iterated stochastic integral with respect to N. It is known that the integral $I_{N,n}(f_n)$ has the isometry property

$$\|I_{N,n}(f_n)\|^2_{L^2(\Omega,\mathcal{A},P)} = (n!)^2 \int_{\Delta^n} |f_n(t_1, \dots, t_n)|^2\, dt_1 \dots dt_n = \|f_n\|^2_{L^2([0,T])^{\odot n}} n!,$$

and, moreover, the orthogonality property

$$(I_{N,n}(f_n), I_{N,m}(f_m))_{L^2(\Omega,\mathcal{A},P)} = \begin{cases} 0, & n \neq m; \\ \|f_n\|^2_{L^2([0,T])^{\odot n}} n!, & n = m. \end{cases}$$

Hence, the mapping

$$I_N \colon \mathcal{F} \to L^2(\Omega, \mathcal{A}, P), \quad f = (f_n)_{n=0}^{\infty} \mapsto I_N f := \sum_{n=0}^{\infty} I_{N,n}(f_n), \tag{2.10}$$

is a well-defined isometry. When $I_N : \mathcal{F} \to L^2(\Omega, \mathcal{A}, P)$ is a unitary operator (i.e., I_N isometrically maps the whole space $\mathcal{F}$ onto whole $L^2(\Omega, \mathcal{A}, P)$) one says that N possesses the Chaotic Representation Property (CRP). In this case the unique decomposition of $F \in L^2(\Omega, \mathcal{A}, P)$ as $F = \sum_{n=0}^{\infty} I_{N,n}(f_n)$ is called the chaotic expansion of F. We observe that the Brownian motion, the compensated Poisson process and some Azéma martingales are examples of normal martingales which possess the CRP, see for instance [20], [14], [13], [19], [3].

Let N be a normal martingale with CRP. It is not difficult to show that the normal martingale N is the I_N-image of the $\mathcal{F}$-valued martingale

$$Z \colon [0,T] \to \mathcal{F}, \quad t \mapsto Z_t := (0, 1\!\!1_{[0,t]}, 0, 0, \dots).$$

More exactly,

$$N_t = I_N Z_t, \quad t \in [0,T].$$

Moreover, according to [25] (Lemma 6.1) the resolution of identity

$$E : [0,T] \to \mathcal{L}(\mathcal{H}), \quad t \mapsto E_t := \mathbb{E}[\,\cdot\,|\mathcal{A}_t],$$

is the I_N-image of the resolution of identity

$$\mathrm{Exp}\,1\!\!1 \colon [0,T] \to \mathcal{L}(\mathcal{F}), \quad t \mapsto \mathrm{Exp}\,1\!\!1_{[0,t]} := \mathrm{id} \oplus \bigoplus_{n=1}^{\infty} 1\!\!1_{[0,t]^n}.$$

Namely,

$$\mathrm{Exp}\,1\!\!1_{[0,t]} = I_N^{-1} E_t I_N, \quad t \in [0,T].$$

Before establishing the relationship between the classical Itô integral and the Itô integral on the Fock space $\mathcal{F}$ we note that the spaces $L^2([0,T] \times \Omega)$ and $L^2([0,T]; \mathcal{F})$ can be interpreted as tensor products $L^2([0,T]) \otimes L^2(\Omega, \mathcal{A}, P)$ and $L^2([0,T]) \otimes \mathcal{F}$ respectively. Therefore

$$1 \otimes I_N : L^2([0,T]; \mathcal{F}) \to L^2([0,T] \times \Omega)$$

is a well-defined unitary operator. From [25] (Theorem 6.1) we get the following result.

Theorem 2.3. *We have* $L^2_a([0,T] \times \Omega) = (1 \otimes I_N) L^2_a([0,T]; \mathcal{F})$ *and*

$$I_N \left(\int_{[0,T]} f(t)\, dZ_t \right) = \int_{[0,T]} I_N f(t)\, dN_t$$

for every $f \in L^2_a([0,T]; \mathcal{F})$.

Remark 2.1. It should be stressed that for any normal martingale N with CRP the I_N^{-1}-image of the Itô integral with respect to N concides with the integral $\mathbb{I}$ on $\mathcal{F}$.

Remark 2.2. Comparing (2.7) and (2.8) one can see the relationship between the Wick multiplication $\lozenge$ on $\mathcal{F}$ and the ordinary multiplication on $L^2(\Omega, \mathcal{A}, P)$. Namely, suppose $t \in [0,T]$ and $F \in L^2(\Omega, \mathcal{A}, P)$ is an $\mathcal{A}_t$-adapted function. Then, for each interval $(s_1, s_2] \subset (t,T]$, we have

$$\int_{[0,T]} F \mathbb{1}_{(s_1,s_2]}(t)\, dN_t = F(N_{s_2} - N_{s_1}).$$

On the other hand

$$\int_{[0,T]} F \mathbb{1}_{(s_1,s_2]}(t)\, dN_t = I_N\left(\int_{[0,T]} I_N^{-1}(F)\mathbb{1}_{(s_1,s_2]}(t)\, dZ_t\right) = I_N\Big(I_N^{-1}(F)\lozenge(Z_{s_2} - Z_{s_1})\Big).$$

Since $N_t = I_N Z_t$ we conclude that

$$I^{-1}(F(N_{s_2} - N_{s_1})) = I_N^{-1}(F)\lozenge\Big(I_N^{-1}(N_{s_2}) - I_N^{-1}(N_{s_1})\Big).$$

However it can be shown that in general case the I_N-image of the Wick multiplication $\lozenge$ distinguishes from the ordinary multiplication.

Remark 2.3. Since $L_a^2([0,T]\times\Omega) = (1 \otimes I_N)L_a^2([0,T];\mathcal{F})$, for any $F \in L_a^2([0,T]\times\Omega)$ there exists a uniquely defined vector $f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L_a^2([0,T];\mathcal{F})$ such that

$$F(t) = I_N f(t) = \sum_{n=0}^\infty I_{N,n}(f_n(t))$$

for almost all $t \in [0,T]$. Hence, by (2.9), (2.10) and Theorem 2.3, we immediately get the "Fock–space" representation of the classical Itô integral

$$\begin{aligned}\int_{[0,T]} F(t)\, dN_t &= I_N\left(\int_{[0,T]} f(t)\, dZ_t\right)\\ &= I_N(0, \hat{f}_1, \ldots, \hat{f}_n, \ldots) = \sum_{n=1}^\infty I_{N,n}(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P).\end{aligned}$$

3. Hitsuda–Skorokhod integral on a Fock space

In this section we recall some standard facts about the Hitsuda–Skorokhod integral (a natural generalization of the Itô integral $\mathbb{I}$) on a Fock space and its rigging. For the proof we refer the reader to e.g. [19, 1, 17].

The most naive and natural idea is to define a generalization of the Itô integral $\mathbb{I}$ by formula (2.9) for all functions $f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L^2([0,T];\mathcal{F})$ such that $(0, \widehat{f}_0, \widehat{f}_1, \ldots)$ belongs to $\mathcal{F}$. Namely, we accept the following definition.

Definition 3.1. For a function $f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L^2([0,T];\mathcal{F})$ such that

$$(0, \widehat{f}_0, \widehat{f}_1, \ldots) \in \mathcal{F} \quad \text{or, equivalently,} \quad \sum_{n=0}^\infty \|\widehat{f}_n\|^2_{L^2([0,T])^{\odot n+1}}(n+1)! < \infty \tag{3.11}$$

the Hitsuda–Skorokhod integral on $\mathcal{F}$ is defined by the formula

$$\mathbb{I}_{\mathrm{ext}}(f) := (0, \widehat{f}_0, \widehat{f}_1, \ldots).$$

Remark 3.1. Let N be a normal martingale with CRP. Applying the unitary operator I_N to $\mathbb{I}_{\mathrm{ext}}$, we obtain a definition of the Hitsuda–Skorokhod integral on $L^2(\Omega, \mathcal{A}, P)$, see e.g. [19]. Note that, in the case when N is a Brownian motion, exactly in such a way the extended stochastic integral was defined by Hitsuda [12] and Skorokhod [23, 16].

It is known that the integral $\mathbb{I}_{\mathrm{ext}}$ can be extended to a Bochner one. Before formulation the corresponding result, let as first look at the following heuristic argumentation.

According to (2.8) for a simple Itô integrable function

$$f(\cdot)=\sum_{k=0}^{n-1} f_{(k)}\,\mathbb{1}_{(t_k,t_{k+1}]}(\cdot)\in L^2_a([0,T];\mathcal{F})$$

we have

$$\mathbb{I}(f)=\sum_{k=0}^{n-1} f_{(k)}\Diamond(0,\mathbb{1}_{(t_k,t_{k+1}]},0,0,\dots).$$

Using this equality and the formal representation

$$(0,\mathbb{1}_{(t_k,t_{k+1}]},0,0,\dots)=\int_{(t_k,t_{k+1}]}(0,\delta_t,0,0,\dots)\,dt$$

(here δ_t denotes the Dirac delta function at t) we obtain (at least formally)

$$\begin{aligned}
\mathbb{I}(f)&=\sum_{k=0}^{n-1} f_{(k)}\Diamond(0,\mathbb{1}_{(t_k,t_{k+1}]},0,0,\dots)=\sum_{k=0}^{n-1} f_{(k)}\Diamond\int_{(t_k,t_{k+1}]}(0,\delta_t,0,0,\dots)\,dt\\
&=\sum_{k=0}^{n-1}\int_{(t_k,t_{k+1}]} f_{(k)}\Diamond(0,\delta_t,0,0,\dots)\,dt\\
&=\int_{[0,T]}\Big(\sum_{k=0}^{n-1} f_{(k)}\,\mathbb{1}_{(t_k,t_{k+1}]}(t)\Big)\Diamond(0,\delta_t,0,0,\dots)\,dt\\
&=\int_{[0,T]} f(t)\Diamond(0,\delta_t,0,0,\dots)\,dt.
\end{aligned}$$

Since the delta-function δ_t is not a square integrable one, the last formula can not be accepted as an extension of $\mathbb{I}$ on $L^2([0,T];\mathcal{F})$. However it can be shown that this formula holds in some bigger space than $L^2([0,T];\mathcal{F})$. To this end, it is necessary to introduce an appropriate rigging of the Fock space $\mathcal{F}$.

Namely, let us fix a rigging of $L^2([0,T])$,

$$W^2_{-p}\supset L^2([0,T])\supset W^2_p, \tag{3.12}$$

where $W^2_p:=W^2_p([0,T],dt)$, $p\in\mathbb{N}$, is the Sobolev space. By definition, W^2_p is the closure of $C^\infty([0,T])$ in the norm $\|\cdot\|_{W^2_p}$ generated by the scalar product

$$(\varphi,\psi)_{W^2_p}=\sum_{k=0}^{p}\Big(\int_{[0,T]}\Big(\frac{d^k\varphi}{dt^k}\Big)(t)\Big(\frac{d^k\overline{\psi}}{dt^k}\Big)(t)\,dt\Big).$$

Here $C^\infty([0,T])$ denotes the set of all real–valued infinite differentiable functions on $[0,T]$.

It is well known (see, e.g., [7], [8]) that for all $p\in\mathbb{N}$ the space W^2_p is densely and continuously embedded into the space $L^2([0,T])$, and this embedding is quasinuclear, i.e. of Hilbert–Schmidt type. The space W^2_{-p} is dual to W^2_p with respect to the zero space H_0 (see, e.g., [7], [8] for more details). We denote by $\langle\cdot,\cdot\rangle$ the dual pairing between elements of W^2_{-p} and W^2_p inducted by the scalar product in $L^2([0,T])$. We preserve the notation $\langle\cdot,\cdot\rangle$ for the dual pairings in tensor powers of chain (3.12).

Now we are ready to introduce the so-called Kondratiev–type Fock spaces $\mathcal{F}(p,q)$. Namely, for $p,q\in\mathbb{N}$ we set

$$\mathcal{F}(p,q):=\bigoplus_{n=0}^{\infty}(W^2_p)^{\odot n}(n!)^2 2^{qn},\quad \mathcal{F}_+:=\operatorname*{pr\,lim}_{p,q\in\mathbb{N}}\mathcal{F}(p,q),$$

where $\mathcal{F}(p,q)$ denotes a complex Hilbert space of sequences $f = (f_n)_{n=0}^{\infty}$ such that $f_n \in (W_p^2)^{\odot n}$ ($(W_p^2)^{\odot 0} := \mathbb{C}$) and

$$\|f\|^2_{\mathcal{F}(p,q)} := \sum_{n=0}^{\infty} \|f_n\|^2_{(W_p^2)^{\odot n}} (n!)^2 2^{qn} < \infty.$$

It can be shown that, for all $p, q \in \mathbb{N}$, the embedding $\mathcal{F}(p,q) \hookrightarrow \mathcal{F}$ is dense and quasinuclear (see, e.g., [7] for details) and one can construct a rigging

$$\mathcal{F}(-p,-q) \supset \mathcal{F} \supset \mathcal{F}(p,q),$$

where the space

$$\mathcal{F}(-p,-q) = \bigoplus_{n=0}^{\infty} (W_{-p}^2)^{\odot n} 2^{-qn}$$

is dual one of $\mathcal{F}(p,q)$ with respect to the zero space $\mathcal{F}$. We denote by $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle_{\mathcal{F}}$ the dual pairing between elements of $\mathcal{F}(-p,-q)$ and $\mathcal{F}(p,q)$ inducted by the scalar product in $\mathcal{F}$.

From results of [1] the correctness of the following definition follows.

Definition 3.2. The *Hitsuda–Skorokhod integral* of a function $\xi \in L^2([0,T]; \mathcal{F}(-p,-q))$ is defined as a Bochner one in the space $\mathcal{F}(-p,-q)$ by the formula

$$\widehat{\mathbb{I}}_{\text{ext}}(\xi) := \int_{[0,T]} \xi(t) \Diamond (0, \delta_t, 0, 0, \ldots)\, dt \in \mathcal{F}(-p,-q).$$

Remark 3.2. By direct calculations we have (see, e.g., [1], Theorem 3.1)

$$\mathbb{I}_{\text{ext}}(f) = \widehat{\mathbb{I}}_{\text{ext}}(f), \quad f \in \text{Dom}\,(\mathbb{I}_{\text{ext}}).$$

This result explains the same name for the integrals $\mathbb{I}_{\text{ext}}$ and $\widehat{\mathbb{I}}_{\text{ext}}$.

Remark 3.3. For each $t \in \mathbb{R}_+$ we define the *annihilation operator* $a_-(\delta_t)$ on $\mathcal{F}(p,q)$ and the *creation operator* $a_+(\delta_t)$ on $\mathcal{F}(-p,-q)$ by setting "on coordinates"

$$(a_-(\delta_t)\varphi_n)(t_1, \ldots, t_{n-1}) := n\varphi_n(t_1, \ldots, t_{n-1}, t), \quad (a_+(\delta_t)\xi_n) := \delta_t \odot \xi_n.$$

It is easy to show (see, e.g., [9]) that $a_-(\delta_t)$ and $a_+(\delta_t)$ are continuous operators on $\mathcal{F}(p,q)$ and $\mathcal{F}(-p,-q)$ respectively, and $a_+(\delta_t)$ is the dual operator of $a_-(\delta_t)$ in the sense that for all $\xi \in \mathcal{F}(-p,-q)$ and $\varphi \in \mathcal{F}(p,q)$

$$\langle\!\langle a_+(\delta_t)\xi, \varphi \rangle\!\rangle_{\mathcal{F}} = \langle\!\langle \xi, a_-(\delta_t)\varphi \rangle\!\rangle_{\mathcal{F}}.$$

It is obvious that

$$a_+(\delta_t)\xi = \xi \Diamond (0, \delta_t, 0, 0, \ldots), \quad \xi \in \mathcal{F}(-p,-q),$$

and, for all $\xi \in L^2([0,T]; \mathcal{F}(-p,-q))$,

$$\widehat{\mathbb{I}}_{\text{ext}}(\xi) = \int_{[0,T]} a_+(\delta_t)\xi(t)\, dt \in \mathcal{F}(-p,-q).$$

4. The H-stochastic integral as a Bochner one

4.1. **Definition and properties.** Let $\mathcal{H}$ be a complex separable Hilbert space and

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+$$

be a rigging of $\mathcal{H}$ such that the embedding operator $O : \mathcal{H}_+ \to \mathcal{H}$ is a Hilbert–Schmidt type. It can be shown that the adjoint (with respect to $\mathcal{H}$) operator $O^+ : \mathcal{H} \to \mathcal{H}_-$ is also a Hilbert–Schmidt one and embeds $\mathcal{H}$ into $\mathcal{H}_-$. The space of all Hilbert–Schmidt operators from $\mathcal{H}_+$ to $\mathcal{H}_-$ will be denoted by $\mathcal{S}_2 = \mathcal{S}_2(\mathcal{H}_+ \to \mathcal{H}_-)$.

Let $E \colon \mathcal{B}([0,T]) \to \mathcal{L}(\mathcal{H})$ be a resolution of the identity in $\mathcal{H}$ and $M : [0,T] \to \mathcal{H}$ be an $\mathcal{H}$-valued martingale with respect to E. Suppose that $M_T \in \mathcal{H}_+$ and the measure $\mu(\alpha) := \|M(\alpha)\|^2_{\mathcal{H}}$ is equivalent to E, i.e., μ is absolutely continuous with respect to E

and vice versa. Then due to the Berezansky–Gel′fand–Kostyuchenko theorem (see, e.g., [5], [11]) the operator–valued measure

$$\theta : \mathcal{B}([0,T]) \to \mathcal{S}_2(\mathcal{H}_+ \to \mathcal{H}_-), \quad \alpha \mapsto \theta(\alpha) := O^+E(\alpha)O,$$

is differentiable with respect to μ. More exactly, there exists a weakly μ-measurable non-negative operator–valued function $[0,T] \ni t \mapsto P(t)$ with values in $\mathcal{S}_2(\mathcal{H}_+ \to \mathcal{H}_-)$ such that

$$O^+E(\alpha)O = \int_\alpha P(s)\, d\mu(s), \quad \alpha \in \mathcal{B}([0,T]), \tag{4.13}$$

where the latter integral exists as a Bochner one in the space $\mathcal{S}_2(\mathcal{H}_+ \to \mathcal{H}_-)$. Moreover, $\|P(s)\|_{\mathcal{S}_2} \le C$ for μ-almost all $s \in [0,T]$ and some $C > 0$. The function $P(t)$ is called the Radon–Nikodym derivative $(d\theta/d\rho)(t) = P(t)$.

Since $M_t = E_tM_T$ for all $t \in [0,T]$ and $M_T \in \mathcal{H}_+$, using (4.13) we get

$$O^+M_t = O^+E_tOM_T = \int_{[0,t]} P(s)M_T\, d\mu(s), \quad t \in [0,T]. \tag{4.14}$$

Let us define an integral

$$\int_{[0,T]} A(t)\, dO^+M_t$$

for a certain class of functions $[0,T] \ni t \mapsto A(t)$ whose values are linear operators in $\mathcal{H}_-$. A construction of such integral we give step-by-step, beginning with the simplest class of operator–valued functions. Let us introduce the required class of simple functions.

For each point $t \in [0,T]$, we denote by

$$\mathcal{H}_{O^+M}(t) := \overline{\mathrm{span}}\{O^+M((s_1,s_2]) \,|\, (s_1,s_2] \subset (t,T]\} \subset \mathcal{H}_-$$

the closed linear span of the set $\{O^+M((s_1,s_2]) \,|\, (s_1,s_2] \subset (t,T]\}$ in $\mathcal{H}_-$ and by

$$\mathcal{L}_{O^+M}(t) = \mathcal{L}(\mathcal{H}_{O^+M}(t) \to \mathcal{H}_-)$$

the set of all linear operators in $\mathcal{H}_-$ that continuously act from $\mathcal{H}_{O^+M}(t)$ to $\mathcal{H}_-$. It is clear that $\mathcal{H}_M(t) \subset \mathcal{H}_{O^+M}(t)$ and $P(t)M_T \in \mathcal{H}_{O^+M}(t)$ for μ-almost all $t \in [0,T]$.

Definition 4.1. A family $\{A(t)\}_{t\in[0,T]}$ of linear operators in $\mathcal{H}_-$ is called an O^+M-*adapted* operator–valued function $[0,T] \ni t \mapsto A(t)$ if $A(t) \in \mathcal{L}_{O^+M}(t)$ for every $t \in [0,T]$.

An O^+M-adapted operator–valued function $[0,T] \ni t \mapsto A(t)$ is called simple if there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that

$$A(t) = \sum_{k=0}^{n-1} A_k 1\!\!1_{(t_k,t_{k+1}]}(t), \quad t \in [0,T]. \tag{4.15}$$

On the space $S_- = S_-(O^+M)$ of all simple O^+M-adapted operator–valued functions $[0,T] \ni t \mapsto A(t)$ we introduce a seminorm

$$\|A\|_{S_{-,2}} := \left(\int_{[0,T]} \|A(t)\|^2_{\mathcal{L}_{O^+M}(t)}\, d\mu(t)\right)^{\frac{1}{2}} := \left(\sum_{k=0}^{n-1} \|A_k\|^2_{\mathcal{L}_{O^+M}(t_k)}\, \mu((t_k,t_{k+1}])\right)^{\frac{1}{2}}.$$

The corresponding Banach space we denote by $S_{-,2} = S_{-,2}(O^+M)$.

For $A \in S_-$ of kind (4.15) we define an extended H-*stochastic integral* with respect to O^+M as an element of $\mathcal{H}_-$ given by

$$\int_{[0,T]} A(t)\, dO^+M_t := \sum_{k=0}^{n-1} A_k O^+M((t_k,t_{k+1}]). \tag{4.16}$$

Since $A \in S_-$ is an O^+M-adapted operator–valued functions, taking into account (4.14) and the properties of Bochner integral, we can rewrite (4.16) in the form

$$\begin{aligned}
\int_{[0,T]} A(t)\, dO^+M_t &= \sum_{k=0}^{n-1} A_k O^+ M((t_k, t_{k+1}]) = \sum_{k=0}^{n-1} A_k O^+ E((t_k, t_{k+1}]) O M_T \\
&= \sum_{k=0}^{n-1} A_k \int_{(t_k, t_{k+1}]} P(t) M_T \, d\mu(t) \\
&= \int_{[0,T]} \sum_{k=0}^{n-1} A_k 1\!\!1_{(t_k, t_{k+1}]}(t) P(t) M_T \, d\mu(t) \\
&= \int_{[0,T]} A(t) P(t) M_T \, d\mu(t).
\end{aligned}$$

So, integral (4.16) of a simple function $A \in S_-$ can be regarded as an ordinary Bochner integral of vector–valued function $[0,T] \ni t \mapsto A(t)P(t)M_T \in \mathcal{H}_-$. This suggests to us to take the following definition of the extended H-stochastic integral.

Definition 4.2. For $A \in S_{-,2}$, an integral of A with respect to O^+M is defined as an element of $\mathcal{H}_-$ given by

$$\int_{[0,T]} A(t)\, dO^+M_t := \int_{[0,T]} A(t)P(t)M_T \, d\mu(t),$$

where in the right-hand side we have a Bochner integral of the vector–valued function $[0,T] \ni t \mapsto A(t)P(t)M_T \in \mathcal{H}_-$.

The correctness of this definition follows from the following statement.

Lemma 4.1. *If $A \in S_{-,2}$ then the function $[0,T] \ni t \mapsto A(t)P(t)M \in \mathcal{H}_-$ is integrable in the Bochner sense with respect to μ on $[0,T]$.*

Proof. Since μ is a finite measure, $A \in S_{-,2}$ and $\|P(t)\|_{HS} \le C$ for μ-almost all $t \in [0,T]$ and some $C > 0$, we get

$$\begin{aligned}
\int_{[0,T]} \|A(t)P(t)M_T\|_{\mathcal{H}_-} d\mu(t) &\le \int_{[0,T]} \|A(t)\|_{\mathcal{L}_{O^+M}(t)} \|P(t)M_T\|_{\mathcal{H}_-} d\mu(t) \\
&\le \Big(\int_{[0,T]} \|A(t)\|^2_{\mathcal{L}_{O^+M}(t)} \mu(t) \Big)^{\frac{1}{2}} \Big(\int_{[0,T]} \|P(t)M_T\|^2_{\mathcal{H}_-} d\mu(t) \Big)^{\frac{1}{2}} \\
&= L\|A\|_{S_{-,2}} < \infty, \quad L := \Big(\int_{[0,T]} \|P(t)M_T\|^2_{\mathcal{H}_-} d\mu(t) \Big)^{\frac{1}{2}},
\end{aligned}$$

whence the necessary statement follows. □

Let us show that $\int_{[0,T]} A(t)\, dO^+M_t$ coincides with $\int_{[0,T]} A(t)\, dM_t$ for all $A \in S_{-2} \cap S_2$.

Theorem 4.1. *Let $A \in S_{-2} \cap S_2$. Then*

$$\int_{[0,T]} A(t)P(t)M_T \, d\mu(t) = \int_{[0,T]} A(t)\, dM_t$$

in the space $\mathcal{H}_-$.

Proof. It follows from the proof of Lemma 4.1 that

$$\Big\| \int_{[0,T]} A(t)\, dO^+M_t \Big\|_{\mathcal{H}_-} \le L\|A\|_{S_{-,2}}, \quad A \in S_{-,2}.$$

On the other hand, due to (2.4) and (2.5), we have

$$\Big\| \int_{[0,T]} A(t)\, dM_t \Big\|_{\mathcal{H}} \leq \|A\|_{S_2}, \quad A \in S_2.$$

Therefore, taking into account the definitions of the integrals, it is sufficient to prove the statement only for simple functions $A \in S_- \cap S$. But in this case the statement directly follows from the definitions of the integrals. □

4.2. **The Hitsuda–Skorokhod integral as the H-stochastic one.** We set

$$\begin{array}{ccccc} \mathcal{F}(-p,-q) & \supset & \mathcal{F} & \supset & \mathcal{F}(p,q) \\ \shortparallel & & \shortparallel & & \shortparallel \\ \mathcal{H}_- & & \mathcal{H} & & \mathcal{H}_+ \end{array}$$

and consider in the Fock space $\mathcal{F}$ the resolution of identity

$$\mathrm{Exp}\, 1\!\!1 \colon [0,T] \to \mathcal{L}(\mathcal{F}), \quad t \mapsto E([0,t]) := \mathrm{Exp}\, 1\!\!1_{[0,t]} = \mathrm{id} \oplus \bigoplus_{n=1}^{\infty} 1\!\!1_{[0,t]^n},$$

and the martingale

$$Z \colon [0,T] \to \mathcal{F}, \quad t \mapsto Z_t := (0, 1\!\!1_{[0,t]}, 0, 0, \ldots).$$

It is easy to see that $Z_T \in \mathcal{F}(p,q)$ and the Lebesgue measure $\mu([0,t]) := \|Z_t\|^2_{\mathcal{F}} = t$ is equivalent to $E([0,t]) := \mathrm{Exp}\, 1\!\!1_{[0,t]}$. Hence, according to (4.13) we get

$$O^+ \mathrm{Exp}\, 1\!\!1_{[0,t]} O = \int_{[0,t]} P(s)\, ds, \quad t \in [0,T]. \tag{4.17}$$

Now our purpose is to find an explicit expression of the operator–valued function $t \mapsto P(t)$. To this end, for each $t \in [0,T]$, we introduce a linear continuous operator $\tilde{\delta}_t : W^2_p \to W^2_{-p}$ by setting

$$\tilde{\delta}_t f := a_+(\delta_t) a_-(\delta_t) f = f(t)\delta_t, \quad f \in W^2_p.$$

Theorem 4.2. *For m-almost all $t \in [0,T]$, the operator $P(t) : \mathcal{F}(p,q) \to \mathcal{F}(-p,-q)$ from representation (4.17) has the form*

$$P(t) = \mathrm{id} \oplus \bigoplus_{n=1}^{\infty} \tilde{\delta}_t \otimes 1\!\!1_{[0,t]^{n-1}} n, \tag{4.18}$$

i.e., for all $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}(p,q)$,

$$P(t) f = (f_0, \tilde{\delta}_t f_1, \ldots, \tilde{\delta}_t \otimes 1\!\!1_{[0,t]^{n-1}} n f_n, \ldots).$$

In particular,

$$P_t Z_T = (0, \delta_t, 0, 0, \ldots).$$

Proof. In order to prove (4.18), it is sufficient to show that

$$\langle\!\langle \mathrm{Exp}\, 1\!\!1_{[0,t]} f^{\otimes n}, g^{\otimes n} \rangle\!\rangle_{\mathcal{F}} = \int_{[0,t]} \langle\!\langle \tilde{\delta}_s \otimes 1\!\!1_{[0,s]^{n-1}} n f^{\otimes n}, g^{\otimes n} \rangle\!\rangle_{\mathcal{F}}\, ds \tag{4.19}$$

for any $f, g \in W^2_p$, $t \in [0,T]$ and $n \in \mathbb{N}$ (here and below in this proof we identify f_n with $(0, \ldots, 0, f_n, 0, 0, \ldots)$, where f_n standing at the n-th position).

On the one hand, we have

$$\begin{aligned} \langle\!\langle \mathrm{Exp}\, 1\!\!1_{[0,t]} f^{\otimes n}, g^{\otimes n} \rangle\!\rangle_{\mathcal{F}} &= n! \langle 1\!\!1_{[0,t]^n} f^{\otimes n}, g^{\otimes n} \rangle \\ &= n! \int_{[0,t]^n} f^{\otimes n}(t_1, \ldots, t_n) \overline{g^{\otimes n}(t_1, \ldots, t_n)}\, dt_1 \ldots dt_n. \end{aligned}$$

Taking into account that the functions $f^{\otimes n}$ and $g^{\otimes n}$ are symmetric with respect to all variables, we get

$$\begin{aligned}\int_{[0,t]} \langle\!\langle \tilde{\delta}_s \otimes \mathbb{I}_{[0,s]^{n-1}} n f^{\otimes n}, g^{\otimes n}\rangle\!\rangle_{\mathcal{F}}\, ds &= n n! \int_{[0,t]} \langle (\tilde{\delta}_s \otimes \mathbb{I}_{[0,s]^{n-1}}) f^{\otimes n}, g^{\otimes n}\rangle\, ds \\ &= n n! \int_{[0,t]} \langle \tilde{\delta}_s f, g\rangle \langle \mathbb{I}_{[0,s]^{n-1}} f^{\otimes n-1}, g^{\otimes n-1}\rangle\, ds \\ &= n n! \int_{[0,t]} \left(f(s)\overline{g(s)} \int_{[0,s]^{n-1}} f^{\otimes n-1}(t_1,\dots,t_{n-1})\overline{g^{\otimes n-1}(t_1,\dots,t_{n-1})}\, dt_1 \dots dt_{n-1}\right) ds \\ &= (n!)^2 \int_0^t \int_0^s \int_0^{t_{n-1}} \cdots \int_0^{t_2} f^{\otimes n}(s,t_1,\dots,t_{n-1})\overline{g^{\otimes n}(s,t_1,\dots,t_{n-1})}\, dt_1 \dots dt_{n-1} ds \\ &= n! \int_{[0,t]^n} f^{\otimes n}(t_1,\dots,t_n)\overline{g^{\otimes n}(t_1,\dots,t_n)}\, dt_1 \dots dt_n.\end{aligned}$$

Hence, formula (4.19) holds. □

The following theorem shows that the Hitsuda–Skorokhod integral $\widehat{\mathbb{I}}_{\mathrm{ext}}(\xi)$ of the function $\xi \in L^2([0,T];\mathcal{F}(-p,-q))$ can be regarded as the extended H-stochastic integral.

Theorem 4.3. *Let ξ belongs to the space $L^2([0,T];\mathcal{F}(-p,-q))$ and $[0,T] \ni t \mapsto A_\xi(t)$ be the corresponding operator–valued function whose values are operators $A_\xi(t)$ of Wick multiplication by $\xi(t) \in \mathcal{F}(-p,-q)$ in the Fock space $\mathcal{F}(-p,-q)$, i.e.,*

$$\mathcal{F}(-p,-q) \supset \mathrm{Dom}(A_\xi(t)) \ni \eta \mapsto A_\xi(t)\eta := \xi(t)\Diamond\eta \in \mathcal{F}(-p,-q),$$
$$\mathrm{Dom}(A_\xi(t)) := \{\eta \in \mathcal{F}(-p,-q) \,|\, \xi(t)\Diamond\eta \in \mathcal{F}(-p,-q)\}.$$

Then, $A_\xi \in S_{-2}$ and

$$\widehat{\mathbb{I}}_{\mathrm{ext}}(\xi) := \int_{[0,T]} \xi(t)\Diamond(0,\delta_t,0,0,\dots)\, dt = \int_{[0,T]} A_\xi(t) P_t Z_T\, dt.$$

Proof. Let $\xi \in L^2([0,T];\mathcal{F}(-p,-q))$. According to [9] (Lemma 11.2) we have

$$\|A_\xi(t)\eta\|_{\mathcal{F}(-p,-q)} = \|\xi(t)\Diamond\eta\|_{\mathcal{F}(-p,-q)} \le 2^{-\frac{q}{2}} \|\xi(t)\|_{\mathcal{F}(-p,-q)} \|\eta\|_{W^2_{-p}}$$

for m-almost all $t \in [0,T]$ and $\eta \in W^2_{-p}$. Hence

$$\|A_\xi(t)\|_{\mathcal{L}_{O+M}(t)} \le 2^{-\frac{q}{2}} \|\xi(t)\|_{\mathcal{F}(-p,-q)}$$

and $A_\xi \in S_{-2}$. The last part of the statement is obvious since $P_t Z_T = (0,\delta_t,0,0,\dots)$. □

Bibliography

1. S. Albeverio, Yu. M. Berezansky, and V. Tesko, *A generalization of an extended stochastic integral*, Ukrainian Math. J. **59** (2007), no. 5, 645–677.
2. S. Attal, *Quantum noises. Open quantum systems. II.*, Lecture Notes in Mathematics, Springer, Berlin, vol. 1881, 2006, pp. 79–147.
3. S. Attal and A. Belton, *The chaotic-representation property for a class of normal martingales*, Probab. Theory Related Fields **139** (2007), no. 3-4, 543–562.
4. S. Attal and J. M. Lindsay, *Quantum stochastic calculus with maximal operator domains*, Ann. Probab. **32** (2004), no. 1A, 488–529.
5. Yu. M. Berezansky, *Expansions in eigenfunctions of selfadjoint operators*, Transl. Math. Monogr., vol. 17, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965).
6. Yu. M. Berezansky, N. W. Zhernakov, and G. F. Us, *A spectral approach to quantum stochastic integrals*, Rep. Math. Phys. **28** (1989), no. 3, 347–360.
7. Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral Methods in Infinite Dimensional Analysis*, Vols. 1, 2, Kluwer Academic Publishers, Dordrecht–Boston–London, 1995 (Russian edition: Naukova Dumka, Kiev, 1988).
8. Yu. M. Berezansky, Z. G. Sheftel, and G. F. Us, *Functional Analysis*, Vols. 1, 2, Birkhäuser Verlag, Basel–Boston–Berlin, 1996 (Russian edition: Vyshcha Shkola, Kiev, 1990).

9. Yu. M. Berezansky and V. A. Tesko, *Spaces of test and generalized functions related to generalized translation operators*, Ukrainian Math. J. **55** (2003), no. 12, 1907-1979.
10. C. Dellacherie, B. Maisonneuve, and P.A. Meyer, *Probabilites et Potentiel. Ch XVII a XXIV. Processus de Markov (fin). Complements de Calcul Stochastique*, Hermann, Paris, 1992.
11. I.M. Gel′fand and I.Ya. Vilenkin, *Generalized functions. Vol. 4. Applications of harmonic analysis*, Academic Press, New York–London, 1964 (Russian edition: Fizmatgiz, Moscow, 1961).
12. M. Hitsuda, *Formula for Brownian partial derivatives*, Second Japan–USSR Symp. Probab. **2** (1972), 111–114.
13. M. Emery, *On the Azema martingales*, Lecture Notes in Mathematics, Springer, Berlin, vol. 1372, 1989, pp. 66–87.
14. M. Emery, *On the chaotic representation property for martingales*, Probability theory and mathematical statistics, eds. I.A. Ibragimov and A.Yu. Zaitsev, Gordon and Breach, 1996, pp. 155–166.
15. I. I. Gikhman and A. V. Skorokhod, *Theory of Random Processes*, Vol. 3, Nauka, Moscow, 1975 (Russian).
16. Yu.M. Kabanov and A.V. Skorohod, *Extended stochastic integrals*, in Proceedings the school-seminar on the theory of stoch. proc., Vilnus, Inst. Phys. Math., **I**, 1975, 123-167 (Russian).
17. N. A. Kachanovsky and V. A. Tesko, *Stochastic integral of Hitsuda–Skorokhod type on the extended Fock space*, Ukrainian Math. J. **61** (2009), no. 6, 733–764.
18. R. S. Lipcer and A. N. Shirjaev, *Statistics of random processes*, Nauka, Moscow, 1974 (in Russian).
19. J. Ma, P. Protter, and J. S. Martin, *Anticipating Integrals for a Class of Martingales*, Bernoulli, **4** (1998), no. 1, 81–114.
20. P. A. Meyer, *Quantum probability for probabilists*, Lecture Notes in Mathematics, Springer–Verlag, Berlin, vol. 1538, 1993, x+287 pp.
21. D. Nualart, *The Malliavin Calculus and Related Topics* Probability and its Applications (New York). Springer-Verlag, New York, 1995, xii+266 pp.
22. Ph. E. Protter, *Stochastic integration and differential equations.* Second edition. Springer-Verlag, Berlin, 2004, xiv+415 pp.
23. A. V. Skorokhod, *On a generalization of the stochastic integral*, Teor. Verojatnost. i Primenen. **20** (1975), no. 2, 223–238 (Russian).
24. V. Tesko, *A stochastic integral of operator-valued functions*, Methods Funct. Anal. Topology **14** (2008), no. 2, 132–141.
25. V. Tesko, *Hilbert space-valued integral of operator-valued functions*, Confluentes Mathematici **2** (2010), no. 1, 135–157.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

E-mail address: tesko@imath.kiev.ua