Ukrainian Mathematical Journal, Vol. 56, No. 7, 2004

SPACES APPEARING IN THE CONSTRUCTION OF INFINITE-DIMENSIONAL ANALYSIS ACCORDING TO THE BIORTHOGONAL SCHEME

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We study properties of annihilation operators of infinite order that act in spaces of test functions. The results obtained are used for establishing the coincidence of spaces of test functions.

1. Introduction

The biorthogonal approach to the construction of the theory of generalized functions of an infinite-dimensional variable x with special spaces of test functions and coupling given by integration with respect to a certain probability measure $d\rho(x)$ was proposed in the early 1990s and has been extensively developed since then (see [1–20]).

Within the framework of this approach, one constructs spaces of test functions using a certain system of functions. The scheme of the construction of these spaces can most simply be explained for a model one-dimensional case.

Let $h(x, \lambda)$ be a given function that, for every $x \in Q$ (Q is a separable metric space) admits the expansion

$$h(x, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} h_n(x), \qquad B_h = \{\lambda \in \mathbb{C}^1 \mid |x| < R_h\}, \tag{0.1}$$

with coefficients $h_n(x) \in \mathbb{C}^1$. In other words, for every $x \in Q$, the function $\mathbb{C}^1 \ni \lambda \mapsto h(x, \lambda) \in \mathbb{C}^1$ is analytic in a certain neighborhood of $0 \in \mathbb{C}^1$.

For fixed $q \in \mathbb{N}$ and $K > 1$, using the coefficients $h_n(x)$ of this function as an orthogonal basis, we construct the Hilbert space

$$H^h(q) := \left\{ f(x) = \sum_{n=0}^{\infty} f_n h_n(x),\ f_n \in \mathbb{C}^1 \,\middle|\, \|f_n\|^2_{H^h(q)} = \sum_{n=0}^{\infty} |f_n|^2 (n!)^2 K^{qn} < \infty \right\} \tag{0.2}$$

with the corresponding scalar product.

Let ρ be a Borel probability measure on Q. Under certain assumptions imposed on the function $h(x, \lambda)$ and measure ρ (see [12]), the space $H^h(q)$ (for sufficiently large $K > 1$) can be interpreted as positive with respect to the zero space $(L^2) := L^2(Q, d\rho(x))$. This enables us to construct the following rigging of the space (L^2) by the positive and negative spaces $H^h(q)$ and $H^h(-q)$:

$$H^h(-q) \supset (L^2) \supset H^h(q); \tag{0.3}$$

as a result, we construct the theory of test and generalized functions of the variable $x \in Q$.

Institute of Mathematics, Ukrainian Academy of Sciences, Kyiv. Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 56, No. 7, pp. 977–990, July, 2004. Original article submitted May 20, 2003.

 0041–5995/04/5607–1166

Parallel with the space $H^h(q)$, it is reasonable to consider the space $H^\kappa(q)$ constructed according to rule (0.2) on the basis of the coefficients $\kappa_n(x)$ of the function

$$\kappa(x, \lambda) = \ell(\lambda) h(x, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \kappa_n(x), \qquad \lambda \in B_\kappa \subset B_h,$$

where $\ell(\lambda)$ is a function analytic in the neighborhood of $0 \in \mathbb{C}^1$ and such that $\ell(0) \neq 0$.

It was established in [12] (Theorem 5.1) that the imbeddings

$$H^\kappa(q) \hookrightarrow H^h(q-1), \qquad H^h(q) \hookrightarrow H^\kappa(q-1), \tag{0.4}$$

where $q \in \mathbb{N}_3 := \{3, 4, \ldots\}$ and the constant $K > 1$ is sufficiently large and common for the spaces $H^h(q)$ and $H^\kappa(q)$, are dense and continuous.

By virtue of relations (0.4) and (0.3), the space $H^\kappa(q)$ can also be interpreted as positive with respect to the zero space (L^2). As a result, one can construct a rigging of the space (L^2) by taking either $H^h(q)$ or $H^\kappa(q)$ as a positive space.

In the present paper, we study certain properties of annihilation operators of infinite order that act in spaces of test functions. Using properties of these operators, we prove the main result, according to which, *for sufficiently large* $K > 1$, $H^h(q)$ *and* $H^\kappa(q)$, $q \in \mathbb{N}_3$, *coincide as topological spaces.*

The lemma below is important for the proof of the coincidence of the topological spaces $H^h(q)$ and $H^\kappa(q)$.

Lemma 0.1. *Let a linear set* $\mathcal{P}$ *be dense in Banach spaces* E_1 *and* E_2 *with norms* $\|\cdot\|_{E_1}$ *and* $\|\cdot\|_{E_2}$, *respectively. Assume that linear operators* A *and* B *are defined on the set* $\mathcal{P}$ *and are such that, as the operators*

$$E_2 \supset \mathcal{P} \ni \varphi \mapsto A\varphi \in E_2, \qquad E_1 \supset \mathcal{P} \ni \varphi \mapsto B\varphi \in E_1,$$

they are continuous, and as the operators

$$E_1 \supset \mathcal{P} \ni \varphi \mapsto A\varphi \in E_2, \qquad E_2 \supset \mathcal{P} \ni \varphi \mapsto B\varphi \in E_1,$$

they are isometric.

Then the operator $\mathrm{U}: E_1 \to E_2$ *that is the extension of the operator*

$$E_1 \supset \mathcal{P} \ni \varphi \mapsto U\varphi = \varphi \in \mathcal{P} \subset E_2 \tag{0.5}$$

by continuity realizes a topological isomorphism between E_1 *and* E_2.

Proof. To prove the lemma, it suffices to establish the equivalence of the norms $\|\cdot\|_{E_1}$ and $\|\cdot\|_{E_2}$ on the set $\mathcal{P}$, or, more precisely, to verify that there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|\varphi\|_{E_1} \le \|\varphi\|_{E_2} \le c_2\|\varphi\|_{E_1}, \quad \varphi \in \mathcal{P}. \tag{0.6}$$

Indeed, if estimate (0.6) is true, then the operator U given by (0.5) is defined and continuous. Further, let $f \in E_1$ and let $(\varphi_n)_{n=0}^{\infty}$, $\varphi_n \in \mathcal{P}$, be an arbitrary sequence that converges to f in E_1. Then $\varphi_n \to \mathrm{U}f$ as $n \to \infty$ in the topology of the space E_2. By virtue of (0.6), for each $\varphi_n \in \mathcal{P}$, $n \in \mathbb{N}_0$, the following estimate is true:

$$c_1\|\varphi_n\|_{E_1} \le \|\varphi_n\|_{E_2} \le c_2\|\varphi_n\|_{E_1}. \tag{0.7}$$

Passing to the limit as $n \to \infty$ in (0.7), we get

$$c_1\|f\|_{E_1} \le \|\mathrm{U}f\|_{E_2} \le c_2\|f\|_{E_1}, \quad f \in E_1. \tag{0.8}$$

By virtue of (0.8), the range of values $\mathrm{Ran}\,(\mathrm{U})$ of the operator U is closed in the topology of the space E_2. Since $\mathcal{P} = \mathrm{Ran}\,(U) \subset \mathrm{Ran}\,(\mathrm{U})$ and the set $\mathcal{P}$ is dense in E_2, we have $\mathrm{Ran}\,(\mathrm{U}) = E_2$. Taking the last relation and inequalities (0.8) into account, we conclude that the operator $\mathrm{U}: E_1 \to E_2$ is a one-to-one mutually continuous mapping between the spaces E_1 and E_2, i.e., it realizes a topological isomorphism between these spaces.

Let us verify the existence of estimate (0.6). Since the operator A is continuous as an operator in $E_2(\mathrm{Dom}\,(A) = \mathcal{P})$, there exists a constant $c_3 > 0$ such that

$$\|A\varphi\|_{E_2} \le c_3\|\varphi\|_{E_2}, \quad \varphi \in \mathcal{P}. \tag{0.9}$$

By analogy, there exists a constant $c_4 > 0$ such that

$$\|B\varphi\|_{E_1} \le c_4\|\varphi\|_{E_1}, \quad \varphi \in \mathcal{P}. \tag{0.10}$$

In addition, by virtue of the isometry of the operators A and B, we have

$$\|A\varphi\|_{E_2} = \|\varphi\|_{E_1}, \quad \|B\varphi\|_{E_1} = \|\varphi\|_{E_2}, \quad \varphi \in \mathcal{P}. \tag{0.11}$$

Using relation (0.9) and the first equality in (0.11), we get

$$\|\varphi\|_{E_1} \le c_3\|\varphi\|_{E_2}, \quad \varphi \in \mathcal{P}. \tag{0.12}$$

By analogy, using relation (0.10) and the second equality in (0.11), we obtain

$$\|\varphi\|_{E_2} \le c_4\|\varphi\|_{E_1}, \quad \varphi \in \mathcal{P}.$$

Taking into account the last relation and (0.12), we easily obtain (0.6).

1. General Facts about Basis Functions and Spaces Associated with Them

Consider the fixed chain

$$\mathcal{N}' := \operatorname*{ind\,lim}_{p\in\mathbb{N}} N_{-p} \supset \ldots \supset N_{-p} \supset \ldots \supset N_0 \supset \ldots \supset N_p \supset \ldots \supset \operatorname*{pr\,lim}_{p\in\mathbb{N}} N_p =: \mathcal{N} \tag{1.1}$$

of real separable Hilbert spaces with coupling $(\cdot,\cdot)_{N_0} =: \langle\cdot,\cdot\rangle$ with respect to the zero space N_0, where $N_{-p} := (N_p)'$. *Assume that* $\|\cdot\|_{N_0} \le \|\cdot\|_{N_1} \le \ldots$ *and the imbedding* $N_{p+1} \hookrightarrow N_p$, $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ *is quasinuclear* (i.e., a Hilbert–Schmidt imbedding (the Hilbert–Schmidt norm is denoted by $\|\cdot\|_{HS}$).

Complexifying the spaces of chain (1.1), i.e., passing from N_p, $\mathcal{N}$ to $N_{p,\mathbb{C}}$, $\mathcal{N}_{\mathbb{C}}$ and taking their symmetric tensors of power $\hat{\otimes}$ (see [21]), for each $n \in \mathbb{N}_0$, we construct the nuclear chain

$$\left(\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}\right)' := \operatorname*{ind\,lim}_{p\in\mathbb{N}} N_{-p,\mathbb{C}}^{\hat{\otimes}n} \supset N_{-p,\mathbb{C}}^{\hat{\otimes}n} \supset N_{0,\mathbb{C}}^{\hat{\otimes}n} \supset N_{p,\mathbb{C}}^{\hat{\otimes}n} \supset \operatorname*{pr\,lim}_{p\in\mathbb{N}} N_{p,\mathbb{C}}^{\hat{\otimes}n} =: \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}$$

(for $n = 0$ all spaces coincide with $\mathbb{C}^1$). The complex coupling $(\cdot,\cdot)_{N_{p,\mathbb{C}}^{\hat{\otimes}n}}$ coincides with the real coupling $(\cdot,\bar{\cdot})_{N_p^{\hat{\otimes}n}} =: \langle\cdot,\bar{\cdot}\rangle$.

Let $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ denote the space of germs of functions $\phi : \mathcal{N}_{\mathbb{C}} \to \mathbb{C}^1$ analytic in $0 \in \mathcal{N}_{\mathbb{C}}$. We equip it with inductive topology given by a family of norms

$$\|\phi\|_{p,l} := \sup_{\|\lambda\|_{N_{p,\mathbb{C}}} \le K^{-l}} |\phi(\lambda)|, \qquad p, l \in \mathbb{N},$$

for fixed $K > 1$. It is clear that $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ is a commutative algebra with respect to ordinary multiplication of functions.

According to [12] (Theorem 2.1) and [20], the following equality of topological spaces is true:

$$\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}) = \operatorname*{ind\,lim}_{p,q\in\mathbb{N}} \mathrm{Hol}(N_{p,\mathbb{C}}, q),$$

where $\mathrm{Hol}(N_{p,\mathbb{C}}, q)$, $p, q \in \mathbb{N}$, is the Hilbert space of functions analytic in $0 \in N_{p,\mathbb{C}}$:

$$\mathrm{Hol}(N_{p,\mathbb{C}}, q) = \left\{ \phi(\lambda) = \sum_{n=0}^{\infty} \left\langle \lambda^{\otimes n}, \xi_n \right\rangle,\ \xi_n \in N_{-p,\mathbb{C}}^{\hat{\otimes}n},\ \lambda \in B_p(K^{-q/2}) \,\middle|\, \|\phi\|^2_{\mathrm{Hol}(N_{p,\mathbb{C}},q)} = \sum_{n=0}^{\infty} \|\xi_n\|^2_{N_{-p,\mathbb{C}}^{\hat{\otimes}n}} K^{-qn} < \infty \right\}, \tag{1.2}$$

$$B_p(K^{-q/2}) = \left\{ \lambda \in N_{p,\mathbb{C}} \,\middle|\, \|\lambda\|_{N_{p,\mathbb{C}}} < K^{-q/2} \right\}.$$

Let Q be a separable metric space of points $x, y, \ldots$ and let $C(Q)$ be the linear space of all complex-valued locally bounded (i.e., bounded on every ball in Q) continuous functions on Q. We assume that $C(Q)$ is a topological space with uniform convergence on every ball in Q.

Let U_0 be a certain neighborhood of zero in the space $N_{1,\mathbb{C}}$ and let $Q \times U_0 \ni \{x, \lambda\} \mapsto h(x, \lambda) \in \mathbb{C}^1$ be a given function. *We assume that, for every x, this function is a function of λ analytic in $0 \in N_{1,\mathbb{C}}$, for every $\lambda \in U_0$ we have $h(\cdot, \lambda) \in C(Q)$, and, furthermore, this function is locally uniformly bounded with respect to λ from an arbitrary closed ball in U_0.* We fix a function h possessing the properties indicated.

Since the function $h(x, \lambda)$ is analytic in λ, for every $x \in Q$ it can expanded in the series (see Sec. 3 in [12])

$$h(x, \lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, h_n(x) \rangle, \qquad \lambda \in B_h = \{ \lambda \in N_{2,\mathbb{C}} \mid \|\lambda\|_{N_{2,\mathbb{C}}} < R_h \}, \tag{1.3}$$

which uniformly converges (in λ) in every closed ball from B_h. *We assume that there exists a neighborhood B_h, common for all $x \in Q$, in which expansion (1.3) is true.*

Note that, for every $x \in Q$, the coefficients (basis functions) $h_n(x)$ belong to $N_{-2,\mathbb{C}}^{\hat{\otimes} n}$ and the elementary functions $Q \ni x \mapsto \langle f_n, h_n(x) \rangle \in \mathbb{C}^1$ $(f_n \in N_{p,\mathbb{C}}^{\hat{\otimes} n}, n \in \mathbb{N}_0, p \in \mathbb{N}_3 := \{3, 4, \ldots\})$ are continuous and locally bounded (see [12]).

For fixed $p \in \mathbb{N}_3$, $q \in \mathbb{N}$, and $K > 1$, we consider the Hilbert space of formal series

$$H^h(p, q) := \left\{ f(x) = \sum_{n=0}^{\infty} \langle f_n, h_n(x) \rangle,\ f_n \in N_{p,\mathbb{C}}^{\hat{\otimes} n},\ x \in Q \,\middle|\, \|f\|_{H^h(p,q)}^2 = \sum_{n=0}^{\infty} \|f_n\|_{N_{p,\mathbb{C}}^{\hat{\otimes} n}}^2 (n!)^2 K^{qn} < \infty \right\} \tag{1.4}$$

with the corresponding scalar product.

Using Lemma 4.2 from [12], one can easily verify that, for sufficiently large $K > 1$, the first series in (1.4) converges uniformly on each ball from Q to a continuous locally bounded function (we fix K for which this is true). At the same time, different elements of the space $H^h(p, q)$ may coincide as functions of $x \in Q$, i.e., $\sum_{n=0}^{\infty} \langle f_n, h_n(\cdot) \rangle \neq \sum_{n=0}^{\infty} \langle g_n, h_n(\cdot) \rangle$ in $H^h(p, q)$, but $\sum_{n=0}^{\infty} \langle f_n, h_n(x) \rangle = \sum_{n=0}^{\infty} \langle g_n, h_n(x) \rangle$ for any $x \in Q$. To eliminate this difficulty, *we assume that the system of basis functions $(h_n(x))_{n=0}^{\infty}$ is minimal in the sense that if the series $\sum_{n=0}^{\infty} \langle f_n, h_n(x) \rangle = f(x)$, $f_n \in N_{p,\mathbb{C}}^{\hat{\otimes} n}$, converges in $H^h(p, q)$, $p \in \mathbb{N}_3$, $q \in \mathbb{N}$, and its sum is equal to 0 for all $x \in Q$, then $f_n = 0$, $n \in \mathbb{N}_0$. In other words, if $f \in H^h(p, q)$ and $f(x) = 0$ $\forall x \in Q$, then $f = 0$ in the topology of the space $H^h(p, q)$.*

Remark 1.1. If, under this minimality assumption, a function $f \in C(Q)$ admits the representation

$$f(x) = \sum_{n=0}^{\infty} \langle f_n, h_n(x) \rangle, \quad x \in Q, \qquad \sum_{n=0}^{\infty} \|f_n\|_{N_{p,\mathbb{C}}^{\hat{\otimes} n}}^2 (n!)^2 K^{qn} < \infty, \quad p \in \mathbb{N}_3, \quad q \in \mathbb{N}, \tag{1.5}$$

then this representation is unique. Therefore, it is quite natural to interpret the Hilbert space $H^h(p, q)$, $p \in \mathbb{N}_3$,

$q \in \mathbb{N}$, as the space of continuous locally bounded functions f on Q that admit representation (1.5) with the corresponding Hilbert norm $\|\cdot\|_{H^h(p,q)}$ given on this space, i.e.,

$$H^h(p,q) = \left\{ f \in C(Q) \,\middle|\, f(x) = \sum_{n=0}^{\infty} \langle f_n, h_n(x) \rangle,\ \|f\|^2_{H^h(p,q)} = \sum_{n=0}^{\infty} \|f_n\|^2_{N_{p,\mathbb{C}}^{\hat{\otimes} n}} (n!)^2 K^{qn} < \infty \right\}. \tag{1.6}$$

Remark 1.2. It is clear that the set

$$\mathcal{P}(Q) := \left\{ \varphi \in C(Q) \,\middle|\, \varphi(x) = \sum_{n=0}^{s} \langle \varphi_n, h_n(x) \rangle,\ x \in Q, \varphi_n \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, s \in \mathbb{N}_0 \right\} \tag{1.7}$$

is dense in the Hilbert space $H^h(p,q)$, $p \in \mathbb{N}_3$, $q \in \mathbb{N}$.

Remark 1.3. The space $H^h(p,q)$, $p \in \mathbb{N}_3$, $q \in \mathbb{N}$, is unitarily isomorphic to the weighted Fock space $\mathcal{F}(N_p, \gamma(q))$, $\gamma(q) = ((n!)^2 K^{qn})_{n=0}^{\infty}$, of sequences

$$\begin{gathered} f = (f_n)_{n=0}^{\infty}, \quad f_n \in \mathcal{F}_n(N_p) := N_{p,\mathbb{C}}^{\hat{\otimes} n}, \\ \|f\|^2_{\mathcal{F}(N_p,\gamma(q))} = \sum_{n=0}^{\infty} \|f_n\|^2_{N_{p,\mathbb{C}}^{\hat{\otimes} n}} (n!)^2 K^{qn} < \infty. \end{gathered} \tag{1.8}$$

This isomorphism is given by the mapping

$$\mathcal{F}(N_p, \gamma(q)) \ni f = (f_n)_{n=0}^{\infty} \mapsto (I^h f)(\cdot) = \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot) \rangle \in H^h(p,q). \tag{1.9}$$

Parallel with the space $H^h(p,q)$, using the same rule (1.4) we construct a space on the basis of the coefficients determined from an expansion of the type (1.3) with a modified left-hand side closely related to $h(x,\lambda)$.

Let $\ell : N_{1,\mathbb{C}} \to \mathbb{C}^1$ *be a function analytic in* $0 \in N_{1,\mathbb{C}}$ *and let* $\ell(0) \neq 0$ (in what follows, we fix this function). Then, for every $x \in Q$, the function $\kappa(x,\lambda) := \ell(\lambda) h(x,\lambda)$ is analytic in $0 \in N_{1,\mathbb{C}}$ and admits a representation of the form (1.3):

$$\begin{gathered} \kappa(x,\lambda) = \ell(\lambda) h(x,\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, \kappa_n(x) \rangle, \\ \lambda \in B_\kappa = \{ \lambda \in N_{2,\mathbb{C}} \mid \|\lambda\|_{N_{2,\mathbb{C}}} < R_\kappa \}. \end{gathered} \tag{1.10}$$

For every $x \in Q$, the coefficients (basis functions associated with $\kappa(x,\lambda)$) $\kappa_n(x) \in N_{-2,\mathbb{C}}^{\hat{\otimes} n}$ and elementary functions $Q \ni x \mapsto \langle f_n, \kappa_n(x) \rangle \in \mathbb{C}^1$ ($f_n \in N_{p,\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$, $p \in \mathbb{N}_3$) are continuous and locally bounded.

For fixed $p \in \mathbb{N}_3$, $q \in \mathbb{N}$, and $K > 1$, we construct the Hilbert space of formal series

$$H^{\kappa}(p,q) := \left\{ f(x) = \sum_{n=0}^{\infty} \langle f_n, \kappa_n(x) \rangle,\ f_n \in N_{p,\mathbb{C}}^{\hat{\otimes} n},\ x \in Q \,\middle|\, \|f\|^2_{H^{\kappa}(p,q)} = \sum_{n=0}^{\infty} \|f_n\|^2_{N_{p,\mathbb{C}}^{\hat{\otimes} n}} (n!)^2 K^{qn} < \infty \right\}. \quad (1.11)$$

It is easy to see that, for sufficiently large $K > 1$, the first series in (1.11) converges uniformly on every ball from Q to a continuous locally bounded function. Furthermore, by virtue of Lemma 5.3 in [12], the system of basis functions $(\kappa_n(x))_{n=0}^{\infty}$ is minimal in the sense that *if the series*

$$\sum_{n=0}^{\infty} \langle f_n, \kappa_n(x) \rangle = f(x), \qquad f_n \in N_{p,\mathbb{C}}^{\hat{\otimes} n},$$

converges in $H^{\kappa}(p,q)$, $p, q \in \mathbb{N}_3$, and its sum is equal to 0 for all $x \in Q$, then $f_n = 0$, $n \in \mathbb{N}_0$. In other words, if $f \in H^{\kappa}(p,q)$ and $f(x) = 0$ $\forall x \in Q$, then $f = 0$ in the topology of the space $H^{\kappa}(p,q)$. For this reason, it is quite natural to interpret the space $H^{\kappa}(p,q)$, $p, q \in \mathbb{N}_3$, as follows:

$$H^{\kappa}(p,q) = \left\{ f \in C(Q) \,\middle|\, f(x) = \sum_{n=0}^{\infty} \langle f_n, \kappa_n(x) \rangle,\ \|f\|^2_{H^{\kappa}(p,q)} = \sum_{n=0}^{\infty} \|f_n\|^2_{N_{p,\mathbb{C}}^{\hat{\otimes} n}} (n!)^2 K^{qn} < \infty \right\}. \quad (1.12)$$

In what follows, the constant $K > 1$ is chosen to be sufficiently large and common for the spaces $H^h(p,q)$ and $H^{\kappa}(p,q)$.

Remark 1.4. The space $H^{\kappa}(p,q)$, $p, q \in \mathbb{N}_3$, is unitarily isomorphic to the weighted Fock space $\mathcal{F}(N_p, \gamma(q))$ defined by (1.8) with weight $\gamma(q) = ((n!)^2 K^{qn})_{n=0}^{\infty}$. This isomorphism is given by the mapping

$$\mathcal{F}(N_p, \gamma(q)) \ni f = (f_n)_{n=0}^{\infty} \mapsto (I^{\kappa} f)(\cdot) = \sum_{n=0}^{\infty} \langle f_n, \kappa_n(\cdot) \rangle \in H^{\kappa}(p,q). \quad (1.13)$$

Using the fact that the functions $h(x,\lambda)$ and $\kappa(x,\lambda)$ are closely related, one can find formulas for the representation of the basis functions $h_n(x)$ and $\kappa_n(x)$ in terms of one another. Namely, since the function $\ell(\lambda)$ is analytic and $\ell(0) \neq 0$, the function $1/\ell(\lambda)$ is also analytic in $0 \in N_{1,\mathbb{C}}$. Therefore, these functions can be represented in the form

$$\ell(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, \alpha_n \rangle, \qquad \frac{1}{\ell(\lambda)} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \lambda^{\otimes n}, \beta_n \rangle,$$
$$\alpha_n, \beta_n \in N_{-2,\mathbb{C}}^{\hat{\otimes} n}, \quad n \in \mathbb{N}_0, \qquad \lambda \in B_{\ell} = \{ \lambda \in N_{2,\mathbb{C}} \mid \|\lambda\|_{N_{2,\mathbb{C}}} < R_{\ell} \}. \quad (1.14)$$

Expansion (1.3) [respectively, (1.10)] is the product of (1.10) [respectively, (1.3)] and the second (respectively, the first) expansion in (1.14). Equating the coefficients, we obtain (see Sec. 5 in [12])

$$\kappa_n(x) = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}\alpha_{n-m}\hat{\otimes} h_m(x),$$

$$h_n(x) = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}\beta_{n-m}\hat{\otimes}\kappa_m(x), \quad x \in Q, \quad n \in \mathbb{N}_0. \tag{1.15}$$

Recall the following fact [see, e.g., equality (5.15) in [12]): if $\xi_k \in N_{-p,\mathbb{C}}^{\hat{\otimes}k}$ and $f_m \in N_{p,\mathbb{C}}^{\hat{\otimes}m}$, $m \geq k$, then there exists $f_m^{\xi_k} \in N_{p,\mathbb{C}}^{\hat{\otimes}(m-k)}$, $p \in \mathbb{N}_0$, such that

$$\langle f_m, \xi_k \hat{\otimes} \eta_{m-k} \rangle = \langle f_m^{\xi_k}, \eta_{m-k} \rangle, \quad \eta_{m-k} \in N_{-p,\mathbb{C}}^{\hat{\otimes}(m-k)}. \tag{1.16}$$

Using relations (1.15) and (1.16), we can give another description of the set $\mathcal{P}(Q)$.

Lemma 1.1 [12, Remark 5.3]**.** *For any function* $\varphi \in \mathcal{P}(Q)$, *there exists a single-valued representation*

$$\varphi(x) = \sum_{n=0}^{s} \langle \varphi_n, \kappa_n(x) \rangle, \quad \varphi_n \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}, \quad x \in Q, \quad s \in \mathbb{N}_0, \tag{1.17}$$

and, conversely, an arbitrary function of the form (1.17) belongs to $\mathcal{P}(Q)$.

Remark 1.5. It is obvious that the set $\mathcal{P}(Q)$ is dense in the Hilbert space $H^{\kappa}(p, q)$, $p, q \in \mathbb{N}_3$.

Let us prove a lemma necessary for what follows.

Lemma 1.2. *For sufficiently large* $K > 1$, *the functions* ℓ *and* $1/\ell$ *defined by (1.14) belong to the Hilbert space* $\mathrm{Hol}(N_{3,\mathbb{C}}, 1)$.

Proof. Let us show that $\ell \in \mathrm{Hol}(N_{3,\mathbb{C}}, 1)$ (the fact that the function $1/\ell$ belongs to $\mathrm{Hol}(N_{3,\mathbb{C}}, 1)$ is proved by analogy). Using the first expansion in (1.14), we obtain the estimate [see estimate (5.7) in [12]]

$$\|\alpha_n\|_{N_{-3,\mathbb{C}}^{\hat{\otimes}n}} \leq \frac{n! e^n \|O_{3,2}\|_{HS}^n}{r^n} \sup_{\|\lambda\|_{N_{2,\mathbb{C}}} = r} |\ell(\lambda)|, \tag{1.18}$$

$$\lambda \in N_{3,\mathbb{C}}, \quad n \in \mathbb{N}_0, \quad r \in (0, R_\ell),$$

where $\|O_{3,2}\|_{HS}$ is the Hilbert–Schmidt norm of the imbedding operator $O_{3,2} : N_3 \to N_2$.

Further, using the definition (1.2) of the space $\mathrm{Hol}(N_{3,\mathbb{C}}, 1)$ and relations (1.14) and (1.18), we get

$$\|\ell\|^2_{\mathrm{Hol}(N_{3,\mathbb{C}},1)} = \sum_{n=0}^{\infty} \left\| \frac{1}{n!}\alpha_n \right\|^2_{N_{-3,\mathbb{C}}^{\hat{\otimes}n}} K^{-n} \leq \left(\sup_{\|\lambda\|_{N_{2,\mathbb{C}}=r}} |\ell(\lambda)| \right)^2 \sum_{n=0}^{\infty} \frac{e^{2n} \|O_{3,2}\|_{HS}^{2n}}{r^{2n} K^n}. \tag{1.19}$$

We fix $r \in (0, R_\ell)$ and choose $K > 1$ so large that $e^2 \|O_{3,2}\|_{HS}^2 r^{-2} K^{-1}$ is smaller than 1. Then, using (1.19), we obtain

$$\|\ell\|^2_{\mathrm{Hol}(N_{3,\mathbb{C}},1)} \le c\left(\sup_{\|\lambda\|_{N_{2,\mathbb{C}}}=r} |\ell(\lambda)|\right)^2 < \infty,$$

whence $\ell \in \mathrm{Hol}(N_{3,\mathbb{C}}, 1)$.

Note that, in order that Lemma 1.2 be true, K must satisfy the condition

$$K > \max\left\{1,\ e^2 R_\ell^{-2} \|O_{3,2}\|^2_{HS}\right\}$$

(in the proof of Lemma 1.2, one must take $r = R_\ell - \varepsilon$ with a sufficiently small fixed $\varepsilon > 0$).

2. Annihilation Operators of Infinite Order

In this section, we introduce annihilation operators of infinite order (by analogy with [16, 17]), establish several properties of these operators, and show that $H^h(p,q)$ and $H^\kappa(p,q)$ coincide as topological spaces.

The spaces $H^h(p,q)$ and $H^\kappa(p,q)$ are unitarily isomorphic to the weighted Fock space $\mathcal{F}(N_p, \gamma(q))$ (Remarks 1.3 and 1.4), on which the annihilation operator is defined. For this reason, it is quite natural to define the annihilation operator as the image of the annihilation operator that acts in the Fock space $\mathcal{F}(N_p, \gamma(q))$.

It is known that the annihilation operator $a_-(\xi_m)$ with coefficient $\xi_m \in N_{-p,\mathbb{C}}^{\hat{\otimes} m}$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, is defined in the Fock space $\mathcal{F}(N_{p'}, \gamma(q))$, $p' \ge p$, $q \in \mathbb{N}$, as the linear continuous operator that acts in every n-particle subspace $\mathcal{F}_n(N_{p'})$ according to the rule

$$\mathcal{F}_n(N_{p'}) \ni f_n \mapsto a_-(\xi_m) f_n = \begin{cases} n\ldots(n-m+1) f_n^{\xi_m} \in \mathcal{F}_{n-m}(N_{p'}) & \text{if} \quad n \in \{m, m+1, \ldots\}, \\ a_-(\xi_m) f_n = 0 & \text{if} \quad n = 0, \ldots, m-1, \end{cases}$$

where $f_n^{\xi_m}$ is determined from (1.16).

In the space $H^h(p', q)$, $p' \ge p \in \mathbb{N}_3$, $q \in \mathbb{N}$, we introduce an annihilation operator $\partial_h(\xi_m)$ with coefficient $\xi_m \in N_{-p,\mathbb{C}}^{\hat{\otimes} m}$, $m \in \mathbb{N}_0$, by setting

$$\partial_h(\xi_m) := I^h a_-(\xi_m)(I^h)^{-1}. \tag{2.1}$$

By analogy, in the space $H^\kappa(p', q)$, $p' \ge p \in \mathbb{N}_3$, $q \in \mathbb{N}_3$, we introduce the annihilation operator

$$\partial_\kappa(\xi_m) := I^\kappa a_-(\xi_m)(I^\kappa)^{-1} \tag{2.2}$$

[recall that the mappings I^h and I^κ are defined by (1.9) and (1.13), respectively].

It is clear that the operator $\partial_h(\xi_m)$, $\xi_m \in N_{-p,\mathbb{C}}^{\hat{\otimes}m}$, $p \in \mathbb{N}_3$, acts continuously in every space $H^h(p', q)$, and the operator $\partial_\kappa(\xi_m)$ acts continuously in every space $H^\kappa(p', q)$, $p' \geq p$, $q \in \mathbb{N}_3$.

It was established in [12] (Lemma 12.2) that

$$\partial_h(\xi_m) \upharpoonright \mathcal{P}(Q) = \partial_\kappa(\xi_m) \upharpoonright \mathcal{P}(Q) =: \partial(\xi_m) \tag{2.3}$$

and the operator $\partial(\xi_m)$, $\xi_m \in N_{-p,\mathbb{C}}^{\hat{\otimes}m}$, $m \in \mathbb{N}_0$, acts on the elementary functions $\langle \varphi_n, h_n(x) \rangle$, $\varphi_n \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}$, $n \in \mathbb{N}_0$, as follows:

$$\forall x \in Q: \quad \partial(\xi_m)\langle \varphi_n, h_n(x) \rangle = \begin{cases} \dfrac{n!}{(n-m)!} \left\langle \varphi_n, \xi_m \hat{\otimes} h_{n-m}(x) \right\rangle & \text{if} \quad n \geq m, \\ 0 & \text{if} \quad n < m. \end{cases} \tag{2.4}$$

The action of the operator $\partial(\xi_m)$ on the elementary functions $\langle \varphi_n, \kappa_n(x) \rangle$ is analogous [with h replaced by κ in (2.4)].

Let $\phi(\cdot) = \sum_{m=0}^{\infty} \left\langle \cdot^{\otimes m}, \xi_m \right\rangle \in \mathrm{Hol}(N_{p,\mathbb{C}}, q)$, $p \in \mathbb{N}_3$, $q \in \mathbb{N}$. On the set $\mathcal{P}(Q)$, we define an annihilation operator of infinite order by setting

$$\phi(\partial) := \sum_{m=0}^{\infty} \partial(\xi_m). \tag{2.5}$$

It is clear that this definition is correct (because, for $\varphi \in \mathcal{P}(Q)$, the series $\phi(\partial)\varphi$ has only finitely many terms) and the operator $\phi(\partial)$ is linear.

Lemma 2.1. *Let* $\phi(\cdot) = \sum_{m=0}^{\infty} \left\langle \cdot^{\otimes m}, \xi_m \right\rangle \in \mathrm{Hol}(N_{p,\mathbb{C}}, q-1)$, $p, q \in \mathbb{N}_3$. *Then* $\phi(\partial)$ *is continuous as the operator*

$$H^h(p,q) \supset \mathcal{P}(Q) \ni \varphi \mapsto \phi(\partial), \quad \varphi \in H^h(p,q), \tag{2.6}$$

and as the operator

$$H^\kappa(p,q) \supset \mathcal{P}(Q) \ni \varphi \mapsto \phi(\partial), \quad \varphi \in H^\kappa(p,q). \tag{2.7}$$

Proof. We prove the continuity of operator (2.6) [the continuity of operator (2.7) is proved by analogy]. In view of the fact that the operator $\phi(\partial)$ is linear, it suffices to verify the following estimate:

$$\exists\, c > 0: \quad \| \phi(\partial)\varphi \|_{H^h(p,q)} \leq c \| \phi \|_{\mathrm{Hol}(N_{p,\mathbb{C}}, q-1)} \| \varphi \|_{H^h(p,q)}, \qquad \varphi \in \mathcal{P}(Q). \tag{2.8}$$

By virtue of Lemma 12.1 in [12], the operator $\partial(\xi_m)$, $\xi_m \in N_{-p,\mathbb{C}}^{\hat{\otimes}m}$, is such that

$$\|\partial(\xi_m)\varphi\|_{H^h(p,q)} \le K^{-\frac{qm}{2}}\|\xi_m\|_{N_{-p,\mathbb{C}}^{\hat{\otimes}m}}\|\varphi\|_{H^h(p,q)}, \qquad \varphi\in\mathcal{P}(Q).$$

Using the last estimate and relation (2.5), we obtain

$$\forall\,\varphi\in\mathcal{P}(Q):\ \|\phi(\partial)\varphi\|_{H^h(p,q)} = \left\|\sum_{m=0}^{\infty}\partial(\xi_m)\varphi\right\|_{H^h(p,q)} \le \sum_{m=0}^{\infty}\|\partial(\xi_m)\varphi\|_{H^h(p,q)}$$

$$\le \sum_{m=0}^{\infty}K^{\frac{-qm}{2}}\|\xi_m\|_{N_{-p,\mathbb{C}}^{\hat{\otimes}m}}\|\varphi\|_{H^h(p,q)} \le \|\varphi\|_{H^h(p,q)}\sum_{m=0}^{\infty}K^{\frac{-qm}{2}}\|\xi_m\|_{N_{-p,\mathbb{C}}^{\hat{\otimes}m}}. \quad (2.9)$$

Since $\phi\in\mathrm{Hol}(N_{p,\mathbb{C}},q-1)$, we have

$$\|\phi\|^2_{\mathrm{Hol}(N_{p,\mathbb{C}},q-1)} = \sum_{m=0}^{\infty}\|\xi_m\|^2_{N_{-p,\mathbb{C}}^{\hat{\otimes}m}}K^{-(q-1)m} < \infty.$$

Hence,

$$\|\xi_m\|_{N_{-p,\mathbb{C}}^{\hat{\otimes}m}} \le \|\phi\|_{\mathrm{Hol}(N_{p,\mathbb{C}},q-1)}K^{\frac{(q-1)m}{2}}$$

and, therefore (recall that $K>1$),

$$\sum_{m=0}^{\infty}K^{-\frac{qm}{2}}\|\xi_m\|_{N_{-p,\mathbb{C}}^{\hat{\otimes}m}} \le \|\phi\|_{\mathrm{Hol}(N_{p,\mathbb{C}},q-1)}\sum_{m=0}^{\infty}K^{\frac{(q-1)m}{2}}K^{-\frac{qm}{2}} \le c\|\phi\|_{\mathrm{Hol}(N_{p,\mathbb{C}},q-1)}, \quad c = \frac{\sqrt{K}}{\sqrt{K}-1} < \infty.$$

Taking the last relation into account, we obtain (2.8) from (2.9).

The lemma is proved.

Corollary 2.1. *By virtue of Lemma 1.2, the functions* ℓ *and* $\frac{1}{\ell}$ *in (1.14) belong to the space* $\mathrm{Hol}(N_{3,\mathbb{C}},1)\subset\mathrm{Hol}(N_{p,\mathbb{C}},q-1)$, $p,q\in\mathbb{N}_3$. *Hence,* $\ell(\partial)=\sum_{m=0}^{\infty}\frac{1}{m!}\partial(\alpha_m)$ *and* $\frac{1}{\ell}(\partial) = \sum_{m=0}^{\infty}\frac{1}{m!}\partial(\beta_m)$ *are continuous. as the operators*

$$H^h(p,q)\supset\mathcal{P}(Q)\ni\varphi\mapsto\ell(\partial)\varphi\in H^h(p,q),$$

$$H^h(p,q)\supset\mathcal{P}(Q)\ni\varphi\mapsto\frac{1}{\ell}(\partial)\varphi\in H^h(p,q),$$

$$H^\kappa(p,q)\supset\mathcal{P}(Q)\ni\varphi\mapsto\ell(\partial)\varphi\in H^\kappa(p,q),$$

$$H^\kappa(p,q)\supset\mathcal{P}(Q)\ni\varphi\mapsto\frac{1}{\ell}(\partial)\varphi\in H^\kappa(p,q).$$

Remark 2.1. By analogy with the proof of Lemma 2.1, we can show that the operators

$$\phi(\partial_h) := \sum_{m=0}^{\infty} \partial_h(\xi_m): H^h(p', q') \to H^h(p', q'),$$
$$\phi(\partial_\kappa) := \sum_{m=0}^{\infty} \partial_\kappa(\xi_m): H^\kappa(p', q') \to H^\kappa(p', q') \tag{2.10}$$

are continuous for $\phi(\cdot) = \sum_{m=0}^{\infty} \langle \cdot^{\otimes m}, \xi_m \rangle \in \mathrm{Hol}(N_{p,\mathbb{C}}, q-1)$, $p, q \in \mathbb{N}_3$, $p' \geq p$, $q' \geq q$. Moreover, it is easy to verify that

$$\phi(\partial_h) \restriction \mathcal{P}(Q) = \phi(\partial_\kappa) \restriction \mathcal{P}(Q) = \phi(\partial).$$

Let us determine the action of the operators $\ell(\partial)$ and $\frac{1}{\ell}(\partial)$ on functions from the set $\mathcal{P}(Q)$.

Lemma 2.2. *The operator* $\ell(\partial)\colon \mathcal{P}(Q) \to \mathcal{P}(Q)$ *maps the entire set* $\mathcal{P}(Q)$ *onto the entire set* $\mathcal{P}(Q)$ (*i.e.,* $\mathrm{Ran}(\ell(\partial)) = \mathcal{P}(Q)$) *and acts on functions* $\varphi \in \mathcal{P}(Q)$ *so that if* φ *is represented in the form*

$$\varphi(x) = \sum_{n=0}^{s} \langle \varphi_n, h_n(x) \rangle, \tag{2.11}$$

then

$$\ell(\partial)\varphi(x) = \sum_{n=0}^{s} \langle \varphi_n, \kappa_n(x) \rangle, \qquad x \in Q, \quad \varphi_n \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \quad s \in \mathbb{N}_0. \tag{2.12}$$

As a result, the operator $\ell(\partial)$ *is isometric as the operator*

$$H^h(p,q) \supset \mathcal{P}(Q) \ni \varphi \mapsto \ell(\partial)\varphi \in H^\kappa(p,q), \quad p, q \in \mathbb{N}_3. \tag{2.13}$$

Proof. Assume that a function $\varphi \in \mathcal{P}(Q)$ is represented in the form (2.11). Using (2.4) and (2.5) [(1.14) for $\phi = \ell$] and the first relation in (1.15), we get

$$\ell(\partial)\varphi(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \partial(\alpha_m) \sum_{n=0}^{s} \langle \varphi_n, h_n(x) \rangle = \sum_{m=0}^{s} \frac{1}{m!} \sum_{n=m}^{s} \frac{n!}{(n-m)!} \left\langle \varphi_n, h_{n-m}(x) \hat{\otimes} \alpha_m \right\rangle$$
$$= \sum_{n=0}^{s} \left\langle \varphi_n, \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} h_{n-m}(x) \hat{\otimes} \alpha_m \right\rangle = \sum_{n=0}^{s} \langle \varphi_n, \kappa_n(x) \rangle.$$

The lemma is proved.

Remark 2.2. By analogy, one can show that the operator $\frac{1}{\ell}(\partial)$ acts on functions from $\mathcal{P}(Q)$ as follows: if a function $\varphi \in \mathcal{P}(Q)$ is represented in the form

$$\varphi(x) = \sum_{n=0}^{s} \langle \varphi_n, \kappa_n(x) \rangle,$$

then

$$\frac{1}{\ell}(\partial)\varphi(x) = \sum_{n=0}^{s} \langle \varphi_n, h_n(x) \rangle, \qquad x \in Q, \quad \varphi_n \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \quad s \in \mathbb{N}_0.$$

It is clear that $\operatorname{Ran}\left(\frac{1}{\ell}(\partial)\right) = \mathcal{P}(Q)$ and the operator $\frac{1}{\ell}(\partial)$ is isometric as the operator

$$H^{\kappa}(p,q) \supset \mathcal{P}(Q) \ni \varphi \mapsto \frac{1}{\ell}(\partial)\varphi \in H^{h}(p,q), \quad p, q \in \mathbb{N}_3.$$

Corollary 2.2. *The operator* $\ell(\partial)\colon \mathcal{P}(Q) \to \mathcal{P}(Q)$ *is invertible, and, furthermore,* $(\ell(\partial))^{-1} = \frac{1}{\ell}(\partial)$.

We can now pass to the proof of the main result of the paper.

Theorem 2.1. *For* $K > \max\left\{1, e^2 R^{-2} \| O_{3,2} \|_{HS}^2\right\}$ $(R := \min\{R_h, R_\kappa, R_l\})$, *the following equality of topological spaces is true:*

$$H^{h}(p,q) = H^{\kappa}(p,q) =: H(p,q), \quad p, q \in \mathbb{N}_3. \tag{2.14}$$

Proof. By virtue of the minimality assumption, the spaces $H^h(p,q)$ and $H^\kappa(p,q)$ (for the given choice of K) are spaces of continuous locally bounded functions on Q. Therefore, to prove the theorem, it suffices to show that the operator that associates every function $f \in H^\kappa(p,q)$ with the same function f regarded as an element of the space $H^h(p,q)$ is defined and that it realizes a topological isomorphism between $H^\kappa(p,q)$ and $H^h(p,q)$.

We fix $p, q \in \mathbb{N}_3$. It follows from Corollary 2.1, Lemma 2.2, Remark 2.2, and Lemma 0.1, where $E_1 = H^\kappa(p,q)$, $E_2 = H^h(p,q)$, $\mathcal{P} = \mathcal{P}(Q)$, $A = \frac{1}{\ell}(\partial)$, and $B = \ell(\partial)$, that the operator $\mathrm{U}\colon H^\kappa(p,q) \to H^h(p,q)$ that is the extension of the operator

$$H^{\kappa}(p,q) \supset \mathcal{P}(Q) \ni \varphi \mapsto U\varphi = \varphi \in \mathcal{P}(Q) \subset H^{h}(p,q)$$

by continuity realizes a topological isomorphism between $H^\kappa(p,q)$ and $H^h(p,q)$.

Furthermore, this operator is such that

$$\forall\, f \in H^{\kappa}(p,q)\colon \quad f(x) = (\mathrm{U} f)(x), \quad x \in Q. \tag{2.15}$$

Indeed, by virtue of Lemma 4.2 in [12], for every ball $V \subset Q$, there exist constants $c_3 = c_3(V) > 0$ and $c_4 = c_4(V) > 0$ such that, for any $x \in V$, the following relations are true:

$$
\begin{aligned}
|f(x)| &\leq c_3 \| f \|_{H^h(p,q)}, \quad f \in H^h(p,q), \\
|g(x)| &\leq c_4 \| g \|_{H^\kappa(p,q)}, \quad g \in H^\kappa(p,q).
\end{aligned}
\tag{2.16}
$$

Let $f \in H^\kappa(p,q)$ and let $\mathcal{P}(Q) \ni \varphi_n \to f$ in $H^\kappa(p,q)$. Then $\mathcal{P}(Q) \ni \varphi_n \to \mathrm{U}f$ in $H^h(p,q)$. Using (2.16), for any $x \in V \subset Q$ we get

$$
\begin{aligned}
|f(x) - (\mathrm{U}f)(x)| &= |f(x) - \varphi_n(x) + \varphi_n(x) - (\mathrm{U}f)(x)| \leq |f(x) - \varphi_n(x)| + |\varphi_n(x) - (\mathrm{U}f)(x)| \\
&\leq c_4 \| f - \varphi_n \|_{H^\kappa(p,q)} + c_3 \| \varphi_n - \mathrm{U}f \|_{H^h(p,q)} \to 0
\end{aligned}
\tag{2.17}
$$

as $n \to \infty$. Since relation (2.17) is true for x from an arbitrary ball $V \subset Q$, we have $f(x) = (\mathrm{U}f)(x) \quad \forall\, x \in Q$.
The theorem is proved.

Remark 2.3. As a set, the space $H(p,q) := H^h(p,q) = H^\kappa(p,q)$ consists of functions continuous and locally bounded on Q that can be expanded in series both in the basis functions $h_n(x)$ and in the basis functions $\kappa_n(x)$. More precisely, if a function f belongs to the set $H(p,q)$, then it can be uniquely represented both in the form

$$
f(x) = \sum_{n=0}^{\infty} \langle f_n, h_n(x) \rangle, \quad x \in Q, \quad \sum_{n=0}^{\infty} \| f_n \|^2_{N_{p,\mathbb{C}}^{\hat{\otimes} n}} (n!)^2 K^{qn} < \infty
$$

and in the form

$$
f(x) = \sum_{n=0}^{\infty} \left\langle \tilde{f}_n, \kappa_n(x) \right\rangle, \quad x \in Q, \quad \sum_{n=0}^{\infty} \left\| \tilde{f}_n \right\|^2_{N_{p,\mathbb{C}}^{\hat{\otimes} n}} (n!)^2 K^{qn} < \infty.
$$

Moreover, the following estimate is true: There exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$
c_1 \| f \|_{H^h(p,q)} \leq \| f \|_{H^\kappa(p,q)} \leq c_2 \| f \|_{H^h(p,q)}, \qquad f \in H(p,q).
$$

This estimate guarantees the topological equality of the spaces $H^h(p,q)$ and $H^\kappa(p,q)$.

Taking into account that the operator $\partial_h(\xi_m)$, $\xi_m \in N_{-p,\mathbb{C}}^{\hat{\otimes} m}$, $p \in \mathbb{N}_3$, acts continuously in every space $H^h(p',q)$, the operator $\partial_\kappa(\xi_m)$ acts continuously in every space $H^\kappa(p',q)$, $p' \geq p$, $q \in \mathbb{N}_3$, and equality (2.3) is true and using Theorem 2.1, we establish the following corollary:

Corollary 2.3. *The operators* $\partial_h(\xi_m)$ *(2.1) and* $\partial_\kappa(\xi_m)$ *(2.2)* ($\xi_m \in N_{-p,\mathbb{C}}^{\hat{\otimes} m}$, $p \in \mathbb{N}_3$, $m \in \mathbb{N}_0$) *act continuously in every topological space* $H(p',q)$, $p' \geq p$, $q \in \mathbb{N}_3$, *and*

$$
\partial_h(\xi_m) = \partial_\kappa(\xi_m) =: \boldsymbol{\partial}(\xi_m).
$$

Theorem 2.1 and Remark 2.1 yield the following corollary:

Corollary 2.4. *Suppose that* $\phi(\cdot) = \sum_{m=0}^{\infty} \langle \cdot^{\otimes m}, \xi_m \rangle \in \mathrm{Hol}(N_{p,\mathbb{C}}, q-1)$, $p, q \in \mathbb{N}_3$. *The operators* $\phi(\partial_h)$ *and* $\phi(\partial_\kappa)$ *defined by (2.10) act continuously in every topological space* $H(p', q')$, $p' \geq p$, $q' \geq q$, *and coincide:*

$$\phi(\partial_h) = \phi(\partial_\kappa) = \phi(\partial) := \sum_{m=0}^{\infty} \partial(\xi_m). \tag{2.18}$$

Lemmas 1.2 and 2.1, Remark 2.2, and Corollary 2.4 yield the following corollary:

Corollary 2.5. *The operator* $\ell(\partial) = \sum_{m=0}^{\infty} \frac{1}{m!} \boldsymbol{\partial}(\alpha_m)\colon H^h(p,q) \to H^\kappa(p,q)$, $p, q \in \mathbb{N}_3$, *is unitary and acts as follows:*

$$H^h(p,q) \ni f(\cdot) = \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot) \rangle \mapsto \ell(\partial) f(\cdot) = \sum_{n=0}^{\infty} \langle f_n, \kappa_n(\cdot) \rangle \in H^\kappa(p,q). \tag{2.19}$$

The operator inverse to it is the unitary operator $\frac{1}{\ell}(\partial) = \sum_{m=0}^{\infty} \frac{1}{m!} \boldsymbol{\partial}(\beta_m)$ *that acts from* $H^\kappa(p,q)$ *into* $H^h(p,q)$ *as follows:*

$$H^\kappa(p,q) \ni f(\cdot) = \sum_{n=0}^{\infty} \langle f_n, \kappa_n(\cdot) \rangle \mapsto \frac{1}{\ell}(\partial) f(\cdot) = \sum_{n=0}^{\infty} \langle f_n, h_n(\cdot) \rangle \in H^h(p,q).$$

Remark 2.4. For a certain fixed Borel measure ρ on Q (for details, see [12]), the spaces $H^h(p,q)$ and $H^\kappa(p,q)$ can be interpreted as positive with respect to the zero space $L^2(Q, d\rho(x))$. It is clear that the coincidence of the positive spaces $H^h(p,q)$ and $H^\kappa(p,q)$ (see Theorem 2.1) leads to the coincidence of the corresponding negative spaces $H^h(-p,-q)$ and $H^\kappa(-p,-q)$, $p, q \in \mathbb{N}_3$.

The author is grateful to Yu. M. Berezans'kyi and M. O. Kachanovs'kyi for helpful discussions and remarks.

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