A CONSTRUCTION OF GENERALIZED TRANSLATION OPERATORS

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Abstract. We reconstruct a family of generalized translation operators from the function which generates a given theory of generalization function.

1. Introduction

The biorthogonal approach to a construction of the theory of generalized functions of an infinite number of variables was inspired by [3], proposed in [1] and developed in [1-17] (the paper [11] contains a fairly complete bibliography). The most general results obtained in [4-8], where characters of some family of generalized translation operators were used instead of exponents. Spaces of test function in [4-8] were constructed by its characters.

In [9] the inverse problem is solved in a model one-dimensional case. Namely, for a given function $h(x, \lambda)$ which generate the theory of generalized functions (this function must satisfy assumptions given in Section 2) it was constructed a family of generalized translation operators for which the function h is a character.

This article is devoted to solving a corresponding problem in the infinite-dimensional case. We claim that a generalized translation operator is the operator $h_x(\partial)$ (the so-called annihilation operator of infinite order) associated with the function $h(x, \lambda)$. Note than such operators were investigated in [13,14] for a special function $h(x, \lambda) = \gamma(\lambda)\chi(\langle x, \alpha(\lambda)\rangle)$, where $\chi : \mathbb{C}^1 \to \mathbb{C}^1$ is an entire function, $\gamma : \mathcal{N}_{\mathbb{C}} \to \mathbb{C}^1$ and $\alpha : \mathcal{N}_{\mathbb{C}} \to \mathcal{N}_{\mathbb{C}}$ is a function analytic at $0 \in \mathcal{N}_{\mathbb{C}}$.

2. The spaces of test functions

We use the following notation:

$$\mathbb{N}_p := \{p, p+1, \dots\}, \quad p \in \mathbb{Z},$$

where $\mathbb{Z} := \{\dots, -1, 0, 1, \dots\}$.

Let Q be a separable complete metric space of points $x, y, \dots$. We denote by $C(Q)$ the linear space of all complex-valued locally bounded (i.e. bounded on

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every ball in Q) continuous functions on Q. We will understand $C(Q)$ as a linear topological space with convergence uniform on every ball from Q.

For any $p \in \mathbb{N}_1$ we consider a fixed chain of real separable Hilbert spaces,

$$\mathcal{N}' := \operatorname*{ind\,lim}_{\tilde{p}\in\mathbb{N}_1} N_{-\tilde{p}} \supset N_{-p} \supset N_0 \supset N_p \supset \operatorname*{pr\,lim}_{\tilde{p}\in\mathbb{N}_1} N_{\tilde{p}} =: \mathcal{N},$$

where N_{-p} is the space negative with respect to the positive space N_p and the zero space N_0. We will suppose that the embedding $N_{p+1} \hookrightarrow N_p,\ p \in \mathbb{N}_0$ is quasinuclear (i.e. the inclusion operator is of the Hilbert-Schmidt type) and, moreover, $\|\cdot\|_{N_p} \le \|\cdot\|_{N_{p+1}}$. Let us denote by $\langle\cdot\,,\cdot\rangle$ the real pairing between N_{-p} and N_p, inducted by the scalar product in N_0. We will preserve these notations for tensor powers and complexifications of spaces.

For any $p \in \mathbb{Z}$ and a weight $\gamma = (\gamma_n)_{n=0}^\infty,\ \gamma_n > 0$, we can construct *a symmetric weighted Fock space*

$$\begin{aligned}\mathcal{F}(N_p,\gamma) &:= \bigoplus_{n=0}^{\infty} \mathcal{F}_n(N_p)\gamma_n \\ &= \Big\{ f = (f_n)_{n=0}^\infty \mid f_n \in \mathcal{F}_n(N_p),\ \|f\|^2_{\mathcal{F}(N_p,\gamma)} = \sum_{n=0}^\infty \|f_n\|^2_{\mathcal{F}_n(N_p)}\gamma_n < \infty \Big\},\end{aligned}$$

with the corresponding inner product. Here the n-particle subspace $\mathcal{F}_n(N_p),\ p \in \mathbb{Z}$ is equal to the n-th symmetric tensor power $\hat{\otimes}$ of the complexification $N_{p,\mathbb{C}}$ of the space N_p, $\mathcal{F}_n(N_p) := N_{p,\mathbb{C}}^{\hat{\otimes}n}$, $N_{p,\mathbb{C}}^{\hat{\otimes}0} := \mathbb{C}^1$.

In what follows, *we will consider the family $(\mathcal{F}(N_p,\gamma(q)))_{p,q\in\mathbb{N}_1}$ of weighted Fock spaces $\mathcal{F}(N_p,\gamma(q))$ with the weight*

$$\gamma(q) = (\gamma_n(q))_{n=0}^\infty, \quad \gamma_n(q) = (n!)^2 K^{qn}, \quad K > 1. \tag{1}$$

Let B_0 be some neighborhood of 0 in the space $N_{1,\mathbb{C}}$ and

$$Q \times B_0 \ni \{x,\lambda\} \mapsto h(x,\lambda) \in \mathbb{C}^1 \tag{2}$$

be a given function. *Suppose that for each $x \in Q$ $h(x,\cdot)$ is analytic at $0 \in N_{1,\mathbb{C}}$, and, for each $\lambda \in B_0$, $h(\cdot,\lambda) \in C(Q)$. Moreover, $h(\cdot,\lambda)$ is locally bounded uniformly with respect to λ from any closed ball inside of B_0.*

It follows from the analyticity ([11], Subsections 2–3) that, for each point $x \in Q$, there exists a neighborhood of zero

$$B(x) := \{\lambda \in N_{2,\mathbb{C}} \mid \|\lambda\|_{N_{2,\mathbb{C}}} < R(x),\ R(x) > 0\} \subset B_0,$$

such that

$$h(x,\lambda) = \sum_{n=0}^\infty \frac{1}{n!}\langle \lambda^{\otimes n}, h_n(x)\rangle, \quad h_n(x) \in \mathcal{F}_n(N_{-2}), \tag{3}$$

for all λ from $B(x)$. Moreover, the last series converges uniformly on any closed ball from $B(x)$. *Suppose that for all $x \in Q$ there exists a general neighborhood of zero*

$$B := \{\lambda \in N_{2,\mathbb{C}} \mid \|\lambda\|_{N_{2,\mathbb{C}}} < R,\ R > 0\} \subset B_0$$

with this property.

In accordance with [11] the function

$$Q \ni x \mapsto \langle f_n, h_n(x)\rangle \in \mathbb{C}^1$$

belongs to $C(Q)$ for all $f_n \in \mathcal{F}_n(N_p),\ n \in \mathbb{N}_0,\ p \in \mathbb{N}_3$. Moreover ([11], Lemma 4.2), if $K > 1$ (here K from (1)) is sufficiently large, then the series

$$\sum_{n=0}^{\infty} \langle f_n, h_n(\cdot)\rangle, \quad (f_n)_{n=0}^{\infty} \in \mathcal{F}(N_p, \gamma(q)), \quad p \in \mathbb{N}_3,\ q \in \mathbb{N}_1$$

converges in the topology of $C(Q)$ to some function $f \in C(Q)$.

In what follows, we take $K > 1$ sufficiently large. For such fixed $K > 1$ and $p \in \mathbb{N}_3, q \in \mathbb{N}_1$ we can consider the mapping

$$\mathcal{F}(N_p, \gamma(q)) \ni f = (f_n)_{n=0}^{\infty} \mapsto (I(p,q)f)(\cdot) := \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot)\rangle \in C(Q). \tag{4}$$

Suppose that for $p = 3,\ q = 1$ *the mapping (4) is injective.* Then it is obvious that the mapping $I(p,q) : \mathcal{F}(N_p, \gamma(q)) \to C(Q)$ is injective for any $p \in \mathbb{N}_3, q \in \mathbb{N}_1$.

Applying the mapping $I(p,q)$ we can define the family $(H(p,q))_{p\in\mathbb{N}_3, q\in\mathbb{N}_1}$ of Hilbert spaces

$$\begin{aligned} H(p,q) &:= I(p,q)(\mathcal{F}(N_p, \gamma(q))) \\ &= \{f \in C(Q) \,|\, \exists (f_n)_{n=0}^{\infty} \in \mathcal{F}(N_p, \gamma(q)) :\ f(x) = \sum_{n=0}^{\infty} \langle f_n, h_n(x)\rangle,\ x \in Q\} \end{aligned}$$

with the Hilbert norm

$$\|f\|_{H(p,q)} = \|\sum_{n=0}^{\infty} \langle f_n, h_n(\cdot)\rangle\|_{H(p,q)} := \|(f_n)_{n=0}^{\infty}\|_{\mathcal{F}(N_p,\gamma(q))}.$$

Remark. We note that the spaces $H(p,q)$ are the test functions spaces in a generalization of the white noise analysis (see [11] for more details).

3. Annihilation operators

An annihilation operator $a_-(\xi_m)$ with a coefficient $\xi_m \in \mathcal{F}_m(N_{-p}),\ m \in \mathbb{N}_0$, is defined in the Fock space $\mathcal{F}(N_p, \gamma(q)),\ p \in \mathbb{N}_3, q \in \mathbb{N}_1$ as linear continuous operator acting by the rule (see [11]): for any $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}(N_p, \gamma(q))$

$$a_-(\xi_m)f = a_-(\xi_m)(f_0, f_1, \dots) := (m! f_m^{\xi_m}, \dots, \frac{n!}{(n-m)!} f_n^{\xi_m}, \dots) \in \mathcal{F}(N_p, \gamma(q)),$$

where $f_n^{\xi_m} \in \mathcal{F}_{n-m}(N_p),\ n \geq m$ is defined by

$$\langle f_n, \xi_m \hat{\otimes} \eta_{n-m}\rangle = \langle f_n^{\xi_m}, \eta_{n-m}\rangle$$

for all $\eta_{n-m} \in \mathcal{F}_{n-m}(N_{-p})$.

Using the unitary operator

$$\mathcal{F}(N_p,\gamma(q)) \ni f=(f_n)_{n=0}^{\infty} \mapsto (I(p,q)f)(\cdot)=\sum_{n=0}^{\infty}\langle f_n,h_n(\cdot)\rangle \in H(p,q)$$

we transfer the annihilation operator $a_-(\xi_m)$ into the operator

$$\partial(\xi_m):=I(p,q)a_-(\xi_m)I^{-1}(p,q):H(p,q)\to H(p,q).$$

A simple calculation gives its action on elementary functions $\langle f_n,h_n(\cdot)\rangle \in H(p,q)$, $n\in\mathbb{N}_0$: for all $m\in\mathbb{N}_0$ and $x\in Q$

$$(\partial(\xi_m)\langle f_n,h_n(\cdot)\rangle)(x):=\begin{cases}\frac{n!}{(n-m)!}\langle f_n,\xi_m\hat{\otimes}h_{n-m}(x)\rangle & n\in\mathbb{N}_m;\\ 0 & n=0,\dots,m-1.\end{cases}\tag{5}$$

Let $\ell:N_{1,\mathbb{C}}\to\mathbb{C}^1$ be an analytic function at $0\in N_{1,\mathbb{C}}$. Then in some neighborhood of $0\in N_{2,\mathbb{C}}$ there exists an expansion

$$\ell(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\langle\lambda^{\otimes n},\alpha_n\rangle,\quad \alpha_n\in\mathcal{F}_n(N_{-2}).$$

In accordance with [17] the function ℓ generates a linear continuous operator (the so-called annihilation operator of infinite order)

$$H(p,q)\ni f\mapsto \ell(\partial)f:=\sum_{n=0}^{\infty}\frac{1}{n!}\partial(\alpha_n)f\in H(p,q),\quad p,q\in\mathbb{N}_3.$$

Thus, *the function $h(x,\lambda)$ generates a family $h(\partial)=(h_x(\partial))_{x\in Q}$ of linear continuous operators*

$$H(p,q)\ni f\mapsto h_x(\partial)f:=\sum_{n=0}^{\infty}\frac{1}{n!}\partial(h_n(x))f\in H(p,q),\quad p,q\in\mathbb{N}_3.$$

3. Generalized translation operators

Let a family $T=(T_x)_{x\in Q}$ of linear operators $T_x:C(Q)\to C(Q)$ be given. Such a family T is, by definition (see [8,9,11]), a family of generalized translation operators if

(a) $(T_xf)(y)=(T_yf)(x)$ for any $f\in C(Q)$ and $x,y\in Q$ (commutativity);
(b) there exists a point $e\in Q$ (basis unity) such that $T_e=id$;
(c) for any $x,y\in Q$ the mapping $C(Q)\ni f\mapsto (T_xf)(y)\in\mathbb{C}^1$ is continuous (continuity).

Note, that axioms (a)–(c) are only some part of axioms for generalized translation operators from theory of commutative hypercomplex systems and hypergroups, see [10].

Because the embedding $H(3,3)\hookrightarrow C(Q)$ is continuous (see [11], Theorem 4.1), we can generalize the definition of T. In what follows, *we will call $T=(T_x)_{x\in Q}$ a family of generalized translation operators* if the operators T_x act from the space $H(3,3)$ into $C(Q)$ and the following axioms are satisfied:

(a′) $(T_xf)(y)=(T_yf)(x)$ for any $f\in H(3,3)$ and $x,y\in Q$ (commutativity);
(b′) there exists a point $e\in Q$ (basis unity) such that $T_e=id$;
(c′) for any $x,y\in Q$ the mapping $H(3,3)\ni f\mapsto (T_xf)(y)\in\mathbb{C}^1$ is continuous (continuity).

We say that a non-zero function $\chi \in H(3,3))$ *is a character of the family* T if

$$(T_x\chi)(y) = \chi(x)\chi(y), \quad x, y \in Q.$$

Without loss of generality one can consider that

$$h(o, \lambda) = 1,$$

for some point $o \in Q$ and all $\lambda \in V := \{\lambda \in N_{3,\mathbb{C}} \mid \|\lambda\|_{N_{3,\mathbb{C}}} < r,\ r > 0\}$ (here $r > 0$ sufficiently small). In what follows, we fixed a such point $o \in Q$.

Theorem. *The family* $h(\partial) = (h_x(\partial))_{x\in Q}$ *of linear continuous operators*

$$h_x(\partial) := \sum_{n=0}^{\infty} \frac{1}{n!}\partial(h_n(x)) : H(3,3) \to C(Q)$$

is a family of generalized translation operators. For each fixed $\lambda \in V$ *the function* $Q \ni x \mapsto h(x, \lambda) \in \mathbb{C}^1$ *is a character of the family* $h(\partial)$.

If $h(\cdot, \lambda)$ *is a character of some family* $T = (T_x)_{x\in Q}$ *of generalized translation operators for all* $\lambda \in V$, *then*

$$T_x = h_x(\partial) : H(3,3) \to C(Q),$$

for all $x \in Q$.

Proof. Axioms (a′), (b′), (c′) are fulfilled for $h(\partial)$.

Indeed, since $h(o, \lambda) = 1$ for $\lambda \in V$ we conclude that $h_o(\partial) = id$ and axiom (b′) is fulfilled. The embedding operator $O : H(3,3) \hookrightarrow C(Q)$ and operator $h_x(\partial) : H(3,3) \to H(3,3)$ are continuous. Therefore, the operator $h_x(\partial) : H(3,3) \to C(Q)$ is continuous and axiom (c′) is also fulfilled. The axiom (a′) follows from (6) (see below) and axiom (c′).

We have to prove that $h(\cdot, \lambda)$ is a character of family $(h_x(\partial))_{x\in Q}$ for all $\lambda \in V$.

Due to (5), the action of the operator $h_x(\partial)$ on $\langle f_n, h_n(\cdot)\rangle \in H(3,3)$, $n \in \mathbb{N}_0$ is given by

$$\begin{aligned}(h_x(\partial)\langle f_n, h_n(\cdot)\rangle)(y) &= \sum_{m=0}^{\infty} \frac{1}{m!}(\partial(h_m(x))\langle f_n, h_n(\cdot)\rangle)(y) \\ &= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}\langle f_n, h_m(x)\hat{\otimes} h_{n-m}(y)\rangle \\ &= \langle f_n, \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} h_m(x)\hat{\otimes} h_{n-m}(y)\rangle,\end{aligned} \tag{6}$$

for all $x, y \in Q$.

The series $h(\cdot, \lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle \lambda^{\otimes n}, h_n(\cdot)\rangle$, $\lambda \in V$ converges in the topology of $H(3,3)$ ([11], Proposition 4.1) and operator $h_x(\partial) : H(3,3) \to C(Q)$ is continuous, therefore, for any $x, y \in Q$, by (6)

$$
\begin{aligned}
(h_x(\partial)h(\cdot,\lambda))(y) &= \Big(h_x(\partial)\Big(\sum_{n=0}^{\infty}\frac{1}{n!}\langle\lambda^{\otimes n}, h_n(\cdot)\rangle\Big)\Big)(y) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(h_x(\partial)\langle\lambda^{\otimes n}, h_n(\cdot)\rangle)(y) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\langle\lambda^{\otimes n}, \sum_{m=0}^{n}\frac{n!}{m!(n-m)!}h_m(x)\hat{\otimes}h_{n-m}(y)\rangle \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\langle\lambda^{\otimes m}, h_m(x)\rangle\langle\lambda^{\otimes(n-m)}, h_{n-m}(y)\rangle \\
&= \Big(\sum_{n=0}^{\infty}\frac{1}{n!}\langle\lambda^{\otimes n}, h_n(x)\rangle\Big)\Big(\sum_{n=0}^{\infty}\frac{1}{n!}\langle\lambda^{\otimes n}, h_n(y)\rangle\Big) = h(x,\lambda)h(y,\lambda).
\end{aligned}
$$

Now we prove that if the function $h(\cdot,\lambda)$ is a character of some family $T = (T_x)_{x\in Q}$ of generalized translation operators for all $\lambda \in V$, that

$$T_x = h_x(\partial) : H(3,3) \to C(Q).$$

for all $x \in Q$.

The mappings

$$H(3,3) \ni f \to (h_x(\partial)f)(y) \in \mathbb{C}^1, \quad H(3,3) \ni f \to (T_x f)(y) \in \mathbb{C}^1 \tag{7}$$

are linear and continuous for all $x, y \in Q$. Therefore, it is enough to show that

$$
\begin{aligned}
(T_x\langle f_n, h_n(\cdot)\rangle)(y) &= (h_x(\partial)\langle f_n, h_n(\cdot)\rangle)(y) \\
&= \langle f_n, \sum_{m=0}^{n}\frac{n!}{m!(n-m)!}h_m(x)\hat{\otimes}h_{n-m}(y)\rangle,
\end{aligned}
$$

for any $\langle f_n, h_n(\cdot)\rangle \in H(3,3)$, $n \in \mathbb{N}_0$ and all $x, y \in Q$.

Fix $x, y \in Q$. It follows from the continuoity of the second mapping in (7) that there exists a constant $c > 0$ such that

$$|(T_x f)(y)| \le c\|f\|_{H(3,3)}, \quad f \in H(3,3).$$

Therefore, for $f(\cdot) = \langle f_n, h_n(\cdot)\rangle \in H(3,3)$, $n \in \mathbb{N}_0$ we have

$$|(T_x\langle f_n, h_n(\cdot)\rangle)(y)| \le c\|\langle f_n, h_n(\cdot)\rangle\|_{H(3,3)} = c\|f_n\|_{\mathcal{F}_n(N_3)}.$$

From this estimate we conclude that there exists a unique vector

$$k_n(x,y) \in \mathcal{F}_n(N_{-3})$$

such that

$$(T_x\langle f_n, h_n(\cdot)\rangle)(y) = \langle f_n, k_n(x,y)\rangle,$$

for all $f_n \in \mathcal{F}_n(N_3)$.

Now it is sufficient to prove that

$$k_n(x,y) = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} h_m(x)\hat{\otimes} h_{n-m}(y). \tag{8}$$

The series $h(\cdot,\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle \lambda^{\otimes n}, h_n(\cdot)\rangle$ converges in the topology of $H(3,3)$ for all $\lambda \in V$ and second mapping in (7) is linear and continuous, therefore, for all $x, y \in Q$ and $\lambda \in V$ we have

$$(T_x h(\cdot,\lambda))(y) = \sum_{n=0}^{\infty} \frac{1}{n!}(T_x\langle \lambda^{\otimes n}, h_n(\cdot)\rangle)(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle \lambda^{\otimes n}, k_n(x,y)\rangle. \tag{9}$$

On the other hand, according to (3), for all $x, y \in Q$ and $\lambda \in V$ we have

$$\begin{aligned} (T_x h(\cdot\,\lambda))(y) = h(x,\lambda)h(y,\lambda) &= \sum_{n,m=0}^{\infty} \frac{1}{n!m!}\langle \lambda^{\otimes n}, h_n(x)\rangle\langle \lambda^{\otimes m}, h_m(y)\rangle \\ &= \sum_{n,m=0}^{\infty} \frac{1}{n!m!}\langle \lambda^{\otimes (n+m)}, h_n(x)\hat{\otimes} h_m(y)\rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\langle \lambda^{\otimes n}, \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} h_m(x)\hat{\otimes} h_{n-m}(y)\rangle. \end{aligned} \tag{10}$$

Let $z \in \mathbb{C}^1$ be sufficiently small and $\varphi \in N_{3,\mathbb{C}}$, $\|\varphi\|_{N_{3,\mathbb{C}}} = 1$. By substituting $\lambda = z\varphi$ in (9), (10) and comparing the coefficients before z^n, we get for $x, y \in Q$

$$\langle \varphi^{\otimes n}, k_n(x,y)\rangle = \langle \varphi^{\otimes n}, \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} h_m(x)\hat{\otimes} h_{n-m}(y)\rangle.$$

The last equality, polarization identity and linearity with respect to $\varphi^{\otimes n}$ give (8). □

Remark. It is not difficult to prove that, for all $x, y, z \in Q$ and $f \in H(3,3)$, the following relation of associativity holds:

$$(h_z^y(\partial)(h_y(\partial)f))(x) = (h_y^x(\partial)(h_z(\partial)f))(x),$$

where the notation $(h_z^y(\partial)(h_y(\partial)f))(x)$ means that the operator $h_z(\partial)$ acts on the function $(h_y(\partial)f)(x)$ depending on two variables y and x with respect to the variable y.

Remark. Let $T = (T_x)_{x\in Q}$ be a family of generalized translation operators. If $h(\cdot,\lambda)$, $\lambda \in V$ is a character of the family T, then for each $p, q \in \mathbb{N}_3$ the Hilbert space $H(p,q)$ is invariant with respect to the action of the operator T_x. Moreover, the following equality of operators holds:

$$T_x = h_x(\partial) : H(p,q) \to H(p,q).$$

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