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ORTHOGONAL APPROACH TO THE CONSTRUCTION OF THE THEORY OF GENERALIZED FUNCTIONS OF INFINITELY MANY VARIABLES AND THE POISSON ANALYSIS OF WHITE NOISE

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We develop an orthogonal approach to the construction of the theory of generalized functions of infinitely many variables (without using Jacobi fields) and apply it to the construction and investigation of the Poisson analysis of white noise.

0. Introduction

In our previous survey [1], we described the general scheme of the construction of the theory of generalized functions of infinitely many variables based on the notion of generalized translation. In particular, we described the so-called biorthogonal approach, according to which spaces of test functions are constructed on the basis of a certain system of functions, and spaces of generalized functions are associated with the corresponding biorthogonal system.

In the present paper, which, in fact, is a continuation of [1], we study the case where the indicated system of functions is orthogonal. This is the so-called orthogonal (or spectral) approach to the construction of the theory of generalized functions. The previous works in this direction were cited in [1].

Note that, in the present paper, we do not consider a purely spectral approach related to the spectral theory of Jacobi fields, where a mapping that realizes an isomorphism between spaces from the rigging of the Fock space and constructed spaces is given by the corresponding Fourier transformation in the decomposition in common generalized eigenvectors of the field. Roughly speaking, we assume that the biorthogonal system used in [1] is orthogonal and obtain the corresponding consequences.

An example of the orthogonal approach is the classical Brownian analysis of white noise (the corresponding works were cited in [1]). In the present paper, we consider another example, namely, the Poisson analysis of white noise [i.e., the corresponding theory of generalized functions (see [2–10])] and show how the results (both known and new) of this theory are obtained on the basis of the general approach described in [1]. Note that this approach to the Poisson analysis was announced in [11].

In the present paper, we consider a Poisson measure for which the intensity measure σ is a Lebesgue measure on $\mathbb{R}^1$. However, the results obtained can easily be generalized to any nonatomic measure σ; for this purpose, it is necessary to use the corresponding generalized Sobolev spaces (for the definition of these spaces, see, e.g., [12, 13]).

Let us make several additional remarks. Spaces of test and generalized functions in the orthogonal case are constructed in Secs. 3 and 4. In Sec. 5, we study operators of second quantization in terms of the corresponding space L^2. Note that, in different cases, these operators were studied in [14, 15, 6, 10, 16]. The Poisson analysis on the Sobolev space of generalized functions is given in Secs. 7–11, and its modification for the configuration space is presented in Sec. 12.

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1. Fock Space and Its Rigging

Denote

$$\mathbb{N}_p := \{p, p+1, \ldots\}, \quad p \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of all integers.

Consider a fixed family $(N_p)_{p\in\mathbb{N}_0}$ of real separable Hilbert spaces N_p such that, for all $p \in \mathbb{N}_0,$ the space N_{p+1} is topologically (densely and continuously) and quasinuclearly (the imbedding operator is a Hilbert–Schmidt operator) is imbedded into the space N_p and, furthermore, $\|\cdot\|_{N_p} \le \|\cdot\|_{N_{p+1}}.$

We construct a nuclear chain (see [14, 17])

$$\mathcal{N}' := \operatorname*{ind\,lim}_{\tilde{p}\in\mathbb{N}_1} N_{-\tilde{p}} \supset N_{-p} \supset N_0 \supset N_p \supset \operatorname*{pr\,lim}_{\tilde{p}\in\mathbb{N}_1} N_{\tilde{p}} =: \mathcal{N}, \tag{1.1}$$

where $N_{-p},\ p \in \mathbb{N}_1,$ is a negative space with respect to a zero space N_0 and a positive space N_p. Denote the coupling between N_{-p} and N_p generated by the scalar product $(\cdot\,,\cdot)_{N_0}$ in the space N_0 by $\langle\cdot\,,\cdot\rangle,$ omitting the subscript $N_0.$

For each $p \in \mathbb{Z},$ we construct a weighted symmetric Fock space $\mathcal{F}(N_p,\tau)$ with weight $\tau = (\tau_n)_{n=0}^{\infty},$ $\tau_n > 0,$ by setting

$$\mathcal{F}(N_p,\tau) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(N_p)\tau_n$$

$$= \left\{ f = (f_n)_{n=0}^{\infty} \mid f_n \in \mathcal{F}_n(N_p), \quad \|f\|^2_{\mathcal{F}(N_p,\tau)} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{F}_n(N_p)}\tau_n < \infty \right\},$$

where the n-particle Fock space $\mathcal{F}_n(N_p)$ is the nth symmetric tensor power of the complexification $N_{p,\mathbb{C}}$ of the space $N_p,$ i.e., $\mathcal{F}_n(N_p) := N_{p,\mathbb{C}}^{\hat{\otimes}n}$ ($N_{p,\mathbb{C}}^{\hat{\otimes}0} := \mathbb{C}^1$). It is clear that the set $\mathcal{F}_{\text{fin}}(N_p)$ of finite sequences from $\mathcal{F}(N_p,\tau)$ is dense in this space.

We fix $K > 1$ and consider the family $(\tau(q))_{q\in\mathbb{N}_1}$ of weights

$$\tau(q) = (\tau_n(q))_{n=0}^{\infty}, \quad \tau_n(q) = (n!)^2 K^{qn}. \tag{1.2}$$

Using chain (1.1) and the indicated family of weights, we construct the nuclear chain

$$\begin{gathered} \mathcal{F}(\mathcal{N}') \supset \mathcal{F}(N_{-p},\tau_F(q)) \supset F(N_0) \supset \mathcal{F}(N_p,\tau(q)) \supset \mathcal{F}(\mathcal{N}), \\ \mathcal{F}(\mathcal{N}) := \operatorname*{pr\,lim}_{\tilde{p},\,\tilde{q}\in\mathbb{N}_1} \mathcal{F}(N_{\tilde{p}},\tau(\tilde{q})), \qquad \mathcal{F}(\mathcal{N}') := \operatorname*{ind\,lim}_{\tilde{p},\,\tilde{q}\in\mathbb{N}_1} \mathcal{F}(N_{-\tilde{p}},\tau_F(\tilde{q})). \end{gathered} \tag{1.3}$$

Here, $\mathcal{F}(N_{-p},\tau_F(q)),$ $\tau_F(q) := (K^{-qn})_{n=0}^{\infty},$ is a negative space with respect to the zero space $F(N_0) := \mathcal{F}(N_0,(n!)_{n=0}^{\infty})$ and the positive space $\mathcal{F}(N_p,\tau(q)).$ *In what follows, the weight* $\tau(q)$ *is understood as weight (1.2), and the weight* $\tau_F(q)$ *is understood as the weight* $(K^{-qn})_{n=0}^{\infty}$.

"Coordinatewise," the coupling $\langle \cdot\,,\cdot \rangle_{F(N_0)}$ between $\mathcal{F}(N_{-p}, \tau_F(q))$ and $\mathcal{F}(N_p, \tau(q))$ generated by the scalar product $(\cdot\,,\cdot)_{F(N_0)}$ in the space $F(N_0)$ admits the following representation: for any $\xi = (\xi_n)_{n=0}^{\infty} \in \mathcal{F}(N_{-p}, \tau_F(q))$ and $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}(N_p, \tau(q))$, one has

$$\langle \xi, f \rangle_{F(N_0)} = \sum_{n=0}^{\infty} \langle \xi_n, f_n \rangle_{\mathcal{F}_n(N_0)} n!,$$

where $\langle \cdot\,,\cdot \rangle_{\mathcal{F}_n(N_0)}$ denotes the complex coupling between $\mathcal{F}_n(N_{-p})$ and $\mathcal{F}_n(N_p)$ generated by the scalar product $(\cdot\,,\cdot)_{\mathcal{F}_n(N_0)}$ in the space $\mathcal{F}_n(N_0)$. Parallel with the complex coupling $\langle \cdot\,,\cdot \rangle_{\mathcal{F}_n(N_0)}$, we use the real coupling $\langle \cdot\,,\cdot \rangle := \langle \cdot\,,\bar{\cdot} \rangle_{\mathcal{F}_n(N_0)}$, where the overbar denotes complex conjugation.

We introduce the following notation necessary for what follows: for a given linear subset $D \subset N_0$, let $\mathring{\mathcal{F}}_n(D) \subset \mathcal{F}_n(N_0)$, $n \in \mathbb{N}_1$, denote the linear span of the set of vectors

$$\Big\{ \varphi_1 \hat{\otimes} \ldots \hat{\otimes} \varphi_n \,|\, \varphi_i \in D_{\mathbb{C}}, \quad i = 1, \ldots, n \Big\}$$

($D_{\mathbb{C}}$ is the complexification of D) and let $\mathring{\mathcal{F}}_{\mathrm{fin}}(D) \subset \mathcal{F}_{\mathrm{fin}}(N_0)$ denote the subset of finite vectors with components from $\mathring{\mathcal{F}}_n(D)$. By virtue of the polarization identity, we have

$$\mathring{\mathcal{F}}_n(D) := \text{l.s.}\Big\{ \varphi_1 \hat{\otimes} \ldots \hat{\otimes} \varphi_n \,|\, \varphi_i \in D_{\mathbb{C}}, \quad i = 1, \ldots, n \Big\} = \text{l.s.}\Big\{ \varphi^{\otimes n} \,|\, \varphi \in D_{\mathbb{C}} \Big\}$$

for all $n \in \mathbb{N}_1$.

2. Annihilation and Creation Operators

On the set $\mathcal{F}_{\mathrm{fin}}(N_p)$, $p \in \mathbb{Z}$, we define a linear operator (a so-called creation operator) $a_+(\xi_m)$ with coefficient $\xi_m \in \mathcal{F}_m(N_s)$, $m \in \mathbb{N}_0$, $s \in \mathbb{N}_p$, by setting

$$a_+(\xi_m)\eta = a_+(\xi_m)(\eta_0, \eta_1, \ldots) := \Big(\underbrace{0, \ldots, 0}_{m}, \xi_m \hat{\otimes} \eta_0, \xi_m \hat{\otimes} \eta_1, \ldots \Big),$$

$$(a_+(\xi_m)\eta)_n := \begin{cases} \xi_m \hat{\otimes} \eta_{n-m} \in \mathcal{F}_n(N_p) & \text{if} \quad n \in \mathbb{N}_m, \\ 0 \in \mathcal{F}_n(N_p) & \text{if} \quad n = 0, \ldots, m-1, \end{cases} \tag{2.1}$$

for any $\eta = (\eta_n)_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{fin}}(N_p)$.

In addition to operator (2.1), on the set $\mathcal{F}_{\mathrm{fin}}(N_p)$, $p \in \mathbb{N}_0$, we also define a linear operator (a so-called annihilation operator) $a_-(\xi_m)$ with coefficient $\xi_m \in \mathcal{F}_m(N_{-p})$, $m \in \mathbb{N}_0$, by setting

$$a_-(\xi_m)f = a_-(\xi_m)(f_0, f_1, \ldots) := \left(m! f_m^{\xi_m}, \ldots, \frac{n!}{(n-m)!} f_n^{\xi_m}, \ldots \right),$$

$$(a_-(\xi_m)f)_n := \frac{(n+m)!}{n!} f_{n+m}^{\xi_m} \in \mathcal{F}_n(N_p), \quad n \in \mathbb{N}_0, \tag{2.2}$$

for any $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\text{fin}}(N_p)$. The element $f_n^{\xi_m} \in \mathcal{F}_{n-m}(N_p)$, $n \in \mathbb{N}_m$, in (2.2) is uniquely determined from the equality (see, e.g., [1], Sec. 5)

$$\langle f_n, \xi_m \hat{\otimes} \eta_{n-m} \rangle = \langle f_n^{\xi_m}, \eta_{n-m} \rangle, \tag{2.3}$$

which is valid for any $\eta_{n-m} \in \mathcal{F}_{n-m}(N_{-p})$.

The following statement is true (see, e.g., [1], Lemmas 11.1–11.3):

Proposition 2.1. *For fixed* $\xi_m \in \mathcal{F}_m(N_{-p})$, $m \in \mathbb{N}_0$, $p \in \mathbb{N}_1$, *the mappings*

$$\mathcal{F}(N_p, \tau(q)) \supset \mathcal{F}_{\text{fin}}(N_p) \ni f \mapsto a_-(\xi_m) f \in \mathcal{F}(N_p, \tau(q)),$$

$$\mathcal{F}(N_{-p}, \tau_F(q)) \supset \mathcal{F}_{\text{fin}}(N_{-p}) \ni \eta \mapsto a_+(\xi_m)\eta \in \mathcal{F}(N_{-p}, \tau_F(q))$$

are continuous, and, after their closure by continuity, they are linear continuous operators (we preserve the notation $a_-(\xi_m)$ *and* $a_+(\xi_m)$ *for the corresponding closures) adjoint with respect to chain (1.3). More exactly,*

$$\langle a_+(\bar{\xi}_m)\eta, f \rangle_{F(N_0)} = \langle \eta, a_-(\xi_m) f \rangle_{F(N_0)}$$

for any $\eta \in \mathcal{F}(N_{-p}, \tau_F(q))$ *and* $f \in \mathcal{F}(N_p, \tau(q))$.

Remark 2.1. If $\xi_m \in \mathcal{F}_m(N_{-p})$, $m \in \mathbb{N}_0$, $p \in \mathbb{N}_1$, then $\xi_m \in \mathcal{F}_m(N_{-p'})$ for all $p' \in \mathbb{N}_p$. Therefore, the operator $a_-(\xi_m)$ acts continuously in every space $\mathcal{F}(N_{p'}, \tau(q))$, $q \in \mathbb{N}_1$, and, furthermore, it satisfies the estimate (see [1], Lemma 11.1)

$$\|a_-(\xi_m) f\|_{\mathcal{F}(N_{p'}, \tau(q))} \le K^{-\frac{qm}{2}} \|\xi_m\|_{\mathcal{F}_m(N_{-p'})} \|f\|_{\mathcal{F}(N_{p'}, \tau(q))}, \quad f \in \mathcal{F}(N_{p'}, \tau(q)). \tag{2.4}$$

The adjoint operator $a_+(\xi_m)$ $\big($with respect to $F(N_0)\big)$ acts continuously in the negative spaces $\mathcal{F}(N_{-p'}, \tau_F(q))$ and satisfies an analogous estimate.

3. Operators of Generalized Translation and Related Objects

Let Q be a separable metric space of points $x, y, \ldots$. Let $C(Q)$ denote the linear space of all complex-valued locally bounded (i.e., bounded on every ball in Q) continuous functions on Q. It is convenient to consider $C(Q)$ as a topological space with uniform convergence on every sphere from Q.

Assume that, in the space $C(Q)$, a family $T = (T_x)_{x \in Q}$ of linear operators (so-called operators of generalized translation) with the following properties is given:

(a) $(T_x f)(y) = (T_y f)(x)$, $x, y \in Q$, for an arbitrary function $f \in C(Q)$ ("commutativity");

(b) there exists a point $e \in Q$ ("basis unit") such that $T_e = \mathrm{id}$ (id is an identity operator);

(c) for any $x, y \in Q$, there exists a ball $W_{x,y} \subset Q$ such that, for an any function $f \in C(Q)$, the values of $(T_x f)(y)$ are independent of the values of $f(s)$ for $s \in Q \backslash W_{x,y}$ ("locality");

(d) for any $x, y \in Q$, the linear mapping $C(Q) \ni f \mapsto (T_x f)(y) \in \mathbb{C}^1$ is continuous ("continuity").

A function $\chi \in C(Q)$ that is not identically equal to zero is called a character of the family T if it possesses the property

$$(T_x \chi)(y) = \chi(x)\chi(y), \quad x, y \in Q.$$

We assume that the function $\chi(x) = 1,\ x \in Q,$ is a character (identity character).

Let B_0 be a certain neighborhood of zero in the space $N_{1,\mathbb{C}}$ and let

$$Q \times B_0 \ni \{x, \lambda\} \mapsto \chi(x, \lambda) \in \mathbb{C}^1$$

be a given function. *Assume that, for every $x \in Q$, $\chi(x, \cdot)$ is a function of λ analytic at the zero of the space $N_{1,\mathbb{C}}$, and, for every $\lambda \in B_0$, $\chi(\cdot, \lambda)$ is a character of the family T. In addition, assume that $\chi(\cdot, \lambda)$ is locally bounded uniformly in λ from an arbitrary closed ball from B_0 and $\chi(x, 0) = 1$ for all x from Q.*

By virtue of analyticity (see, e.g., [1], Secs. 2 and 3), for every point $x \in Q$ there exists a neighborhood

$$B_\chi(x) = \Big\{\lambda \in N_{2,\mathbb{C}} \mid \|\lambda\|_{N_{2,\mathbb{C}}} < R_\chi(x),\ R_\chi(x) > 0\Big\} \subset B_0$$

such that

$$\chi(x, \lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle \lambda^{\otimes n}, \chi_n(x)\rangle \tag{3.1}$$

for all λ from $B_\chi(x)$. Furthermore, series (3.1) converges uniformly on every closed ball from $B_\chi(x)$. The coefficients $\chi_n(x) \in \mathcal{F}_n(N_{-2})$ are called *Delsarte characters.*

Assume that, for all x from Q, there exists a common neighborhood

$$B_\chi = \Big\{\lambda \in N_{2,\mathbb{C}} \mid \|\lambda\|_{N_{2,\mathbb{C}}} < R_\chi,\ \ R_\chi > 0\Big\} \subset B_0$$

in which the function $\chi(x, \cdot)$ can be represented in the form (3.1).

Note that, for this function χ, every vector $f_n \in \mathcal{F}_n(N_p),\ n \in \mathbb{N}_0,\ p \in \mathbb{N}_3,$ generates a function

$$Q \ni x \mapsto \langle f_n, \chi_n(x)\rangle \in \mathbb{C}^1$$

from the space $C(Q)$ (see, e.g., [1], Lemma 3.2), which can be expressed in terms of this function. To this end, it is necessary to use the polarization identity, the continuity of the coupling $\langle \cdot\,, \cdot\rangle$, and the obvious relation

$$\langle \varphi^{\otimes n}, \chi_n(x)\rangle = \frac{d^n}{dz^n}\chi(x, z\varphi)\Big|_{z=0}, \tag{3.2}$$

which is true for any φ from $N_{2,\mathbb{C}}$.

Let ρ be a fixed Borel probability measure on Q and let $(L^2_\rho) := L^2(Q, d\rho(x))$ be the corresponding L^2-space. *Assume that the generalized Laplace transformation*

$$N_{1,\mathbb{C}} \ni \lambda \mapsto \widehat{\rho}\,(\lambda) := \int_Q \chi(x, \lambda) d\rho(x) \in \mathbb{C}^1$$

is defined and is an analytic function at $0 \in N_{1,\mathbb{C}}$.

Since $\widehat{\rho}(0) = \rho(Q) = 1,$ we conclude that $N_{1,\mathbb{C}} \ni \lambda \mapsto \dfrac{1}{\widehat{\rho}(\lambda)} \in \mathbb{C}^1$ is an analytic function at $0 \in N_{1,\mathbb{C}}.$ Therefore, for every $x \in Q,$ the function $\omega(x,\lambda) := \dfrac{\chi(x,\lambda)}{\widehat{\rho}(\lambda)}$ is analytic with respect to the variable λ at the zero of the space $N_{1,\mathbb{C}}$ and admits the representation

$$\omega(x,\lambda) := \frac{\chi(x,\lambda)}{\widehat{\rho}(\lambda)} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, \omega_n(x) \rangle, \tag{3.3}$$

$$\lambda \in B_\omega = \Big\{ \lambda \in N_{2,\mathbb{C}} \mid \| \lambda \|_{N_{2,\mathbb{C}}} < R_\omega, \quad R_\omega > 0 \Big\} \subset B_\chi,$$

with coefficients $Q \ni x \mapsto \omega_n(x) \in \mathcal{F}_n(N_{-2}),$ which are called *Appell characters*.

Proposition 3.1. *The orthogonality relation*

$$\int_Q \langle \varphi_n, \omega_n(x) \rangle \overline{\langle \psi_m, \omega_m(x) \rangle} d\rho(x) = \delta_{n,m} n! \langle \varphi_n, \overline{\psi_n} \rangle, \tag{3.4}$$

$$\varphi_n \in \mathcal{F}_n(\mathcal{N}), \quad \psi_m \in \mathcal{F}_m(\mathcal{N}), \qquad n,m \in \mathbb{N}_0,$$

is true if and only if there exists $p \in \mathbb{N}_2$ *such that*

$$\Big\| \| \omega_n(\cdot) \|_{\mathcal{F}_n(N_{-p})} \Big\|_{(L^2)} \leq L C^n n! \quad \textit{for certain} \quad C > 0 \ \textit{and} \ L > 0 \ \textit{and all} \ n \in \mathbb{N}_0, \tag{3.5}$$

and, for any $\varphi, \psi \in \mathcal{N}_\mathbb{C},$ $\|\varphi\|_{N_{p,\mathbb{C}}}, \|\psi\|_{N_{p,\mathbb{C}}} < \min\{R_\omega, C^{-1}\},$ *one has*

$$\int_Q \omega(x,\varphi) \overline{\omega(x,\psi)} d\rho(x) = \exp\langle \varphi, \overline{\psi} \rangle. \tag{3.6}$$

Proof. *Necessity*. By virtue of Proposition 7.2 in [1], if the orthogonality relation (3.4) is true, then estimate (3.5) is also true. Furthermore, if this relation holds for all λ from $B_\omega,$ then the series $\omega(\cdot,\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, \omega_n(\cdot) \rangle$ converges in the topology of the space $(L^2_\rho).$ Taking into account this fact and the orthogonality relation (3.4), for $\varphi, \psi \in \mathcal{N}_\mathbb{C},$ $\|\varphi\|_{N_{p,\mathbb{C}}}, \|\psi\|_{N_{p,\mathbb{C}}} < \min\{R_\omega, C^{-1}\},$ we get

$$\int_Q \omega(x,\varphi)\overline{\omega(x,\psi)} d\rho(x) = (\omega(\cdot,\varphi), \omega(\cdot,\psi))_{(L^2_\rho)}$$

$$= \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \Big(\langle \varphi^{\otimes n}, \omega_n(\cdot) \rangle, \langle \psi^{\otimes m}, \omega_m(\cdot) \rangle \Big)_{(L^2_\rho)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varphi, \overline{\psi} \rangle^n = \exp\langle \varphi, \overline{\psi} \rangle.$$

Sufficiency. Assume that there exists $p \in \mathbb{N}_2$ such that estimate (3.5) and equality (3.6) are true. Let the vectors φ and ψ be the same as in the proposition. We represent them in the form

$$\varphi = z_1\tilde{\varphi}, \quad \psi = z_2\tilde{\psi},$$

$$\tilde{\varphi}, \tilde{\psi} \in \mathcal{N}_{\mathbb{C}}, \quad \|\tilde{\varphi}\|_{N_{p,\mathbb{C}}} = \|\tilde{\psi}\|_{N_{p,\mathbb{C}}} = 1,$$

$$z_1, z_2 \in \mathbb{C}^1, \quad |z_1|, |z_2| < \min\{R_\omega, C^{-1}\},$$

and take into account that, for $\lambda = \varphi, \psi,$ series (3.3) converges in the topology of the space (L^2_ρ) to $\omega(\cdot, \varphi)$ and $\omega(\cdot, \psi),$ respectively (see [1], Lemma 4.1). This yields

$$\int_Q \omega(x, z_1\tilde{\varphi})\overline{\omega(x, z_2\tilde{\psi})}d\rho(x) = (\omega(\cdot, z_1\tilde{\varphi}), \omega(\cdot, z_2\tilde{\psi}))_{(L^2_\rho)}$$

$$= \sum_{n,m=0}^{\infty} \frac{z_1^n \bar{z}_2^m}{n!m!}\Big(\langle\tilde{\varphi}^{\otimes n}, \omega_n(\cdot)\rangle, \langle\tilde{\psi}^{\otimes m}, \omega_m(\cdot)\rangle\Big)_{(L^2_\rho)}$$

$$= \sum_{n,m=0}^{\infty} \frac{z_1^n \bar{z}_2^m}{n!m!}\int_Q \langle\tilde{\varphi}^{\otimes n}, \omega_n(x)\rangle\overline{\langle\tilde{\psi}^{\otimes m}, \omega_m(x)\rangle}d\rho(x). \tag{3.7}$$

On the other hand, using relation (3.6), we obtain

$$\int_Q \omega(x, z_1\tilde{\varphi})\overline{\omega(x, z_2\tilde{\psi})}d\rho(x) = \exp(z_1\bar{z}_2\langle\tilde{\varphi}, \overline{\tilde{\psi}}\rangle) = \sum_{n=0}^{\infty} \frac{z_1^n \bar{z}_2^n}{n!}\Big\langle\tilde{\varphi}^{\otimes n}, \overline{\tilde{\psi}}^{\otimes n}\Big\rangle. \tag{3.8}$$

Thus, we have two representations for the function of z_1 and z_2 on the left-hand side of relation (3.7), namely (3.7) and (3.8). Comparing the coefficients of these representations, we get

$$\int_Q \langle\tilde{\varphi}^{\otimes n}, \omega_n(x)\rangle\overline{\langle\tilde{\psi}^{\otimes m}, \omega_m(x)\rangle}d\rho(x) = \delta_{n,m}n!\Big\langle\tilde{\varphi}^{\otimes n}, \overline{\tilde{\psi}}^{\otimes n}\Big\rangle.$$

Using the polarization identity, linearity with respect to $\tilde{\varphi}^{\otimes n},$ antilinearity with respect to $\tilde{\psi}^{\otimes n},$ and the continuity of the scalar product $\big($and also the coupling $\langle\cdot\,,\cdot\rangle\big),$ we establish that the orthogonality relation (3.4) holds for any vectors $\varphi_n \in \mathcal{F}_n(\mathcal{N})$ and $\psi_m \in \mathcal{F}_m(\mathcal{N}).$

The proposition is proved.

Remark 3.1. Let $Q = N_{-1}.$ A Borel probability measure ρ on Q is called analytic if its Laplace transform

$$l_\rho(\lambda) = \int_{N_{-1}} \exp\langle x, \lambda\rangle d\rho(x), \quad \lambda \in N_{1,\mathbb{C}},$$

is an analytic function at the zero of the space $N_{1,\mathbb{C}}.$

If the measure ρ is analytic and

$$\omega(x,\lambda)=\frac{\exp\langle x,\alpha(\lambda)\rangle}{l_\rho(\alpha(\lambda))},\qquad x\in Q=N_{-1},\quad \lambda\in N_{1,\mathbb{C}}$$

$\big($the function $\alpha\colon N_{1,\mathbb{C}}\to N_{1,\mathbb{C}}$ is analytic and invertible in the neighborhood of $0\in N_{1,\mathbb{C}}$ and $\alpha(0)=0\big)$, estimate (3.5) is automatically satisfied (see [18, 19]). Furthermore, the linear span of the functions (in this case, continuous polynomials) $Q\ni x\mapsto\langle\varphi_n,\omega_n(x)\rangle\in\mathbb{C}^1$, $\varphi_n\in\mathcal{F}_n(\mathcal{N})$, $n\in\mathbb{N}_0$, is dense in the space (L^2_ρ) (see [20], Chap. 2, Sec. 10, Theorem 1).

4. Spaces of Test and Generalized Functions

Assume that functions $Q\ni x\mapsto\langle\varphi_n,\omega_n(x)\rangle\in\mathbb{C}^1$, $\varphi_n\in\mathcal{F}_n(\mathcal{N})$, $n\in\mathbb{N}_0$, *satisfy the orthogonality relation (3.4) and their linear span is dense in the space* (L^2_ρ).

By virtue of the last assumption, the mapping

$$F(N_0)\ni f=(f_n)_{n=0}^\infty\mapsto(I_\rho f)(\cdot):=\sum_{n=0}^\infty\langle f_n,\omega_n(\cdot)\rangle\in(L^2_\rho) \tag{4.1}$$

is defined and is a unitary operator [1] (Sec. 7). Here,

$$\langle f_n,\omega_n(\cdot)\rangle:=\lim_{k\to\infty}\langle\varphi_n^{(k)},\omega_n(\cdot)\rangle\in(L^2_\rho),\qquad f_n\in\mathcal{F}_n(N_0),\quad n\in\mathbb{N}_0, \tag{4.2}$$

where $(\varphi_n^{(k)})_{k=0}^\infty\subset\mathcal{F}_n(\mathcal{N})$ is an arbitrary sequence convergent to f_n in the topology of the space $\mathcal{F}_n(N_0)$ [the limit in (4.2) is understood as a limit in (L^2_ρ)].

Since the operator I_ρ (4.1) realizes a unitary isomorphism between the Fock space $F(N_0)$ and the space (L^2_ρ), it is natural to construct a rigging of the space (L^2_ρ) (i.e., spaces of test and generalized functions) as the image of the rigging (1.3) of the space $F(N_0)$ under the mapping I_ρ. More exactly, the image $I_\rho(\mathcal{F}(N_p,\tau(q)))=:$ $H^\omega(p,q)\subset(L^2_\rho)$ with topology $\mathcal{F}(N_p,\tau(q))$ is a Hilbert space densely and continuously imbedded into (L^2_ρ) and generating the following nuclear rigging: for any $p,\ q\in\mathbb{N}_1$,

$$\begin{gathered}(\Phi^\omega)'\supset H^\omega(-p,-q)\supset(L^2_\rho)\supset H^\omega(p,q)\supset\Phi^\omega,\\ \Phi^\omega:=\operatorname*{pr\,lim}_{\tilde p,\,\tilde q\in\mathbb{N}_1}H^\omega(\tilde p,\tilde q),\qquad(\Phi^\omega)':=\operatorname*{ind\,lim}_{\tilde p,\,\tilde q\in\mathbb{N}_1}H^\omega(-\tilde p,-\tilde q),\end{gathered} \tag{4.3}$$

where $H^\omega(-p,-q)$ is a negative space with respect to the zero space (L^2_ρ) and the positive space $H^\omega(p,q)$. By definition,

$$\begin{aligned}H^\omega(p,q)&:=I_\rho(\mathcal{F}(N_p,\tau(q)))\\ &=\left\{f\in(L^2_\rho)\,|\,\exists(f_n)_{n=0}^\infty\in\mathcal{F}(N_p,\tau(q))\colon f(\cdot)=\sum_{n=0}^\infty\langle f_n,\omega_n(\cdot)\rangle\right\}\end{aligned}$$

is a Hilbert space with the Hilbert norm

$$\|f\|_{H^\omega(p,q)} = \left\| \sum_{n=0}^{\infty} \langle f_n, \omega_n(\cdot) \rangle \right\|_{H^\omega(p,q)} := \|(f_n)_{n=0}^{\infty}\|_{\mathcal{F}(N_p,\tau(q))}. \tag{4.4}$$

Remark 4.1. The presence of the generating function ω (with the properties indicated above) for the Appell characters $\omega_n(x)$ enables us to state (see [1], Sec. 7) that every series [the constant $K > 1$ from (1.2) is chosen sufficiently large]

$$\sum_{n=0}^{\infty} \langle f_n, \omega_n(\cdot) \rangle, \qquad (f_n)_{n=0}^{\infty} \in \mathcal{F}(N_p, \tau(q)), \quad p \in \mathbb{N}_3, \quad q \in \mathbb{N}_1,$$

converges in the topology of the space $C(Q)$ to a continuous locally bounded function $f \in C(Q)$. Furthermore, for every ball $U \subset C(Q)$, there exists a constant $c = c(U) > 0$ such that

$$|f(x)| \le c\|f\|_{H^\omega(p,q)}, \quad x \in U, \quad f \in H^\omega(p,q). \tag{4.5}$$

As a result, *the space* $H^\omega(p,q)$, $p \in \mathbb{N}_3$, $q \in \mathbb{N}_1$, *is continuously imbedded into the space* $C(Q)$ and can be interpreted as the set of continuous functions

$$H^\omega(p,q) = \left\{ f \in C(Q) \mid \exists (f_n)_{n=0}^{\infty} \in \mathcal{F}(N_p, \tau(q))\colon f(x) = \sum_{n=0}^{\infty} \langle f_n, \omega_n(x) \rangle,\ x \in Q \right\} \tag{4.6}$$

with the Hilbert norm (4.4).

It is easy to see that the mapping

$$\mathcal{F}(N_{-p}, \tau_F(q)) \supset F(N_0) \ni f = (f_n)_{n=0}^{\infty} \ \mapsto\ (I_\rho f)(\cdot) = \sum_{n=0}^{\infty} \langle f_n, \omega_n(\cdot) \rangle \ \in H^\omega(-p,-q)$$

is isometric, and, after closure by continuity, it realizes a unitary isomorphism between $\mathcal{F}(N_{-p}, \tau_F(q))$ and $H^\omega(-p,-q)$, p, $q \in \mathbb{N}_1$. As a result, the negative space of generalized functions $H^\omega(-p,-q)$, $p, q \in \mathbb{N}_1$, admits the representation

$$H^\omega(-p,-q) = \Bigg\{ \xi(\cdot) = \sum_{n=0}^{\infty} \langle \xi_n, \omega_n(\cdot) \rangle,$$

$$(\xi_n)_{n=0}^{\infty} \in \mathcal{F}(N_{-p}, \tau_F(q)) \,\Bigg|\, \|\xi\|_{H^\omega(-p,-q)} = \|(\xi_n)_{n=0}^{\infty}\|_{\mathcal{F}(N_{-p},\tau_F(q))} \Bigg\}.$$

Here,

$$\langle \xi_n, \omega_n(\cdot) \rangle := \lim_{k \to \infty} \langle f_n^{(k)}, \omega_n(\cdot) \rangle \in H^\omega(-p,-q), \quad n \in \mathbb{N}_0, \tag{4.7}$$

where $(f_n^{(k)})_{k=0}^{\infty} \subset \mathcal{F}_n(N_0)$ is an arbitrary sequence convergent to $\xi_n \in \mathcal{F}_n(N_{-p})$ in the topology of the space $\mathcal{F}_n(N_{-p})$ $\big($the limit in (4.7) is understood as a limit in $H^\omega(-p,-q)\big)$.

"Coordinatewise," the coupling $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ between $H^\omega(-p,-q)$ and $H^\omega(p,q)$, $p,q\in\mathbb{N}_1$, generated by the scalar product $(\cdot\,,\cdot)_{(L^2_\rho)}$ in the space (L^2_ρ) admits the representation

$$\langle\!\langle \xi, f\rangle\!\rangle = \Big\langle (\xi_n)_{n=0}^\infty, (f_n)_{n=0}^\infty \Big\rangle_{F(N_0)} = \sum_{n=0}^\infty \langle \xi_n, f_n\rangle_{\mathcal{F}_n(N_0)} n! \tag{4.8}$$

for any

$$\xi(\cdot) = \sum_{n=0}^\infty \langle \xi_n, \omega_n(\cdot)\rangle \in H^\omega(-p,-q), \qquad f(\cdot) = \sum_{n=0}^\infty \langle f_n, \omega_n(\cdot)\rangle \in H^\omega(p,q).$$

Remark 4.2. In the space $H^\omega(p,q)$, $p,q\in\mathbb{N}_1$, the annihilation operator $\partial(\xi_n)$ with coefficient $\xi_n\in\mathcal{F}_n(N_{-p})$, $n\in\mathbb{N}_0$, is defined as follows:

$$\partial(\xi_n) := I_\rho a_-(\xi_n) I_\rho^{-1}\colon H^\omega(p,q)\to H^\omega(p,q), \tag{4.9}$$

where $a_-(\xi_n)\colon \mathcal{F}(N_p,\tau(q))\to\mathcal{F}(N_p,\tau(q))$ acts according to rule (2.2). The creation operator

$$\partial^+(\bar\xi_n) := I_\rho a_+(\bar\xi_n) I_\rho^{-1} : H^\omega(-p,-q)\to H^\omega(-p,-q)$$

is adjoint to the operator $\partial(\xi_n)$ with respect to chain (4.3).

By virtue of Lemma 12.5 and Theorem 12.2 in [1], the space $H^\omega(-p,-q)$, $p,q\in\mathbb{N}_1$, admits the representation

$$H^\omega(-p,-q) = \Bigg\{ \xi = \sum_{n=0}^\infty \mathcal{Q}(\xi_n), \\ (\xi_n)_{n=0}^\infty \in \mathcal{F}(N_{-p},\tau_F(q)) \,\Bigg|\, \|\xi\|_{H^\omega(-p,-q)} = \|(\xi_n)_{n=0}^\infty\|_{\mathcal{F}(N_{-p},\tau_F(q))} \Bigg\} \tag{4.10}$$

with the Appell cocharacters

$$\mathcal{Q}(\xi_n) := \partial^+(\bar\xi_n)1 \in H^\omega(-p,-q), \quad n\in\mathbb{N}_0,$$

associated with the functions $Q\ni x\mapsto \langle f_m,\omega_m(x)\rangle\in\mathbb{C}^1$, $f_m\in\mathcal{F}_m(N_p)$, $m\in\mathbb{N}_0$, by the biorthogonality relation

$$\langle\!\langle\, \mathcal{Q}(\xi_n), \langle f_m,\omega_m(\cdot)\rangle \,\rangle\!\rangle = \delta_{n,m} n! \langle \xi_n, \overline{f_n}\rangle. \tag{4.11}$$

Comparing relations (4.8) and (4.11), we conclude that

$$\mathcal{Q}(\xi_n) = \langle \xi_n, \omega_n(\cdot)\rangle, \quad \xi_n\in\mathcal{F}_n(N_{-p}),$$

as elements of the space $H^\omega(-p,-q)$, $p,q\in\mathbb{N}_1$.

Parallel with $H^{\omega}(p,q)$, as spaces of test functions, one can use the spaces $H^{\chi}(p,q)$ constructed on the basis of the Delsarte characters $\chi_n(x)$. More exactly, for fixed $p,q\in\mathbb{N}_3$ and sufficiently large $K>1$ [where K is from (1.2)], the mapping

$$\mathcal{F}(N_p,\tau(q))\ni f=(f_n)_{n=0}^{\infty}\mapsto(I^{\chi}f)(\cdot):=\sum_{n=0}^{\infty}\langle f_n,\chi_n(\cdot)\rangle\in C(Q) \tag{4.12}$$

is defined and injective (see [1], Sec. 5). Therefore,

$$H^{\chi}(p,q):=I^{\chi}(\mathcal{F}(N_p,\tau(q)))=$$

$$=\left\{f\in C(Q)\,|\,\exists(f_n)_{n=0}^{\infty}\in\mathcal{F}(N_p,\tau(q))\colon f(x)=\sum_{n=0}^{\infty}\langle f_n,\chi_n(x)\rangle,\ \ x\in Q\right\} \tag{4.13}$$

is a Hilbert space with the Hilbert norm

$$\|f\|_{H^{\chi}(p,q)}:=\left\|\sum_{n=0}^{\infty}\langle f_n,\chi_n(\cdot)\rangle\right\|_{H^{\chi}(p,q)}=\|(f_n)_{n=0}^{\infty}\|_{\mathcal{F}(N_p,\tau(q))}.$$

Further, we choose a sufficiently large constant $K>1$ common for the spaces $H^{\omega}(p,q)$ and $H^{\chi}(p,q)$.

By virtue of Theorem 2.1 in [21], $H^{\omega}(p,q)$ (4.6) and $H^{\chi}(p,q)$, $p,q\in\mathbb{N}_3$, *coincide as topological spaces*. As a result, along with rigging (4.3), we can construct the following rigging: for any $p,q\in\mathbb{N}_3$,

$$(\Phi^{\chi})'\supset H^{\chi}(-p,-q)\supset(L^2_{\rho})\supset H^{\chi}(p,q)\supset\Phi^{\chi},$$

$$\Phi^{\chi}:=\operatorname*{pr\,lim}_{\tilde p,\,\tilde q\in\mathbb{N}_3}H^{\chi}(\tilde p,\tilde q),\quad(\Phi^{\chi})':=\operatorname*{ind\,lim}_{\tilde p,\,\tilde q\in\mathbb{N}_3}H^{\chi}(-\tilde p,-\tilde q),$$

where $H^{\chi}(-p,-q)$ is a negative space with respect to the zero space (L^2_{ρ}) and the positive space $H^{\chi}(p,q)$.

The negative space of generalized functions $H^{\chi}(-p,-q)$, $p,q\in\mathbb{N}_3$, admits the representation ([1], Lemma 12.3 and Theorem 12.1)

$$H^{\chi}(-p,-q)=\left\{\xi=\sum_{n=0}^{\infty}\theta(\xi_n),\right.$$

$$\left.(\xi_n)_{n=0}^{\infty}\in\mathcal{F}(N_{-p},\tau_F(q))\,\middle|\,\|\xi\|_{H^{\chi}(-p,-q)}=\|(\xi_n)_{n=0}^{\infty}\|_{\mathcal{F}(N_{-p},\tau_F(q))}\right\} \tag{4.14}$$

with the Delsarte cocharacters

$$\theta(\xi_n):=\partial^{+}(\bar\xi_n)\delta_e\in H^{\chi}(-p,-q),\quad n\in\mathbb{N}_0,$$

where δ_e is the δ-function concentrated at the point $e\in Q$ (e is the basis unit of the family T), i.e., $\langle\!\langle\delta_e,f\rangle\!\rangle=\overline{f(e)}$, $f\in H^{\chi}(p,q)$, $\delta_e\in H^{\chi}(-p,-q)$.

The Delsarte cocharacters $\theta(\xi_n)$, $n \in \mathbb{N}_0$, are associated with the functions $Q \ni x \mapsto \langle f_m, \chi_m(x) \rangle \in \mathbb{C}^1$, $f_m \in \mathcal{F}_m(N_p)$, $m \in \mathbb{N}_0$, by the biorthogonality relation

$$\langle\!\langle\, \theta(\xi_n), \langle f_m, \chi_m(\cdot) \rangle \,\rangle\!\rangle = \delta_{n,m} n! \langle \xi_n, \overline{f_n} \rangle. \tag{4.15}$$

As a result, "coordinatewise," the coupling $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ between $H^\chi(-p,-q)$ and $H^\chi(p,q)$ admits the representation

$$\langle\!\langle \xi, f \rangle\!\rangle = \langle (\xi)_{n=0}^\infty, (f)_{n=0}^\infty \rangle_{F(N_0)} = \sum_{n=0}^\infty \langle \xi_n, f_n \rangle_{\mathcal{F}_n(N_0)} n!$$

for any

$$\xi = \sum_{n=0}^\infty \theta(\xi_n) \in H^\chi(-p,-q), \quad f = \sum_{n=0}^\infty \langle f_n, \chi_n(\cdot) \rangle \in H^\chi(p,q).$$

5. Operators of Second Quantization

Let A be a self-adjoint positive operator in N_0 with domain of definition $\operatorname{Dom}(A)$. It can naturally be extended to an analogous operator in $\mathcal{F}_1(N_0) = N_{0,\mathbb{C}}$. For the extended operator, we use the same notation A. For each $n \in \mathbb{N}_1$, A_n denotes the operator in $\mathcal{F}_n(N_0)$ defined by the formula

$$A_n := A \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes A \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes A$$

on $\mathring{\mathcal{F}}_n(D)$ ($D \subset \operatorname{Dom}(A)$ is a fixed linear set in N_0).

The operator in $F(N_0)$ *defined by the formula*

$$d\operatorname{Exp} A := \bigoplus_{n=0}^\infty A_n, \quad A_0 := 0, \tag{5.1}$$

on $\mathring{\mathcal{F}}_{\mathrm{fin}}(D)$ *is called the second quantization of the operator* A. By virtue of [14] (Chap. 6, Sec. 1), this operator is an Hermite operator. Moreover, if D is the domain of essential self-adjointness, then this operator is essentially self-adjoint.

Let $H_\rho^A := I_\rho d\operatorname{Exp} A I_\rho^{-1}$ denote the image of the operator $d\operatorname{Exp} A$ under mapping (4.1). The operator H_ρ^A is also called the second quantization of the operator A. It is clear that if the set D is the domain of essential self-adjointness of the operator A, then the set $I_\rho(\mathring{\mathcal{F}}_{\mathrm{fin}}(D))$ is the domain of essential self-adjointness of the operator H_ρ^A.

Let us introduce operators ∂_x, $x \in Q$, necessary for the construction of the symmetric bilinear form of the operator H_ρ^A.

For fixed $x \in Q$, we denote by ∂_x a linear continuous operator that acts from $H^\omega(p,q)$ ($p \in \mathbb{N}_3$ and $q \in \mathbb{N}_1$ are fixed) into $\mathcal{F}_1(N_0) = N_{0,\mathbb{C}}$ and satisfies the equality

$$(\partial_x f, \xi_1)_{\mathcal{F}_1(N_0)} = (\partial(\bar{\xi}_1) f)(x), \qquad \xi_1 \in \mathcal{F}_1(N_0), \quad f \in H^\omega(p,q), \tag{5.2}$$

which uniquely defines this operator.

The existence of this operator is guaranteed by the estimate [here, we use relations (4.5), (4.9), and (2.4)]

$$|(\partial(\xi_1)f)(x)| \le c\|\partial(\xi_1)f\|_{H^\omega(p,q)} \le cK^{-\frac{q}{2}}\|\xi_1\|_{\mathcal{F}_1(N_{-p})}\|f\|_{H^\omega(p,q)}$$

$$\le cK^{-\frac{q}{2}}\|\xi_1\|_{\mathcal{F}_1(N_0)}\|f\|_{H^\omega(p,q)},$$

which is true for a certain $c = c(x) > 0$ and all $f \in H^\omega(p,q)$ and $\xi_1 \in \mathcal{F}_1(N_0)$.

To explicitly determine the action of the operator $\partial_x\colon H^\omega(p,q) \to \mathcal{F}_1(N_0)$ ($p \in \mathbb{N}_3$ and $q \in \mathbb{N}_1$ are fixed) on the functions

$$\langle \varphi_1^{\otimes n}, \omega_n(\cdot)\rangle \in H^\omega(p,q), \quad \varphi_1 \in \mathcal{F}_1(N_p), \quad n \in \mathbb{N}_0,$$

we use relations (5.2), (4.9), (2.2), and (2.3). Namely, for arbitrary $\xi_1 \in \mathcal{F}_1(N_0)$, we have

$$\Big(\partial_x\langle \varphi_1^{\otimes n}, \omega_n\rangle, \xi_1\Big)_{\mathcal{F}_1(N_0)} = \Big(\partial(\bar{\xi}_1)\langle \varphi_1^{\otimes n}, \omega_n(\cdot)\rangle\Big)(x)$$

$$= \begin{cases} n\langle \varphi_1^{\otimes n}, \omega_{n-1}(x)\hat{\otimes}\bar{\xi}_1\rangle & \text{if } n \ge 0, \\ 0 & \text{if } n = 0, \end{cases}$$

$$= \begin{cases} (n\langle \varphi_1^{\otimes(n-1)}, \omega_{n-1}(x)\rangle\varphi_1, \xi_1)_{\mathcal{F}_1(N_0)} & \text{if } n \ge 0, \\ 0 & \text{if } n = 0. \end{cases}$$

This yields

$$\partial_x\langle \varphi_1^{\otimes n}, \omega_n\rangle = \begin{cases} n\langle \varphi_1^{\otimes(n-1)}, \omega_{n-1}(x)\rangle\varphi_1 & \text{if } n \ge 0, \\ 0 & \text{if } n = 0. \end{cases} \tag{5.3}$$

The following theorem is true:

Theorem 5.1. *Suppose that $\mathcal{N} \subset \mathrm{Dom}\, A$. The symmetric bilinear form of the operator H_ρ^A admits the representation*

$$(H_\rho^A\varphi, \psi)_{(L_\rho^2)} = \int\limits_Q (A\partial_x\varphi, \partial_x\psi)_{\mathcal{F}_1(N_0)}d\rho(x) \tag{5.4}$$

for all $\varphi, \psi \in I_\rho(\mathring{\mathcal{F}}_{\mathrm{fin}}(\mathcal{N}))$.

Proof. It suffices to prove equality (5.4) for the functions

$$\varphi(\cdot) = \langle \varphi_1^{\otimes n}, \omega_n(\cdot)\rangle, \quad \psi(\cdot) = \langle \psi_1^{\otimes n}, \omega_n(\cdot)\rangle, \quad \varphi_1, \psi_1 \in \mathcal{F}_1(\mathcal{N}), \quad n, m \in \mathbb{N}_1.$$

On the one hand, using relation (4.1), for ρ-almost all $x \in Q$ we get

$$\Big(H^A_\rho\langle\varphi_1^{\otimes n},\omega_n(\cdot)\rangle\Big)(x)=\Big(I_\rho d\,\mathrm{Exp}\,AI_\rho^{-1}\langle\varphi_1^{\otimes n},\omega_n(\cdot)\rangle\Big)(x)=\Big(I_\rho((d\,\mathrm{Exp}\,A)\varphi_1^{\otimes n})\Big)(x)$$

$$=\Big(I_\rho(A\varphi_1\otimes\varphi_1\otimes\ldots\otimes\varphi_1+\ldots+\varphi_1\otimes\ldots\otimes\varphi_1\otimes A\varphi_1)\Big)(x)$$

$$=\Big(I_\rho(nA\varphi_1\hat{\otimes}\varphi_1^{\otimes(n-1)}\Big)(x)=n\langle A\varphi_1\hat{\otimes}\varphi_1^{\otimes(n-1)},\omega_n(x)\rangle.$$

Hence, using (3.4), we obtain

$$(H^A_\rho\varphi,\psi)_{(L^2_\rho)}=\int_Q n\langle A\varphi_1\hat{\otimes}\varphi_1^{\otimes(n-1)},\omega_n(x)\rangle\overline{\langle\psi_1^{\otimes m},\omega_m(x)\rangle}d\rho(x)$$

$$=\delta_{n,m}nn!\langle A\varphi_1\hat{\otimes}\varphi_1^{\otimes(n-1)},\bar{\psi}_1^{\otimes n}\rangle=\delta_{n,m}nn!\langle\varphi_1,\bar{\psi}_1\rangle^{n-1}\langle A\varphi_1,\bar{\psi}_1\rangle. \tag{5.5}$$

On the other hand, using (5.3), for any $x\in Q$ we get

$$(A\partial_x\varphi,\partial_x\psi)_{\mathcal{F}_1(N_0)}=\Big(A\partial_x\langle\varphi_1^{\otimes n},\omega_n\rangle,\partial_x\langle\psi_1^{\otimes m},\omega_m\rangle\Big)_{\mathcal{F}_1(N_0)}$$

$$=nm\langle\varphi_1^{\otimes(n-1)},\omega_{n-1}(x)\rangle\overline{\langle\psi_1^{\otimes(m-1)},\omega_{m-1}(x)\rangle}\langle A\varphi_1,\bar{\psi}_1\rangle.$$

This and the orthogonality relation (3.4) yield

$$\int_Q(A\partial_x\varphi,\partial_x\psi)_{\mathcal{F}_1(N_0)}d\rho(x)$$

$$=nm\langle A\varphi_1,\bar{\psi}_1\rangle\int_Q\langle\varphi_1^{\otimes(n-1)},\omega_{n-1}(x)\rangle\overline{\langle\psi_1^{\otimes(m-1)},\omega_{m-1}(x)\rangle}d\rho(x)$$

$$=\delta_{n,m}nn!\langle\varphi_1,\bar{\psi}_1\rangle^{n-1}\langle A\varphi_1,\bar{\psi}_1\rangle. \tag{5.6}$$

Comparing (5.5) and (5.6), we obtain the required result. The theorem is proved.

6. Taylor–Delsarte Translation

On functions f from the space $C(Q)$, we consider a linear operation $\mathcal{L}(\xi_n)$ with coefficient $\xi_n\in\mathcal{F}_n(\mathcal{N})$, $n\in\mathbb{N}_0$, such that, for every $x\in Q$, the mapping

$$C(Q)\ni f\mapsto(\mathcal{L}(\xi_n)f)(x)\in\mathbb{C}^1$$

is defined and linear. Assume that the character $\chi(x,\lambda)$ is an “eigenfunction” of $\mathcal{L}(\xi_n)$ in the following sense:

$$(\mathcal{L}(\xi_n)\chi(\cdot,\lambda))(x)=\langle\lambda^{\otimes n},\xi_n\rangle\chi(x,\lambda),\qquad x\in Q,\quad\lambda\in B_\chi. \tag{6.1}$$

Operators of generalized translation with indicated operations $\mathcal{L}(\xi_n)$ are called Taylor–Delsarte operators of generalized translation. Note that the operations $\mathcal{L}(\xi_n)$ are often encountered independently of the constructions presented in previous sections (see [22, 23]).

Proposition 6.1 ([1], Theorem 13.1)**.** *Let $\xi_n \in \mathcal{F}_n(\mathcal{N})$, $n \in \mathbb{N}_0$. Suppose that the linear operation $\mathcal{L}(\xi_n)$ possesses property (6.1) and is a continuous operator that acts from $H^\omega(p,q)$, $p,q \in \mathbb{N}_3$, into $C(Q)$. Then the space $H^\omega(p,q)$ is invariant under this operation, and the operation $\mathcal{L}(\xi_n)$, as an operator in $H^\omega(p,q)$, is continuous. Moreover, the operator $\mathcal{L}(\xi_n)\colon H^\omega(p,q) \to H^\omega(p,q)$ coincides with the annihilation operator $\partial(\xi_n)\colon H^\omega(p,q) \to H^\omega(p,q)$.*

7. Class of Weighted Sobolev Spaces

Let σ be the Lebesgue measure on the axis $\mathbb{R}^1$, i.e., $d\sigma(t) = dt$. Consider the real Sobolev space

$$S_p(\mathbb{R}^1) := W_p^2(\mathbb{R}^1, (1+t^2)^p d\sigma(t)), \quad p \in \mathbb{N}_0, \tag{7.1}$$

which is the complement of the set $C^\infty_{\text{fin}}(\mathbb{R}^1)$ ($C^\infty_{\text{fin}}(\mathbb{R}^1)$ is the space of real infinitely differentiable finite functions on $\mathbb{R}^1$) with respect to the Hilbert norm

$$\|\varphi\|^2_{S_p} := \sum_{n=0}^{p} \int_{\mathbb{R}^1} \Big((D^n\varphi)(t)\Big)^2 (1+t^2)^p d\sigma(t), \quad \varphi \in C^\infty_{\text{fin}}(\mathbb{R}^1) \tag{7.2}$$

$\big($it is clear that $S_0(\mathbb{R}^1) = L^2(\mathbb{R}^1, d\sigma(t)) =: L^2(\mathbb{R}^1)\big)$. It is known (see, e.g., [17], Chap. 14, Sec. 4.3) that the space $S_p(\mathbb{R}^1)$, $p \in \mathbb{N}_1$, is continuously imbedded into the Banach space $C_{\text{b}}(\mathbb{R}^1)$ of functions continuous and bounded on $\mathbb{R}^1$ $\big($with norm $\|f\|_{C_{\text{b}}(\mathbb{R}^1)} := \sup_{t\in\mathbb{R}^1} |f(t)|\big)$. Furthermore, it is known that a Sobolev space on a bounded domain is an algebra (see, e.g., [24], Chap. 1, Sec. 1.7). A similar result is also true for $S_p(\mathbb{R}^1)$.

Theorem 7.1. *The space $S_p(\mathbb{R}^1)$, $p \in \mathbb{N}_1$, is a Banach algebra with respect to the ordinary (pointwise) multiplication of functions.*

Proof. It is necessary to show that

$$\exists c := c_p > 0: \quad \|fg\|_{S_p} \le c\|f\|_{S_p}\|g\|_{S_p}, \quad f, g \in S_p(\mathbb{R}^1).$$

To this end, it suffices to prove the estimate

$$\exists c = c_p > 0: \quad \|\varphi\psi\|_{S_p} \le c\|\varphi\|_{S_p}\|\psi\|_{S_p}, \quad \varphi, \psi \in C^\infty_{\text{fin}}(\mathbb{R}^1). \tag{7.3}$$

Using relation (7.2), the Leibniz formula, and the obvious inequalities

$$C_n^m := \frac{n!}{m!(n-m)!} \le 2^p, \qquad n \in \{0,\ldots,p\}, \quad \mathbb{N}_0 \ni m \le n,$$

$$(a_1 + \ldots + a_n)^2 \le n(a_1^2 + \ldots + a_n^2), \qquad a_1,\ldots,a_n \in \mathbb{R}^1, \quad n \in \mathbb{N}_1,$$

for arbitrary $\varphi, \psi \in C^{\infty}_{\text{fin}}(\mathbb{R}^1)$ we get

$$\|\varphi\psi\|^2_{S_p} = \sum_{n=0}^{p} \int_{\mathbb{R}^1} ((D^n(\varphi\psi))(t))^2 (1+t^2)^p d\sigma(t)$$

$$= \sum_{n=0}^{p} \int_{\mathbb{R}^1} \left(\sum_{m=0}^{n} C_n^m (D^m\varphi)(t)(D^{n-m}\psi)(t) \right)^2 (1+t^2)^p d\sigma(t)$$

$$\leq 4^p \sum_{n=0}^{p} (n+1) \int_{\mathbb{R}^1} \sum_{m=0}^{n} \Big((D^m\varphi)(t)(D^{n-m}\psi)(t) \Big)^2 (1+t^2)^p d\sigma(t)$$

$$\leq (p+1)4^p \sum_{m=0}^{p} \mathbb{J}_m, \tag{7.4}$$

where

$$\mathbb{J}_m := \sum_{n=m}^{p} \int_{\mathbb{R}^1} \Big((D^m\varphi)(t)(D^{n-m}\psi)(t) \Big)^2 (1+t^2)^p d\sigma(t).$$

Let us estimate $\mathbb{J}_m,\ m \in \{0, \ldots, p\}$. According to [17] (Chap. 14, Sec. 4.3), for every $p \in \mathbb{N}_1$ there exists a constant $d = d_p > 0$ such that

$$|(D^m\varphi)(t)| \leq d\|\varphi\|_{S_p}, \quad \varphi \in C^{\infty}_{\text{fin}}(\mathbb{R}^1), \tag{7.5}$$

for all $m \in \{0, \ldots, p-1\}$ and $t \in \mathbb{R}^1$. With regard for the last result, for $m \in \{0, \ldots, p-1\}$ we get

$$\mathbb{J}_m \leq d^2 \|\varphi\|^2_{S_p} \sum_{n=m}^{p} \int_{\mathbb{R}^1} \Big((D^{n-m}\psi)(t) \Big)^2 (1+t^2)^p d\sigma(t) \leq d_p^2 \|\varphi\|^2_{S_p} \|\psi\|^2_{S_p}. \tag{7.6}$$

For $\mathbb{J}_p$, we obtain

$$\mathbb{J}_p = \int_{\mathbb{R}^1} \Big((D^p\varphi)(t)\psi(t) \Big)^2 (1+t^2)^p d\sigma(t) \leq d^2 \|\varphi\|^2_{S_p} \|\psi\|^2_{S_p}. \tag{7.7}$$

Estimates (7.4), (7.6), and (7.7) yield (7.3).

The theorem is proved.

Remark 7.1. Theorem 7.1 remains true for the complex Sobolev spaces $S_{p,\mathbb{C}}(\mathbb{R}^1),\ p \in \mathbb{N}_1$.

8. Poisson Measure on the Space of Generalized Functions

We use the known representation of the Schwartz space

$$\mathcal{S}(R^1) = \operatorname*{pr\,lim}_{p\in\mathbb{N}_0} S_p(\mathbb{R}^1)$$

in the form of the projective limit of the family $(S_p(\mathbb{R}^1))_{p\in\mathbb{N}_1}$ of the Sobolev spaces $S_p(\mathbb{R}^1)$ (7.1). Recall [17] (Chap. 14, Theorem 4.4) that, for every $p \in \mathbb{N}_0$, the imbedding $S_{p+1}(\mathbb{R}^1) \hookrightarrow S_p(\mathbb{R}^1)$ is quasinuclear. Therefore, the space $\mathcal{S}(\mathbb{R}^1)$ is a nuclear space. Let $S_{-p}(\mathbb{R}^1)$, $p \in \mathbb{N}_1$, denote the space dual to $S_p(\mathbb{R}^1)$ with respect to $S_0(\mathbb{R}^1) = L^2(\mathbb{R}^1)$. Consider the nuclear chain

$$\mathcal{S}'(\mathbb{R}^1) = \operatorname*{ind\,lim}_{\tilde{p}\in\mathbb{N}_1} S_{-\tilde{p}}(\mathbb{R}^1) \supset S_{-p}(\mathbb{R}^1)$$

$$\supset L^2(\mathbb{R}^1) \supset S_p(\mathbb{R}^1) \supset \operatorname*{pr\,lim}_{\tilde{p}\in\mathbb{N}_1} S_{\tilde{p}}(\mathbb{R}^1) = \mathcal{S}(\mathbb{R}^1) \tag{8.1}$$

with coupling $\langle \cdot\,, \cdot\rangle$ generated by the scalar product $(\cdot\,, \cdot)_{L^2(\mathbb{R}^1)}$ in the space $L^2(\mathbb{R}^1)$. By $\mathcal{B}(E)$ we always denote the σ-algebra of Borel sets of the topological space E.

The Borel probability measure π on $\mathcal{S}'(\mathbb{R}^1)$, which, by virtue of the Minlos theorem (see, e.g., [14]), is uniquely determined by its Fourier transform

$$\int_{\mathcal{S}'(\mathbb{R}^1)} e^{i\langle\lambda,x\rangle} d\pi(x) = \exp\biggl(\int_{\mathbb{R}^1} (e^{i\lambda(t)} - 1) d\sigma(t)\biggr), \quad \lambda \in \mathcal{S}(\mathbb{R}^1), \tag{8.2}$$

is called a *Poisson measure* with Lebesgue intensity measure σ.

Remark 8.1. For other alternative methods for the introduction of a Poisson measure on the σ-algebra $\mathcal{B}(\mathcal{S}'(\mathbb{R}^1))$, see, e.g., [5, 6, 8–10]. The name "Poisson measure" can be explained as follows:

The classical Poisson measure π_a (with parameter $a > 0$) on $(\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1))$ is a probability measure concentrated on $\mathbb{N}_0$ and such that, for all $\alpha \in \mathcal{B}(\mathbb{R}^1)$, one has

$$\pi_a(\alpha) = \sum_{k\in\alpha\cap\mathbb{N}_0} \frac{e^{-a}a^k}{k!} \in \mathbb{R}^1.$$

The measure π_a has the Fourier transform

$$\int_{\mathbb{R}^1} e^{i\lambda x} d\pi_a(x) = \exp(a(e^{i\lambda} - 1)), \quad \lambda \in \mathbb{R}^1.$$

A Poisson measure π_a $\bigl($with parameter $a = (a_1, \ldots, a_n) \in \mathbb{R}^d$, $a_j > 0\bigr)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $d \in \mathbb{N}_1$, can naturally be understood as the probability measure

$$\mathcal{B}(\mathbb{R}^d) \ni \alpha \mapsto \pi_a(\alpha) := \mathop{\times}_{j=1}^{d} \pi_{a_j}(\alpha) \in \mathbb{R}^1.$$

This measure has the Fourier transform

$$\int\limits_{\mathbb{R}^d} e^{i\langle\lambda,x\rangle_d} d\pi_a(x) = \exp\left(\sum_{j=1}^{d} a_j(e^{i\lambda_j} - 1)\right), \tag{8.3}$$

$$\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d, \quad \langle\lambda, x\rangle_d := x_1\lambda_1 + \ldots + x_d\lambda_d.$$

Identifying $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ with the function

$$\mathbb{R}^1 \ni t \mapsto \lambda(t) := \sum_{j=1}^{d} \lambda_j k_{\tau_j}(t)$$

$\big($here, $\tau_1, \ldots, \tau_d$ are fixed nonempty Borel sets that form a decomposition of the axis $\mathbb{R}^1$, i.e., $\tau_j \cap \tau_k = \varnothing$ $(j \neq k)$, $\bigcup_{j=1}^{d} \tau_j = \mathbb{R}^1$, and k_{τ_j} is the indicator of the set $\tau_j\big)$, we can interpret the sum $\sum_{j=1}^{d} a_j(e^{i\lambda_j} - 1)$ as the Lebesgue integral $\int_{\mathbb{R}^1} (e^{i\lambda(t)} - 1) d\mu(t)$, where μ is a finite Borel measure on $\mathbb{R}^1$ such that $\mu(\tau_j) = a_j$, $j = 1, \ldots, d$. As a result, the Fourier transform (8.3) can be represented in the form

$$\int\limits_{\mathbb{R}^d} e^{i\langle\lambda,x\rangle} d\pi_a(x) = \exp\bigg(\int\limits_{\mathbb{R}^1} (e^{i\lambda(t)} - 1) d\mu(t)\bigg),$$

whence we conclude that the measure π (8.2) can naturally be called a "Poisson measure."

Using Theorem 7.1, we establish that the mapping

$$\mathcal{S}(\mathbb{R}^1) \ni \lambda \mapsto e^{i\lambda} - 1 = \sum_{n=1}^{\infty} \frac{(i\lambda)^n}{n!} \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^1),$$

where $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^1)$ is the complexification of $\mathcal{S}(\mathbb{R}^1)$, is continuous. Therefore, we can represent the right-hand side of (8.2) in the form

$$\exp\bigg(\int\limits_{\mathbb{R}^1} (e^{i\lambda(t)} - 1) d\sigma(t)\bigg) = \exp\langle 1, e^{i\lambda} - 1\rangle, \quad \lambda \in \mathcal{S}(\mathbb{R}^1).$$

Since the mapping $S_1(\mathbb{R}^1) \ni \lambda \mapsto \exp\langle 1, e^{i\lambda} - 1\rangle \in \mathbb{C}^1$ is continuous (this follows from Theorem 7.1) and the imbedding $S_2(\mathbb{R}^1) \hookrightarrow S_1(\mathbb{R}^1)$ is quasinuclear, by virtue of the Minlos–Sazonov theorem we can modify the measure π to a Borel probability measure on $S_{-2}(\mathbb{R}^1)$. We denote the measure π modified with respect to $S_{-2}(\mathbb{R}^1)$ by the same symbol π and call it a Poisson measure. This measure has the Fourier transform

$$\int\limits_{S_{-2}(\mathbb{R}^1)} e^{i\langle\lambda,x\rangle} d\pi(x) = \exp\langle 1, e^{i\lambda} - 1\rangle, \quad \lambda \in S_2(\mathbb{R}^1). \tag{8.4}$$

Recall the meaning of a modification of a measure (see, e.g., [14], Chap. 3, Sec. 1.9). Consider measurable spaces $(R, \mathcal{R})$ and $(R', \mathcal{R}')$ and assume that the space R contains R', and the mapping

$$\mathcal{R} \ni \alpha \mapsto \alpha' = \alpha \cap R' \in \mathcal{R}'$$

is a mapping of the entire σ-algebra $\mathcal{R}$ onto the entire σ-algebra $\mathcal{R}'$.

Let μ be a fixed measure on $\mathcal{R}$. Assume that the set R' is of full outer measure μ (i.e., an arbitrary set $\mathcal{R} \ni \alpha \supset R'$ is of full measure μ). Then μ generates a measure μ' on $\mathcal{R}'$ according to the rule

$$\mathcal{R}' \ni \alpha' \mapsto \mu'(\alpha') = \mu'(\alpha \cap R') := \mu(\alpha) \in \mathbb{C}^1,$$

where α is a certain set from $\mathcal{R}$ determined from the equality $\alpha' = \alpha \cap R'$. The measure μ' is called the modification of a measure μ with respect to R'. If a function $R \ni x \mapsto f(x) \in \mathbb{C}^1$ is measurable with respect to the σ-algebra $\mathcal{R}$, then its restriction $f \upharpoonright R'$ is measurable with respect to the σ-algebra $\mathcal{R}'$. The function f is summable with respect to the measure μ if and only if $f \upharpoonright R'$ is summable with respect to μ', and, furthermore,

$$\int\limits_R f(x)d\mu(x) = \int\limits_{R'} f(x)d\mu'(x).$$

9. Spaces of Test and Generalized Functions in Poisson Analysis

We now pass to the realization of the general procedure of the construction of spaces of test and generalized functions described in Secs. 3–6 in the case where Q is a linear real space $S_{-2}(\mathbb{R}^1)$ with operation of addition $+$ and ρ is a Poisson measure on $\mathcal{B}(Q)$. Note that, for a different choice of the space Q $\big($where it is either $S'(\mathbb{R}^1)$ or $D'(\mathbb{R}^1)\big)$, the main results presented in this section are known (see [3–10] and the bibliography therein).

In the linear space $C(Q)$, we introduce a family $T = (T_x)_{x \in Q}$ of operators of generalized translation T_x by setting

$$(T_x f)(y) = f(x + y), \quad y \in Q, \quad f \in C(Q) \tag{9.1}$$

(i.e., we have an ordinary translation by $C(Q)$ with basis unit $e = 0 \in Q$). It is obvious that the family $T = (T_x)_{x \in Q}$ satisfies axioms (a)–(d) from Sec. 3.

As (1.1), we use chain (8.1). More exactly, in view of the fact that $Q = S_{-2}(\mathbb{R}^1)$, *we assume that* $N_0 = S_0(\mathbb{R}^1) = L^2(\mathbb{R}^1)$ *and* $N_p = S_{p+1}(\mathbb{R}^1)$, $p \in \mathbb{N}_1$.

Let

$$B_0 := \Big\{\lambda \in N_{1,\mathbb{C}} \mid \|\lambda\|_{N_{1,\mathbb{C}}} < R, \quad 0 < R < 1\Big\}.$$

By virtue of Theorem 7.1, the mapping

$$N_{1,\mathbb{C}} \supset B_0 \ni \lambda \mapsto \log(1 + \lambda) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda^n}{n} \in N_{1,\mathbb{C}}$$

and, hence, the function

$$Q \times B_0 \ni \{x, \lambda\} \mapsto \chi(x, \lambda) := \exp\langle x, \log(1 + \lambda)\rangle \in \mathbb{C}^1 \tag{9.2}$$

are defined. It is easy to see that function (9.2) satisfies the assumptions from Sec. 3, and, hence, it admits representation (3.1). The Delsarte characters $\chi_n(x) \in \mathcal{F}_n(N_{-2})$, $n \in \mathbb{N}_1$, for this function are determined from the recurrence relation

$$\langle \varphi^{\otimes n}, \chi_n(x) \rangle = \sum_{m=0}^{n-1} (-1)^{n-m-1} \frac{(n-1)!}{m!} \langle \varphi^{n-m}, x \rangle \langle \varphi^{\otimes m}, \chi_m(x) \rangle,$$

which is true for all $\varphi \in N_{2,\mathbb{C}}$ and $x \in Q$ $(\chi_0(x) = 1,\ x \in Q)$. To obtain this relation, it is necessary to use (3.2) and the formula

$$\frac{d}{dz}\chi(x, z\varphi) = \left\langle x, \frac{\varphi}{1+z\varphi} \right\rangle \chi(x, z\varphi), \qquad x \in Q, \quad \varphi \in N_{2,\mathbb{C}}$$

($z \in \mathbb{C}^1$ and $|z|$ is sufficiently small), which follows directly from (9.2).

Let us determine the generalized Laplace transform of the measure π. Since the Laplace transform has the form

$$l_\pi(\lambda) = \int_Q \exp\langle x, \lambda \rangle d\pi(x) = \exp\langle 1, e^\lambda - 1 \rangle, \quad \lambda \in N_{1,\mathbb{C}}, \tag{9.3}$$

we have

$$\widehat{\pi}(\lambda) = \int_Q \chi(x, \lambda) d\pi(x) = \int_Q \exp\langle x, \log(1+\lambda) \rangle d\pi(x) = \exp\langle 1, \lambda \rangle$$

for arbitrary λ from B_0.

It is obvious that the generalized Laplace transform $\widehat{\pi}$ is an analytic function at the zero of the space $N_{1,\mathbb{C}}$. Therefore, the function

$$\omega(x, \lambda) := \frac{\chi(x, \lambda)}{\widehat{\pi}(\lambda)} = \exp\Big(\langle x, \log(1+\lambda) \rangle - \langle 1, \lambda \rangle \Big) \tag{9.4}$$

admits an expansion in series (3.3). The Appell characters $\omega_n(x) \in \mathcal{F}_n(N_{-2})$ that correspond to it are customarily called the *Charlier polynomials* (see, e.g., [4–10]). Similarly to Delsarte characters, these polynomials are determined from the recurrence relation (see also [3])

$$\langle \varphi^{\otimes n}, \omega_n(x) \rangle = \sum_{m=0}^{n-1} (-1)^{n-m-1} \frac{(n-1)!}{m!} \langle \varphi^{n-m}, x \rangle \langle \varphi^{\otimes m}, \omega_m(x) \rangle - \langle 1, \varphi \rangle \langle \varphi^{\otimes (n-1)}, \omega_{n-1}(x) \rangle,$$

which is true for all $\varphi \in N_{2,\mathbb{C}}$ and $x \in Q$ $(\omega_0(x) = 1,\ x \in Q)$.

To use the results presented in Sec. 4, it remains to show that the functions $Q \ni x \mapsto \langle \varphi_n, \omega_n(x) \rangle \in \mathbb{C}^1$, $\varphi_n \in \mathcal{F}_n(\mathcal{N}) = \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$, satisfy the orthogonality relation (3.4) and their linear span is dense in the space $(L^2_\pi) := L^2(Q, d\pi(x))$.

Since the Laplace transform l_π of the measure π is an analytic function of the variable λ at the zero of the space $N_{1,\mathbb{C}}$, the linear span of the functions $Q \ni x \mapsto \langle \varphi_n, \omega_n(x) \rangle \in \mathbb{C}^1$, $\varphi_n \in \mathcal{F}_n(\mathcal{N})$, $n \in \mathbb{N}_0$, is dense in the space (L^2_π) (see Remark 3.1). As for the required orthogonality (3.4), the following statement is true (for a different presentation, see [3]):

Proposition 9.1. *For* $\varphi_n \in \mathcal{F}_n(\mathcal{N}) = \mathcal{S}_{\mathbb{C}}^{\hat{\otimes}n}$ *and* $\psi_m \in \mathcal{F}_m(\mathcal{N}) = \mathcal{S}_{\mathbb{C}}^{\hat{\otimes}m}$, $n, m \in \mathbb{N}_0$, *the following orthogonality relation is true:*

$$\int\limits_Q \langle \varphi_n, \omega_n(x)\rangle \overline{\langle \psi_m, \omega_m(x)\rangle} d\pi(x) = \delta_{n,m} n! \langle \varphi_n, \overline{\psi_n}\rangle. \tag{9.5}$$

Proof. It suffices to obtain estimate (3.5) and prove equality (3.6).

Since the Laplace transform l_π of the measure π is an analytic function of the variable λ at the zero of the space $N_{1,\mathbb{C}}$, by virtue of Remark 3.1 there exists $p \in \mathbb{N}_2$ such that estimate (3.5) is true.

Let us prove equality (3.6). Using relations (9.3) and (9.4), for any $\varphi, \psi \in \mathcal{N}_{\mathbb{C}} = \mathcal{S}_{\mathbb{C}}$, $\|\varphi\|_{N_{p,\mathbb{C}}}, \|\psi\|_{N_{p,\mathbb{C}}} < \min\{R_\omega, C^{-1}\}$, we obtain

$$\int\limits_Q \omega(x,\varphi)\overline{\omega(x,\psi)} d\pi(x) = \exp(-\langle 1, \varphi + \overline{\psi}\rangle) \int\limits_Q \exp\langle x, \log(1+\varphi)\overline{(1+\psi)}\rangle d\pi(x)$$

$$= \exp(-\langle 1, \varphi + \overline{\psi}\rangle) \exp\langle 1, (1+\varphi)\overline{(1+\psi)} - 1\rangle$$

$$= \exp\langle 1, \varphi\overline{\psi}\rangle = \exp \int\limits_{\mathbb{R}^1} \varphi(t)\overline{\psi(t)} dt = \exp\langle \varphi, \overline{\psi}\rangle.$$

The proposition is proved.

Using these properties, we can formulate the following theorem:

Theorem 9.1. *The results presented in Secs.* 3–6 *are true for the Poisson analysis with space* $Q = S_{-2}(\mathbb{R}^1)$, *Poisson measure* $\rho = \pi$, *translation (9.1), and Delsarte and Appell characters (Charlier polynomials) generated by functions (9.2) and (9.4), respectively.*

In particular, for all $p, q \in \mathbb{N}_3$, *the spaces* $H^\chi(p,q)$ *(4.13) and* $H^\omega(p,q)$ *(4.6) converge in the topological sense, and, in addition, they are continuously imbedded into the space* $C(Q)$. *Moreover, these spaces are densely and continuously imbedded into the space* $(L^2_\pi) = L^2(Q, d\pi(x))$, *which enables one to construct the nuclear chains*

$$\begin{array}{ccccccccc} (\Phi^\chi)' & \supset \cdots \supset & H^\chi(-p,-q) & \supset \cdots \supset & (L^2_\pi) & \supset \cdots \supset & H^\chi(p,q) & \supset \cdots \supset & \Phi^\chi \\ \| & & \| & & \| & & \| & & \| \\ (\Phi^\omega)' & \supset \cdots \supset & H^\omega(-p,-q) & \supset \cdots \supset & (L^2_\pi) & \supset \cdots \supset & H^\omega(p,q) & \supset \cdots \supset & \Phi^\omega \end{array}$$

with the coupling $\langle\langle \cdot, \cdot \rangle\rangle$ *generated by the scalar product in the space* (L^2_π).

Remark 9.1. It can be shown that the unitary isomorphism

$$F(N_0) \ni f = (f_n)_{n=0}^\infty \mapsto (I_\pi f)(\cdot) := \sum_{n=0}^\infty \langle f_n, \omega_n(\cdot)\rangle \in (L^2_\pi) \tag{9.6}$$

is the Fourier transformation with respect to the common eigenvectors of a certain family $A = (A(\varphi))_{\varphi \in S_2(\mathbb{R}^1)}$ of commuting self-adjoint operators $A(\varphi)$ that act in the Fock space $F(N_0)$, $N_0 = L^2(\mathbb{R}^1)$, and have the Jacobi structure. More exactly, the family $A = (A(\varphi))_{\varphi \in S_2(\mathbb{R}^1)}$ forms a so-called Poisson field (for the definition and properties of this field, see [25]).

10. Annihilation Operators and Operators of Second Quantization in Poisson Analysis

On functions f from the space $C(Q)$, we introduce a linear difference operation $\mathcal{L}(\varphi_n)$ with finite coefficient $\varphi_n = \varphi_n(t_1, \ldots, t_n) \in \mathcal{F}_n(\mathcal{N}) = \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}(\mathbb{R}^1)$, $n \in \mathbb{N}_1$, by setting

$$(\mathcal{L}(\varphi_n)f)(x) := \int_{\mathbb{R}^n} (\ell(\delta_{t_1}) \ldots \ell(\delta_{t_n})f)(x)\varphi_n(t_1, \ldots, t_n) d\sigma(t_1) \ldots d\sigma(t_n), \tag{10.1}$$

$$(\ell(\delta_t)f)(x) := f(x + \delta_t) - f(x), \quad x \in Q = S_{-2}(\mathbb{R}^1), \quad t \in \mathbb{R}^1, \tag{10.2}$$

where δ_t is the δ-function concentrated at the point t. It is known that $\delta_t \in S_{-2}(\mathbb{R}^1)$ (see [17], Chap. 14, Sec. 4).

For every $x \in Q$, the function

$$\mathbb{R}^n \ni \{t_1, \ldots, t_n\} \mapsto f\left(x + \sum_{m=1}^{n} \delta_{t_m}\right) \in \mathbb{C}^1, \quad n \in \mathbb{N}_1, \tag{10.3}$$

is continuous and bounded. Therefore, for f and φ_n mentioned above, the right-hand side of (10.1) is defined for all $x \in Q$.

Indeed, since $f \in C(Q)$ and the mapping

$$\mathbb{R}^n \ni \{t_1, \ldots, t_n\} \mapsto \sum_{m=1}^{n} \delta_{t_m} \in Q$$

is continuous (see [17], Chap. 14, Sec. 4), mapping (10.3) is also continuous. The estimate

$$\exists c > 0: \quad \|\delta_t\|_{S_{-2}(\mathbb{R}^1)} \leq \frac{c}{\sqrt{1+t^2}}, \quad t \in \mathbb{R}^1, \tag{10.4}$$

and the local boundedness of the function $f \in C(Q)$ guarantee the boundedness of function (10.3).

Applying the difference operation (10.2) to character (9.2), for all $x \in Q$ and $\lambda \in B_\chi$ we get

$$\begin{aligned}(\ell(\delta_t)\chi(\cdot, \lambda))(x) &= \chi(x + \delta_t, \lambda) - \chi(x, \lambda) \\ &= \exp\langle x + \delta_t, \log(1+\lambda)\rangle - \exp\langle x, \log(1+\lambda)\rangle \\ &= \exp\langle x, \log(1+\lambda)\rangle(\exp\langle \delta_t, \log(1+\lambda)\rangle - 1) \\ &= \exp\langle x, \log(1+\lambda)\rangle\lambda(t) = \lambda(t)\chi(x, \lambda), \quad t \in \mathbb{R}^1.\end{aligned}$$

Using this result, we get

$$(\mathcal{L}(\varphi_n)\chi(\cdot, \lambda))(x) = \langle \lambda^{\otimes n}, \varphi_n\rangle\chi(x, \lambda), \quad x \in Q, \quad \lambda \in B_\chi,$$

for an arbitrary finite function $\varphi_n \in \mathcal{F}_n(\mathcal{N})$, $n \in \mathbb{N}_1$.

Thus, the translation T_x (9.1) is a Taylor–Delsarte translation.

Theorem 10.1. *For every finite function* $\varphi_n \in \mathcal{F}_n(\mathcal{N}) = \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}(\mathbb{R}^1)$, $n \in \mathbb{N}_1$, *the mapping*

$$H^{\omega}(p,q) \ni f \mapsto \mathcal{L}(\varphi_n) f \in C(Q)$$

is defined for all $p, q \in \mathbb{N}_3$ *and is linear and continuous.*

Proof. It suffices to establish the following estimate: for an arbitrary ball $U \subset Q$, there exists a constant $c = c(U) > 0$ such that

$$|(\mathcal{L}(\varphi_n) f)(x)| \leq c \|f\|_{H^{\omega}(p,q)}, \quad x \in U, \quad f \in H^{\omega}(p,q). \tag{10.5}$$

Since the function $\varphi_n \in \mathcal{F}_n(\mathcal{N}) = \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}(\mathbb{R}^1)$ is finite, it suffices to prove that, for an arbitrary ball $U \subset Q$, there exists a constant $a = a(U) > 0$ such that

$$|(\ell(\delta_{t_1}) \ldots \ell(\delta_{t_n}) f)(x)| \leq a \|f\|_{H^{\omega}(p,q)}, \quad x \in U, \quad f \in H^{\omega}(p,q),$$

for any $t_1, \ldots, t_n \in \mathbb{R}^1$.

The last estimate follows directly from estimate (4.5). To this end, it is necessary to use relation (10.2) and take into account that, for an arbitrary ball $U \subset Q$, there exists a ball $U' \subset Q$ such that $x + \sum_{m=k}^{n} \delta_{t_m}$ belongs to U' for all $t_k, \ldots, t_n \in \mathbb{R}^1$, $k \in \{1, \ldots, n\}$, and $x \in U$ [estimate (10.4) guarantees the existence of the ball $U' \subset Q$].

The theorem is proved.

Corollary 10.1. *It follows from Proposition 6.1 and Theorem 10.1 that the space* $H^{\omega}(p,q)$ *is invariant under the action of* $\mathcal{L}(\varphi_n)$ (φ_n *is a finite function from* $\mathcal{F}_n(\mathcal{N}) = \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}(\mathbb{R}^1)$, $n \in \mathbb{N}_1$). *Moreover, the operator* $\mathcal{L}(\varphi_n) :$ $H^{\omega}(p,q) \to H^{\omega}(p,q)$ *is continuous and coincides with the annihilation operator* $\partial(\varphi_n) \colon H^{\omega}(p,q) \to H^{\omega}(p,q)$.

Now we consider operators of second quantization that act in the space $(L^2_{\pi}.)$ As before, let A be a self-adjoint positive operator in $N_0 = L^2(\mathbb{R}^1)$ with domain of definition $\operatorname{Dom}(A)$. Let $H_{\pi}^{A} := I_{\pi} d \operatorname{Exp} A I_{\pi}^{-1}$ denote the image of the operator of second quantization $d \operatorname{Exp} A$ under the mapping I_{π} (9.6). Applying Theorem 5.1 to H_{π}^{A}, we obtain a known statement (see, e.g., [6, 9, 10]).

Theorem 10.2. *Let* $\mathcal{N} = \mathcal{S}(R^1) \subset \operatorname{Dom} A$. *Then the symmetric bilinear form of the operator* H_{π}^{A} *admits the representation*

$$(H_{\pi}^{A} \varphi, \psi)_{(L^2_{\pi})} = \int\limits_{Q} (A \partial_x \varphi, \partial_x \psi)_{L^2_{\mathbb{C}}(\mathbb{R}^1)} d\pi(x) \tag{10.6}$$

for all $\varphi, \psi \in I_{\pi}(\mathring{\mathcal{F}}_{\text{fin}}(\mathcal{N}))$.

Remark 10.1. It is easy to determine the action of the operator $\partial_x \colon H^{\omega}(p,q) \to \mathcal{F}_1(N_0) = L^2_{\mathbb{C}}(\mathbb{R}^1)$ ($p, q \in \mathbb{N}_3$ are fixed): for almost all $t \in \mathbb{R}^1$,

$$(\partial_x f)(t) = (\ell(\delta_t) f)(x) = f(x + \delta_t) - f(x), \quad f \in H^{\omega}(p,q). \tag{10.7}$$

Indeed, by virtue of equality (5.2), for an arbitrary function $\xi_1 \in \mathcal{F}_1(N_0) = L^2_{\mathbb{C}}(\mathbb{R}^1)$ we have

$$(\partial(\xi_1) f)(x) = (\partial_x f, \bar{\xi}_1)_{L^2_{\mathbb{C}}(\mathbb{R}^1)}, \quad f \in H^{\omega}(p,q). \tag{10.8}$$

On the other hand, by virtue of relation (10.1), for any finite function $\varphi_1 \in \mathcal{F}_1(\mathcal{N}) = \mathcal{S}_{\mathbb{C}}(\mathbb{R}^1)$ we have

$$(\partial(\varphi_1)f)(x) = \int_{\mathbb{R}^1} (\ell(\delta_t)f)(x)\varphi_1(t)d\sigma(t), \quad f \in H^\omega(p,q). \tag{10.9}$$

Comparing relations (10.8) and (10.9) and taking into account that the set of finite functions from $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^1)$ is dense in the space $L^2_{\mathbb{C}}(\mathbb{R}^1)$, we obtain equality (10.7).

11. Lebesgue–Poisson Measure

The configuration space $\Gamma = \Gamma(\mathbb{R}^1)$ over $\mathbb{R}^1$ is understood (see, e.g., [3–10]) as the collection of all locally finite subsets (configurations) of $\mathbb{R}^1$, i.e.,

$$\Gamma := \Big\{\gamma \subset \mathbb{R}^1 \mid |\gamma \cap \Lambda| < \infty \text{ for an arbitrary compact set } \Lambda \subset \mathbb{R}^1\Big\},$$

where $|X|$ denotes the number of points of the set $X \subset \mathbb{R}^1$.

For each $n \in \mathbb{N}_1$, we consider a subset $\Gamma^{(n)} = \Gamma^{(n)}(\mathbb{R}^1)$ of the space Γ that consists of all n-point configurations $\eta = \{t_1, \ldots, t_n\}$, i.e.,

$$\Gamma^{(n)} := \{\eta \subset \mathbb{R}^1 \big| |\gamma| = n\}.$$

Let

$$\widehat{\mathbb{R}}^n := \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_k \neq t_j \text{ if } k \neq j\}.$$

The mapping

$$\widehat{\mathbb{R}}^n \ni (t_1, \ldots, t_n) \mapsto \{t_1, \ldots, t_n\} = \eta \in \Gamma^{(n)} \tag{11.1}$$

defines the Hausdorff topology on $\Gamma^{(n)}$ (on $\widehat{\mathbb{R}}^n$, we consider the topology induced by the topology of $\mathbb{R}^n$). The Borel σ-algebra $\mathcal{B}(\Gamma^{(n)})$ corresponding to this topology is the image of the Borel σ-algebra $\mathcal{B}(\widehat{\mathbb{R}}^n)$ under mapping (11.1).

Parallel with Γ, we consider the space of finite configurations, i.e., the disjunctive sum of the topological spaces $\Gamma^{(n)}$:

$$\Gamma_0 = \Gamma_0(\mathbb{R}^1) := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}, \quad \Gamma^{(0)} := \varnothing.$$

The space Γ_0 has the ordinary topology of a disjunctive union and the Borel σ-algebra $\mathcal{B}(\Gamma_0)$ corresponding to this topology. Regarded as a set,

$$\Gamma_0 = \bigcup_{n=0}^{\infty} \Gamma^{(n)} = \{\gamma \in \Gamma \mid |\gamma| < \infty\}$$

is a subset of the space Γ. In what follows, as a rule, we denote finite configurations, i.e., points from Γ_0, by η and arbitrary points from Γ by γ.

Using the Lebesgue measure σ on $\mathbb{R}^1$, we construct a measure on Γ_0. For arbitrary $n \in \mathbb{N}_1$, we consider the product measure $\sigma^{\otimes n}$ on $\mathcal{B}(\mathbb{R}^n)$. Since $\sigma^{\otimes n}(\mathbb{R}^n \backslash \widehat{\mathbb{R}}^n) = 0$, we can regard $\sigma^{\otimes n}$ as a σ-finite measure on

$\mathcal{B}(\widehat{\mathbb{R}}^n)$. Let σ_n denote the σ-finite Borel measure on $\Gamma^{(n)}$ that is the image of the measure $\sigma^{\otimes n}$ under mapping (11.1). The *Lebesgue–Poisson measure* on $\mathcal{B}(\Gamma_0)$ with intensity measure σ is understood as the σ-finite measure

$$\nu_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!}\sigma_n, \quad \sigma_0(\varnothing) := 1.$$

Let us establish a natural unitary isomorphism between the Fock space $F(N_0)$, $N_0 = L^2(\mathbb{R}^1)$, and the Hilbert space $L^2_\nu(\Gamma_0) := L^2(\Gamma_0, d\nu_\sigma(\eta))$ of complex-valued functions square summable with respect to the measure ν_σ.

First of all, recall that, for every configuration $\eta = \{t_1, \ldots, t_n\} \in \Gamma^{(n)}$, the order of points t_j is inessential. Therefore, to define a certain function $\Gamma^{(n)} \ni \eta = \{t_1, \ldots, t_n\} \mapsto f(\eta) = f(\{t_1, \ldots, t_n\}) \in \mathbb{C}^1$ is the same as to define a function $\widehat{\mathbb{R}}^n \ni (t_1, \ldots, t_n) \mapsto f(t_1, \ldots, t_n) \in \mathbb{C}^1$ symmetric with respect to the variables $t_1, \ldots, t_n$:

$$f(\{t_1, \ldots, t_n\}) = f(t_1, \ldots, t_n).$$

Since $\sigma^{\otimes n}\big(\mathbb{R}^n \backslash \widehat{\mathbb{R}}^n\big) = 0$, for an arbitrary Borel function $\Gamma^{(n)} \ni \eta \mapsto f(\eta) \in \mathbb{C}^1$ (i.e., for a symmetric Borel function $\widehat{\mathbb{R}}^n \ni (t_1, \ldots, t_n) \mapsto f(t_1, \ldots, t_n) \in \mathbb{C}^1$) we have

$$\int_{\Gamma^{(n)}} f(\eta) d\sigma_n(\eta) = \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) d\sigma^{\otimes n}(t_1, \ldots, t_n)$$

$$= \int_{\mathbb{R}^n} f(t_1, \ldots, t_n) d\sigma^{\otimes n}(t_1, \ldots, t_n). \tag{11.2}$$

Taking into account relation (11.2) and the fact that the n-particle Fock space $\mathcal{F}_n(N_0)$ coincides with the space $\widehat{L}^2_{\mathbb{C}}\big(\mathbb{R}^n, d\sigma^{\otimes n}(t)\big)$ of all symmetric functions from $L^2_{\mathbb{C}}\big(\mathbb{R}^n, d\sigma^{\otimes n}(t)\big)$, we get

$$L^2\big(\Gamma^{(n)}, d\sigma_n(\eta)\big) = \widehat{L}^2_{\mathbb{C}}\big(\mathbb{R}^n, d\sigma^{\otimes n}(t)\big) = \mathcal{F}_n(N_0).$$

This enables us to interpret the space $L^2_\nu(\Gamma_0)$ as the image of the Fock space $F(N_0)$ under the unitary mapping

$$F(N_0) \ni f = (f_n)_{n=0}^{\infty} \mapsto (I_\nu f)(\cdot) := F(\cdot) = \sum_{n=0}^{\infty} F_n(\cdot) \in L^2_\nu(\Gamma_0),$$

where

$$\Gamma_0 \ni \eta \mapsto F_0(\eta) := \begin{cases} f_0 & \text{if } \eta = \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Gamma_0 \ni \eta \mapsto F_n(\eta) := \begin{cases} n! f_n(t_1, \ldots, t_n) & \text{if } \eta = \{t_1, \ldots, t_n\} \in \Gamma^{(n)}, \\ 0 & \text{otherwise,} \end{cases} \qquad n \in \mathbb{N}_1.$$

In particular, the space $L^2_\nu(\Gamma_0)$ admits the representation

$$L^2_\nu(\Gamma_0) = \bigoplus_{n=0}^{\infty} L^2\Big(\Gamma^{(n)}, d\sigma_n(\eta)\Big)\frac{1}{n!}\,.$$

Under the action of the mapping $I_\nu,$ the nuclear rigging (1.3) transforms into the nuclear rigging of the space $L^2_\nu(\Gamma_0)$ (recall that $N_0 = L^2(\mathbb{R}^1) = S_0(\mathbb{R}^1)$ and $N_p = S_{p+1}(\mathbb{R}^1),$ $p \in \mathbb{N}_1$). More exactly, we have

$$(\Phi_\nu)' \supset H_\nu(-p,-q) \supset L^2_\nu(\Gamma_0) \supset H_\nu(p,q) \supset \Phi_\nu, \tag{11.3}$$

$$\Phi_\nu := \operatorname*{pr\,lim}_{\tilde p,\tilde q\in\mathbb{N}_1} H_\nu(\tilde p,\tilde q), \qquad (\Phi_\nu)' := \operatorname*{ind\,lim}_{\tilde p,\tilde q\in\mathbb{N}_1} H_\nu(-\tilde p,-\tilde q),$$

where $H_\nu(-p,-q),$ $p,q\in\mathbb{N}_1,$ is a negative space with respect to the zero space $L^2_\nu(\Gamma_0)$ and the positive Hilbert space

$$H_\nu(p,q) := I_\nu\Big(\mathcal{F}(N_p,\tau(q))\Big)$$

$$= \Big\{F \in L^2_\nu(\Gamma_0)\,|\,\exists (f_n)_{n=0}^\infty \in \mathcal{F}(N_p,\tau(q))\colon F = I_\nu(f_n)_{n=0}^\infty\Big\}$$

with the Hilbert norm

$$\|F\|^2_{H_\nu(p,q)} = \big\|I_\nu(f_n)_{n=0}^\infty\big\|^2_{H_\nu(p,q)} := \sum_{n=0}^\infty \|f_n\|^2_{\mathcal{F}_n(N_0)}K^{qn}, \quad F\in H_\nu(p,q).$$

The spaces (L^2_π) and $L^2_\nu(\Gamma_0)$ are different functional realizations of the Fock space $F(N_0),$ $N_0 = L^2(\mathbb{R}^1).$ The mapping

$$I_{\nu\pi} := I_\pi I_\nu^{-1}\colon L^2_\nu(\Gamma_0) \to (L^2_\pi)$$

defines a unitary isomorphism between these spaces.

It is clear that, realizing the Fock space $F(N_0)$ as the space $L^2_\nu(\Gamma_0),$ we can construct the rigging of the space (L^2_π) as the image of rigging (11.3). Using the unitary mapping

$$I_{\nu\pi} := I_\pi I_\nu^{-1}\colon H_\nu(p,q) \to H^\omega(p,q), \quad p,q\in\mathbb{N}_1,$$

we obtain the rigging of the space (L^2_π) constructed on the basis of Appell characters. Using the unitary mapping

$$I^\chi_\nu := I^\chi I_\nu^{-1}\colon H_\nu(p,q) \to H^\chi(p,q), \quad p,\ q\in\mathbb{N}_3,$$

we obtain the rigging of the space (L^2_π) constructed on the basis of Delsarte characters [the mapping I^χ (4.12) is considered as a unitary operator that acts from $\mathcal{F}(N_p,\tau(q))$ into $H^\chi(p,q)$].

12. Poisson Analysis on Configuration Space

Let $\mathcal{D}(\mathbb{R}^1) := C^\infty_{\text{fin}}(\mathbb{R}^1)$ be the space of test functions with classical topology and let $\mathcal{D}'(\mathbb{R}^1)$ be its dual space of generalized functions with weak topology and the σ-algebra $C_\sigma(\mathcal{D}')$ of its subsets generated by the cylindrical sets

$$\{x \in \mathcal{D}'(\mathbb{R}^1) \,|\, (\langle x, \varphi_1\rangle, \ldots, \langle x, \varphi_1\rangle) \in \beta\}, \qquad \varphi_i \in \mathcal{D}(\mathbb{R}^1), \quad \beta \in \mathcal{B}(\mathbb{R}^n), \quad n \in \mathbb{N}_1.$$

It is easy to see that the mapping

$$\Gamma \ni \gamma \mapsto O\gamma := x = \sum_{t\in\gamma} \delta_t \in \mathcal{D}'(\mathbb{R}^1) \tag{12.1}$$

is defined and injective (δ_t is the δ-function concentrated at a point $t \in \mathbb{R}^1$). Let $\widetilde{\Gamma} = \widetilde{\Gamma}(\mathbb{R}^1)$ denote the collection of all configurations $\gamma \in \Gamma$ that are transformed under the mapping O into generalized functions from the space $S_{-2}(\mathbb{R}^1) \subset \mathcal{D}'(\mathbb{R}^1)$ $\big($i.e., $O\gamma = \sum_{t\in\gamma} \delta_t \in S_{-2}(\mathbb{R}^1)$ for $\gamma \in \widetilde{\Gamma}\big)$.

Let $\widetilde{Q}$ be the image of the space $\widetilde{\Gamma}$ under mapping (12.1), i.e.,

$$\widetilde{Q} := \left\{ \sum_{t\in\gamma} \delta_t \in \mathcal{D}'(\mathbb{R}^1) \mid \gamma \in \Gamma \right\} \cap S_{-2}(\mathbb{R}^1) = \left\{ \sum_{t\in\gamma} \delta_t \in S_{-2}(\mathbb{R}^1) \mid \gamma \in \widetilde{\Gamma} \right\}.$$

The norm in $S_{-2}(\mathbb{R}^1)$ induces a metric in $\widetilde{Q}$ and transforms $\widetilde{Q}$ into a separable metric space. The set $\widetilde{Q} \subset S_{-2}(\mathbb{R}^1)$ is of full outer Poisson measure π (8.4), which follows from the fact that the sets $\Gamma \subset \mathcal{D}'(\mathbb{R}^1)$ and $S_{-2}(\mathbb{R}^1) \subset \mathcal{D}'(\mathbb{R}^1)$ are of full outer measure of the measure π extended from $\mathcal{B}(S_{-2})$ to $C_\sigma(\mathcal{D}')$ (see, e.g., [6, 8–10, 14]), and, therefore, π can be modified to a Borel probability measure on $\widetilde{Q}$. We denote the measure π modified with respect to $\widetilde{Q}$ also by π and call it a Poisson measure. This measure has the Fourier transform

$$\int_{\widetilde{Q}} e^{i\langle\lambda, x\rangle} d\pi(x) = \exp\langle 1, e^{i\lambda} - 1\rangle, \quad \lambda \in N_1 = S_2(\mathbb{R}^1). \tag{12.2}$$

It is clear that the results presented in Secs. 9–11 remain valid for the space $Q = \widetilde{Q}$, measure $\rho = \pi$, and functions $\chi(x,\lambda)$ (9.2) and $\omega(x,\lambda)$ (9.4), which, in the case considered, can be represented as follows: for all $x \in \widetilde{Q}$ and $\lambda \in B_\omega$,

$$\chi(x,\lambda) = \exp\langle x, \log(1+\lambda)\rangle = \prod_{t\in\gamma}(1+\lambda(t)),$$

$$\omega(x,\lambda) = \exp\Big(\langle x, \log(1+\lambda)\rangle - \langle 1, \lambda\rangle\Big) = \exp(-\langle 1, \lambda\rangle) \prod_{t\in\gamma}(1+\lambda(t)),$$

where $\gamma = O^{-1}(x) \in \widetilde{\Gamma}$ [here and in what follows, the mapping O (12.1) is regarded as an invertible mapping $O\colon \widetilde{\Gamma} \to \widetilde{Q}$ with inverse $O^{-1}\colon \widetilde{Q} \to \widetilde{\Gamma}$].

According to the results presented in Sec. 11, the rigging of the space $(L^2_\pi) := L^2(\widetilde{Q}, d\pi(x))$ can be constructed as the image of rigging (11.3) of the space $L^2_\nu(\Gamma_0)$ under the mapping $I_{\nu\pi}$ or I^χ_ν. It turns out that the unitary isomorphism $I^\chi_\nu\colon H_\nu(p,q) \to H^\chi(p,q)$ has a simple combinatorial interpretation. Prior to the formulation of the corresponding statement, we first recall the definition of the so-called K-transformation between functions on Γ_0 and $\widetilde{Q}$, which was introduced in [26] and studied in [26, 27].

By definition,

$$(\mathsf{K}f)(x) := \sum_{\eta \subset \gamma} f(\eta), \quad x \in \widetilde{Q}, \tag{12.3}$$

where $\gamma = O^{-1}(x) \in \widetilde{\Gamma}$ and $f\colon \Gamma_0 \to \mathbb{C}^1$ is an arbitrary function for which the right-hand side of (12.3) is meaningful. The summation in the last relation is carried out over all finite subconfigurations of the configuration $\gamma = O^{-1}(x) \in \widetilde{\Gamma}$. Note that, at least for vectors f from the set $I_\nu\Big(\mathring{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D}(\mathbb{R}^1))\Big)$ dense in the space $L^2_\nu(\Gamma_0)$, the K-transformation is defined [sum (12.3) is finite].

Theorem 12.1. *For all* $p, q \in \mathbb{N}_3$, *the mapping*

$$H_\nu(p,q) \supset I_\nu(\mathring{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D}(\mathbb{R}^1))) \ni f \mapsto \mathsf{K}f \in H^\chi(p,q)$$

is defined, linear, and continuous. After the closure by continuity, the operator K *is unitary. Moreover, the following operator equality is true:*

$$\mathsf{K} = I^\chi_\nu \colon H_\nu(p,q) \to H^\chi(p,q).$$

Proof. The statement of the theorem follows directly from the fact that, on the vectors

$$e_\nu(\varphi) := I_\nu\left(\frac{\varphi^{\otimes n}}{n!}\right)_{n=0}^{\infty} \in H_\nu(p,q), \quad \varphi \in \mathcal{D}_{\mathbb{C}}(\mathbb{R}^1)$$

(the norm $\|\varphi\|_{\mathcal{F}_1(N_p)}$ is sufficiently small), the K-transformation is defined [sum (12.3) is finite] and

$$(\mathsf{K}e_\nu(\varphi))(x) = \sum_{\eta \subset \gamma} (e_\nu(\varphi))(\eta) = \prod_{t \in \gamma}(1 + \varphi(t))$$

$$= \chi(x,\varphi) = (I^\chi_\nu e_\nu(\varphi))(x), \quad \gamma = O^{-1}(x) \in \widetilde{\Gamma},$$

for all $x \in \widetilde{Q}$.

The theorem is proved.

Corollary 12.1. *If* $\varphi \in \mathcal{D}_{\mathbb{C}}(\mathbb{R}^1)$, *then the following relation holds for all* $x \in \widetilde{Q}$ *and* $n \in \mathbb{N}_1$:

$$\langle \varphi^{\otimes n}, \chi_n(x)\rangle = (I^\chi_\nu(I_\nu\varphi^{\otimes n}))(x) = (\mathsf{K}(I_\nu\varphi^{\otimes n}))(x) = n! \sum_{\{t_1,\ldots,t_n\} \subset \gamma} \prod_{i=1}^{n} \varphi(t_i),$$

where $\gamma = O^{-1}(x) \in \widetilde{\Gamma}$.

It is easy to see that, on functions $f \in I_\pi(\mathcal{F}_{\mathrm{fin}}(\mathcal{N}))$, $\mathcal{N} = \mathcal{S}(\mathbb{R}^1)$, the creation operator $\partial^+(\varphi)$, $\varphi \in \mathcal{F}_1(\mathcal{N}) = \mathcal{S}_{\mathbb{C}}(\mathbb{R}^1)$, acts according to the following rule: for π-almost all $x \in \widetilde{Q}$,

$$(\partial^+(\varphi)f)(x) = \sum_{t\in\gamma} f(x-\delta_t)\varphi(t) - f(x)\int_{\mathbb{R}^1}\varphi(t)d\sigma(t), \quad \gamma = O^{-1}(x) \in \widetilde{\Gamma}. \tag{12.4}$$

Indeed, let $g \in I_\pi(\mathcal{F}_{\mathrm{fin}}(\mathcal{N}))$. Then, by virtue of (10.9), we have

$$(\partial(\bar{\varphi})g, f)_{(L^2_\pi)} = \int_{\tilde{Q}}\int_{\mathbb{R}^1} g(x+\delta_t)\overline{f(x)\varphi(t)}d\sigma(t)d\pi(x) - \left(\int_{\tilde{Q}} g(x)\overline{f(x)}d\pi(x)\right)\left(\int_{\mathbb{R}^1}\overline{\varphi(t)}d\sigma(t)\right). \tag{12.5}$$

Applying the Mecke identity [28]

$$\int_{\tilde{Q}}\sum_{t\in O^{-1}(x)} k(x,t)d\pi(x) = \int_{\tilde{Q}}\int_{\mathbb{R}^1} k(x+\delta_t,t)d\sigma(t)d\pi(x)$$

to the first integral on the right-hand side of (12.5), we get

$$(\partial(\bar{\varphi})g, f)_{(L^2_\pi)} = \int_{\tilde{Q}} g(x)\sum_{t\in O^{-1}(x)}\overline{f(x-\delta_t)\varphi(t)}\,d\pi(x) - \left(\int_{\tilde{Q}} g(x)\overline{f(x)}d\pi(x)\right)\left(\int_{\mathbb{R}^1}\overline{\varphi(t)}d\sigma(t)\right),$$

which guarantees the validity of (12.4).

Note that the Mecke identity presented above has a simple operator interpretation in terms of a Poisson field whose spectral measure is a Poisson measure (see [25]).

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