An approach to a generalization of white noise analysis

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This paper is dedicated to M. G. Krein.

Abstract. In this article we review some recent developments in white noise analysis and its generalizations. In particular, we describe the main idea of the biorthogonal approach to a generalization of white noise analysis connected with the theory of hypergroups.

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1. Introduction

The classical white noise analysis (Gaussian white noise analysis) it is possible to understand as a theory of generalized functions of infinite many variables with pairing between test and generalized functions provided by integration with respect to the Gaussian measure. It is well known that there exist several approaches to the construction of such theory of generalized functions: the Berezansky-Samoilenko approach [19] and the Hida approach [24]. In the Berezansky-Samoilenko approach spaces of test and generalized functions are constructed as infinite tensor products of one-dimensional spaces. The Hida approach consists in the construction of some rigging of a Fock space with subsequent application to the spaces of this rigging the Wiener-Itô-Segal isomorphism.

After a number of years it became clear that the Hida approach is more convenient and in most cases all investigations in white noise analysis and its generalizations are based on this approach. There exist many works dedicated to white noise analysis development:

- Works deal with the investigation of spaces of test and generalized functions and operators acting in these spaces using the Wiener-Itô-Segal isomorphism and various riggings of the Fock space. For more information, see the books [24, 12, 25, 39], surveys [40, 41] and the references therein.

- Works deal with the so-called *Jacobi fields approach* to a generalization of white noise analysis. In these works the role of the Wiener-Itô-Segal isomorphism is played a unitary Fourier transform which is defined by the Jacobi field, i.e., by some family of commuting selfadjoint operators that act in the Fock space and have a Jacobi structure. The theory of Jacobi fields was created by Berezansky under the influence of the works of M. G. Krein (see e.g. [37, 38]) about Jacobi matrices. A detailed study of general commutative Jacobi fields in the Fock space and a corresponding spectral measure was carried out in the works by Berezansky and his collaborators, see e.g. [4], [6]–[10], [15]–[18], [42]–[46]. Note that the Wiener-Itô-Segal isomorphism it is possible to understand as the Fourier transform of a certain Jacobi field, so-called free field. This result was obtained by Koshmanenko and Samoilenko in [36], see also [12].
- Works are devoted the *biorthogonal approach* to a generalization of white noise analysis. In this approach, one replaces the system of Hermite polynomials, that are orthogonal with respect to the Gaussian measure, with a certain biorthogonal system. The biorthogonal approach was inspired by [22], proposed in [3] and developed in [51, 2, 13, 14, 29, 30, 20, 21] (see survey [20] for the complete bibliography). Note that in [13, 14] was first observed that the biorthogonal approach is deep connected with the theory of hypergroups.

There exists the deep analogy between the above mentioned works. In all these works spaces of test function are constructed as image of positive spaces from some rigging of the Fock space. But in the first series of works is used the Wiener-Itô-Segal isomorphism, in the second series is used a Fourier transform and in the third series is used a certain biunitary map.

This survey is devoted to the biorthogonal approach to a generalization of classical white noise analysis. In the first part of the survey we recall the main idea of Hida approach to the construction of classical white noise analysis. In the second part we give the basic idea of the biorthogonal approach. In order to make it more simple, at first we will consider the corresponding theory of generalized functions for model one-dimensional case and than briefly for infinite-dimensional. For the details and proofs we refer the reader to the surveys [20, 21].

2. Gaussian white noise analysis

Let us shortly recall some basic results of Gaussian white noise analysis, for details see e.g. [12, 25]. We consider a rigging of the real Hilbert space $L^2(\mathbb{R}) := L^2(\mathbb{R}, dt)$,

$$\mathcal{S}' \supset L^2(\mathbb{R}) \supset \mathcal{S},$$

where $\mathcal{S}$ is the Schwartz space of infinite differentiable, rapidly decreasing function on $\mathbb{R}$, $\mathcal{S}'$ is the Schwartz space of distributions dual of $\mathcal{S}$ with respect to the zero space $L^2(\mathbb{R})$. We denote by $\langle \cdot\,,\cdot \rangle$ the dual pairing between elements of $\mathcal{S}'$ and $\mathcal{S}$ inducted by the scalar product in $L^2(\mathbb{R})$, i.e., for any $f \in L^2(\mathbb{R})$ and any $\varphi \in \mathcal{S}$

$$\langle f, \varphi \rangle := (f, \varphi)_{L^2(\mathbb{R})}.$$

We will preserve this notation for tensor powers and complexifications of spaces.

Let ρ_G be a probability measure on the Borel σ-algebra $\mathcal{B}(S')$ such that

$$\int_{\mathcal{S}'} e^{i\langle x,\varphi\rangle} d\rho_G(x) = e^{-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R})}}, \qquad \varphi \in \mathcal{S}. \tag{2.1}$$

By Minlos theorem the measure ρ_G is completely characterized by (2.1). This measure ρ_G is called the *Gaussian measure*.

Note that elements $x \in S'$ can be thought of as paths of the derivative of Brownian motion, i.e., as white noise. More precisely, it follows from (2.1) that

$$\int_{\mathcal{S}'} \langle x, \varphi \rangle^2 d\rho_G(x) = \|\varphi\|^2_{L^2(\mathbb{R})}, \qquad \varphi \in \mathcal{S}.$$

Hence, extending the mapping

$$L^2(\mathbb{R}) \supset \mathcal{S} \ni \varphi \mapsto \langle \cdot, \varphi \rangle \in L^2(\mathcal{S}', \rho_G)$$

by continuity, we obtain a random variable $\langle \cdot, f \rangle \in L^2(\mathcal{S}', \rho_G)$ for each $f \in L^2(\mathbb{R})$. Thus, we can define the stochastic process $\{B_t\}_{t\in\mathbb{R}}$,

$$B_t(\cdot) := \begin{cases} \langle \cdot, \varkappa_{[0,t]} \rangle, & t \geq 0, \\ -\langle \cdot, \varkappa_{[t,0]} \rangle, & t < 0 \end{cases}$$

($\varkappa_\alpha$ is the indicator function of a set α). It is easily seen that $\{B_t\}_{t\in\mathbb{R}}$ is a version of Brownian motion, i.e., finite-dimensional distributions of the process $\{B_t\}_{t\in\mathbb{R}}$ coincide with those of Brownian motion. We now informally have, for all $t \in \mathbb{R}$,

$$B_t(x) = \int_0^t x(s) ds, \quad \text{so that} \quad \frac{d}{dt} B_t(x) = x(t).$$

The main technical tool of the construction and study of spaces of test and generalized functions in Gaussian white noise analysis is the *Wiener-Itô-Segal isomorphism*

$$I_G : F(L^2(\mathbb{R})) \to L^2(\mathcal{S}', \rho_G)$$

between the symmetric Fock space $F(L^2(\mathbb{R}))$ and the complex space $L^2(\mathcal{S}', \rho_G)$. Let us recall that the symmetric Fock space $F(L^2(\mathbb{R}))$ over $L^2(\mathbb{R})$ is defined as

$$F(L^2(\mathbb{R})) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(L^2(\mathbb{R}))n!,$$

where the n-particle Fock space

$$\mathcal{F}_n(L^2(\mathbb{R})) := (L^2_{\mathbb{C}}(\mathbb{R}))^{\widehat{\otimes} n} \qquad ((L^2_{\mathbb{C}}(\mathbb{R}))^{\widehat{\otimes} 0} := \mathbb{C})$$

is equal to the n-th symmetric tensor power $\widehat{\otimes}$ of the complexification $L^2_{\mathbb{C}}(\mathbb{R})$ of the real space $L^2(\mathbb{R})$ (here and subsequently, the lower index $\mathbb{C}$ denotes complexification of a real space). Thus, for each $f = (f_n)_{n=0}^{\infty} \in F(L^2(\mathbb{R}))$,

$$\|f\|^2_{F(L^2(\mathbb{R}))} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{F}_n(L^2(\mathbb{R}))} n! < \infty.$$

The isomorphism I_G is completely characterized by its following properties:

1. $I_G : F(L^2(\mathbb{R})) \to L^2(\mathcal{S}', \rho_G)$ is the unitary operator.
2. $I_G(f_0, 0, 0, \ldots) = f_0$ for all $f_0 \in \mathbb{C}$.
3. For each $n \in \mathbb{N}$ and any disjoint Borel sets $\alpha_1, \ldots, \alpha_n \in \mathcal{B}(\mathbb{R})$ of finite Lebesgue measure,

$$\big(I_G(\underbrace{0, \ldots, 0}_{n}, \varkappa_{\alpha_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\alpha_n}, 0, 0, \ldots)\big)(\cdot) = \langle \cdot, \varkappa_{\alpha_1} \rangle \ldots \langle \cdot, \varkappa_{\alpha_n} \rangle.$$

There are several equivalent ways of the construction of such isomorphism:

- Using multiple stochastic integrals. In this case I_G is constructed by representation of any function from $L^2(\mathcal{S}', \rho_G)$ as an infinite sum of pairwise orthogonal multiple stochastic integrals with respect to the Brownian motion $\{B_t\}_{t \in \mathbb{R}}$, see e.g. [24, 27, 25].
- Using Jacobi fields approach. Now I_G is the Fourier transform of the free field, i.e., a certain family of commuting selfadjoint operators that act in the Fock space $F(L^2(\mathbb{R}))$ and have the Jacobi structure, see for instance [36, 12].
- Using the system of infinite-dimensional Hermite polynomials which are orthogonal (in terms of the Fock space $F(L^2(\mathbb{R}))$, see below) with respect to the Gaussian measure ρ_G, see e.g. [12, 25].

Our investigation is connected with the third way of the construction of the Wiener-Itô-Segal isomorphism I_G. Let us look at this way more closely.

We consider the function

$$H(x, \varphi) := e^{\langle x, \varphi \rangle - \frac{1}{2}\|\varphi\|^2_{L^2_{\mathbb{C}}(\mathbb{R})}}, \qquad x \in \mathcal{S}', \quad \varphi \in \mathcal{S}_{\mathbb{C}}.$$

It is well known that H is the *generating function* for the *infinite-dimensional Hermite polynomials* $H_n(x) \in (\mathcal{S}')^{\widehat{\otimes} n}$ which are defined from the decomposition

$$H(x, \varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varphi^{\otimes n}, H_n(x) \rangle,$$

where symbol $\otimes$ denotes the tensor power. The polynomials $H_n(x)$ are *orthogonal in the space* $L^2(\mathcal{S}', \rho_G)$ in terms of the Fock space $F(L^2(\mathbb{R}))$,

$$\int_{\mathcal{S}'} \langle \varphi_n, H_n(x) \rangle \overline{\langle \psi_m, H_m(x) \rangle} \, d\rho_G(x) = \delta_{n,m} n! (\varphi_n, \psi_n)_{\mathcal{F}_n(L^2(\mathbb{R}))}, \tag{2.2}$$

$$\varphi_n \in \mathcal{S}_{\mathbb{C}}^{\widehat{\otimes} n}, \quad \psi_m \in \mathcal{S}_{\mathbb{C}}^{\widehat{\otimes} m}, \qquad n, m \in \mathbb{Z}_+,$$

and the mapping

$$F(L^2(\mathbb{R})) \supset \mathcal{F}_{\text{fin}}(\mathcal{S}) \ni \varphi = (\varphi_n)_{n=0}^{\infty} \mapsto (I_G\varphi)(\cdot) := \sum_{n=0}^{\infty} \langle \varphi_n, H_n(\cdot)\rangle \in L^2(\mathcal{S}^{'}, \rho_G)$$

after being extended by continuity to the whole space $F(L^2(\mathbb{R}))$ is the Wiener-Itô-Segal isomorphism. Here $\mathcal{F}_{\text{fin}}(\mathcal{S})$ denotes the set of all finite sequences $(\varphi_n)_{n=0}^{\infty}$ such that each φ_n belongs to $\mathcal{S}_{\mathbb{C}}^{\widehat{\otimes} n}$.

With the help of the Wiener–Itô–Segal isomorphism I_G, spaces of test and generalized functions are constructed and investigated. These spaces are obtained as the I_G-image of some rigging of the Fock space $F(L^2(\mathbb{R}))$:

$$\begin{array}{ccccc} \mathcal{F}_- & \supset & F(L^2(\mathbb{R})) & \supset & \mathcal{F}_+ \\ & & \downarrow I_G & & \downarrow I_G \\ \mathcal{H}_- & \supset & L^2(\mathcal{S}^{'}, \rho_G) & \supset & \mathcal{H}_+. \end{array}$$

Here $\mathcal{F}_+$ is a certain Fock space densely and continuously embedded into $F(L^2(\mathbb{R}))$, $\mathcal{F}_-$ is the negative space with respect to the positive space $\mathcal{F}_+$ and the zero space $F(L^2(\mathbb{R}))$. By definition the space of test functions $\mathcal{H}_+ := I_G\mathcal{F}_+$ is the I_G-image of the Fock space $\mathcal{F}_+$ with topology inducted by the topology of $\mathcal{F}_+$, the space of generalized functions $\mathcal{H}_- := (\mathcal{H}_+)^{'}$ is the dual of $\mathcal{H}_+$ with respect to $L^2(\mathcal{S}^{'}, \rho_G)$. Note that we can extend the isomorphism $I_G : F(L^2(\mathbb{R})) \to L^2(\mathcal{S}^{'}, \rho_G)$ to the isomorphism between the negative Fock space $\mathcal{F}_-$ and the space of generalized functions $\mathcal{H}_-$.

3. Biorthogonal approach

In this section we give the basic idea of the biorthogonal approach. In order to make it more simple, at first we will consider the corresponding theory of generalized functions for a model one-dimensional case and than briefly for infinite-dimensional.

3.1. One-dimensional case

At first we consider one-dimensional analogue of the Gaussian white noise analysis. Then we describe the biorthogonal approach to a generalization of such analysis.

3.1.1. Gaussian case. Let ρ_G be a *Gaussian measure* on on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, its Fourier transform has the form

$$\int_{\mathbb{R}} e^{ix\lambda} d\rho_G(x) = e^{-\frac{1}{2}\lambda^2}, \qquad \lambda \in \mathbb{R}.$$

In this case we have the well-known *generating function*

$$H(x,\lambda) := e^{x\lambda - \frac{1}{2}\lambda^2} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(x), \qquad x \in \mathbb{R}, \quad \lambda \in \mathbb{C},$$

for the *Hermite polynomials* H_n that are orthogonal with respect to ρ_G. More precisely, we have, for all $n, m \in \mathbb{Z}_+$,

$$\int_{\mathbb{R}} H_n(x)\overline{H_m(x)}\, d\rho_G(x) = \delta_{n,m} n!.$$

Now the role of the Fock space $F(L^2(\mathbb{R}))$ plays the l^2-space

$$l^2 := \Big\{ f = (f_n)_{n=0}^{\infty},\ f_n \in \mathbb{C} \,\Big|\, \|f\|_{l^2}^2 := \sum_{n=0}^{\infty} |f_n|^2 n! < \infty \Big\}$$

and an analogue of the Wiener-Itô-Segal isomorphism has the form

$$l^2 \ni f = (f_n)_{n=0}^{\infty} \mapsto (I_G f)(\cdot) = \sum_{n=0}^{\infty} f_n H_n(\cdot) \in L^2(\mathbb{R}, \rho_G).$$

By analogy with the infinite-dimensional situation using the space l^2 instead of the Fock space $F(L^2(\mathbb{R}))$ and the unitary map $I_G : l^2 \to L^2(\mathbb{R}, \rho_G)$ instead of the Wiener-Itô-Segal isomorphism we obtain spaces of test and generalized functions of variables $x \in \mathbb{R}$ as the I_G-image of a rigging of the space l^2.

Namely, for fixed $K > 1$ and $q \in \mathbb{N}$ we denote

$$l_+^2(q) := \Big\{ f = (f_n)_{n=0}^{\infty},\ f_n \in \mathbb{C} \,\Big|\, \|f\|_{l_+^2(q)}^2 := \sum_{n=0}^{\infty} |f_n|^2 (n!)^2 K^{qn} < \infty \Big\},$$

$$l_+^2 := \operatorname*{pr\,lim}_{q\in\mathbb{N}} l_+^2(q).$$

Then the dual spaces of $l_+^2(q)$ and l_+^2 with respect to the zero space l^2 are

$$l_-^2(q) := \big(l_+^2(q)\big)' = \Big\{ f = (f_n)_{n=0}^{\infty},\ f_n \in \mathbb{C} \,\Big|\, \|f\|_{l_-^2(q)}^2 := \sum_{n=0}^{\infty} |f_n|^2 K^{-qn} < \infty \Big\},$$

$$l_-^2 := \big(l_+^2\big)' = \operatorname*{ind\,lim}_{q\in\mathbb{N}} l_-^2(q),$$

respectively. Thus, for each $q \in \mathbb{N}$, we get a rigging

$$l_-^2 \supset l_-^2(q) \supset l^2 \supset l_+^2(q) \supset l_+^2.$$

Using the unitary operator I_G, one defines spaces of test functions

$$\mathcal{H}_+(q) := I_G l_+^2(q), \qquad \mathcal{H}_+ := I_G l_+^2 = \operatorname*{pr\,lim}_{q\in\mathbb{N}} \mathcal{H}_+(q),$$

and their dual (with respect to the space $L^2(\mathbb{R}, \rho_G)$) spaces of generalized functions

$$\mathcal{H}_-(q) := \big(\mathcal{H}_+(q)\big)', \qquad \mathcal{H}_- := \big(\mathcal{H}_+\big)' = \operatorname*{ind\,lim}_{q\in\mathbb{N}} \mathcal{H}_-(q).$$

Hence, for each $q \in \mathbb{N}$, we have a rigging

$$\mathcal{H}_- \supset \mathcal{H}_-(q) \supset L^2(\mathbb{R}, \rho_G) \supset \mathcal{H}_+(q) \supset \mathcal{H}_+$$

with pairing between test and generalized functions provided by integration with respect to the Gaussian measure ρ_G on $\mathbb{R}$.

3.1.2. Biorthogonal case. Let ρ be a Borel probability measure on $\mathbb{R}$, $L^2(\mathbb{R},\rho)$ be the corresponding L^2-space. Our purpose is to construct some class of test and generalized functions on $\mathbb{R}$ with pairing provided by integration with respect to ρ. We will try to construct these classes functions on $\mathbb{R}$ in a way parallel to the Gaussian case, but using a certain biunitary mapping instead of the Wiener-Itô-Segal isomorphism.

Let us consider instead of $H(x,\lambda)$ a fixed function

$$\mathbb{R}\times\mathbb{C}\ni\{x,\lambda\}\mapsto h(x,\lambda)\in\mathbb{C}$$

such that for each λ from some neighborhood B_0 of zero in $\mathbb{C}$ the function $\mathbb{R}\ni x\mapsto h(x,\lambda)\in\mathbb{C}$ is continuous and for every $x\in\mathbb{R}$

$$h(x,\lambda)=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}h_n(x),\qquad \lambda\in B_0.$$

We additionally assume that $h(\cdot,\lambda)$ is locally bounded uniformly with respect to λ on any closed ball inside of B_0, and that $h(x,0)=1$ for all x from $\mathbb{R}$. In our consideration the role of the function $h(x,\lambda)$ will be same as the role of the generating function $H(x,\lambda)=e^{x\lambda-\frac{1}{2}\lambda^2}$ for the Hermite polynomials in the Gaussian case.

We denote by $C(\mathbb{R})$ the linear space of all complex-valued locally bounded (i.e. bounded on every ball in $\mathbb{R}$) continuous functions on $\mathbb{R}$. It follows from the properties of h that for every $n\in\mathbb{Z}_+$ the function $\mathbb{R}\ni x\mapsto h_n(x)\in\mathbb{C}$ belongs to the space $C(\mathbb{R})$ and the mapping

$$l^2_+(q)\ni f=(f_n)_{n=0}^{\infty}\mapsto(I^hf)(\cdot):=\sum_{n=0}^{\infty}f_nh_n(\cdot)\in C(\mathbb{R})$$

is well-defined for each $q\in\mathbb{N}$ and sufficiently large $K>1$ (we recall that K is used in the definition of the space $l^2_+(q)$). In what follows, we fix such $K>1$.

From the general results, one has (see e.g. [20]).

Theorem 3.1. *Let the above mentioned function h be such that*

- $\|h_n\|_{L^2(\mathbb{R},\rho)}\le C^nn!$ *for some $C>0$ and all $n\in\mathbb{Z}_+$.*
- *The linear span of the functions $\{h_n\}_{n=0}^{\infty}$ is dense in the space $L^2(\mathbb{R},\rho)$.*
- $\|I^hf\|_{L^2(\mathbb{R},\rho)}=0$ *if and only if $f=0$ in $l^2_+(q)$, $q\in\mathbb{N}$.*

Then the I^h-image

$$\mathcal{H}^h_+(q):=I^h(l^2_+(q))=\Big\{f\in C(\mathbb{R})\,\Big|\,\exists(f_n)_{n=0}^{\infty}\in l^2_+(q),f(x)=\sum_{n=0}^{\infty}f_nh_n(x)\Big\}$$

of the space $l^2_+(q)$, $q\in\mathbb{N}$, is a Hilbert space of continuous functions with topology inducted by the topology of $l^2_+(q)$. Moreover $\mathcal{H}^h_+(q)$ densely and continuously embedded in $L^2(\mathbb{R},\rho)$ and we can construct a rigging

$$\mathcal{H}^h_-\supset\mathcal{H}^h_-(q)\supset L^2(\mathbb{R},\rho)\supset\mathcal{H}^h_+(q)\supset\mathcal{H}^h_+,$$

$$\mathcal{H}_+^h := I^h l_+^2 = \operatorname*{pr\,lim}_{q\in\mathbb{N}} \mathcal{H}_+^h(q), \qquad \mathcal{H}_-^h := (\mathcal{H}_+^h)' = \operatorname*{ind\,lim}_{q\in\mathbb{N}} \mathcal{H}_-^h(q).$$

Let all requirements of Theorem 3.1 be fulfilled. It follows from [5] that for the unitary operator $I^h : l_+^2(q) \to \mathcal{H}_+^h(q)$ there exists a unique determined unitary operator $I_-^h : l_-^2(q) \to \mathcal{H}_-^h(q)$ such that

$$(I_-^h \xi, I^h \varphi)_{L^2(\mathbb{R},\rho)} = (\xi, \varphi)_{l^2}, \qquad \xi \in l_-^2(q), \quad \varphi \in l_+^2(q).$$

The pair $\{I_-^h, I^h\}$ is called a *biunitary map*. This biunitary map transfers the rigging of the space l^2 to a rigging of the space $L^2(\mathbb{R},\rho)$:

$$\begin{array}{ccccc} l_-^2(q) & \supset & l^2 & \supset & l_+^2(q) \\ \downarrow I_-^h & & & & \downarrow I^h \\ \mathcal{H}_-^h(q) & \supset & L^2(\mathbb{R},\rho) & \supset & \mathcal{H}_+^h(q). \end{array}$$

Thus, in biorthogonal case the spaces of test and generalized functions are constructed in a way parallel to the Gaussian case (as the image of the rigging of the space l^2), but using the biunitary map $\{I_-^h, I^h\}$ instead of the Wiener-Itô-Segal isomorphism. This give us a possible to develop the biorthogonal white noise analysis by analogy to the Gaussian analysis. In particular, we can give an inner description of the spaces of test and generalized functions, construct in general situation an S-transform, Wick multiplication etc, see e.g. [20] for more details.

Note that there arises *a natural question*, — under which conditions on h the biunitary map $\{I_-^h, I^h\}$ is the unitary map, i.e., the system of functions $\{h_n\}_{n=0}^\infty$ constitutes an *orthogonal basis* in the space $L^2(\mathbb{R},\rho)$?

The answer is the following [21].

Theorem 3.2. *The system of the functions $\{h_n\}_{n=0}^\infty$ with the generating function h constitutes an orthogonal basis in the space $L^2(\mathbb{R},\rho)$ if and only if the following conditions hold:*

- $\|h_n\|_{L^2(\mathbb{R},\rho)} \le C^n n!$ *for some $C > 0$ and all $n \in \mathbb{Z}_+$.*
- *The linear span of the functions $\{h_n\}_{n=0}^\infty$ is dense in the space $L^2(\mathbb{R},\rho)$.*
- *For each λ, μ from some neighborhood of zero in $\mathbb{C}$*

$$\int_{\mathbb{R}} h(x,\lambda)\overline{h(x,\mu)}\,d\rho(x) = e^{\lambda\bar{\mu}}.$$

It is possible to prove that if all conditions of Theorem 3.2 take place then all conditions of Theorem 3.1 also will take place (see [21]). In other words the orthogonal situation is a particular case of the biorthogonal situation.

Consider more special situation.

Example. Let ρ be a Borel probability measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} e^{\varepsilon|x|}\,d\rho(x) < \infty \quad \text{for some} \quad \varepsilon > 0,$$

$h(x,\lambda)$ be a generating function for the *Schefer polynomials* $h_n(x)$ (in another terminology, the generalized Appel polynomials), that is,

$$h(x,\lambda) := \gamma(\lambda)e^{\alpha(\lambda)x} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} h_n(x), \qquad x \in \mathbb{R}, \quad \lambda \in \mathbb{C}, \tag{3.1}$$

where γ and α are fixed analytic functions in some neighborhood of $0 \in \mathbb{C}$ such that $\alpha(0) = 0$, $\alpha^{'}(0) = 1$ and $\gamma(0) = 1$.

In this case the estimate

$$\|h_n\|_{L^2(\mathbb{R},\rho)} \leq C^n n! \quad \text{for some} \quad C > 0 \quad \text{and all} \quad n \in \mathbb{Z}_+$$

is automatically satisfied and the linear span of the functions $\{h_n\}_{n=0}^{\infty}$ is dense in the space $L^2(\mathbb{R},\rho)$, see e.g. [33, 35]. Hence, if the mapping

$$l^2_+(q) \ni f = (f_n)_{n=0}^{\infty} \mapsto (I^h f)(\cdot) := \sum_{n=0}^{\infty} f_n h_n(\cdot) \in L^2(\mathbb{R},\rho)$$

is injective then all requirements of Theorem 3.1 are fulfilled and we can construct the corresponding theory of generalized functions.

The next question is: Which of the Schefer polynomials are orthogonal?

The answer was given by Meixner [47] in 1934 (see also [43, 48] for more details). There exist exactly five type of orthogonal Schefer polynomials: the Hermite, Charlier, Laguerre, Meixner and Meixner–Pollaczek polynomials, which are orthogonal with respect to the Gaussian, Poissonian, Gamma, Pascal and Meixner measures respectively.

3.1.3. Some useful tools in biorthogonal analysis. Let ρ be a Borel probability measure on $\mathbb{R}$ and

$$h(x,\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} h_n(x), \qquad x \in \mathbb{R}, \quad \lambda \in \mathbb{C},$$

be a fixed function such that Theorem 3.1 takes place.

• *Annihilation and creation operators.* The annihilation operator ∂ acts continuously in the space of test functions $\mathcal{H}^h_+$ by the formula

$$\partial h_n := n h_{n-1}, \qquad \partial h_0 := 0.$$

The creation operator ∂^+ is by definition the adjoint to ∂ with respect to the zero space $L^2(\mathbb{R},\rho_G)$ and acts continuously in the space of generalized functions $\mathcal{H}^h_-$.

These operators play an essential role in our considerations. Using them we investigate the spaces of test and generalized functions, construct generalized translation operators, extended stochastic integral (in infinite-dimensional case) etc (see e.g. [20, 21, 49, 50, 1] and references therein). Note that in the Gaussian case the annihilation operator ∂ is the derivative and $\partial + \partial^+$ as operator in the space $L^2(\mathbb{R},\rho_G)$ is the operator of multiplication by x.

• *S-transform.* For each $\xi \in \mathcal{H}^h_-$, the S-transform is defined by the formula

$$(S\xi)(\lambda) := \langle\langle \xi, h(\cdot,\overline{\lambda}) \rangle\rangle,$$

where λ belongs to a neighborhood of zero in $\mathbb{C}$, $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ is the dual pairing between elements of $\mathcal{H}_-^h$ and $\mathcal{H}_+^h$ generated by the scalar product in the space $L^2(\mathbb{R},\rho)$.

Each generalized function $\xi \in \mathcal{H}_-^h$ is uniquely determined by its S-transform. More exactly, let $\mathrm{Hol}_0(\mathbb{C})$ denotes the set of all (germs) of functions which are holomorphic in a neighborhood of zero in $\mathbb{C}$. According to [14] the S-transform is a one-to-one map between $\mathcal{H}_-^h$ and $\mathrm{Hol}_0(\mathbb{C})$.

• *Wick multiplication.* Taking into account that $\mathrm{Hol}_0(\mathbb{C})$ is an algebra of analytic functions with ordinary algebraic operations, we can define a Wick product $\xi \Diamond \eta$ of $\xi, \eta \in \mathcal{H}_-^h$ through the formula

$$\xi \Diamond \eta := S^{-1}(S\xi \cdot S\eta)$$

and make $\mathcal{H}_-^h$ an algebra with such multiplication.

Using this multiplication we can construct the elements of Wick calculus. In Gaussian white noise analysis such calculus has found numerous applications, in particular, in fluid mechanics and financial mathematics, see e.g. [23, 26] for more details.

3.2. Infinite-dimensional case

Now we start with a fixed family $(H_p)_{p\in\mathbb{Z}_+}$ of *real* separable Hilbert spaces H_p such that for all $p \in \mathbb{Z}_+$ the space H_{p+1} is densely embedded in H_p, and this embedding is quasinuclear, i.e., the Hilbert-Schmidt type. So, one can construct the rigging of the space H_0,

$$\Phi^{'} := \operatorname*{ind\,lim}_{p\in\mathbb{Z}_+} H_{-p} \supset H_{-p} \supset H_0 \supset H_p \supset \operatorname*{pr\,lim}_{p\in\mathbb{Z}_+} H_p =: \Phi, \tag{3.2}$$

where H_{-p} is the dual space to H_p with respect to the zero space H_0. We denote by $\langle \cdot\,,\cdot \rangle$ the dual pairing between elements of H_{-p} and H_p inducted by the scalar product in H_0. As earlier we will preserve this notation for tensor powers and complexifications of spaces.

For each $p \in \mathbb{Z}$ we introduce a weighted symmetric Fock space $\mathcal{F}(H_p,\tau)$ over H_p with a fixed weight $\tau = (\tau_n)_{n=0}^\infty$, $\tau_n > 0$, by setting

$$\begin{aligned}\mathcal{F}(H_p,\tau) :&= \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H_p)\tau_n \\ :&= \Big\{ f = (f_n)_{n=0}^\infty,\ f_n \in \mathcal{F}_n(H_p) \,\Big|\, \|f\|^2_{\mathcal{F}(H_p,\tau)} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{F}_n(H_p)}\tau_n < \infty \Big\},\end{aligned}$$

where the n-particle Fock space

$$\mathcal{F}_n(H_p) := H_{p,\mathbb{C}}^{\widehat{\otimes} n} \qquad (H_{p,\mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C})$$

is equal to the n-th symmetric tensor power $\widehat{\otimes}$ of the complexification $H_{p,\mathbb{C}}$ of the real space H_p. Using rigging (3.2) and the weight

$$\tau(q) = ((n!)^2 K^{qn})_{n=0}^\infty, \qquad q \in \mathbb{N},$$

with fix $K > 1$ we construct the rigging of the Fock space $F(H_0) := \mathcal{F}(H_0, (n!)_{n=0}^{\infty})$,

$$\mathcal{F}(-p,-q) \supset F(H_0) \supset \mathcal{F}(p,q),$$

where

$$\mathcal{F}(-p,-q) := \mathcal{F}(H_{-p}, (K^{-qn})_{n=0}^{\infty}), \qquad \mathcal{F}(p,q) := \mathcal{F}(H_p, ((n!)^2 K^{qn})_{n=0}^{\infty})$$

are dual with respect to the zero space $F(H_0)$.

Let ρ be a fixed Borel probability measure on $\Phi^{'}$, $L^2(\Phi^{'},\rho)$ be the corresponding space of square integrable functions. Our goal is to construct some class of test and generalized functions on $\Phi^{'}$ with pairing generated by the scalar product in $L^2(\Phi^{'},\rho)$. We will try to construct this class functions on $\Phi^{'}$ in a way parallel to the Gaussian and above stated one-dimensional cases.

Let B_0 be a neighborhood of zero in the space $\Phi_{\mathbb{C}}$. Instead of the generating function for the infinite-dimensional Hermite polynomials we will take a function

$$\Phi^{'} \times B_0 \ni \{x, \varphi\} \mapsto h(x,\varphi) \in \mathbb{C}$$

such that for each $\varphi \in B_0$ the function $h(\cdot,\varphi)$ is continuous and for each $x \in \Phi^{'}$ the function $h(x,\cdot)$ is analytic in B_0, i.e., has the representation

$$h(x,\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varphi^{\otimes n}, h_n(x) \rangle, \qquad h_n(x) \in (\Phi^{'}_{\mathbb{C}})^{\widehat{\otimes} n},$$

for all φ from B_0. We additionally assume that $h(\cdot,\varphi)$ is locally bounded uniformly with respect to φ on any closed ball inside of B_0, and that $h(x,0) = 1$ for all x from $\Phi^{'}$ (see [20], Sections 2.3, for more details).

Due to such properties of h one can show that for each $\varphi_n \in \Phi_{\mathbb{C}}^{\widehat{\otimes} n}$ the functions

$$\Phi^{'} \ni x \mapsto \langle \varphi_n, h_n(x) \rangle \in \mathbb{C}$$

belong to the space $C(\Phi^{'})$ of all complex-valued locally bounded continuous functions on $\Phi^{'}$. Moreover, there exist $p, q \in \mathbb{N}$ and $K > 1$ (we recall that K is used in the definition of space $\mathcal{F}(p,q)$) such that the mapping

$$\mathcal{F}(p,q) \ni f = (f_n)_{n=0}^{\infty} \mapsto (I^h f)(\cdot) := \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot) \rangle \in C(\Phi^{'})$$

is well-defined. In what follows, we fix such $p, q \in \mathbb{N}$ and $K > 1$.

According to [20] we have.

Theorem 3.3. *Let the above mentioned function h be such that*

- $\| \|h_n(\cdot)\|_{\mathcal{F}_n(H_{-p})} \|_{L^2(\Phi',\rho)} \le C^n n!$ *for some $C > 0$ and all $n \in \mathbb{Z}_+$.*
- *The linear span of the set of functions*

  $$\{\langle \varphi_n, h_n(\cdot) \rangle \in L^2(\Phi^{'},\rho) \,|\, \varphi_n \in \Phi_{\mathbb{C}}^{\widehat{\otimes} n},\, n \in \mathbb{Z}_+\}$$

  is dense in the space $L^2(\Phi^{'},\rho)$.
- $\|I^h f\|_{L^2(\Phi',\rho)} = 0$ *if and only if $f = 0$ in $\mathcal{F}(p,q)$.*

Then the I^h-image

$$\mathcal{H}^h(p,q) := I^h(\mathcal{F}(p,q))$$
$$= \Big\{ f \in C(\Phi') \,\Big|\, \exists (f_n)_{n=0}^\infty \in \mathcal{F}(p,q),\ f(x) = \sum_{n=0}^\infty \langle f_n, h_n(x)\rangle \Big\}$$

of the space $\mathcal{F}(p,q)$ is a Hilbert space of continuous functions with topology inducted by the topology of $\mathcal{F}(p,q)$. Moreover $\mathcal{H}^h(p,q)$ densely and continuously embedded in $L^2(\Phi',\rho)$ and we can construct a rigging

$$\mathcal{H}^h(-p,-q) \supset L^2(\Phi',\rho) \supset \mathcal{H}^h(p,q)$$

with pairing between elements of $\mathcal{H}^h(-p,-q) := (\mathcal{H}^h(p,q))'$ and $\mathcal{H}^h(p,q)$ provided by integration with respect to the measure ρ.

Under the conditions of Theorem 3.3, for the unitary operator

$$I^h : \mathcal{F}(p,q) \to \mathcal{H}^h(p,q)$$

there exists a unique determined unitary operator

$$I^h_- : \mathcal{F}(-p,-q) \to \mathcal{H}^h(-p,-q)$$

such that

$$(I^h_-\xi, I^h\varphi)_{L^2(\Phi',\rho)} = (\xi,\varphi)_{F(H_0)}, \qquad \xi \in \mathcal{F}(-p,-q), \quad \varphi \in \mathcal{F}(p,q).$$

So, we have a *biunitary map* $\{I^h_-, I^h\}$. This map transfers the rigging of the space $F(H_0)$ to a rigging of the space $L^2(\Phi',\rho)$:

$$\begin{array}{ccccc} \mathcal{F}(-p.-q) & \supset & F(H_0) & \supset & \mathcal{F}(p,q) \\ \downarrow I^h_- & & & & \downarrow I^h \\ \mathcal{H}^h(-p,-q) & \supset & L^2(\Phi',\rho) & \supset & \mathcal{H}^h(p,q). \end{array}$$

Thus, in biorthogonal case the spaces of test and generalized functions are constructed in a way parallel to the Gaussian case (as the image of the rigging of the Fock space $F(H_0)$), but using the biunitary map $\{I^h_-, I^h\}$ instead of the Wiener-Itô-Segal isomorphism I_G.

The infinite-dimensional analogue of Theorem 3.2 holds, see [20, 21].

Theorem 3.4. *The mapping*

$$F(H_0) \ni f = (f_n)_{n=0}^\infty \mapsto (I^h f)(\cdot) := \sum_{n=0}^\infty \langle f_n, h_n(\cdot)\rangle \in L^2(\Phi',\rho) \qquad (3.3)$$

is well-defined and unitary if and only if the following three conditions hold:

- $\| \, \|h_n(\cdot)\|_{\mathcal{F}_n(H_{-p})} \|_{L^2(\Phi',\rho)} \le C^n n!$ *for some $C > 0$ and all $n \in \mathbb{Z}_+$.*

- *The linear span of the set of functions*

$$\{\langle\varphi_n, h_n(\cdot)\rangle \in L^2(\Phi^{'}, \rho) \,|\, \varphi_n \in \Phi_{\mathbb{C}}^{\widehat{\otimes} n},\, n \in \mathbb{Z}_+\}$$

is dense in the space $L^2(\Phi^{'}, \rho)$.
- *For each* φ, ψ *from some neighborhood of zero in* $\Phi_{\mathbb{C}}$

$$\int_{\Phi^{'}} h(x,\varphi)\overline{h(x,\psi)}d\rho(x) = e^{(\varphi,\psi)_{H_{0,\mathbb{C}}}}.$$

Note that under the assumptions of Theorem 3.4 the functions

$$\Phi^{'} \ni x \mapsto \langle\varphi_n, h_n(x)\rangle \in \mathbb{C}, \qquad \Phi^{'} \ni x \mapsto \langle\psi_m, h_m(x)\rangle \in \mathbb{C},$$

$$\varphi_n \in \Phi_{\mathbb{C}}^{\widehat{\otimes} n}, \quad \psi_m \in \Phi_{\mathbb{C}}^{\widehat{\otimes} m}, \qquad n, m \in \mathbb{Z}_+,$$

are orthogonal in the space $L^2(\Phi^{'}, \rho)$ in terms of the Fock space $F(H_0)$,

$$\int_{\Phi^{'}} \langle\varphi_n, h_n(x)\rangle\overline{\langle\psi_m, h_m(x)\rangle}d\rho(x) = \delta_{n,m} n!(\varphi_n, \psi_n)_{\mathcal{F}_n(H_0)},$$

and all requirements of Theorem 3.3 are fulfilled.

Let us consider the special case when $\Phi = \mathcal{S}$, $H_0 = L^2(\mathbb{R})$ and as a consequence $\Phi^{'} = \mathcal{S}^{'}$. Let the function h satisfies all conditions of Theorem 3.4 and

$$\left.\frac{\partial^n h(x, z_1\varphi_1 + \cdots + z_n\varphi_n)}{\partial z_1 \ldots \partial z_n}\right|_{z_1=\ldots=z_n=0\in\mathbb{C}} = \left.\frac{\partial}{\partial z_1} h(x, z_1\varphi_1)\right|_{z_1=0\in\mathbb{C}} \cdots \left.\frac{\partial}{\partial z_n} h(x, z_n\varphi_n)\right|_{z_n=0\in\mathbb{C}} \tag{3.4}$$

for all $x \in \mathcal{S}^{'}$ and all $\varphi_1, \ldots, \varphi_n \in \mathcal{S}$ such that $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \varnothing$ if $j \neq i$, $i, j \in \{1, \ldots, n\}$, $n \in \mathbb{N}$. Then according to [1] the mapping (3.3) is completely characterized by the following properties:

1. $I^h : F(L^2(\mathbb{R})) \to L^2(\mathcal{S}^{'}, \rho)$ is the unitary operator.
2. $I^h(f_0, 0, 0, \ldots) = f_0$ for all $f_0 \in \mathbb{C}$.
3. For each $n \in \mathbb{N}$ and any disjoint sets $\alpha_1, \ldots, \alpha_n \in \mathcal{B}(\mathbb{R})$ of finite Lebesgue measure,

$$\big(I^h(\underbrace{0, \ldots, 0}_{n}, \varkappa_{\alpha_1} \widehat{\otimes} \cdots \widehat{\otimes} \varkappa_{\alpha_n}, 0, 0, \ldots)\big)(\cdot) = \langle h_1(\cdot), \varkappa_{\alpha_1}\rangle \ldots \langle h_1(\cdot), \varkappa_{\alpha_n}\rangle.$$

Note that in the case of the Gaussian measure $\rho := \rho_G$ on $\mathcal{B}(\mathcal{S}^{'})$ the function

$$h(x,\varphi) := H(x,\varphi) = e^{\langle x,\varphi\rangle - \frac{1}{2}\|\varphi\|^2_{L^2_{\mathbb{C}}(\mathbb{R})}}, \qquad x \in \mathcal{S}', \quad \varphi \in \mathcal{S}_{\mathbb{C}},$$

satisfies all conditions of Theorem 3.4 and equality (3.4).

Now we have an analog of the Example from Subsection 3.1.

Example. Let ρ be a Borel probability measure on $\Phi^{'}$ such that

$$\int_{\Phi^{'}} e^{\varepsilon\|x\|_{H_{-p}}} d\rho(x) < \infty \quad \text{for some} \quad \varepsilon > 0 \quad \text{and} \quad p \in \mathbb{N}.$$

Schefer polynomials (3.1) have the corresponding infinite-dimensional counterpart and are defined by the Taylor expansion of the generating function

$$h(x, \varphi) := \gamma(\varphi) e^{\langle \alpha(\varphi), x\rangle} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \varphi^{\otimes n}, h_n(x)\rangle,$$

where γ and α are fixed analytic functions in some neighborhood of zero in $\Phi_{\mathbb{C}}$ such that $\alpha(0) = 0$, $\gamma(0) = 1$ and α is invertible in a neighborhood of zero.

In this case the estimate

$$\| \, \|h_n(\cdot)\|_{\mathcal{F}_n(H_{-p})} \|_{L^2(\Phi', \rho)} \leq C^n n! \quad \text{for some} \quad C > 0 \quad \text{and all} \quad n \in \mathbb{Z}_+$$

is automatically satisfied and the linear span of the functions

$$\{\langle \varphi_n, h_n(\cdot)\rangle \in L^2(\Phi^{'}, \rho) \,|\, \varphi_n \in \Phi_{\mathbb{C}}^{\widehat{\otimes} n}, \, n \in \mathbb{Z}_+\}$$

is dense in the space $L^2(\mathbb{R}, \rho)$, see e.g. [33, 35]. Hence, if the mapping

$$\mathcal{F}(p, q) \ni f = (f_n)_{n=0}^{\infty} \mapsto (I^h f)(\cdot) := \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot)\rangle \in L^2(\Phi^{'}, \rho)$$

is injective then all requirements of Theorem 3.3 are fulfilled and we can construct the corresponding theory of generalized functions (see [3, 51, 2, 46, 33, 34, 32, 11, 31, 48, 45] for more detailed acount).

For the infinite-dimensional counterpart of the Hermite, Charlier, Laguerre, Meixner and Meixner–Pollaczek polynomials the orthogonality preserves in the following sense:

- In the Gaussian and Poisson cases the orthogonality of the Hermite and Charlier polynomials respectively is given in terms of the Fock space (relation type (2.2)), see e.g. the books [24, 12, 25] and articles [28, 51, 34, 10, 31, 21]. Note that the study of Poisson white noise analysis was initiated by Y. Ito and I. Kubo [28] in 1988. They were the first to construct spaces of test and generalized functions of Poisson white noise, to study them and some operators acting in these spaces.
- In the Gamma, Pascal and Meixner cases the orthogonality of the corresponding polynomials is more complicated and is given in terms of the so-called "extended Fock space", see for instance [34, 32, 17, 11, 16, 18, 43, 44, 48, 45].

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