A GENERALIZATION OF AN EXTENDED STOCHASTIC INTEGRAL

SERGIO ALBEVERIO[1,2,3,4], YURIJ M. BEREZANSKY[*], VOLODYMYR TESKO[**]

Abstract. In this work we propose a generalization of an extended stochastic integral in the case of integration with respect to a wide class of random processes. In particular we obtain conditions for the coincidence of our integral with the classical Itô stochastic integral.

[1] Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn (Germany)

[2] SFB 611, Bonn, BiBoS, Bielefeld–Bonn (Germany)

[3] CERFIM, Locarno and Acc. Arch., USI (Switzerland)

[4] IZKS, Bonn (Germany)

E-mail: albeverio@uni-bonn.de

[*] Institute of Mathematics NAS of Ukraine, Tereshchenkivs'ka str. 3, 01601 Kyiv (Ukraine)

E-mail: berezan@mathber.carrier.kiev.ua

[**] Institute of Mathematics NAS of Ukraine, Tereshchenkivs'ka str. 3, 01601 Kyiv (Ukraine)

E-mail: tesko@imath.kiev.ua

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1. Introduction

It is well-known that the extended (Hitsuda–Skorohod) stochastic integral (that is a natural generalization of the classical Itô integral) plays an important role in the Gaussian and Poissonian analysis. The notion of such integrals was introduced approximately at the same time in the works of several mathematicians: M. Hitsuda [1], Yu. L. Daletsky and S. N. Paramonova [2, 3], A .V. Skorokhod [4], Yu. M. Kabanov and A .V. Skorokhod [5], Yu. M. Kabanov [6] and later for the Gamma-process by N. A. Kachanovsky [7, 8]. The definitions of the extended stochastic integral proposed in the mentioned works are equivalent but their forms are different (see, e.g., [9, 10] for details).

In this work the notion of an extended stochastic integral is introduced in terms of a rigging of a Fock space. Under a functional realization of Fock space using a Wiener–Itô–Segal-type isomorphism (see, e.g., [11, 12]) we obtain a general definition of an extended stochastic integral in terms of an L^2-space and its rigging. Note that in the Gaussian and Poissonian cases this definition coincides with the corresponding definitions given in [1, 4–6].

Such an approach to the construction of the extended stochastic integral is, on the one hand, simple, and, on other hand, very general and applicable to many stochastic processes. It is based on the theory of generalized functions of infinitely many variables (see the corresponding surveys [11, 12] and, in particular, the papers [13–15]).

One of the main ingredients is the realization of the conditional expectations as orthogonal projectors in an L^2-space (see, e.g., [16]). In this way for a square integrable martingale $M(t)$ one can write $M(t) = E(t)M$, $t \in [0, \infty)$, where $E(t)$ is some resolution of identity in the L^2-space [17–20] and M is a fixed vector from L^2. Since in the theory of stochastic processes it is an accepted assumption that $M(t)$ is right-continuous, we will assume that $E(t)$ is right-continuous (instead usual for functional analysis of left-continuous).

In the last part of this paper we find conditions under which the extended stochastic integral is an extension of the Itô integral. Here we also recall the theory of multiple spectral integrals for the symmetric complex-valued functions (this theory is based on some general results of spectral theory, see [20]) and describe an interconnection of such integrals with multiple Itô integrals.

The authors hope that a similar construction can be developed for the more complicated cases of stochastic integration connected with Gamma, Pascal, and Meixner processes. But in these cases it is necessary to use more complicated "extended Fock spaces"; some results connected with such spaces are given in [21–30].

Let us describe the general idea of our construction. Let $(\Omega, \mathcal{A}, P)$ be a probability space with a flow of σ-subalgebras $\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$. Let $\{M(t)\}_{t\in\mathbb{R}_+}$ be a normal martingale with respect to the flow

$\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$ with the chaotic representation property. This property means that the mapping

$$\mathcal{F} \ni f = (f_n)_{n=0}^{\infty} \mapsto If = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega, \mathcal{A}, P) =: L^2$$

is well-defined and unitary. Here $\mathcal{F}$ is a Fock space constructed over $L^2(\mathbb{R}_+, dt)$ (dt is the Lebesgue measure) and $I_n(f_n)$ is an n-multiple stochastic integral with respect to M.

By definition the Itô integral $\int_{\mathbb{R}_+} F(t)dM(t)$ of a simple $\mathcal{A}_t$-adapted function (constructed using the characteristic functions $\varkappa_{\Delta_j}$ of sets Δ_j)

$$F(t) = \sum_{j=1}^{n} F_j \varkappa_{\Delta_j}(t), \qquad \Delta_j = (s_j, t_j], \quad F_j \in L^2,$$

is defined by the equality

$$\int_{\mathbb{R}_+} F(t)dM(t) := \sum_{j=1}^{n} F_j(M(t_j) - M(s_j)) \in L^2.$$

It is easy to verify that the I^{-1}-image of this integral has the form

$$I^{-1}\Big(\int_{\mathbb{R}_+} F(t)dM(t)\Big) = \sum_{j=1}^{n} I^{-1}(F_j)\lozenge\varkappa_{\Delta_j} \in \mathcal{F},$$

where $\lozenge$ is the Wick multiplication in the space $\mathcal{F}$ (see, for example, [11]).

According to this equality it is reasonable to define the "Itô integral" of a simple function

$$f(t) = \sum_{j=1}^{n} f_j \varkappa_{\Delta_j}(t), \qquad f_j \in \mathcal{F},$$

on the Fock space $\mathcal{F}$ by the formula

$$\mathbb{S}_{\mathrm{I}}(f) := \sum_{j=1}^{n} f_j \lozenge \varkappa_{\Delta_j} \in \mathcal{F}.$$

Using the heuristic representation

$$\varkappa_{\Delta} = \int_{\Delta} \delta_t dt,$$

we obtain (at least heuristically)

$$\begin{aligned}\mathbb{S}_{\mathrm{I}}(f) &= \sum_{j=1}^{n} f_j \lozenge \varkappa(\Delta_j) = \sum_{j=1}^{n} f_j \lozenge \int_{\Delta_j} \delta_t dt = \sum_{j=1}^{n} \int_{\Delta_j} f_j \lozenge \delta_t dt \\ &= \int_0^{\infty} \Big(\sum_{j=1}^{n} f_j \varkappa_{\Delta_j}(t)\Big)\lozenge\delta_t dt = \int_0^{\infty} f(t)\lozenge\delta_t dt = \int_0^{\infty} a_+(\delta_t) f(t) dt.\end{aligned}$$

Here $a_+(\delta_t)$ is the creation operator in a Fock space, δ_t is the δ-function concentrated at $t \in \mathbb{R}_+$.

This heuristic construction gives a reason to take the formula

$$\int_0^{\infty} a_+(\delta_t) f(t) dt \tag{1.1}$$

as the definition of an extended stochastic integral in a Fock space. In Section 3 we prove that this integral exists as a Bochner integral of the vector-valued function $\mathbb{R}_+ \ni t \mapsto a_+(\delta_t)f(t)$ with values in some negative Fock space $\mathcal{F}_- \supset \mathcal{F}$. In Section 5 we show that the image of this integral under several functional realizations of Fock space $\mathcal{F}$ is an extension of the Itô integral.

This paper presents the above described results. Other results connected with subject of this article are given, e.g., in [9, 10, 31–36] (see also references therein). The preliminary version of this paper was published in the preprint [37].

2. Preliminaries

In this Section we recall some well known objects (a Fock space and its riggings) and functional realizations of these objects: Jacobi fields acting on a Fock space and the general theory of generalized functions of infinitely many variables (see, e.g., [39, 20, 40, 11, 12] for more details).

2.1. **A general symmetric Fock space and its rigging.** A more detailed account of the results of this subsection is contained in [38, 11]. In what follows we will use the notation

$$\mathbb{N}_p := \{p, p+1, \ldots\}, \qquad p \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of all entire numbers.

We consider a fixed family $(H_p)_{p\in\mathbb{N}_0}$ of *real* separable Hilbert spaces H_p; H_0 will also be denoted by H. This family is such that for all $p \in \mathbb{N}_0$ the space H_{p+1} is densely embedded in H_p, and this embedding is quasinuclear, i.e. of Hilbert-Schmidt type (the Hilbert–Schmidt norm will be denoted by $\|\cdot\|_{HS}$). Without loss of generality we assume that $\|\cdot\|_{H_p} \le \|\cdot\|_{H_{p+1}}$. We can construct the nuclear rigging of the space H_0

$$\Phi' := \operatorname*{ind\,lim}_{p\in\mathbb{N}_0} H_{-p} \supset H_{-p} \supset H_0 \supset H_p \supset \operatorname*{pr\,lim}_{p\in\mathbb{N}_0} H_p =: \Phi, \tag{2.1}$$

where H_{-p}, $p \in \mathbb{N}_1$, is the dual space to H_p with respect to the zero space H_0. We denote by $\langle \cdot\,,\cdot\rangle$ the dual pairing between the elements of H_{-p} and H_p (this pairing is generated by the scalar product in H_0). It is possible to construct for any $n \in \mathbb{N}_0$ the nuclear chain

$$\begin{array}{ccccccccc} \mathcal{F}_n(\Phi') & \supset & \mathcal{F}_n(H_{-p}) & \supset & \mathcal{F}_n(H_0) & \supset & \mathcal{F}_n(H_p) & \supset & \mathcal{F}_n(\Phi), \\ & & \| & & \| & & \| & & \\ & & H_{-p,\mathbb{C}}^{\widehat{\otimes} n} & & H_{0,\mathbb{C}}^{\widehat{\otimes} n} & & H_{p,\mathbb{C}}^{\widehat{\otimes} n} & & \end{array} \tag{2.2}$$

$$\mathcal{F}_n(\Phi) := \operatorname*{pr\,lim}_{p\in\mathbb{N}_0} \mathcal{F}_n(H_p), \qquad \mathcal{F}_n(\Phi') := \operatorname*{ind\,lim}_{p\in\mathbb{N}_0} \mathcal{F}_n(H_{-p}).$$

Here and below the symbol $\widehat{\otimes}$ denotes the symmetric tensor product ($\otimes$ is the ordinary tensor product), the subindex $\mathbb{C}$ denotes a complexification. We denote by $(\cdot\,,\cdot)_{\mathcal{F}_n(H_0)}$ the *complex* pairing between elements of $\mathcal{F}_n(H_{-p})$ and $\mathcal{F}_n(H_p)$ (for *real* pairings we preserve the notation $\langle\cdot\,,\cdot\rangle$). Note that for $n=0$ all spaces in (2.2) coincide with $\mathbb{C}$.

For each $p\in\mathbb{Z}$ we introduce a weighted symmetric Fock space $\mathcal{F}(H_p,\tau)$ with a fixed weight $\tau=(\tau_n)_{n=0}^\infty,\ \tau_n>0$, by setting

$$\mathcal{F}(H_p,\tau):=\bigoplus_{n=0}^\infty\mathcal{F}_n(H_p)\tau_n=\{f=(f_n)_{n=0}^\infty\,|\,f_n\in\mathcal{F}_n(H_p),\|f\|^2_{\mathcal{F}(H_p,\tau)}=\sum_{n=0}^\infty\|f_n\|^2_{\mathcal{F}_n(H_p)}\tau_n<\infty\}. \tag{2.3}$$

We will often use the following weight: fix $K>1$ and put

$$\tau(q)=((n!)^2K^{qn})_{n=0}^\infty,\qquad q\in\mathbb{N}_0\qquad(0!=1). \tag{2.4}$$

Using rigging (2.1) and the weight (2.4) we construct the nuclear rigging

$$\mathcal{F}(\Phi^{'})\supset\mathcal{F}(-p,-q)\supset F(H_0)\supset\mathcal{F}(p,q)\supset\mathcal{F}(\Phi), \tag{2.5}$$

$$\mathcal{F}(\Phi):=\operatorname*{pr\,lim}_{p,q\in\mathbb{N}_0}\mathcal{F}(p,q),\qquad\mathcal{F}(\Phi^{'}):=\operatorname*{ind\,lim}_{p,q\in\mathbb{N}_0}\mathcal{F}(-p,-q).$$

Here

$$\mathcal{F}(-p,-q):=\mathcal{F}(H_{-p},(K^{-qn})_{n=0}^\infty),\qquad\mathcal{F}(p,q):=\mathcal{F}(H_p,((n!)^2K^{qn})_{n=0}^\infty), \tag{2.6}$$

$$F(H_0):=\mathcal{F}(H_0,(n!)_{n=0}^\infty).$$

The first two spaces from (2.6) are dual with respect to the space $F(H_0)$. We point out that in (2.5) and (2.6) the zero space $F(H_0)$ *is Fock space (2.3) with the weight* $\tau_n=n!$. It is obvious that the set $\mathcal{F}_{\rm fin}(\Phi)$ of all finite sequences $(\varphi_n)_{n=0}^\infty$, $\varphi_n\in\mathcal{F}_n(\Phi)$, is dense in each space of (2.5).

The complex pairing between elements of $\mathcal{F}(-p,-q)$ and $\mathcal{F}(p,q)$ (generated by the scalar product in $F(H_0)$) will be denoted by $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ (or $(\cdot\,,\cdot)_{F(H_0)}$). This pairing is given by the formula

$$\langle\!\langle\xi,f\rangle\!\rangle=\sum_{n=0}^\infty\langle\xi_n,\bar{f}_n\rangle n!, \tag{2.7}$$

$$\xi=(\xi_n)_{n=0}^\infty\in\mathcal{F}(-p,-q),\quad f=(f_n)_{n=0}^\infty\in\mathcal{F}(p,q),$$

where the overbar denotes complex conjugation.

2.2. **Jacobi fields.** We recall some results concerning the theory of Jacobi fields in a Fock space (see [39, 20, 40, 41] for more details). This theory gives a possibility to pass from an abstract Fock space to the functional Hilbert space $L^2(Q,\mathcal{B}(Q),\rho)$ on some space Q with respect to a probability measure ρ on the Borel σ-algebra $\mathcal{B}(Q)$. We recall that the theory of Jacobi fields was created under the influence of the works of M. Krein (see, e.g., [42, 43]) about Jacobi matrices.

Consider Fock space (2.3) with $p=0$ and weight $\tau_n=1$, $n\in\mathbb{N}_0$, i.e., the space

$$\mathcal{F}(H)=\bigoplus_{n=0}^\infty\mathcal{F}_n(H), \tag{2.8}$$

where we set $H = H_0$. As usually $\mathcal{F}_{\mathrm{fin}}(H)$ denotes the set of finite vectors from $\mathcal{F}(H)$; the vector $\Omega = (1, 0, \ldots) \in \mathcal{F}_{\mathrm{fin}}(H)$ is called the vacuum. Let $H_1 = H_+$ be a fixed space from chain (2.1); the embedding $H_+ \hookrightarrow H$ be quasinuclear (as we have demanded above). Consider in the space $\mathcal{F}(H)$ a family $J = (J(\varphi))_{\varphi \in H_+}$ of operator-valued Jacobi matrices

$$J(\varphi) = \begin{pmatrix} b_0(\varphi) & a_0(\varphi) & 0 & 0 & 0 & \ldots \\ a_0(\varphi) & b_1(\varphi) & a_1^*(\varphi) & 0 & 0 & \ldots \\ 0 & a_1(\varphi) & b_2(\varphi) & a_2^*(\varphi) & 0 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \end{pmatrix}, \qquad \varphi \in H_+, \tag{2.9}$$

with entries

$$a_n(\varphi) : \mathcal{F}_n(H) \to \mathcal{F}_{n+1}(H), \qquad b_n(\varphi) = (b_n(\varphi))^* : \mathcal{F}_n(H) \to \mathcal{F}_n(H), \tag{2.10}$$

$$a_n^*(\varphi) = (a_n(\varphi))^* : \mathcal{F}_{n+1}(H) \to \mathcal{F}_n(H); \qquad n \in \mathbb{N}_0.$$

Assume that the following conditions on (2.9) are fulfilled.

a) For any $\varphi \in H_+$ operators (2.10) are bounded and real (i.e., act from real subspaces of $\mathcal{F}_n(H)$, $\mathcal{F}_{n+1}(H)$ into real ones).

b) The dependence of the elements of $J(\varphi)$ on $\varphi \in H_+$ is linear and continuous in the following sense: the operators

$$H_+ \ni \varphi \mapsto a_n(\varphi) f_n \in \mathcal{F}_{n+1}(H_+), \qquad H_+ \ni \varphi \mapsto b_n(\varphi) f_n \in \mathcal{F}_n(H_+), \qquad f_n \in \mathcal{F}_n(H_+), \tag{2.11}$$

$$H_+ \ni \varphi \mapsto a_n^*(\varphi) f_{n+1} \in \mathcal{F}_n(H_+), \qquad f_{n+1} \in \mathcal{F}_{n+1}(H_+), \qquad n \in \mathbb{N}_0,$$

are linear and bounded (this can be seen as a condition of "smoothness" of entries from (2.9): the vectors from H_+ are more "smooth" then vectors from H).

Every matrix (2.9) gives rise to a Hermitian operator $A(\varphi)$ on space $\mathcal{F}(H)$ (2.8): for $f = (f_n)_{n=0}^\infty \in \mathrm{Dom}\,(A(\varphi)) := \mathcal{F}_{\mathrm{fin}}(H_+)$ we put

$$(A(\varphi) f)_n := (J(\varphi) f)_n = a_{n-1}(\varphi) f_{n-1} + b_n(\varphi) f_n + a_n^*(\varphi) f_{n+1}, \tag{2.12}$$

$$n \in \mathbb{N}_0, \qquad a_{-1}(\varphi) = 0.$$

c) The operators $A(\varphi)$, $\varphi \in H_+$, are essentially selfadjoint and their closures $\widetilde{A}(\varphi)$ are strongly commuting.

d) (regularity) For each $n \in \mathbb{N}_1$, a real linear operator $V_{n,n} : \mathcal{F}_n(H_+) \to \mathcal{F}_n(H_+)$ defined by the formula

$$V_{n,n}(\varphi_1 \widehat{\otimes} \cdots \widehat{\otimes} \varphi_n) := (J(\varphi_1) \ldots J(\varphi_n)\Omega)_n = a_{n-1}(\varphi_1) \ldots a_0(\varphi_n) 1, \qquad \varphi_1, \ldots, \varphi_n \in H_+, \tag{2.13}$$

is continuous and invertible; we also put $V_{0,0} := 1$.

Above described family $J = (J(\varphi))_{\varphi \in H_+}$ of matrices is by definition a (commuting) *Jacobi field*. Our first aim is to construct the generalized eigenvector expansion for the family $A = (\widetilde{A}(\varphi))_{\varphi \in H_+}$ of the corresponding selfadjoint operators acting on the Fock space $\mathcal{F}(H)$ (about the general theory of such expansions see, e.g., [38, 44]).

For the investigation of the spectral theory of the family A we start from giving a quasinuclear rigging of real Hilbert spaces

$$H_- \supset H \supset H_+. \tag{2.14}$$

After this we construct the following rigging of the space $\mathcal{F}(H)$, using weighted spaces of form (2.3):

$$\mathcal{F}(H_-, \tau^{-1}) \supset \mathcal{F}(H) \supset \mathcal{F}(H_+, \tau) \supset \mathcal{F}_{\mathrm{fin}}(H_+), \tag{2.15}$$

$$H_- = H_{-1}, \quad \tau = (\tau_n)_{n=0}^{\infty}, \quad \tau_n \geq 1, \quad \tau^{-1} = (\tau_n^{-1})_{n=0}^{\infty}.$$

We suppose that the embedding $\mathcal{F}(H_+, \tau) \hookrightarrow \mathcal{F}(H)$ is quasinuclear, i.e., that the weight is such that

$$\sum_{n=0}^{\infty} \|O\|_{HS}^{2n} \tau_n^{-1} < \infty, \tag{2.16}$$

where O is the embedding operator $H_+ \hookrightarrow H$ [38, 44, 11].

The main result about generalized eigenvector expansion is as follows.

Let $A = (\widetilde{A}(\varphi))_{\varphi \in H_+}$ be a Jacobi field. For A there exists a Borel probability measure ρ on the space H_- (the spectral measure) such that the Fourier transform

$$\begin{aligned} \mathcal{F}(H) \supset \mathcal{F}(H_+, \tau) \ni f = (f_n)_{n=0}^{\infty} \mapsto (Ff)(\cdot) = (f, P(\cdot))_{\mathcal{F}(H)} \\ = \sum_{n=0}^{\infty} (f_n, P_n(\cdot))_{\mathcal{F}_n(H)} \in L^2(H_-, \mathcal{B}(H_-), \rho) =: (L^2_{H_-}) \end{aligned} \tag{2.17}$$

after being extended by continuity to the whole space $\mathcal{F}(H)$ is a unitary operator acting from the space $\mathcal{F}(H)$ to the space $(L^2_{H_-})$.

In (2.17) for any $x \in H_-$, $P(x) = (P_n(x))_{n=0}^{\infty}$, $P_n(x) \in \mathcal{F}_n(H_-)$, is a real-valued sequence that is a joint solution of the system of the following operator-difference equations:

$$(a_{n-1}^*(\varphi))^+ P_{n-1}(x) + (b_n(\varphi))^+ P_n(x) + (a_n(\varphi))^+ P_{n+1}(x) = (x, \varphi)_H P_n(x), \tag{2.18}$$

$$n \in \mathbb{N}_0, \quad x \in H_-, \quad \varphi \in H_+; \quad P_{-1}(x) = 0, \quad P_0(x) = 1.$$

Here we denote by C^+ the operator adjoint to C with respect to the zero spaces H, i.e., if $C : \mathcal{F}_k(H_+) \to \mathcal{F}_j(H_+)$ is continuous, then $C^+ : \mathcal{F}_j(H_-) \to \mathcal{F}_k(H_-)$ and is connected with C by the equality

$$(Cf_k, g_j)_{\mathcal{F}_j(H)} = (f_k, C^+ g_j)_{\mathcal{F}_k(H)}, \qquad f_k \in \mathcal{F}_k(H_+), \quad g_j \in \mathcal{F}_j(H_-), \quad j, k \in \mathbb{N}_0. \tag{2.19}$$

Equality (2.18) and relation (2.19) show that the sequence $P(x) = (P_n(x))_{n=0}^{\infty}$, $P_n(x) \in \mathcal{F}_n(H_-)$ is the solution (in a generalized sense) of the equation

$$J(\varphi)P(x) = (x, \varphi)_H P(x), \qquad \varphi \in H_+, \qquad x \in H_-, \tag{2.20}$$

i.e., is a generalized eigenvector of operator $\widetilde{A}(\varphi)$ with "eigenvalue" $(x, \varphi)_H$. $P_n(x)$ is in some sense "a polynomial of degree n with respect to infinite-dimensional variable $x \in H_-$". For more details on the construction and properties of $P(x)$ see [39, 20, 40].

2.3. **Two classical examples of Jacobi fields.** *1) Free field* (see, e.g., [38, 39, 20, 45]). In this case $\dim H = \infty$, H_+ is arbitrary with quasinuclear embedding $H_+ \hookrightarrow H$. Matrix $J(\varphi)$ (2.9) has the form: for any $\varphi \in H_+$

$$J(\varphi) = J_+(\varphi) + J_-(\varphi), \tag{2.21}$$

where for $f_n \in \mathcal{F}_n(H)$, $n \in \mathbb{N}_0$,

$$J_+(\varphi)f_n = \sqrt{n+1}\varphi \mathbin{\widehat{\otimes}} f_n : \mathcal{F}_n(H) \to \mathcal{F}_{n+1}(H), \qquad J_-(\varphi) = (J_+(\varphi))^*, \tag{2.22}$$

i.e., $J_+(\varphi)$, $J_-(\varphi)$ are classical creation and annihilation operators. The conditions a) - d) are fulfilled and the operators $V_{n,n}$ have the form

$$V_{n,n} = \sqrt{n!}Id, \qquad n \in \mathbb{N}_1. \tag{2.23}$$

The spectral measure ρ is equal to the Gaussian measure g_S on the space H_- with the zero mean and the correlation operator $S = O^+O\mathbb{I} : H_- \to H_-$, where $O : H_+ \hookrightarrow H$, $O^+ : H \hookrightarrow H_-$, $\mathbb{I} : H_- \to H_+$ are canonical operators connected with chain (2.14). Fourier transform (2.17) is the classical Wiener–Itô–Segal transformation.

2) Poisson field ([46, 39, 20, 45, 41, 47]). In this case $H = L^2_{\mathrm{Re}}(R, \mathcal{B}(R), \nu) =: L^2_{\mathrm{Re}}(R, \nu)$, where R is a topological abstract space with a σ-finite Borel measure ν on $\mathcal{B}(R)$. Let H_+ be a certain fixed real Hilbert space embedded into H densely and quasinuclearly (for the construction of such spaces see [44, 12]). Jacobi matrix $J(\varphi)$ (2.9) has now the form

$$J(\varphi) = J_+(\varphi) + B(\varphi) + J_-(\varphi), \qquad \varphi \in H_+, \tag{2.24}$$

i.e., it is equal to some perturbation of matrix (2.21) by a diagonal matrix $B(\varphi)$. This matrix $B(\varphi)$ is equal to the second (differential) quantization of the operator $b(\varphi)$ of multiplication by a bounded function φ in the space H_+, i.e., for any $f_n \in \mathcal{F}_n(H)$

$$\begin{aligned} B(\varphi)f_n = b_n(\varphi)f_n &= (b(\varphi) \otimes Id \otimes \ldots \otimes Id)f_n + (Id \otimes b(\varphi) \otimes Id \otimes \ldots \otimes Id)f_n \\ &+ \cdots + (Id \otimes \ldots \otimes Id \otimes b(\varphi))f_n \in \mathcal{F}_n(H), \qquad n \in \mathbb{N}_1; \quad B(\varphi)f_0 = 0. \end{aligned} \tag{2.25}$$

The conditions a) - d) also are fulfilled. As in example 1) the operator $V_{n,n}$ has form (2.23). The spectral measure ρ is now a centered Poisson measure with intensity ν (ν may be atomic). The measure ρ is defined by its Fourier transform

$$\int_{H_-} e^{i(x,\varphi)_H} d\rho(x) = \exp\Big(\int_R (e^{i\varphi(q)} - 1 - i\varphi(q)) d\nu(q)\Big), \qquad \varphi \in H_+. \tag{2.26}$$

Fourier transform (2.17) is the Wiener–Itô–Segal type transform of Poisson measures.

2.4. **Transfer from the Fock space to a space of functions in the general case.** In Subsections 2.2, 2.3 the unitary operator $F : \mathcal{F}(H) \to L^2(\Omega, \mathcal{A}, P)$ is a Fourier transform generated in $\mathcal{F}(H)$ by some Jacobi field. In this Subsection we will propose some more general construction of the unitary operator $F : \mathcal{F}(H) \to L^2(\Omega, \mathcal{A}, P)$, using the orthogonal approach to the theory of generalized functions of infinitely many variables (see, e.g., [11, 12] and references therein).

Let Q be a (separable) metric space, ρ be a fixed Borel finite measure on $\mathcal{B}(Q)$, and

$$L^2(Q, \mathcal{B}(Q), \rho) =: (L^2_Q)$$

be the corresponding space of square integrable functions. By $C(Q)$ we denote the linear space of all complex-valued locally bounded (i.e. bounded on every ball in Q) continuous functions on Q. We will understand $C(Q)$ as a linear topological space with convergence uniform on every ball from Q.

Let B_0 be a neighborhood of zero in the space $H_{0,\mathbb{C}} = \mathcal{F}_1(H_0)$ and let

$$Q \times B_0 \ni \{x, \lambda\} \mapsto h(x, \lambda) \in \mathbb{C}$$

be a given function. We assume that for each $x \in Q \quad h(x, \cdot)$ is analytic in a neighborhood of zero in $H_{0,\mathbb{C}}$, and, for each $\lambda \in B_0$, $h(\cdot, \lambda) \in C(Q)$. Moreover, $h(\cdot, \lambda)$ is locally bounded uniformly with respect to λ from any closed ball inside of B_0 and $h(x, 0) = 1$ for all x from Q.

It follows from [11], Sections 2.3, that, for each point $x \in Q$, there exists a neighborhood of zero $B_1(x) \subset B_0$ in the space $H_{1,\mathbb{C}}$, such that

$$h(x, \lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, h_n(x) \rangle, \qquad h_n(x) \in \mathcal{F}_n(H_{-1}), \qquad h_0(x) = 1, \tag{2.27}$$

for all λ from $B(x)$. Moreover, the last series converges uniformly on any closed ball from $B(x)$. Suppose that for all $x \in Q$ there exists a general neighborhood of zero $B_1 \subset B_0$ with this property.

It is possible to construct a mapping of type (2.17) using instead of $P_n(x)$ the functions $h_n(x)$ from (2.27). For this aim it is necessary to impose some conditions on h. So, we will assume that for all $n \in \mathbb{N}_0$ the estimate

$$\| \, \|h_n(\cdot)\|_{\mathcal{F}_n(H_{-1})} \|_{(L^2_Q)} \leq L C^n n! \tag{2.28}$$

with some constants $L > 0, \ C > 0$ is fulfilled.

It follows from (2.28) that for any $f_n \in \mathcal{F}_n(H_1)$ the functions

$$Q \ni x \mapsto \langle f_n, h_n(x)\rangle \in \mathbb{C} \tag{2.29}$$

belong to the space (L^2_Q). *We suppose that the linear span of the functions (2.29), where $f_n \in \mathcal{F}_n(H_1)$, $n \in \mathbb{N}_0$, is dense in (L^2_Q) and that they are orthogonal in the following sense*

$$\int_Q \langle f_n, h_n(x)\rangle \overline{\langle g_m, h_m(x)\rangle} d\rho(x) = \delta_{n,m} n! \langle f_n, \bar{g}_n\rangle, \qquad n, m \in \mathbb{N}_0. \tag{2.30}$$

It is possible to prove that condition (2.30) of orthogonality is fulfilled if estimate (2.28) holds with H_{-1} replaced by H_{-p} with some $p \in \mathbb{N}_1$ and the following equality

$$\int_Q h(x, f)\overline{h(x, g)} d\rho(x) = \exp\langle f, \overline{g}\rangle \tag{2.31}$$

is fulfilled. Here $f, g \in \Phi_{\mathbb{C}} \subset H_{1,\mathbb{C}}$ are such that $\|f\|_{H_{p,\mathbb{C}}} < r$, $\|g\|_{H_{p,\mathbb{C}}} < r$, where $r > 0$ is sufficiently small (the proof of the latter results is contained in [12], Section 3, see also [11], Section 7).

Fix function $h(x, \lambda)$ (2.27) with above-mentioned properties and introduce a mapping of the form (2.17) but taking instead of $P_n(x)$ the functions $h_n(x)$. So, we put

$$F(H_0) \supset \mathcal{F}_{\text{fin}}(\Phi) \ni f = (f_n)_{n=0}^{\infty} \mapsto (I_h f)(\cdot) = \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot)\rangle \in (L^2_Q). \tag{2.32}$$

Orthogonality (2.30) and the density of $\mathcal{F}_{\text{fin}}(\Phi)$ in $F(H_0)$ mean that after extending by continuity to the whole space $F(H_0)$ map (2.32) turns into the unitary operator I_h that maps the whole space $F(H_0)$ onto whole (L^2_Q). In this way we get a functional realization of a Fock space.

The map I_h transfers rigging (2.5) onto the following rigging of the space (L^2_Q):

$$\operatorname*{ind\,lim}_{p,q \in \mathbb{N}_0} \mathcal{H}(-p, -q) = (\mathcal{H})' \supset \mathcal{H}(-p, -q) \supset (L^2_Q) \supset \mathcal{H}(p, q) \supset \mathcal{H} = \operatorname*{pr\,lim}_{p,q \in \mathbb{N}_0} \mathcal{H}(p, q). \tag{2.33}$$

Here $\mathcal{H}(p, q) := I_h \mathcal{F}(p, q)$ is a Hilbert space with topology inducted by the topology of $\mathcal{F}(p, q)$, $\mathcal{H}(-p, -q)$ is the negative space with respect to the zero space (L^2_Q) and the positive space $\mathcal{H}(p, q)$.

Remark 2.1. Note that function (2.29) belongs to the space $C(Q)$ (see, e.g., [11], Lemma 3.2) and for

$$f_n = \varphi^{(1)} \widehat{\otimes} \cdots \widehat{\otimes} \varphi^{(n)}, \qquad \varphi^{(1)}, \ldots, \varphi^{(n)} \in H_1,$$

we have:

$$\langle \varphi^{(1)} \widehat{\otimes} \cdots \widehat{\otimes} \varphi^{(n)}, h_n(x)\rangle = \frac{\partial^n}{\partial z_1 \ldots \partial z_n} h(x, z_1\varphi^{(1)} + \cdots + z_n\varphi^{(n)})\Big|_{z_1 = \ldots = z_n = 0}, \tag{2.34}$$

for all $x \in Q$.

Moreover, one can show (see, e.g., [11] for more details) that for $K > 1$ sufficiently large (we recall that K is the constant in (2.3), this constant is used in the definition of $\mathcal{F}(p,q)$) the mapping

$$\mathcal{F}(p,q) \ni (f_n)_{n=0}^{\infty} \mapsto f(\cdot) := \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot)\rangle \in C(Q)$$

is well-defined, continuous and injective. Therefore the space $\mathcal{H}(p,q)$ is embedded in the space $C(Q)$, and one can understand $\mathcal{H}(p,q)$ as the Hilbert space of continuous functions

$$\mathcal{H}(p,q) = \{f \in C(Q) \,|\, \exists (f_n)_{n=0}^{\infty} \in \mathcal{F}(p,q) : \ f(x) = \sum_{n=0}^{\infty} \langle f_n, h_n(x)\rangle, \ x \in Q\}$$

with the Hilbert norm

$$\|f\|_{\mathcal{H}(p,q)} = \|\sum_{n=0}^{\infty} \langle f_n, h_n(\cdot)\rangle\|_{\mathcal{H}(p,q)} = \|(f_n)_{n=0}^{\infty}\|_{\mathcal{F}(p,q)}.$$

Remark 2.2. It is clear that the mapping

$$\mathcal{F}(-p,-q) \supset F(H_0) \ni f = (f_n)_{n=0}^{\infty} \mapsto I_h f = \sum_{n=0}^{\infty} \langle f_n, h_n\rangle \ \in \mathcal{H}(-p,-q)$$

is isometric and after closure by continuity is a unitary isomorphism between $\mathcal{F}(-p,-q)$ and $\mathcal{H}(-p,-q)$ (we preserve the notation I_h for the closure). As a result the space of generalized functions $\mathcal{H}(-p,-q)$ can be presented in the form

$$\begin{aligned} \mathcal{H}(-p,-q) = I_h(\mathcal{F}(-p,-q)) = \big\{\xi = \sum_{n=0}^{\infty} \langle \xi_n, h_n\rangle \ | \\ (\xi_n)_{n=0}^{\infty} \in \mathcal{F}(-p,-q), \|\xi\|_{\mathcal{H}(-p,-q)} = \|(\xi_n)_{n=0}^{\infty}\|_{\mathcal{F}(-p,-q)}\big\}. \end{aligned} \tag{2.35}$$

Here

$$\langle \xi_n, h_n\rangle := \lim_{k\to\infty} \langle f_n^{(k)}, h_n\rangle \in \mathcal{H}(-p,-q), \qquad n \in \mathbb{N}_0, \tag{2.36}$$

where the sequence $(f_n^{(k)})_{k=0}^{\infty} \subset \mathcal{F}_n(H_0)$ converges to $\xi_n \in \mathcal{F}_n(H_{-p})$ in the topology of $\mathcal{F}_n(H_{-p})$ (note that we understand the limit in (2.36) as a limit in $\mathcal{H}(-p,-q)$). One can show (see [12]) that in $\mathcal{H}(-p,-q)$

$$\langle \xi_m, h_m\rangle = \partial^+(\xi_m)1, \qquad \xi_m \in \mathcal{F}_m(H_{-p}), \qquad m \in \mathbb{N}_0,$$

where

$$\partial^+(\xi_m) := I_h a_+(\xi_m) I_h^{-1} : \mathcal{H}(-p,-q) \to \mathcal{H}(-p,-q) \tag{2.37}$$

is a linear continuous operator that is the image of the creation operator

$$a_+(\xi_m) : \mathcal{F}(-p,-q) \to \mathcal{F}(-p,-q), \qquad p,q \in \mathbb{N}_1.$$

We recall that by definition the operator $a_+(\xi_m)$ acts on any vector $\eta = (\eta_n)_{n=0}^{\infty} \in \mathcal{F}(-p,-q)$ by the formula

$$a_+(\xi_m)\eta = a_+(\xi_m)(\eta_0, \eta_1, \ldots) := (\underbrace{0, \ldots, 0}_{m}, \xi_m \mathbin{\widehat{\otimes}} \eta_0, \xi_m \mathbin{\widehat{\otimes}} \eta_1, \ldots); \tag{2.38}$$

$$\|a_+(\xi_m)\eta\|_{\mathcal{F}(-p,-q)} \leq K^{-\frac{qm}{2}} \|\xi_m\|_{\mathcal{F}_m(H_{-p})} \|\eta\|_{\mathcal{F}(-p,-q)} \tag{2.39}$$

(this estimate follows from (2.3) with τ given by (2.4)).

The dual pairing $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ between elements of $\mathcal{H}(-p,-q)$ and $\mathcal{H}(p,q)$, $p,q \in \mathbb{N}_1$, from rigging (2.33) that is generated by the scalar product in (L^2_Q) has the form

$$\langle\!\langle \xi, f \rangle\!\rangle = \langle\!\langle \sum_{n=0}^{\infty} \langle \xi_n, h_n \rangle, \sum_{n=0}^{\infty} \langle f_n, h_n \rangle \rangle\!\rangle = \sum_{n=0}^{\infty} \langle \xi_n, \bar{f}_n \rangle n!, \tag{2.40}$$

$$\xi = \sum_{n=0}^{\infty} \langle \xi_n, h_n \rangle \in \mathcal{H}(-p,-q), \quad f = \sum_{n=0}^{\infty} \langle f_n, h_n \rangle \in \mathcal{H}(p,q).$$

2.5. **Connection between Subsection 2.4 and 2.2.** Let we have Jacobi field (2.9) and construct its spectral representation using chain (2.15). Fourier transform F (2.17) transfers the space $\mathcal{F}(H_0)$ onto the space $L^2(H_-, \mathcal{B}(H_-), \rho) = (L^2_{H_-})$. As it follows from the property of F, the series

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \langle \lambda^{\otimes n}, P_n(x) \rangle, \qquad \lambda \in H_{+,\mathbb{C}},$$

converges in the topology of $(L^2_{H_-})$, and its sum is a holomorphic function with respect to λ. Moreover, according to [39]

$$\| \, \|P_n(\cdot)\|_{\mathcal{F}_n(H_-)} \|_{(L^2_{H_-})} \leq C^n \sqrt{n!}, \qquad n \in \mathbb{N}_0,$$

for some constant $C > 0$. Therefore, if we put

$$h(x,\lambda) := \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, h_n(x) \rangle, \qquad h_n(x) := \sqrt{n!} P_n(x), \tag{2.41}$$

we can understand Fourier transform (2.17) as a particular case of transform (2.32). In this case, it is not necessarily to verify that function (2.41) satisfies all assumptions formulated in Subsection 2.4 because for the sequence $(h_n(x) = \sqrt{n!} P_n(x))_{n=0}^{\infty}$ orthogonality relation (2.30) holds, and this gives a possibility to repeat the corresponding parts of the construction in Subsection 2.4.

Note that it is possible to calculate the generating function $h(x,\lambda)$ for the classical examples of Jacobi fields (2.9) (see, e.g., [14, 11, 12] and Section 6).

3. On extended stochastic integral in a Fock space and in its functional realization

In this Section we give an exact definition of extended stochastic integral (1.1). The probability sense of such integral will be discussed in Section 5.

3.1. **On extended stochastic integral in a Fock space.** Here and below we restrict ourself to special form of rigging (2.1). Namely, fix a constant $T \in (0, \infty)$. Let

$$H_0 := L^2_{\mathrm{Re}}([0,T), \mathcal{B}([0,T)), m) =: L^2_{\mathrm{Re}}([0,T), m),$$

where m is the Lebesgue measure on $[0,T)$, i.e., $dm(t) = dt$. It is clear that the space $\mathcal{F}_n(H_0)$, $n \in \mathbb{N}_1$, is isomorphic to the space $\hat{L}^2([0,T), m^{\otimes n})$ of all complex-valued symmetric functions from $L^2([0,T)^n, m^{\otimes n})$. Now

$$\|f_n\|^2_{\mathcal{F}_n(H_0)} = \int_{[0,T)^n} |f_n(t_1, \dots, t_n)|^2 dt_1 \dots dt_n = n! \int_0^T \int_0^{t_n} \cdots \Big(\int_0^{t_2} |f_n(t_1, \dots, t_n)|^2 dt_1 \Big) \dots dt_{n-1} dt_n.$$

Introduce rigging (2.1) of the form:

$$\Phi' := \operatorname*{ind\,lim}_{p \in \mathbb{N}_0} H_{-p} \supset H_{-p} \supset H_0 \supset H_p \supset \operatorname*{pr\,lim}_{p \in \mathbb{N}_0} H_p =: \Phi, \tag{3.1}$$

where

$$H_p := W^2_p([0,T), m), \qquad p \in \mathbb{N}_0,$$

are the real Sobolev spaces, Φ' and H_{-p} are the spaces dual to Φ and H_p with respect to the zero space H_0 correspondingly (see, e.g., [38, 44] for more details). Using (3.1) we construct two parameter rigging corresponding to (2.5),

$$\mathcal{F}(\Phi') \supset \mathcal{F}(-p,-q) \supset F(H_0) \supset \mathcal{F}(p,q) \supset \mathcal{F}(\Phi). \tag{3.2}$$

Let $\mathcal{K}$ be some Hilbert space. By $L^2([0,T); \mathcal{K})$ we denote the Hilbert space of (vector-valued) functions

$$[0,T) \ni t \mapsto f(t) \in \mathcal{K}, \qquad \|f\|^2_{L^2([0,T);\mathcal{K})} = \int_{[0,T)} \|f(t)\|^2_{\mathcal{K}} dt < \infty,$$

with the corresponding scalar product.

The general definition of an extended stochastic integral is the following:

The extended stochastic integral (in a Fock space) of a function

$$\xi \in L^2([0,T); \mathcal{F}(-p,-q)), \qquad p, q \in \mathbb{N}_1,$$

is defined by the formula

$$\mathbb{S}_{\mathrm{ext}}(\xi) = \int_{[0,T)} a_+(\delta_t) \xi(t) dt \in \mathcal{F}(-p,-q). \tag{3.3}$$

Here we understand the right hand side as a Bochner integral of the vector-valued function

$$[0,T)\ni t\mapsto a_+(\delta_t)\xi(t)\in\mathcal{F}(-p,-q), \tag{3.4}$$

were δ_t is the delta-function concentrated at t.

The correctness of this definition from the following statement follows.

Proposition 3.1. If $\xi\in L^2([0,T);\mathcal{F}(-p,-q))$, $p,q\in\mathbb{N}_1$, then the function (3.4) is integrable in the Bochner sense on $[0,T)$.

Proof. Let $\xi\in L^2([0,T);\mathcal{F}(-p,-q))$. Using (2.39) and the estimate

$$\|\delta_t\|_{\mathcal{F}_1(H_{-p})}\le c,\qquad t\in[0,T),$$

with some $c>0$ (see, e.g., [44]) we obtain

$$\begin{aligned}\int_{[0,T)}\|a_+(\delta_t)\xi(t)\|_{\mathcal{F}(-p,-q)}dt&\le K^{-\frac{q}{2}}\int_{[0,T)}\|\delta_t\|_{\mathcal{F}_1(H_{-p})}\|\xi(t)\|_{\mathcal{F}(-p,-q)}dt\\&\le K^{-\frac{q}{2}}\Big(\int_{[0,T)}\|\delta_t\|^2_{\mathcal{F}_1(H_{-p})}dt\Big)^{\frac12}\Big(\int_{[0,T)}\|\xi(t)\|^2_{\mathcal{F}(-p,-q)}dt\Big)^{\frac12}\\&\le cK^{-\frac{q}{2}}T^{\frac12}\Big(\int_{[0,T)}\|\xi(t)\|^2_{\mathcal{F}(-p,-q)}dt\Big)^{\frac12}<\infty,\end{aligned}$$

whence the necessary statement follows. $\square$

3.2. **Some properties of the introduced integral.** We shall prove the important properties of the extended stochastic integral $\mathbb{S}_{\mathrm{ext}}$. Let $f_n(\cdot;\cdot_1,\dots,\cdot_n)\in\mathcal{F}_1(H_0)\otimes\mathcal{F}_n(H_0)$, $n\in\mathbb{N}_1$. We denote by $\hat f_{n+1}(t_1,\dots,t_{n+1})$ the symmetrization of f_n with respect to $n+1$ variables, i.e.,

$$\hat f_{n+1}(t_1,\dots,t_{n+1}):=\frac{1}{n+1}\sum_{k=1}^{n+1}f_n(t_k;t_1,\dots,\cancel{t_k},\dots,t_{n+1}) \tag{3.5}$$

for $m^{\otimes(n+1)}$-almost all $(t_1,\dots,t_{n+1})\in[0,T)^{n+1}$. We put $\hat f_1(t):=f_0(t)$ for all $t\in\mathbb{R}$.

Theorem 3.1. *Let $f(\cdot)=(f_n(\cdot))_{n=0}^\infty\in L^2([0,T);F(H_0))$ and $\sum_{n=0}^\infty\|\hat f_{n+1}\|^2_{\mathcal{F}_{n+1}(H_0)}(n+1)!<\infty$. Then*

$$\mathbb{S}_{\mathrm{ext}}(f)=\mathbb{S}(f):=(0,\hat f_1,\dots,\hat f_n,\dots) \tag{3.6}$$

in the space $\mathcal{F}(-p,-q)$, $p,q\in\mathbb{N}_1$.

Proof. It is sufficient to show that

$$\langle\!\langle\mathbb{S}_{\mathrm{ext}}(f),\psi\rangle\!\rangle=\langle\!\langle\mathbb{S}(f),\psi\rangle\!\rangle$$

for each $\psi = (\underbrace{0, \ldots, 0}_{k}, \varphi^{\otimes k}, 0, 0, \ldots)$, $\varphi \in \Phi$, $k \in \mathbb{N}_0$. Applying to a function

$$[0,T) \ni t \mapsto f(t) = (f_n(t))_{n=0}^{\infty} \in F(H_0) \subset \mathcal{F}(-p,-q)$$

the operator $a_+(\delta_t)$ and using (2.38) we obtain

$$a_+(\delta_t)f(t) = (0, \delta_t \widehat{\otimes} f_0(t), \delta_t \widehat{\otimes} f_1(t), \ldots) \in \mathcal{F}(-p,-q). \tag{3.7}$$

Using (3.7) we have

$$\begin{aligned}
\langle\!\langle \mathbb{S}_{\mathrm{ext}}(f), \psi \rangle\!\rangle &= \langle\!\langle \int_{[0,T)} a_+(\delta_t) f(t) dt, \psi \rangle\!\rangle = \int_{[0,T)} \langle\!\langle a_+(\delta_t) f(t), \psi \rangle\!\rangle dt \\
&= k! \int_{[0,T)} \langle \delta_t \widehat{\otimes} f_{k-1}(t), \varphi^{\otimes k} \rangle dt = k! \int_{[0,T)} \varphi(t) \langle f_{k-1}(t), \varphi^{\otimes (k-1)} \rangle dt \\
&= k! \int_{[0,T)} \varphi(t) \Big(\int_{[0,T)^{k-1}} f_{k-1}(t; t_1, \ldots, t_{k-1}) \varphi^{\otimes (k-1)}(t_1, \ldots, t_{k-1}) dt_1 \ldots dt_{k-1} \Big) dt \\
&= k! \int_{[0,T)^k} f_{k-1}(t; t_1, \ldots, t_{k-1}) \varphi^{\otimes k}(t, t_1, \ldots, t_{k-1}) dt_1 \ldots dt_{k-1} dt \\
&= k! (f_{k-1}, \varphi^{\otimes k})_{H_{0,\mathbb{C}}^{\otimes k}} = k! (\hat{f}_k, \varphi^{\otimes k})_{H_{0,\mathbb{C}}^{\otimes k}} = k! \langle \hat{f}_k, \varphi^{\otimes k} \rangle = \langle\!\langle \mathbb{S}(f), \psi \rangle\!\rangle.
\end{aligned}$$

□

Let $D \subset L^2([0,T); F(H_0)) \subset L^2([0,T); \mathcal{F}(-p,-q))$, $p, q \in \mathbb{N}_1$, be the class of all functions

$$[0,T) \ni t \mapsto f(t) = (f_n(t))_{n=0}^{\infty} \in F(H_0) \tag{3.8}$$

from $L^2([0,T); F(H_0))$ such that for m-almost all $t \in [0,T)$ and $m^{\otimes n}$-almost all $(t_1, \ldots, t_n) \in [0,T)^n$

$$f_n(t) = f_n(t; t_1, \ldots, t_n) = \varkappa_{(0,t]^n}(t_1, \ldots, t_n) f_n(t; t_1, \ldots, t_n), \qquad n \in \mathbb{N}_1, \tag{3.9}$$

where $\varkappa_\alpha(\cdot)$ is the characteristic function of a Borel set $\alpha \in \mathcal{B}([0,T)^n)$, $\varkappa_{(0,0]^n} := 0$.

Theorem 3.2. *If $f \in D \subset L^2([0,T); F(H_0))$ then*

$$\|\mathbb{S}(f)\|_{\mathcal{F}(H_0)} = \|f\|_{L^2([0,T); F(H_0))}. \tag{3.10}$$

Proof. For $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in D$ we have

$$
\begin{aligned}
\|f\|^2_{L^2([0,T);F(H_0))} &= \int_{[0,T)} \|f(t)\|^2_{F(H_0)} dt = \int_{[0,T)} \sum_{n=0}^{\infty} \|f_n(t)\|^2_{\mathcal{F}_n(H_0)} n! dt \\
&= \sum_{n=0}^{\infty} n! \int_{[0,T)} \|f_n(t)\|^2_{\mathcal{F}_n(H_0)} dt \\
&= \sum_{n=0}^{\infty} n! \int_{[0,T)} \Big(\int_{[0,T)^n} |f_n(t;t_1,\ldots,t_n)|^2 dt_1) \ldots dt_n \Big) dt \\
&= \sum_{n=0}^{\infty} n! \int_{[0,T)} \Big(\int_{[0,t)^n} |f_n(t;t_1,\ldots,t_n)|^2 dt_1 \ldots dt_n \Big) dt \\
&= \sum_{n=0}^{\infty} (n!)^2 \int_0^T \Big(\int_0^t \int_0^{t_n} \cdots \int_0^{t_2} |f_n(t;t_1,\ldots,t_n)|^2 dt_1 \ldots dt_{n-1} dt_n \Big) dt \\
&= \sum_{n=0}^{\infty} ((n+1)!)^2 \int_0^T \int_0^{t_{n+1}} \cdots \int_0^{t_2} |\hat{f}_{n+1}(t_1,\ldots,t_{n+1})|^2 dt_1 \ldots dt_n dt_{n+1} \\
&= \sum_{n=0}^{\infty} (n+1)! \int_{[0,T)^{n+1}} |\hat{f}_{n+1}(t_1,\ldots,t_{n+1})|^2 dt_1 \ldots dt_{n+1} \\
&= \sum_{n=0}^{\infty} \|\hat{f}_{n+1}\|^2_{\mathcal{F}_{n+1}(H_0)} (n+1)! = \|\mathbb{S}(f)\|^2_{F(H_0)}.
\end{aligned}
$$

□

3.3. **On extended stochastic integral in a functional realization of the Fock space.** We will pass now to the construction of the "I_h-image" (the definition of I_h is given by (2.32)) of extended stochastic integral (3.3). We consider instead of rigging (2.5), (2.6) of the Fock space $F(H_0)$ the I_h-image

$$\mathcal{H}(-p,-q) \supset (L^2_Q) \supset \mathcal{H}(p,q)$$

of this rigging (see (2.33)).

Note that the above-mentioned space $L^2([0,T);\mathcal{K})$ ($\mathcal{K}$ is a separable Hilbert space) can be understood as a tensor product $L^2([0,T),m) \otimes \mathcal{K}$, therefore

$$1 \otimes I_h : L^2([0,T);\mathcal{F}(-p,-q)) \to L^2([0,T);\mathcal{H}(-p,-q)), \qquad p,q \in \mathbb{N}_1,$$

is the unitary operator. This remark and (3.3) give the following definition.

The extended stochastic integral of a function

$$\xi \in L^2([0,T);\mathcal{H}(-p,-q)), \qquad p,q \in \mathbb{N}_1, \tag{3.11}$$

is defined by the formula

$$\mathbb{S}_{\mathrm{ext},h}(\xi) = \int_{[0,T)} \partial^+(\delta_t)\xi(t)dt \in \mathcal{H}(-p,-q). \tag{3.12}$$

Here we understand the right hand side as a Bochner integral of the vector-valued function

$$[0,T) \ni t \mapsto \partial^+(\delta_t)\xi(t) \in \mathcal{H}(-p,-q), \tag{3.13}$$

where $\partial^+(\delta_t)$ is the I_h-image of the creation operator $a^+(\delta_t)$, i.e., $\partial^+(\delta_t) := I_h a^+(\delta_t) I_h^{-1}$.

The existence of a Bochner integral in (3.12) follows from Proposition 3.1 because if ξ belongs $L^2([0,T);\mathcal{H}(-p,-q))$ then $(1\otimes I_h)^{-1}\xi$ belongs $L^2([0,T);\mathcal{F}(-p,-q))$ and

$$\begin{aligned}\int_{[0,T)} \|\partial^+(\delta_t)\xi(t)\|^2_{\mathcal{H}(-p,-q)}dt &= \int_{[0,T)} \|I_h a^+(\delta_t) I_h^{-1}\xi(t)\|^2_{\mathcal{H}(-p,-q)}dt \\ &= \int_{[0,T)} \|a^+(\delta_t) I_h^{-1}\xi(t)\|^2_{\mathcal{F}(-p,-q)}dt < \infty.\end{aligned}$$

We also point out that from (3.11), (3.12) and (3.3) we have

$$\mathbb{S}_{\mathrm{ext},h}(\xi) = I_h \mathbb{S}_{\mathrm{ext}}((1\otimes I_h)^{-1}\xi), \qquad \xi \in L^2([0,T);\mathcal{H}(-p,-q)), \quad p,q \in \mathbb{N}_1. \tag{3.14}$$

Assume now that $f(\cdot) = \sum_{n=0}^{\infty} \langle f_n(\cdot), h_n\rangle \in L^2([0,T);(L^2_Q))$ and $\sum_{n=0}^{\infty} \|\hat{f}_{n+1}\|^2_{\mathcal{F}_{n+1}(H_0)}(n+1)! < \infty$. Using (3.14), (3.6) and (2.32) we obtain

$$\mathbb{S}_{\mathrm{ext},h}(f) = I_h \mathbb{S}_{\mathrm{ext}}((1\otimes I_h)^{-1}f) = I_h \mathbb{S}((1\otimes I_h)^{-1}f) = I_h(0,\hat{f}_1,\hat{f}_2,\ldots) = \sum_{n=1}^{\infty}\langle \hat{f}_n, h_n\rangle \in (L^2_Q). \tag{3.15}$$

Moreover, if

$$f \in D_h := (1\otimes I_h)D \subset L^2([0,T);(L^2_Q)) = L^2([0,T),m)\otimes(L^2_Q) \subset L^2([0,T);\mathcal{H}(-p,-q)) \tag{3.16}$$

(D is defined by (3.8), (3.9)) then it follows from (3.10) and (3.15) that

$$\|\mathbb{S}_{\mathrm{ext},h}(f)\|_{(L^2_Q)} = \|f\|_{L^2([0,T);(L^2_Q))}.$$

Formulas (3.11) – (3.16) and corresponding assertions constitute, in particular, the version of Theorem 3.1 and Theorem 3.2 in the language of functional realizations of Fock space.

Remark 3.1. It is easy to understand that the constructions of this Section are preserved for the case $T = \infty$ if we take as H_p the weighted Sobolev space $W^2_p([0,\infty),(1+t^2)^p dm(t))$ (such a construction is described in [37]). Now Φ is the Schwartz space of infinite differentiable rapidly decreasing real-valued functions on $[0,\infty)$.

4. Martingales and their construction. Multiple spectral integrals

4.1. **Resolution of identity and martingales.** We recall at first some generalization of the notion of martingale and the integration with respect to such martingales of scalar-valued functions [17-20].

Let Ω be some space of points ω, endowed by a σ-algebra $\mathcal{A}$ and a probability measure P defined on $\mathcal{A}$, i.e., $(\Omega, \mathcal{A}, P)$ is a probability space. Let $(\mathcal{A}_t)_{t\in[0,T)}$ be a flow of σ-subalgebras $\mathcal{A}_t$ of $\mathcal{A}$ with the properties: $\mathcal{A}_s \subset \mathcal{A}_t$ if $s \leq t,\ s,t \in [0,T)$, and $\bigcap_{t<u<T} \mathcal{A}_u = \mathcal{A}_t,\ T \leq \infty$. All the algebras $\mathcal{A},\ \mathcal{A}_t$ are supposed to be complete with respect to the measure P. So, we have a filtration $(\mathcal{A}_t)_{t\in[0,T)}$, which is right continuous for every $t \in [0,T)$.

Introduce the complex Hilbert space $L^2(\Omega, \mathcal{A}, P) =: L^2$ and its subspaces $L^2(\Omega, \mathcal{A}_t, P) =: L^2_t$, $t \in [0,T)$. Denote by $E(t)$ the orthogonal projector in the space L^2 onto L^2_t:

$$E(t)L^2 = L^2_t, \qquad t \in [0,T). \tag{4.1}$$

For the subspace L^2_t we evidently have:

$$L^2_s \subset L^2_t, \qquad E(s) \leq E(t), \qquad s \leq t, \qquad s,t \in [0,T). \tag{4.2}$$

The inclusion in (4.2) shows that for all $s,t \in [0,T)$

$$E(s)E(t) = E(\min\{s,t\}). \tag{4.3}$$

As a result we constructed the operator-valued function $E(t)$ with the properties of a resolution of identity in L^2. This function we will be called a quasiresolution of the identity because it can be $E([0,T)) < 1$. It is possible to understand $E(t)$ as a projector-valued measure $\mathcal{B}([0,T)) \ni \alpha \mapsto E(\alpha)$ on the σ-algebra $\mathcal{B}([0,T))$ of Borel subsets of $[0,T)$: for this we set $E((s,t]) := E(t) - E(s)$ and extend this definition to all Borel subsets of $[0,T)$. For details on such a procedure see [48], Ch. 6, [49], Ch. 6, [44], Ch. 13, [38], Ch. 3.

Let M_T be some vector from L^2, then the vector-valued function

$$[0,T) \ni t \mapsto M(t) := E(t)M_T \in L^2_t \subset L^2 \tag{4.4}$$

is a uniformly square-integrable martingale on the probability space $(\Omega, \mathcal{A}, P)$ with respect to the filtration $(\mathcal{A}_t)_{t\in[0,T)}$, i.e., $M(t)$ is a martingale with respect to $(\mathcal{A}_t)_{t\in[0,T)}$ and $\|M(t)\|_{L^2} \leq c$, $t \in [0,T)$, for some constant $c > 0$ (see, e.g., [52, 51, 53, 31, 54] for the corresponding definition).

Conversely, every uniformly square-integrable martingale $M(t)$ with respect to $(\mathcal{A}_t)_{t\in[0,T)}$, has form (4.4). In fact, we construct by the filtration $(\mathcal{A}_t)_{t\in[0,T)}$ the corresponding quasiresolution of identity $E(t)$, $t \in [0,T)$, in L^2. Then for each $f \in L^2$ the vector $E(t)f$ is equal to the conditional expectation $E\{f|\mathcal{A}_t\}$, but for a uniformly square-martingale there exists a vector $M_T \in L^2$ such that $M(t) = E\{M_T|\mathcal{A}_t\}$ (see, e.g., [52], Ch. 1, § 1). The latter equality is equivalent to (4.4).

A slight generalization of (4.4) is the following. Let $\mathcal{H}$ be a complex Hilbert space and $E(t)$, $t \in [0,T)$, $T \le \infty$, be some quasiresolution of the identity in $\mathcal{H}$, i.e., a operator-valued function (or the corresponding operator-valued measure $E(\alpha)$) with all properties of right continuous resolutions of identity in $\mathcal{H}$, but for which $E([0,T)) \le 1$. Let $M_T \in \mathcal{H}$ be fixed. *Then the vector-valued function*

$$[0,T) \ni t \mapsto M(t) := E(t)M_T \in \mathcal{H}, \tag{4.5}$$

is by definition, an abstract martingale.

For a Borel function $[0,T) \ni t \mapsto f(t) \in \mathbb{C}$ we introduce an abstract stochastic integral with respect to martingale (4.5) by the formula

$$\int_{[0,T)} f(t)dM(t) := \Big(\int_{[0,T)} f(t)dE(t) \Big) M_T, \tag{4.6}$$

where in the right-hand side we have an ordinary spectral integral. The well-known properties of spectral integrals (see, e.g., [48, 49, 44]) give the corresponding properties of integral (4.6).

Note one simple property of the definitions introduced above. Let U be some unitary operator acting from $\mathcal{H}$ onto another Hilbert space $\mathcal{K}$. Then $Z(t) = UM(t)$, $t \in [0,T)$, is also an abstract martingale in the space $\mathcal{K}$ because

$$Z(t) = UM(t) = G(t)Z_T; \qquad G(t) = UE(t)U^{-1}, \qquad Z_T = UM_T \in \mathcal{K}, \tag{4.7}$$

and $G(t)$ is a quasiresolution of identity in the space $\mathcal{K}$.

Applying the operator U to equality (4.6) we get an abstract stochastic integral with respect to the abstract martingale $Z(t)$:

$$U\Big(\int_{[0,T)} f(t)dM(t) \Big) = \int_{[0,T)} f(t)dZ(t) = \Big(\int_{[0,T)} f(t)dG(t) \Big) Z_T. \tag{4.8}$$

4.2. **On multiple spectral integrals.** In this Subsection we recall a generalization of above constructions (4.5)–(4.8) for the introduction of multiple spectral integrals with respect to an abstract n-dimensional martingale for complex-valued symmetric functions of $n \in \mathbb{N}_1$ variables $t_1, \dots, t_n \in [0,T)$ (see [20] for details).

At first we construct some class of n-dimensional resolutions of identity using a tensor product. Namely, let $E(t)$, $t \in [0,T)$, be some quasiresolution of identity in a complex Hilbert space $\mathcal{H}$. Introduce for each $n \in \mathbb{N}_1$ in the complex Hilbert space $\mathcal{H}^{\otimes n}$ the quasiresolution of identity $E^{\otimes n}$ by setting for Borel rectangles $\Delta_1 \times \cdots \times \Delta_n$

$$E^{\otimes n}(\Delta_1 \times \cdots \times \Delta_n) := E(\Delta_1) \otimes \cdots \otimes E(\Delta_n). \tag{4.9}$$

This projector-valued function of rectangles (projector in the space $\mathcal{H}^{\otimes n}$) can be extended to some quasiresolution of identity $E^{\otimes n}$, which we also denote by $E^{\otimes n}$ (see, e.g., [38]).

Let M_T be some vector from the space $\mathcal{H}$, and $M(t) = E(t)M_T$ be the corresponding martingale. Then we define an *abstract n-dimensional martingale* $\mathcal{B}([0,T)^n) \ni \alpha \mapsto M(\alpha) \in \mathcal{H}^{\otimes n}$ by the formula

$$M_n(\alpha) := E^{\otimes n}(\alpha) M_T^{\otimes n}, \qquad \alpha \in \mathcal{B}([0,T)^n). \tag{4.10}$$

We will pass now to the construction of some n-dimensional quasiresolution of identity $E^{\widehat{\otimes} n}$ acting in the symmetric tensor product $\mathcal{H}^{\widehat{\otimes} n} \subset \mathcal{H}^{\otimes n}$. Denote by $\hat{\mathcal{B}}([0,T)^n) \subset \mathcal{B}([0,T)^n)$ the σ-algebra spanned by all rectangles $\Delta_1 \times \cdots \times \Delta_n$, where $\Delta_1, \ldots, \Delta_n$ are *disjoint* Borel subsets of $[0,T)$. We put for these rectangles $\Delta_1 \times \cdots \times \Delta_n$

$$\begin{aligned} E^{\widehat{\otimes} n}(\Delta_1 \times \cdots \times \Delta_n) = E(\Delta_1) \widehat{\otimes} \cdots \widehat{\otimes} E(\Delta_n) &:= \frac{1}{n!} \sum_{\sigma \in S_n} E^{\otimes n}(\Delta_{\sigma(1)} \times \cdots \times \Delta_{\sigma(n)}) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} E(\Delta_{\sigma(1)}) \otimes \cdots \otimes E(\Delta_{\sigma(n)}), \end{aligned} \tag{4.11}$$

where S_n is the group of all permutation $\sigma(1, \ldots, n) = (\sigma(1), \ldots, \sigma(n))$ of $\{1, \ldots, n\}$ ($n!$ values of the index σ). It is possible to prove [20] that *this projector-valued function of rectangles can be extended to some quasiresolution of identity $E^{\widehat{\otimes} n}(\alpha)$, $\alpha \in \hat{\mathcal{B}}([0,T)^n)$, in the space $\mathcal{H}^{\widehat{\otimes} n}$.* According to (4.11) it is possible to say that the quasiresolution of identity $E^{\widehat{\otimes} n}$, acting in the space $\mathcal{H}^{\widehat{\otimes} n}$, is a symmetrization of the quasiresolution of identity $E^{\otimes n}$.

Introduce the "diagonal" set $d \subset [0,T)^n$:

$$d = \bigcup_{\{j_1, j_2\} \subset \{1, \ldots, n\}} \{t \in [0,T)^n \,|\, t_{j_1} = t_{j_2}\}. \tag{4.12}$$

It follows from (4.9) and (4.11) that for a Borel complex-valued symmetric function $f(t)$, $t \in [0,T)^n$, *vanishing in some neighborhood of the set d*, we have

$$\int_{[0,T)^n} f(t) dE^{\widehat{\otimes} n}(t) = \Big(\int_{[0,T)^n} f(t) dE^{\otimes n}(t) \Big) \restriction \mathcal{H}^{\widehat{\otimes} n}. \tag{4.13}$$

Above described construction gives the possibility to introduce *an abstract symmetric n-dimensional martingale $\hat{M}_n$* similar to (4.10). This martingale will be understood as a vector-valued measure defined on $\hat{\mathcal{B}}([0,T)^n)$ with values in $\mathcal{H}^{\widehat{\otimes} n}$:

$$\hat{\mathcal{B}}([0,T)^n) \ni \alpha \mapsto \hat{M}_n(\alpha) := E^{\widehat{\otimes} n}(\alpha) M_T^{\otimes n} \in \mathcal{H}^{\widehat{\otimes} n}, \tag{4.14}$$

where $M_T \in \mathcal{H}$.

Apply equality (4.13) to $M_T^{\otimes n}$. Using the definition of integral by martingales of type (4.6) and (4.14), (4.10) we find the following relation for an above appearing symmetric function f vanishing in some neighborhood of diagonal d (4.12):

$$\int_{[0,T)^n} f(t) d\hat{M}_n(t) = \int_{[0,T)^n} f(t) dM_n(t). \tag{4.15}$$

The integrals

$$\int_{[0,T)^n} f(t)d\hat{M}_n(t) := \Big(\int_{[0,T)^n} f(t)dE^{\widehat{\otimes} n}(t)\Big)M_T^{\otimes n} \tag{4.16}$$

will be called *multiple spectral integrals with respect to the symmetric n-dimensional martingale $\hat{M}_n$.*

It is possible to apply in case (4.14), (4.16) a construction of the form (4.7), (4.8). Namely, let U be some unitary operator acting from $\mathcal{H}^{\widehat{\otimes} n}$ onto another Hilbert space $\mathcal{K}$. Then

$$\hat{\mathcal{B}}([0,T)^n) \ni \alpha \mapsto \hat{Z}_n(\alpha) := U\hat{M}_n(\alpha) \in \mathcal{K}, \tag{4.17}$$

is an abstract symmetric martingale and for a measurable with respect to $\hat{\mathcal{B}}([0,T)^n)$ functions f vanishing in some neighborhood of diagonal d we have

$$U\Big(\int_{[0,T)^n} f(t)d\hat{M}_n(t)\Big) = \int_{[0,T)^n} f(t)d\hat{Z}_n(t). \tag{4.18}$$

4.3. **The multiple spectral integral in an n-particle Fock space.** The result of Subsection 4.2 is concerned with a general quasiresolution of the identity E acting in a complex Hilbert space $\mathcal{H}$. In this Subsection we will consider a more special situation when $\mathcal{H}$ is equal to

$$\mathcal{H} = L^2([0,T),m) = H_{0,\mathbb{C}}, \qquad 0 < T \le \infty$$

(m is the Lebesgue measure), and the quasiresolution of identity in this space has the form:

$$\mathcal{B}([0,T)) \ni \alpha \mapsto E(\alpha)f := \varkappa_\alpha f \in L^2([0,T),m), \qquad f \in L^2([0,T),m), \tag{4.19}$$

where $\varkappa_\alpha$ denotes the characteristic function of the set α. In other words, our E is the resolution of identity of the operator of multiplication by t in the space $H_{0,\mathbb{C}}$. Construct according to (4.9) and (4.11) the corresponding resolutions of identity $E^{\otimes n}$ and $E^{\widehat{\otimes} n}$. They act in the spaces $\mathcal{H}^{\otimes n} = L^2([0,T)^n, m^{\otimes n})$ and $\mathcal{H}^{\widehat{\otimes} n} = \mathcal{F}_n(H_0) = \hat{L}^2([0,T)^n, m^{\otimes n})$ respectively.

We will prove an essential formula which represents a function from the space $\mathcal{F}_n(H_0)$ as an action of the spectral integral with respect to $E^{\widehat{\otimes} n}$ on a certain function from $\mathcal{F}_n(H_0)$. Namely, let some positive essentially bounded function $M_T \in L^2([0,T),m)$ be fixed. Construct the martingale $\hat{M}_n$ by formulas (4.14) and (4.11) from the one-dimensional resolution of identity (4.19) and M_T.

Lemma 4.1. *For an arbitrary symmetric function $f_n \in \mathcal{F}_n(H_0) = \hat{L}^2([0,T)^n, m^{\otimes n})$ the following representation is valid*

$$f_n(\tau) = \frac{1}{M_T^{\otimes n}(\tau)}\Big(\int_{[0,T)^n} f_n(t)d\hat{M}_n(t)\Big)(\tau) \tag{4.20}$$

for $m^{\otimes n}$-almost all $\tau \in [0,T)^n$. Here the integral is the multiple spectral integral with respect to the symmetric n-dimensional martingale $\hat{M}_n$.

Proof. From (4.16) and (4.13) we conclude that

$$\int_{[0,T)^n} f_n(t)d\hat{M}_n(t) = \Big(\int_{[0,T)^n} f(t)dE^{\widehat{\otimes} n}(t)\Big)M_T^{\otimes n} = \Big(\int_{[0,T)^n} f_n(t)dE^{\otimes n}(t)\Big)M_T^{\otimes n} \tag{4.21}$$

for any $f_n \in \mathcal{F}_n(H_0) = \hat{L}^2([0,T)^n, m^{\otimes n})$, additionally equal to zero in some neighborhood of d. But the Lebesgue measure m is non atomic, therefore the latter functions are dense in the whole space $\mathcal{F}_n(H_0)$. Then equality (4.21) is valid for arbitrary $f_n \in \mathcal{F}_n(H_0)$.

The operator-valued function $E(t)$ is the resolution of identity of the operator of multiplication by t in the space $L^2([0,T),m)$, therefore the spectral integral $\int_{[0,T)^n} f_n(t)dE^{\otimes n}(t)$ is the operator of multiplication by the function $f_n \in \mathcal{F}_n(H_0) = \hat{L}^2([0,T)^n, m^{\otimes n})$ and the right hand side in (4.21) is equal to $f_n(\tau)M_T^{\otimes n}(\tau)$. This gives (4.20). □

Remark 4.1. Assume that $T \in (0,\infty)$, then $m([0,T)) < \infty$ and we can put $M_T = 1$. In this case formula (4.20) have a simpler view: for $m^{\otimes n}$-almost all $\tau \in [0,T)^n$

$$f_n(\tau) = \Big(\int_{[0,T)^n} f_n(t)d\hat{M}_n(t)\Big)(\tau). \tag{4.22}$$

5. The connection of the extended stochastic integral with the classical Itô integral. Multiple Itô integral and its spectral representation

In Section 3 we have defined extended stochastic integral $\mathbb{S}_{\mathrm{ext}} = \mathbb{S}$ (3.6) for vector-valued function $\xi(t)$ with values in the Fock space $F(H_0)$. One can easily "rewrite" this integral in form $\mathbb{S}_{\mathrm{ext},h}$ (3.15), when the values of such function $\xi(t)$ belong to (L^2_Q). We remind that $H_0 = L^2_{\mathrm{Re}}([0,T),m)$, $(Q,\mathcal{B}(Q),\rho)$ is the probability space and $(L^2_Q) = L^2(Q,\mathcal{B}(Q),\rho)$ is the I_h-image of the Fock space $F(H_0)$, where $I_h : F(H_0) \to (L^2_Q)$ is unitary operator (2.32).

In this section we find a condition on $h(x,\lambda)$ (2.27) under which the extended stochastic integral $\mathbb{S}_{\mathrm{ext},h}$ is equal to an ordinary Itô integral constructed by a certain normal martingale. Moreover, we obtain conditions of coincidence of the multiple spectral integral with a multiple Itô integral.

5.1. **Preliminaries.** We will apply the results of Subsection 4.1. Namely, let $T \in (0,\infty)$, $\mathcal{H}$ be equal to $F(H_0)$. The quasiresolution of identity $E(t)$ in this space has the form:

$$[0,T) \in t \mapsto E(t)f = (f_0, \varkappa_{(0,t]}f_1, \ldots, \varkappa_{(0,t]^n}f_n, \ldots) \in F(H_0), \qquad f = (f_n)_{n=0}^\infty \in F(H_0),$$

where $\varkappa_\alpha$, as usual, denotes the characteristic function of the set α; $\varkappa_{(0,0]^n} := 0$, $n \in \mathbb{N}_1$. Let $M_T = (0,1,0,0,\ldots)$ be a fixed vector from $\mathcal{H} = F(H_0)$. Then

$$M(t) := E(t)M_T = (0, \varkappa_{(0,t]}, 0, 0, \ldots), \qquad t \in [0,T), \tag{5.1}$$

is an abstract martingale in the Fock space $F(H_0)$.

If we apply to (5.1) unitary operator (2.32) $U = I_h$, which transfers the Fock space $F(H_0)$ onto (L^2_Q), we get as a result (according to (4.7)) the abstract martingale

$$Z(t) = Z(t,x) = I_h M(t) = \langle \varkappa_{(0,t]}, h_1(x) \rangle, \qquad t \in [0,T),$$

in the space (L^2_Q).

So, we have constructed the required martingale $Z(t)$. Using orthogonality relation (2.30) for h_n and (2.32) it is easy to check that for $0 \le s < t < T$

$$\|Z(t) - Z(s)\|^2_{(L^2_Q)} = \|\langle \varkappa_{(s,t]}, h_1 \rangle\|^2_{(L^2_Q)} = \|\varkappa_{(s,t]}\|^2_{L^2([0,T),m)} = t - s. \tag{5.2}$$

In addition, the condition $h_0(x) = 1$, $x \in Q$, is fulfilled (see (2.27)). Therefore, in accordance with (2.30) for all $t \in [0,T)$

$$\int_Q Z(t,x) d\rho(x) = \int_Q \langle \varkappa_{(0,t]}, h_1(x) \rangle d\rho(x) = \int_Q \langle \varkappa_{(0,t]}, h_1(x) \rangle \overline{\langle \varkappa_{(0,t]}, h_0(x) \rangle} d\rho(x) = 0. \tag{5.3}$$

Let $(\mathcal{A}_t)_{t\in[0,T)}$ be the flow of σ-algebras $\mathcal{A}_t$ generated by the process $\{Z(t) \,|\, t \in [0,T)\}$, i.e., for every $t \in [0,T)$ $\mathcal{A}_t$ is the σ-algebra on Q generated by the sets $\{x \in Q \,|\, Z(s,x) \in \alpha\}$, $\alpha \in \mathcal{B}(\mathbb{C})$, $0 \le s \le t$. This flow is right continuous because $E(t)$ has such a property. We assume that $\mathcal{A}_0$ is complete with respect to the measure ρ and $\mathcal{B}(Q)$ coincides with the smallest σ-algebra generated by $\bigcup_{t\in[0,T)} \mathcal{A}_t$.

In the sequel, we will assume that *the process $\{Z(t) \,|\, t \in [0,T)]\}$ is a normal martingale with respect to the flow of σ-algebras $\mathcal{A}_t$, i.e., that $\{Z(t) \,|\, t \in [0,T)]\}$ and $\{Z^2(t) - t \,|\, t \in [0,T)\}$ are martingales with respect to $(\mathcal{A}_t)_{t\in[0,T)}$.*

Note that if Z has independent increments then Z is a normal martingale. This follows from the properties (5.2), (5.3) and the property of Z having independent increments.

5.2. **The classical Itô integral with respect to the normal martingale Z. Multiple Itô integrals.** Let $T \in (0,\infty)$ be fixed. We denote by D_{I} the set of $\mathcal{B}([0,T)) \times \mathcal{B}(Q)$-measurable functions

$$[0,T) \times Q \ni \{t,x\} \mapsto f(t,x) \in \mathbb{C}, \tag{5.4}$$

which are $\mathcal{A}_t$-adapted and belong to the space $L^2([0,T);(L^2_Q))$. We recall that function (5.4) is $\mathcal{A}_t$-adapted if for each $t \in [0,T)$ the function

$$Q \ni x \mapsto f(t,x) \in \mathbb{C}$$

is $\mathcal{A}_t$-measurable.

We note that in terms of the resolution of identity function (5.4) is $\mathcal{A}_t$-adapted if $f(t) = E(t)f(t)$ for each $t \in [0,T)$, where $E(t)$ is the resolution of identity generated by the σ-algebra $\mathcal{A}_t$ (recalling that $E(t)$ is the projector in the space (L^2_Q) onto its subspace consisting of all functions from (L^2_Q), which are measurable with respect to $\mathcal{A}_t$).

The Itô integral of the integrand $f(t) = f(t,x)$

$$\mathbb{S}_{\mathrm{I}}(f) = \int_0^T f(t)dZ(t) \tag{5.5}$$

with respect to the normal martingale $Z(t)$ is defined as the unique linear isometric mapping

$$L^2([0,T);(L^2_Q)) \supset D_{\mathrm{I}} \ni f \mapsto \mathbb{S}_{\mathrm{I}}(f) \in (L^2_Q) \tag{5.6}$$

such that

$$\mathbb{S}_{\mathrm{I}}(g\varkappa_{(s,t]}) = g(Z(t) - Z(s)) = g\langle \varkappa_{(s,t]}, h_1\rangle, \tag{5.7}$$

$$0 \le s < t < T,$$

for any $\mathcal{A}_s$-measurable function $g \in (L^2_Q)$.

We stress that the isometry of the mapping (5.6) means that the following equality holds:

$$\|\int_0^T f(t)dZ(t)\|^2_{(L^2_Q)} = \int_0^T \|f(t)\|^2_{(L^2_Q)}dt, \qquad f \in D_{\mathrm{I}}. \tag{5.8}$$

For a proof of existence and the properties of such an Itô integral $\mathbb{S}_{\mathrm{I}}(f)$ we refer to the books [52, 51, 53, 54, 19].

Let us recall some results concerning the definition and properties of the multiple Itô integrals for the integrands that are complex-valued symmetric functions. Namely, let $f_n \in \mathcal{F}_n(H_0) = \hat{L}^2([0,T)^n, m^{\otimes n})$, $n \in \mathbb{N}_1$. For such a function f_n we can form the *iterated Itô integral*

$$\int_0^T \int_0^{t_n} \cdots \Big(\int_0^{t_2} f_n(t_1,\ldots,t_n)dZ(t_1)\Big)\ldots dZ(t_{n-1})dZ(t_n) \tag{5.9}$$

because at each Itô integration with respect to $dZ(t_i)$ the integrand is $\mathcal{A}_t$-adapted and square integrable with respect to $d\rho(x) \times dm(t_i)$, $i \in \{1,\ldots,n\}$. Moreover, applying n times equality (5.8) we obtain

$$\begin{aligned}
&\Big\| \int_0^T \Big(\int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1,\dots,t_n) dZ(t_1)\dots dZ(t_{n-1})\Big) dZ(t_n) \Big\|^2_{(L^2_Q)} \\
&\qquad = \int_0^T \Big\| \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1,\dots,t_n) dZ(t_1)\dots dZ(t_{n-1}) \Big\|^2_{(L^2_Q)} dt_n \\
&\qquad = \ \dots\dots\dots\dots \\
&\qquad = \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} |f_n(t_1,\dots,t_n)|^2 dt_1 \dots dt_{n-1} dt_n = \frac{1}{n!} \|f_n\|^2_{\mathcal{F}_n(H_0)}
\end{aligned}$$

Hence, the mapping

$$\mathcal{F}_n(H_0) \ni f_n \mapsto n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1,\dots,t_n) dZ(t_1)\dots dZ(t_{n-1}) dZ(t_n) \in (L^2_Q), \qquad n \in \mathbb{N}_1,$$

is linear and continuous.

For a function $f_n \in \mathcal{F}_n(H_0)$, $n \in \mathbb{N}_1$, we define an *Itô multiple stochastic integral* $\mathbb{S}_n(f_n)$ by

$$\mathbb{S}_n(f_n) := n! \int_0^T \int_0^{t_n} \cdots \Big(\int_0^{t_2} f_n(t_1,\dots,t_n) dZ(t_1)\Big) \dots dZ(t_{n-1}) dZ(t_n). \tag{5.10}$$

The properties of iterated Itô integrals (5.9) give the corresponding properties of the multiple stochastic integrals $\mathbb{S}_n(f_n)$. For more details on the construction and properties of the integrals $\mathbb{S}_n(f_n)$ see [50, 19], in the Gaussian and Poissonian cases see [31, 32, 55, 56]. We note that in the special case

$$f_n = \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n} \in \mathcal{F}_n(H_0), \qquad n \in \mathbb{N}_1,$$

where $\Delta_j = (a_j, b_j] \subset [0,T)$, $j \in \{1,\dots,n\}$, are disjoint, it is possible to calculate the integrals $\mathbb{S}_n(f_n)$ and get:

$$\mathbb{S}_n(\varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}) = \langle \varkappa_{\Delta_1}, h_1\rangle \dots \langle \varkappa_{\Delta_n}, h_1\rangle. \tag{5.11}$$

5.3. **When the image of a multiple spectral integral is an Itô multiple stochastic integral.** We will now continue the investigations of Subsection 4.3 concerning the properties of multiple spectral integrals.

At first, we recall the results of Subsection 4.3. Let $\mathcal{H} = H_{0,\mathbb{C}} = L^2([0,T), m)$, $T < \infty$. Using resolution of identity E (4.19) of the operator of multiplication by t in the space $L^2([0,T), m)$ and the function $M_T = 1$ we construct by (4.14), (4.11) the martingale $\hat{M}_n$ with values in $\mathcal{H}^{\widehat{\otimes} n} = \mathcal{F}_n(H_0) = \hat{L}^2([0,T)^n, m^{\otimes n})$. According to remark 4.1 for an arbitrary function $f_n \in \mathcal{F}_n(H_0)$ the following representation holds

$$f_n = \int_{[0,T)^n} f_n(t) d\hat{M}_n(t). \tag{5.12}$$

Apply operator I_h (2.32) to equality (5.12) (we consider f_n as a vector $(0, \dots, 0, f_n, 0, 0, \dots)$ from $F(H_0)$). Using formulas (2.32) and (4.18) (with $U = I_h$) we get

$$\langle f_n, h_n(x) \rangle = \int_{[0,T)^n} f_n(t) d\hat{Z}_n(t, x), \tag{5.13}$$

in the space (L^2_Q), where

$$\hat{Z}_n(\alpha) = I_h \hat{M}_n(\alpha), \qquad \alpha \in \hat{\mathcal{B}}([0,T)^n),$$

is a symmetric martingale.

Our aim is to find conditions on $h(x, \lambda)$ under which image (5.13) of multiple spectral integral (5.12) coincides with Itô multiple stochastic integral (5.10).

Theorem 5.1. *For an arbitrary function $f_n \in \mathcal{F}_n(H_0)$, $n \in \mathbb{N}_1$, the equality*

$$\mathbb{S}_n(f_n) = \int_{[0,T)^n} f_n(t) d\hat{Z}_n(t) \tag{5.14}$$

holds if and only if

$$I_h(\varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}) = I_h(\varkappa_{\Delta_1}) \dots I_h(\varkappa_{\Delta_n}) \tag{5.15}$$

for all disjoint intervals $\Delta_j = (a_j, b_j]$, $j \in \{1, \dots, n\}$, from $[0, T)$.

Proof. Let us assume that equality (5.15) take place, i.e.

$$\langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}, h_n \rangle = \langle \varkappa_{\Delta_1}, h_1 \rangle \cdots \langle \varkappa_{\Delta_n}, h_1 \rangle \tag{5.16}$$

for all disjoint interval $\Delta_j = (a_j, b_j]$, $j \in \{1, \dots, n\}$, from $[0, T)$. Using (5.16), (5.13) and (5.11) we conclude:

$$\mathbb{S}_n(\varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}) = \int_{[0,T)^n} (\varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n})(t) d\hat{Z}_n(t). \tag{5.17}$$

Since the vectors $\varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}$ form a total set in $\mathcal{F}_n(H_0)$ we obtain (5.14) from (5.17).

Conversely, let (5.14) takes place. Then from (5.13), (2.32) and (5.11) we obtain (5.15). □

We have the following statement.

Theorem 5.2. *If for all $x \in Q$ and for any $\varphi_1, \dots, \varphi_n \in \Phi$ such that* $\operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_j = \varnothing$ *if $j \neq i$, $i, j \in \{1, \dots, n\}$,*

$$\left. \frac{\partial^n h(x, z_1\varphi_1 + \cdots + z_n\varphi_n)}{\partial z_1 \dots \partial z_n} \right|_{z_1 = \ldots = z_n = 0} = \left. \frac{\partial}{\partial z_1} h(x, z_1\varphi_1) \right|_{z_1=0} \cdots \left. \frac{\partial}{\partial z_n} h(x, z_n\varphi_n) \right|_{z_n=0} \tag{5.18}$$

then for all disjoint $\Delta_j = (a_j, b_j] \subset [0, T)$, $j \in \{1, \dots, n\}$,

$$I_h(\varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}) = I_h(\varkappa_{\Delta_1}) \dots I_h(\varkappa_{\Delta_n}), \qquad n \in \mathbb{N}_1.$$

Proof. Let us assume that (5.18) is fulfilled. Then using (2.34) and (5.18) we obtain

$$\langle \varphi_1 \widehat{\otimes} \cdots \widehat{\otimes} \varphi_n, h_n(x)\rangle = \langle \varphi_1, h_1(x)\rangle \dots \langle \varphi_n, h_1(x)\rangle, \qquad n \in \mathbb{N}_1, \tag{5.19}$$

for all $x \in Q$ and any $\varphi_1, \dots, \varphi_n \in \Phi$ (under the conditions of the theorem).

It is well known that for all disjoint $\Delta_1, \dots, \Delta_n \subset [0, T)$ there exist $\varphi_{j,\varepsilon} \in \Phi, \varepsilon > 0$, such that $\operatorname{supp} \varphi_{j,\varepsilon} \subset \Delta_j$ and $\varphi_{j,\varepsilon} \to \varkappa_{\Delta_j}$ in $H_{0,\mathbb{C}} = L^2([0, T), m)$ as $\varepsilon \to 0$. So, using (5.19) we have

$$I_h(\varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}) = \lim_{\varepsilon \to 0} I_h(\varphi_{1,\varepsilon} \widehat{\otimes} \cdots \widehat{\otimes} \varphi_{n,\varepsilon}) = \lim_{\varepsilon \to 0} I_h(\varphi_{1,\varepsilon}) \dots I_h(\varphi_{n,\varepsilon}) = I_h(\varkappa_{\Delta_1}) \dots I_h(\varkappa_{\Delta_n}).$$

□

5.4. **The coincidence of the extended stochastic integral with the Itô integral for adapted processes.** In the classical Gaussian and Poissonian analysis the extended stochastic integral is a generalization of the Itô integral: they are equal for adapted processes. Therefore, there is a natural question about conditions on the unitary map $I_h : F(H_0) \to (L^2_Q)$ such that $\mathbb{S}_{\mathrm{ext},h} = \mathbb{S}_{\mathrm{I}}$. As an answer we have the following statement.

Theorem 5.3. *If the unitary map* $I_h : F(H_0) \to (L^2_Q)$ *is such that*

$$\begin{gathered} I_h(\varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}) = I_h(\varkappa_{\Delta_1}) \dots I_h(\varkappa_{\Delta_n}), \qquad i.e. \\ \langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}, h_n\rangle = \langle \varkappa_{\Delta_1}, h_1\rangle \cdots \langle \varkappa_{\Delta_n}, h_1\rangle, \qquad n \in \mathbb{N}_1, \end{gathered} \tag{5.20}$$

for all disjoint intervals $\Delta_j = (a_j, b_j]$, $j \in \{1, \dots, n\}$, *from* $[0, T)$ *then* $D_{\mathrm{I}} = D_h$ *(the set* D_h *is defined by (3.16)). In this case for the extended integral* $\mathbb{S}_{\mathrm{ext},h}$*, defined by (3.15), we have*

$$\mathbb{S}_{\mathrm{ext},h}(f) = \mathbb{S}_{\mathrm{I}}(f), \qquad f \in D_h = D_{\mathrm{I}}. \tag{5.21}$$

Conversely, if $D_h \subset D_{\mathrm{I}}$ *and (5.21) takes place then (5.20) is fulfilled and* $D_h = D_{\mathrm{I}}$.

Proof. Let (5.20) takes place. In order to prove the equality $D_h = D_{\mathrm{I}}$, it is sufficient to check that

$$E\{\langle f_n, h_n\rangle | \mathcal{A}_t\} = \langle \varkappa_{(0,t]^n} f_n, h_n\rangle, \qquad t \in [0, T), \tag{5.22}$$

for arbitrary $f_n \in \mathcal{F}_n(H_0)$, $n \in \mathbb{N}_1$. Here $E\{f | \mathcal{A}_t\}$ denotes the conditional expectation of a random variable f with respect to the σ-algebras $\mathcal{A}_t$ (note that $E\{\cdot | \mathcal{A}_t\}$ is the projector in the space (L^2_Q) onto its subspace consisting of all functions from (L^2_Q) which are measurable with respect to $\mathcal{A}_t$).

Since the functions

$$f_n = \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}, \qquad \Delta_i \cap \Delta_j = \varnothing, \quad i \neq j,$$

form a total set in $\mathcal{F}_n(H_0)$, it is sufficient to check (5.22) for such functions.

Using (5.20), the equality

$$E\{\langle \varkappa_{(0,s]}, h_1\rangle|\mathcal{A}_t\} = \langle \varkappa_{(0,t]}, h_1\rangle, \qquad 0 < t \le s < T$$

(recall that $Z(t) = \langle \varkappa_{(0,t]}, h_1\rangle$, $t \in [0,T)$ is a martingale) and the properties of the conditional expectation, we get

$$\begin{aligned} E\{f_n|\mathcal{A}_t\} &= E\{\langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}, h_n\rangle|\mathcal{A}_t\} = E\{\langle \varkappa_{\Delta_1}, h_1\rangle \ldots \langle \varkappa_{\Delta_n}, h_1\rangle|\mathcal{A}_t\} \\ &= E\{\prod_{j=1}^{n}(\langle \varkappa_{\Delta_j\cap(0,t]}, h_1\rangle + \langle \varkappa_{\Delta_j\cap(t,T)}, h_1\rangle)|\mathcal{A}_t\} = \langle \varkappa_{\Delta_1\cap(0,t]}, h_1\rangle \ldots \langle \varkappa_{\Delta_n\cap(0,t]}, h_1\rangle \\ &= \langle \varkappa_{\Delta_1\cap(0,t]} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n\cap(0,t]}, h_n\rangle = \langle \varkappa_{(0,t]^n}(\varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}), h_n\rangle = \langle \varkappa_{(0,t]^n} f_n, h_n\rangle. \end{aligned}$$

So, the necessary equality $D_h = D_{\mathrm{I}}$ is proved.

Let us prove (5.21). The mappings

$$L^2([0,T);(L^2_Q)) \supset D_{\mathrm{I}} \ni f \mapsto \mathbb{S}_{\mathrm{I}}(f) \in (L^2_Q), \qquad L^2([0,T);(L^2_Q)) \supset D_h \ni f \mapsto \mathbb{S}_{\mathrm{ext},h}(f) \in (L^2_Q)$$

are linear and continuous. Therefore, it is sufficient to show that (5.21) takes place for the functions

$$f_n(\cdot) = \langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}, h_n\rangle \varkappa_{\Delta}(\cdot) \in D_h = D_{\mathrm{I}}, \qquad n \in \mathbb{N}_1, \tag{5.23}$$

where $\Delta_j = (a_j, b_j] \subset [0,T)$, $j \in \{1,\ldots,n\}$, are disjoint and $\Delta = (a,b] \subset [0,T)$, $a > b_j$, $j \in \{1,\ldots,n\}$ (we note that functions (5.23) form a total set in $D_h = D_{\mathrm{I}}$).

For such Δ_j the function

$$\langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}, h_n\rangle = \langle \varkappa_{\Delta_1}, h_1\rangle \ldots \langle \varkappa_{\Delta_n}, h_1\rangle$$

is $\mathcal{A}_a$-measurable because each functions $\langle \varkappa_{\Delta_j}, h_j\rangle$, $j \in \{1,\ldots,n\}$, is $\mathcal{A}_a$-measurable. Therefore, according to (5.7) and (5.20), for function (5.23) we get

$$\begin{aligned} \mathbb{S}_{\mathrm{I}}(f) &= \int_0^T \langle f_n(t), h_n\rangle dZ(t) = \int_0^T \langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}, h_n\rangle \varkappa_{\Delta}(t) dZ(t) \\ &= \langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n}, h_n\rangle \langle \varkappa_{\Delta}, h_1\rangle = \langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_n} \widehat{\otimes} \varkappa_{\Delta}, h_{n+1}\rangle = \mathbb{S}_{\mathrm{ext},h}(f). \end{aligned}$$

The first part of the theorem is proved.

For the proof of its second part consider the function

$$[0,T) \ni t \mapsto \langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_{n-1}}, h_{n-1}\rangle \varkappa_{\Delta_n}(t) \in (L^2_Q),$$

where $\Delta_j = (a_j, b_j]$, $j \in \{1,\ldots,n\}$, are disjoint and $a_n > \max\{a_1,\ldots,a_{n-1}\}$. Evidently, these functions belong to the set $D_h \subset D_{\mathrm{I}}$ because condition (3.9) is fulfilled: if $t \in [0,T) \setminus \Delta_n$, then the function

$$f_{n-1}(t) := (\varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\Delta_{n-1}}) \varkappa_{\Delta_n}(t) \in \mathcal{F}_{n-1}(H_0)$$

is equal to zero; if $t \in \Delta_n$, then the multiplication by $f_{n-1}(t)$ on $\varkappa_{(0,t]^{n-1}}$ does not change this function ($\Delta_1, \ldots, \Delta_{n-1} \subset (0,t]$). Therefore, according to (3.15) we have:

$$\mathbb{S}_{\mathrm{ext},h}(\langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes}\, \varkappa_{\Delta_{n-1}}, h_{n-1}\rangle \varkappa_{\Delta_n}(\cdot)) = \langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes}\, \varkappa_{\Delta_n}, h_n\rangle. \tag{5.24}$$

On the other hand the function

$$\langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes}\, \varkappa_{\Delta_{n-1}}, h_{n-1}(\cdot)\rangle$$

is $\mathcal{A}_{a_n}$-measurable, therefore using (5.7) we obtain

$$\mathbb{S}_{\mathrm{I}}(\langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes}\, \varkappa_{\Delta_{n-1}}, h_{n-1}\rangle \varkappa_{\Delta_n}(t)) = \langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes}\, \varkappa_{\Delta_{n-1}}, h_{n-1}\rangle \langle \varkappa_{\Delta_n}, h_1\rangle. \tag{5.25}$$

From (5.21), (5.25) and (5.24) we conclude that

$$\langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes}\, \varkappa_{\Delta_{n-1}}, h_{n-1}\rangle \langle \varkappa_{\Delta_n}, h_1\rangle = \langle \varkappa_{\Delta_1} \widehat{\otimes} \cdots \widehat{\otimes}\, \varkappa_{\Delta_n}, h_n\rangle, \qquad n \in \mathbb{N}_2, \tag{5.26}$$

in the space (L^2_Q). Taking in (5.26) $n = 2, 3, \ldots$ we get step by step equality (5.20). □

6. Classical examples

1. Gaussian white noise analysis. Let $Q = H_{-1} = W^2_{-1}(\mathbb{R}_+, (1+t^2)^1 dt)$, $\mathbb{R}_+ = [0, \infty)$, $\rho = \gamma$ be the Gaussian measure, which is completely characterized by its Fourier transform

$$\int_{H_{-1}} \exp(i\langle x, \lambda\rangle) d\gamma(x) = \exp(-\frac{1}{2}\langle \lambda, \lambda\rangle), \qquad \lambda \in H_1 = W^2_1(\mathbb{R}_+, (1+t^2)^1 dt).$$

The function

$$h(x, \lambda) := \exp(\langle x, \lambda\rangle - \frac{1}{2}\langle \lambda, \lambda\rangle) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle \lambda^{\otimes n}, h_n(x)\rangle$$

is the generating function for the Hermite polynomials $h_n(x)$. In this case, the unitary mapping (2.32) is the classical Winer–Itô–Segal isomorphism.

One can verify that the function $h(x, \lambda)$ satisfies (5.18) and the process

$$\{B(t) = B(t, \cdot) := \langle \varkappa_{(0,t]}, h_1(\cdot)\rangle = \langle \varkappa_{(0,t]}, \cdot\rangle \mid t \in \mathbb{R}_+\},$$

is a normal martingale with respect to the flow $(\mathcal{A}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{A}_t$ generated by the set $\{x \in Q \,|\, B(s, x) \in \alpha\}$, $\alpha \in \mathcal{B}(\mathbb{R})$, $0 \le s \le t$. Hence it follows from Theorems 5.3 and 5.2 that the Itô integral coincides with the corresponding extended stochastic integral

$$\mathbb{S}_{\mathrm{I}}(f) = \mathbb{S}_{\mathrm{ext},h}(f) := \sum_{n=1}^{\infty} \langle \hat{f}_n, h_n\rangle.$$

for $f(\cdot) = \sum_{n=0}^{\infty} \langle f_n(\cdot), h_n\rangle \in D_{\mathrm{I}}$.

2. Poissonian white noise analysis. Let $Q = H_{-1} = W^2_{-1}(\mathbb{R}_+, (1+t^2)^1 dt)$, $\rho = \pi$ be the centered Poisson measure with intensity dt, which is completely characterized by its Fourier transform

$$\int_{H_{-1}} \exp(i\langle x, \lambda\rangle) d\pi(x) = \exp\langle 1, e^{i\lambda} - 1 - i\lambda\rangle, \qquad \lambda \in H_1.$$

In this case the function $h(x, \lambda)$ has the form

$$h(x, \lambda) := \exp(\langle x + 1, \log(1+\lambda)\rangle - \langle 1, \lambda\rangle) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, h_n(x)\rangle,$$

where $h_n(x)$ are the Charlier polynomials. It is known that the function $h(x, \lambda)$ satisfies all assumption formulated in Subsection 2.4. Therefore we have the statement that the Itô integral with respect to the process

$$\{C(t) = C(t, \cdot) := \langle \varkappa_{(0,t]}, h_1(\cdot)\rangle = \langle \varkappa_{(0,t]}, \cdot\rangle \mid t \in \mathbb{R}_+\},$$

coincide with the corresponding extended stochastic integral.

3. Let us fix a function

$$\mathbb{R}_+ \ni t \mapsto \theta(t) \in \mathbb{C}$$

such that

$$|\theta(t)| \leq c, \qquad t \in \mathbb{R}_+,$$

for some constant $c > 0$ and

$$\|\theta\varphi\|_{H_p} \leq c_p \|\varphi\|_{H_p}, \qquad \varphi \in H_p := W^2_p(\mathbb{R}_+, (1+t^2)^p dt),$$

for some constant $c_p > 0,\ p \in \mathbb{N}_1$.

We consider the probability measure μ_θ on $\mathcal{B}(H_{-1})$ given by the Fourier transform (see, e.g., [15])

$$\begin{aligned}\int_{S_{-2}} \exp(i\langle x, \lambda\rangle) d\mu_\theta(x) &= \exp\Big(\int_{\mathbb{R}_+} \Big(\sum_{n=2}^{\infty} \frac{(i\lambda(t))^n \theta^{n-2}(t)}{n!}\Big) dt\Big) \\ &= \exp\langle 1, (e^{i\lambda\theta} - 1 - i\lambda\theta)\theta^{-2}\rangle,\end{aligned}$$

where

$$(e^{i\lambda\theta} - 1 - i\lambda\theta)\theta^{-2} := \sum_{n=2}^{\infty} \frac{(i\lambda)^n \theta^{n-2}}{n!} \in H_1.$$

For

$$\theta(t) = 0, \qquad t \in \mathbb{R}_+,$$

μ_0 is the standard Gaussian measure, for

$$\theta(t) = 1, \qquad t \in \mathbb{R}_+,$$

μ_1 is the centered Poissonian measure.

Let us put $Q = H_{-1}$, $\rho = \mu_\theta$. It follows from results of [15] that the function

$$h^\theta(x,\lambda) := \exp(\langle x, \theta^{-1}\log(1+\theta\lambda)\rangle + \langle 1, \theta^{-2}\log(1+\theta\lambda) - \theta^{-1}\lambda\rangle) = \sum_{n=0}^\infty \frac{1}{n!}\langle \lambda^{\otimes n}, h_n^\theta(x)\rangle,$$

$$\theta^{-1}\log(1+\theta\lambda) := \sum_{n=1}^\infty \frac{(-1)^{n-1}\theta^{n-1}\lambda^n}{n} \in H_{-1} = H_-,$$

satisfies all assumptions of Subsection 2.4 required for a function h. Therefore the mapping

$$F(H_0) \ni f = (f_n)_{n=0}^\infty \mapsto (I_{h^\theta} f)(\cdot) := \sum_{n=0}^\infty \langle f_n, h_n^\theta(\cdot)\rangle \in (L^2_{H_-})$$

is well-defined and unitary. Under this mapping rigging (2.5) of the Fock space $F(H_0)$ transforms into a rigging of the corresponding $(L^2_{H_-})$. Note that for $\theta = 0$ the mapping I_{h^θ} is the classical Wiener–Itô–Segal isomorphism. It follows from results of [15] that the function h^θ satisfies (5.18) and the process

$$\{h^\theta(t) = h^\theta(t,\cdot) := \langle \varkappa_{(0,t]}, h_1^\theta(\cdot)\rangle \mid t \in \mathbb{R}_+\},$$

is a normal martingale with respect to the flow $(\mathcal{A}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{A}_t$ generated by $h^\theta(t)$. Hence it follows from Theorems 5.3 and 5.2 that the Itô integral $\mathbb{S}_{\mathrm{I}}(f)$, $f(\cdot) = \sum_{n=0}^\infty \langle f_n(\cdot), h_n^\theta\rangle \in D_{\mathrm{I}}$, coincides with the corresponding extended stochastic integral

$$\mathbb{S}_{h^\theta}(f) := \sum_{n=1}^\infty \langle \hat{f}_n, h_n^\theta\rangle.$$

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