Methods of Functional Analysis and Topology
Vol. 14 (2008), no. 2, pp. 132–141

A STOCHASTIC INTEGRAL OF OPERATOR-VALUED FUNCTIONS

VOLODYMYR TESKO

To Professor M. L. Gorbachuk on the occasion of his 70th birthday.

Abstract. In this note we define and study a Hilbert space-valued stochastic integral of operator-valued functions with respect to Hilbert space-valued measures. We show that this integral generalizes the classical Itô stochastic integral of adapted processes with respect to normal martingales and the Itô integral in a Fock space.

1. Introduction

Here and subsequently, we fix a real number $T > 0$. Let $\mathcal{H}$ be a complex Hilbert space, M be a fixed vector from $\mathcal{H}$ and $[0,T] \ni t \mapsto E_t$ be a resolution of identity in $\mathcal{H}$. Consider the $\mathcal{H}$-valued function (*abstract martingale*)

$$[0,T] \ni t \mapsto M_t := E_t M \in \mathcal{H}.$$

In this paper we construct and study an integral

$$\int_{[0,T]} A(t)\, dM_t \tag{1}$$

for a certain class of operator-valued functions $[0,T] \ni t \mapsto A(t)$ whose values are linear operators in the space $\mathcal{H}$. We define such an integral as an element of the Hilbert space $\mathcal{H}$ and call it a *Hilbert space-valued stochastic integral* (or *H-stochastic integral*). By analogy with the classical integration theory we first define integral (1) for a certain class of simple operator-valued functions and then extend this definition to a wider class.

We illustrate our abstract constructions with a few examples. Thus, we show that the classical Itô stochastic integral is a particular case of the H-stochastic integral. Namely, let $\mathcal{H} := L^2(\Omega, \mathcal{A}, P)$ be a space of square integrable functions on a complete probability space $(\Omega, \mathcal{A}, P)$, $\{\mathcal{A}_t\}_{t\in[0,T]}$ be a filtration satisfying the usual conditions and $\{N_t\}_{t\in[0,T]}$ be a normal martingale on $(\Omega, \mathcal{A}, P)$ with respect to $\{\mathcal{A}_t\}_{t\in[0,T]}$, i.e.,

$$\{N_t\}_{t\in[0,T]} \quad \text{and} \quad \{N_t^2 - t\}_{t\in[0,T]}$$

are martingales for $\{\mathcal{A}_t\}_{t\in[0,T]}$. It follows from the properties of martingales that

$$N_t = \mathbb{E}[N_T|\mathcal{A}_t], \quad t \in [0,T],$$

where $\mathbb{E}[\,\cdot\,|\mathcal{A}_t]$ is a conditional expectation with respect to the σ-algebra $\mathcal{A}_t$. It is well known that $\mathbb{E}[\,\cdot\,|\mathcal{A}_t]$ is the orthogonal projector in the space $L^2(\Omega, \mathcal{A}, P)$ onto its subspace $L^2(\Omega, \mathcal{A}_t, P)$ and, moreover, the corresponding projector-valued function $\mathbb{R}_+ \ni t \mapsto E_t := \mathbb{E}[\,\cdot\,|\mathcal{A}_t]$ is a resolution of identity in $L^2(\Omega, \mathcal{A}, P)$, see e.g. [13, 3, 4, 12, 7]. In this way the normal martingale $\{N_t\}_{t\in[0,T]}$ can be interpreted as an abstract martingale, i.e.,

$$[0,T] \ni t \mapsto N_t = \mathbb{E}[N_T|\mathcal{A}_t] = E_t N_T \in \mathcal{H}.$$

Hence, in the space $L^2(\Omega, \mathcal{A}, P)$ we can construct the H-stochastic integral with respect to the normal martingale N_t. Let $F \in L^2([0,T] \times \Omega, dt \times P)$ be a square integrable

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2000 *Mathematics Subject Classification.* Primary 46G99, 47B15, 60H05.

Key words and phrases. Stochastic integral, resolution of identity, normal martingale, Fock space.

stochastic process adapted to the filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$. We consider the operator-valued function $[0,T] \ni t \mapsto A_F(t)$ whose values are operators $A_F(t)$ of multiplication by the function $F(t) = F(t,\cdot) \in L^2(\Omega,\mathcal{A},P)$ in the space $L^2(\Omega,\mathcal{A},P)$,

$$L^2(\Omega,\mathcal{A},P) \supset \mathrm{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega,\mathcal{A},P).$$

In this paper we prove that the H-stochastic integral of $[0,T] \ni t \mapsto A_F(t)$ coincides with the classical Itô stochastic integral $\int_{[0,T]} F(t)\,dN_t$ of F. That is,

$$\int_{[0,T]} A_F(t)\,dN_t = \int_{[0,T]} F(t)\,dN_t.$$

In the last part of this note we show that the Itô integral in a Fock space is the H-stochastic integral and establish a connection of such an integral with the classical Itô stochastic integral. The corresponding results are given without proofs (the proofs will be given in a forthcoming publication). Note that the Itô integral in a Fock space is a useful tool in the quantum stochastic calculus, see e.g. [2] for more details.

We remark that in [3, 4] the authors gave a definition of the operator-valued stochastic integral

$$B := \int_{[0,T]} A(t)\,dE_t$$

for a family $\{A(t)\}_{t\in[0,T]}$ of commuting normal operators in $\mathcal{H}$. Such an integral was defined using a spectral theory of commuting normal operators. It is clear that for a fixed vector $M \in \mathrm{Dom}(B) \subset \mathcal{H}$ the formula

$$\int_{[0,T]} A(t)\,dM_t := \Big(\int_{[0,T]} A(t)\,dE_t\Big)M$$

can be regarded as a definition of integral (1). In this way we obtain another definition of integral (1) different from the one we have proposed in this paper.

2. The construction of the H-stochastic integral

Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{L}(\mathcal{H})$ be a space of all bounded linear operators in $\mathcal{H}$, $M \neq 0$ be a fixed vector from $\mathcal{H}$ and

$$[0,T] \ni t \mapsto E_t \in \mathcal{L}(\mathcal{H})$$

be a resolution of identity in $\mathcal{H}$, that is a right-continuous increasing family of orthogonal projections in $\mathcal{H}$ such that $E_T = 1$. Note that the resolution of identity E can be regarded as a projector-valued measure $\mathcal{B}([0,T]) \ni \alpha \mapsto E(\alpha) \in \mathcal{L}(\mathcal{H})$ on the Borel σ-algebra $\mathcal{B}([0,T])$. Namely, for any interval $(s,t] \subset [0,T]$ we set

$$E((s,t]) := E_t - E_s, \quad E(\{0\}) := E_0, \quad E(\varnothing) := 0,$$

and extend this definition to all Borel subsets of $[0,T]$, see e.g. [6] for more details.

By definition, the $\mathcal{H}$-valued function

$$[0,T] \ni t \mapsto M_t := E_t M \in \mathcal{H}$$

is an *abstract martingale* in the Hilbert space $\mathcal{H}$.

In this section we give a definition of integral (1) for a certain class of operator-valued functions with respect to the abstract martingale M_t. A construction of such an integral is given step-by-step, beginning with the simplest class of operator-valued functions. Let us introduce the required class of simple functions.

For each point $t \in [0,T]$, we denote by

$$\mathcal{H}_M(t) := \mathrm{span}\{M_{s_2} - M_{s_1} \,|\, (s_1,s_2] \subset (t,T]\} \subset \mathcal{H}$$

the linear span of the set $\{M_{s_2} - M_{s_1} \,|\, (s_1,s_2] \subset (t,T]\}$ in $\mathcal{H}$ and by

$$\mathcal{L}_M(t) = \mathcal{L}(\mathcal{H}_M(t) \to \mathcal{H})$$

the set of all linear operators in $\mathcal{H}$ that continuously act from $\mathcal{H}_M(t)$ to $\mathcal{H}$. The increasing family $\mathcal{L}_M = \{\mathcal{L}_M(t)\}_{t\in[0,T]}$ will play here a role of the filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$ in the classical martingale theory.

For a fixed $t \in [0,T)$, a linear operator A in $\mathcal{H}$ will be called $\mathcal{L}_M(t)$-*measurable* if

(i) $A \in \mathcal{L}_M(t)$ and, for all $s \in [t,T)$,

$$\|A\|_{\mathcal{L}_M(t)} = \|A\|_{\mathcal{L}_M(s)} := \sup\Big\{ \frac{\|Ag\|_{\mathcal{H}}}{\|g\|_{\mathcal{H}}} \,\Big|\, g \in \mathcal{H}_M(s),\, g \neq 0\Big\}.$$

(ii) A is partially commuting with the resolution of identity E. More precisely,

$$AE_s g = E_s A g, \quad g \in \mathcal{H}_M(t), \quad s \in [t,T].$$

Such a definition of $\mathcal{L}_M(t)$-measurability is motivated by a number of reasons:

- $\mathcal{L}_M(t)$-measurability is a natural generalization of the usual $\mathcal{A}_t$-measurability in classical stochastic calculus, see Lemma 1 (Section 3) for more details;
- in some sense, $\mathcal{L}_M(t)$-measurability (for each t) is the minimal restriction on the behavior of a simple operator-valued function $[0,T] \ni t \mapsto A(t)$ that will allow us to obtain an analogue of the Itô isometry property (see inequality (4) below) and to extend the H-stochastic integral from a simple class of functions to a wider one.

In what follows, it is convenient for us to call $\mathcal{L}_M(T)$-measurable all linear operators in $\mathcal{H}$. Evidently, if a linear operator A in $\mathcal{H}$ is $\mathcal{L}_M(t)$-measurable for some $t \in [0,T]$ then A is $\mathcal{L}_M(s)$-measurable for all $s \in [t,T]$.

A family $\{A(t)\}_{t\in[0,T]}$ of linear operators in $\mathcal{H}$ will be called *a simple* $\mathcal{L}_M$*-adapted operator-valued function on* $[0,T]$ if, for each $t \in [0,T]$, the operator $A(t)$ is $\mathcal{L}_M(t)$-measurable and there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that

$$A(t) = \sum_{k=0}^{n-1} A_k \varkappa_{(t_k,t_{k+1}]}(t), \quad t \in [0,T], \tag{2}$$

where $\varkappa_\alpha(\cdot)$ is the characteristic function of the Borel set $\alpha \in \mathcal{B}([0,T])$.

Let $S = S(M)$ denote the space of all simple $\mathcal{L}_M$-adapted operator-valued functions on $[0,T]$. For a function $A \in S$ with representation (2) we define an H*-stochastic integral* of A with respect to the abstract martingale M_t through the formula

$$\int_{[0,T]} A(t)\, dM_t := \sum_{k=0}^{n-1} A_k (M_{t_{k+1}} - M_{t_k}) \in \mathcal{H}. \tag{3}$$

We can show that this definition does not depend on the choice of representation of the simple function A in the space S.

In the space S we introduce a quasinorm by setting

$$\|A\|_{S_2} := \Big(\int_{[0,T]} \|A(t)\|^2_{\mathcal{L}_M(t)}\, d\mu(t)\Big)^{\frac{1}{2}} := \Big(\sum_{k=0}^{n-1} \|A_k\|^2_{\mathcal{L}_M(t_k)} \mu((t_k,t_{k+1}])\Big)^{\frac{1}{2}}$$

for each $A \in S$ with representation (2). Here the measure μ is defined by the formula

$$\mathcal{B}([0,T]) \ni \alpha \mapsto \mu(\alpha) := \|M(\alpha)\|^2_{\mathcal{H}} = (E(\alpha)M, M)_{\mathcal{H}} \in \mathbb{R}_+,$$

where $M(\alpha) := E(\alpha)M$ for all $\alpha \in \mathcal{B}([0,T])$, in particular,

$$M((t_k,t_{k+1}]) := E((t_k,t_{k+1}])M = M_{t_{k+1}} - M_{t_k}, \quad (t_k,t_{k+1}] \subset [0,T].$$

The following statement is fundamental.

Theorem 1. *Let $A, B \in S$ and $a, b \in \mathbb{C}$. Then*

$$\int_{[0,T]} \Big(aA(t) + bB(t)\Big)\, dM_t = a\int_{[0,T]} A(t)\, dM_t + b\int_{[0,T]} B(t)\, dM_t$$

and

$$\Big\| \int_{[0,T]} A(t)\, dM_t \Big\|_{\mathcal{H}}^2 \le \int_{[0,T]} \|A(t)\|^2_{\mathcal{L}_{M(t)}}\, d\mu(t). \tag{4}$$

Proof. The first assertion is trivial.

Let us check inequality (4). Using (i), (ii) and properties of the resolution of identity E, for $A \in S$ with representation (2), we obtain

$$\begin{aligned}
\Big\| \int_{[0,T]} A(t)\, dM_t \Big\|_{\mathcal{H}}^2 &= \Big(\int_{[0,T]} A(t)\, dM_t, \int_{[0,T]} A(t)\, dM_t \Big)_{\mathcal{H}} \\
&= \sum_{k,m=0}^{n-1} \big(A_k M(\Delta_k), A_m M(\Delta_m) \big)_{\mathcal{H}} \\
&= \sum_{k,m=0}^{n-1} \big(A_k E(\Delta_k) M, A_m E(\Delta_m) M \big)_{\mathcal{H}} \\
&= \sum_{k,m=0}^{n-1} \big(E(\Delta_k) A_k E(\Delta_k) M, E(\Delta_m) A_m E(\Delta_m) M \big)_{\mathcal{H}} \\
&= \sum_{k=0}^{n-1} \big(A_k E(\Delta_k) M, A_k E(\Delta_k) M \big)_{\mathcal{H}} = \sum_{k=0}^{n-1} \|A_k M(\Delta_k)\|^2_{\mathcal{H}} \\
&\le \sum_{k=0}^{n-1} \|A_k\|^2_{\mathcal{L}_{M(t_k)}} \|M(\Delta_k)\|^2_{\mathcal{H}} = \sum_{k=0}^{n-1} \|A_k\|^2_{\mathcal{L}_{M(t_k)}} \mu(\Delta_k) \\
&= \int_{[0,T]} \|A(t)\|^2_{\mathcal{L}_{M(t)}}\, d\mu(t),
\end{aligned}$$

where $\Delta_k := (t_k, t_{k+1}]$ for all $k \in \{0, \dots, n-1\}$. □

Inequality (4) enables us to extend the H-stochastic integral to operator-valued functions $[0,T] \ni t \mapsto A(t)$ which are not necessarily simple. Namely, denote by $S_2 = S_2(M)$ a Banach space associated with the quasinorm $\|\cdot\|_{S_2}$. For its construction, it is first necessary to pass from S to the factor space

$$\dot{S} := S / \{A \in S \mid \|A\|_{S_2} = 0\}$$

and then to take the completion of $\dot{S}$. It is not difficult to see that elements of the space S_2 are equivalence classes of operator-valued functions on $[0,T]$ whose values are linear operators in the space $\mathcal{H}$.

An operator-valued function $[0,T] \ni t \mapsto A(t)$ will be called *H-stochastic integrable* with respect to M_t if A belongs to the space S_2. It follows from the definition of the space S_2 that for each $A \in S_2$ there exists a sequence $(A_n)_{n=0}^{\infty}$ of simple operator-valued functions $A_n \in S$ such that

$$\int_{[0,T]} \|A(t) - A_n(t)\|^2_{\mathcal{L}_{M(t)}}\, d\mu(t) \to 0 \quad \text{as} \quad n \to \infty. \tag{5}$$

Due to (4), for such a sequence $(A_n)_{n=0}^{\infty}$, the limit

$$\lim_{n\to\infty} \int_{[0,T]} A_n(t)\, dM_t$$

exists in $\mathcal{H}$ and does not dependent on the choice of the sequence $(A_n)_{n=0}^{\infty} \subset S$ satisfying (5). We denote this limit by

$$\int_{[0,T]} A(t)\, dM_t := \lim_{n\to\infty} \int_{[0,T]} A_n(t)\, dM_t$$

and call it the *H-stochastic integral of* $A \in S_2$ with respect to the abstract martingale M_t. It is clear that for all $A \in S_2$ the assertions of Theorem 1 still hold.

Note one simple property of the integral introduced above. Let U be some unitary operator acting from $\mathcal{H}$ onto another complex Hilbert space $\mathcal{K}$. Then

$$[0,T] \ni t \mapsto G_t := UM_t \in \mathcal{K}$$

is an abstract martingale in the space $\mathcal{K}$ because, for any $t \in [0,T]$,

$$G_t = UM_t = X_t G, \quad X_t := UE_t U^{-1}, \quad G := UM \in \mathcal{K},$$

and X_t is a resolution of identity in the space $\mathcal{K}$.

Let an operator-valued function $[0,T] \ni t \mapsto A(t)$ be H-stochastic integrable with respect to M_t. One can show that the operator-valued function $[0,T] \ni t \mapsto UA(t)U^{-1}$ is H-stochastic integrable with respect to G_t and

$$U\Big(\int_{[0,T]} A(t)\, dM_t\Big) = \int_{[0,T]} UA(t)U^{-1}\, dG_t \in \mathcal{K}.$$

3. The Itô stochastic integral as an H-stochastic integral

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\{\mathcal{A}_t\}_{t\in[0,T]}$ be a right continuous filtration. Suppose that the σ-algebra $\mathcal{A}_0$ contains all P-null sets of $\mathcal{A}$ and $\mathcal{A} = \mathcal{A}_T$. Moreover, we assume that $\mathcal{A}_0$ is trivial, i.e., every set $\alpha \in \mathcal{A}_0$ has probability 0 or 1.

Let $N = \{N_t\}_{t\in[0,T]}$ be a *normal martingale* on $(\Omega, \mathcal{A}, P)$ with respect to $\{\mathcal{A}_t\}_{t\in[0,T]}$. That is, $N_t \in L^2(\Omega, \mathcal{A}_t, P)$ for all $t \in [0,T]$ and

$$\mathbb{E}[N_t - N_s|\mathcal{A}_s] = 0, \quad \mathbb{E}[(N_t - N_s)^2|\mathcal{A}_s] = t - s$$

for all $s, t \in [0,T]$ such that $s < t$. Without loss of generality one can assume that $N_0 = 0$. Note that there are many examples of normal martingales, — the Brownian motion, the compensated Poisson process, the Azéma martingales and others, see for instance [10, 8, 12].

We will denote by $L^2_a([0,T]\times\Omega)$ the set of all functions (equivalence classes), adapted to the filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$, from the space

$$L^2([0,T]\times\Omega) := L^2([0,T]\times\Omega, \mathcal{B}([0,T])\times\mathcal{A}, dt\times P)$$

where dt is the Lebesgue measure on $\mathcal{B}([0,T])$.

Let us show that the *Itô stochastic integral* $\int_{[0,T]} F(t)\, dN_t$ of $F \in L^2_a([0,T]\times\Omega)$ with respect to the normal martingale N can be considered as an H-stochastic integral (see e.g. [15, 16] for the definition and properties of the classical Itô integral). To this end, we set $\mathcal{H} := L^2(\Omega, \mathcal{A}, P)$ and consider, in this space, the resolution of identity

$$[0,T] \ni t \mapsto E_t := \mathbb{E}[\,\cdot\,|\mathcal{A}_t] \in \mathcal{L}(\mathcal{H})$$

generated by the filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$. Let $M := N_T \in L^2(\Omega, \mathcal{A}, P)$, then the corresponding abstract martingale

$$[0,T] \ni t \mapsto N_t := E_t N_T = \mathbb{E}[N_T|\mathcal{A}_t] \in \mathcal{H}$$

is our normal martingale. Note also that

$$\mu([0,t]) = \|N([0,t])\|^2_{L^2(\Omega,\mathcal{A},P)} = \|N_t\|^2_{L^2(\Omega,\mathcal{A},P)} = \mathbb{E}[N_t^2] = \mathbb{E}[N_t^2\,|\,\mathcal{A}_0] = t,$$

i.e., μ is the Lebesgue measure on $\mathcal{B}([0,T])$.

In the context of this section, $\mathcal{L}_M(t)$-measurability is equivalent to the usual $\mathcal{A}_t$-measurability. More precisely, the following result holds.

Lemma 1. *Let $t \in [0,T)$. For given $F \in L^2(\Omega, \mathcal{A}, P)$ the operator A_F of multiplication by the function F in the space $L^2(\Omega, \mathcal{A}, P)$ is $\mathcal{L}_N(t)$-measurable if and only if the function F is $\mathcal{A}_t$-measurable, i.e., $F = \mathbb{E}[F|\mathcal{A}_t]$. Moreover, if $F \in L^2(\Omega, \mathcal{A}, P)$ is an $\mathcal{A}_t$-measurable function then*

$$\|A_F\|_{\mathcal{L}_N(t)} = \|A_F\|_{\mathcal{L}_N(s)} = \|F\|_{L^2(\Omega,\mathcal{A},P)}, \quad s \in [t,T). \tag{6}$$

Proof. Suppose $F \in L^2(\Omega, \mathcal{A}, P)$ is an $\mathcal{A}_t$-measurable function. Let us show that the operator A_F is $\mathcal{L}_N(t)$-measurable.

First, we prove that $A_F \in \mathcal{L}_N(t)$. Taking into account that F is an $\mathcal{A}_t$-measurable function, $\{N_t\}_{t\in[0,T]}$ is the normal martingale and the σ-algebra $\mathcal{A}_0$ is trivial, for any interval $(s_1, s_2] \subset (t, T]$, we obtain

$$\begin{aligned}\|A_F(N_{s_2} - N_{s_1})\|^2_{L^2(\Omega,\mathcal{A},P)} &= \|F(N_{s_2} - N_{s_1})\|^2_{L^2(\Omega,\mathcal{A},P)} = \mathbb{E}[F^2(N_{s_2} - N_{s_1})^2] \\ &= \mathbb{E}[F^2(N_{s_2} - N_{s_1})^2|\mathcal{A}_0] = \mathbb{E}\big[F^2\mathbb{E}[(N_{s_2} - N_{s_1})^2|\mathcal{A}_{s_1}]\big|\mathcal{A}_0\big] \\ &= \mathbb{E}\big[F^2\mathbb{E}[(N_{s_2} - N_{s_1})^2|\mathcal{A}_{s_1}]\big] = \mathbb{E}[F^2](s_2 - s_1) \\ &= \mathbb{E}[F^2]\mathbb{E}[(N_{s_2} - N_{s_1})^2] \\ &= \|F\|^2_{L^2(\Omega,\mathcal{A},P)}\|N_{s_2} - N_{s_1}\|^2_{L^2(\Omega,\mathcal{A},P)}.\end{aligned}$$

We can similarly show that

$$\|A_F G\|^2_{L^2(\Omega,\mathcal{A},P)} = \|F\|^2_{L^2(\Omega,\mathcal{A},P)}\|G\|^2_{L^2(\Omega,\mathcal{A},P)}$$

for all $G \in \mathcal{H}_N(t) = \operatorname{span}\{N_{s_2} - N_{s_1} \,|(s_1, s_2] \subset (t, T]\}$. Hence $A_F \in \mathcal{L}_N(t)$ and, moreover, equality (6) takes place.

Let us check that A_F is partially commuting with E, i.e.,

$$A_F E_s G = E_s A_F G, \quad G \in \mathcal{H}_N(t), \quad s \in [t, T].$$

Since $F \in L^2(\Omega, \mathcal{A}, P)$ is an $\mathcal{A}_t$-measurable function and $FG \in L^2(\Omega, \mathcal{A}, P)$, for any $s \in [t, T]$ and any function $G \in \mathcal{H}_N(t)$, we have

$$A_F E_s G = F E_s G = F\mathbb{E}[G|\mathcal{A}_s] = \mathbb{E}[FG|\mathcal{A}_s] = E_s A_F G.$$

Thus, the first part of the lemma is proved.

Let us prove the converse statement of the lemma: if for a given $F \in L^2(\Omega, \mathcal{A}, P)$ the operator A_F is $\mathcal{L}_N(t)$-measurable then F is an $\mathcal{A}_t$-measurable function.

Since A_F is an $\mathcal{L}_N(t)$-measurable operator, we see that for any $s \in [t, T]$

$$A_F E_s G = E_s A_F G, \quad G \in \mathcal{H}_N(t),$$

or, equivalently,

$$A_F\mathbb{E}[G|\mathcal{A}_s] = \mathbb{E}[A_F G|\mathcal{A}_s], \quad G \in \mathcal{H}_N(t). \tag{7}$$

Let $s \in (t, T)$ and $(s_1, s_2] \subset (t, s]$. We take

$$G := N_{s_2} - N_{s_1} \in \mathcal{H}_N(t).$$

Evidently, G is an $\mathcal{A}_s$-measurable function and

$$A_F\mathbb{E}[G|\mathcal{A}_s] = A_F G = FG, \quad \mathbb{E}[A_F G|\mathcal{A}_s] = \mathbb{E}[FG|\mathcal{A}_s] = G\mathbb{E}[F|\mathcal{A}_s].$$

Hence, using (7), we obtain

$$FG = G\mathbb{E}[F|\mathcal{A}_s].$$

As a result,

$$F = \mathbb{E}[F|\mathcal{A}_s], \quad s \in (t, T].$$

Since the resolution of identity $[0, T] \ni s \mapsto E_s = \mathbb{E}[\,\cdot\,|\mathcal{A}_s] \in \mathcal{L}(\mathcal{H})$ is a right-continuous function, the latter equality still holds for $s = t$, and therefore F is an $\mathcal{A}_t$-measurable function. □

As a simple consequence of Lemma 1 we obtain the following result.

Theorem 2. *Let F belong to $L^2([0, T] \times \Omega)$. The family $\{A_F(t)\}_{t\in[0,T]}$ of the operators $A_F(t)$ of multiplication by $F(t) = F(t, \cdot) \in L^2(\Omega, \mathcal{A}, P)$ in the space $L^2(\Omega, \mathcal{A}, P)$,*

$$L^2(\Omega, \mathcal{A}, P) \supset \operatorname{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, \mathcal{A}, P),$$

is H-stochastic integrable with respect to the normal martingale N (i.e. belongs to S_2) if and only if F belongs to the space $L^2_a([0, T] \times \Omega)$.

The next theorem shows that the Itô stochastic integral with respect to the normal martingale N can be interpreted as an H-stochastic integral.

Theorem 3. *Let $F \in L^2_a([0,T]\times\Omega)$ and $\{A_F(t)\}_{t\in[0,T]}$ be the corresponding family of the operators $A_F(t)$ of multiplication by $F(t)$ in the space $L^2(\Omega,\mathcal{A},P)$. Then*

$$\int_{[0,T]} A_F(t)\,dN_t = \int_{[0,T]} F(t)\,dN_t.$$

Proof. Taking into account Theorem 2, Lemma 1 and the definitions of the integrals

$$\int_{[0,T]} A_F(t)\,dN_t \quad \text{and} \quad \int_{[0,T]} F(t)\,dN_t,$$

it is sufficient to prove Theorem 3 for simple functions $F \in L^2_a([0,T]\times\Omega)$. But in this case Theorem 3 is obvious. □

4. The Itô integral in a Fock space as an H-stochastic integral

Let us recall the definition of the Itô integral in a Fock space, see e.g. [2] for more details. We denote by $\mathcal{F}$ the symmetric Fock space over the real separable Hilbert space $L^2([0,T]) := L^2([0,T],dt)$. By definition (see e.g. [5]),

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{F}_n n!,$$

where $\mathcal{F}_0 := \mathbb{C}$ and, for each $n\in\mathbb{N}$, $\mathcal{F}_n := (L^2_{\mathbb{C}}([0,T]))^{\widehat{\otimes} n}$ is an n-th symmetric tensor power $\widehat{\otimes}$ of the complex Hilbert space $L^2_{\mathbb{C}}([0,T])$. Thus, the Fock space $\mathcal{F}$ is the complex Hilbert space of sequences $f=(f_n)_{n=0}^{\infty}$ such that $f_n\in\mathcal{F}_n$ and

$$\|f\|^2_{\mathcal{F}} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{F}_n} n! < \infty.$$

We denote by $L^2([0,T];\mathcal{F})$ the Hilbert space of all $\mathcal{F}$-valued functions

$$[0,T]\ni t\mapsto f(t)\in\mathcal{F}, \quad \|f\|_{L^2([0,T];\mathcal{F})} := \Big(\int_{[0,T]} \|f(t)\|^2_{\mathcal{F}}\,dt\Big)^{\frac{1}{2}} < \infty$$

with the corresponding scalar product. A function $f(\cdot)=(f_n(\cdot))_{n=0}^{\infty}\in L^2([0,T];\mathcal{F})$ is called *Itô integrable* if, for almost all $t\in[0,T]$,

$$f(t) = (f_0(t), \varkappa_{[0,t]} f_1(t), \ldots, \varkappa_{[0,t]^n} f_n(t), \ldots).$$

We denote by $L^2_a([0,T];\mathcal{F})$ the set of all Itô integrable functions.

Let f belong to the space $L^2_{a,s}([0,T];\mathcal{F})$ of all *simple Itô integrable functions*. That is, f belongs to $L^2_a([0,T];\mathcal{F})$ and there exists a partition $0=t_0<t_1<\cdots<t_n=T$ of $[0,T]$ such that

$$f(t) = \sum_{k=0}^{n-1} f_{(k)} \varkappa_{(t_k,t_{k+1}]}(t) \in \mathcal{F}$$

for almost all $t\in[0,T]$. The *Itô integral* $\mathbb{I}(f)$ of such a function f is defined by the formula

$$\mathbb{I}(f) := \sum_{k=0}^{n-1} f_{(k)} \lozenge (0, \varkappa_{(t_k,t_{k+1}]}, 0, 0, \ldots) \in \mathcal{F},$$

where the symbol $\lozenge$ denotes the Wick product in the Fock space $\mathcal{F}$. Let us recall that for given $f=(f_n)_{n=0}^{\infty}$ and $g=(g_n)_{n=0}^{\infty}$ from $\mathcal{F}$ the Wick product $f\lozenge g$ is defined by

$$f\lozenge g := \Big(\sum_{m=0}^{n} f_m \,\widehat{\otimes}\, g_{n-m}\Big)_{n=0}^{\infty},$$

provided the latter sequence belongs to the Fock space $\mathcal{F}$.

The Itô integral $\mathbb{I}(f)$ of a simple function $f \in L^2_{a,s}([0,T];\mathcal{F})$ has the isometry property

$$\|\mathbb{I}(f)\|^2_{\mathcal{F}} = \int_{[0,T]} \|f(t)\|^2_{\mathcal{F}}\, dt,$$

see e.g. [2, 1]. Hence, extending the mapping

$$L^2_a([0,T];\mathcal{F}) \supset L^2_{a,s}([0,T];\mathcal{F}) \ni f \mapsto \mathbb{I}(f) \in \mathcal{F}$$

by continuity we obtain a definition of the Itô integral $\mathbb{I}(f)$ for each $f \in L^2_a([0,T];\mathcal{F})$ (we keep the same notation $\mathbb{I}$ for the extension).

Let us show that the Itô integral $\mathbb{I}(f)$ of $f \in L^2_a([0,T];\mathcal{F})$ can be considered as an H-stochastic integral. To do this we set $\mathcal{H} := \mathcal{F}$ and consider in this space the resolution of identity

$$[0,T] \ni t \mapsto \mathcal{X}_t f := (f_0, \varkappa_{[0,t]} f_1, \ldots, \varkappa_{[0,t]^n} f_n, \ldots) \in \mathcal{L}(\mathcal{F}), \quad f = (f_n)_{n=0}^{\infty} \in \mathcal{F}.$$

Let $Z := (0,1,0,0,\ldots) \in \mathcal{F}$ and

$$[0,T] \ni t \mapsto Z_t := \mathcal{X}_t Z = (0, \varkappa_{[0,t]}, 0, 0, \ldots) \in \mathcal{F}$$

be the corresponding abstract martingale in the Fock space $\mathcal{F}$. Note that now

$$\mu([0,t]) := \|Z_t\|^2_{\mathcal{F}} = \|\varkappa_{[0,t]}\|^2_{L^2_{\mathbb{C}}([0,T])} = t, \quad t \in [0,T],$$

i.e., μ is the Lebesgue measure on $\mathcal{B}([0,T])$.

We have the following analogues of Theorems 2 and 3.

Theorem 4. *A function $f \in L^2([0,T];\mathcal{F})$ belongs to the space $L^2_a([0,T];\mathcal{F})$ if and only if the corresponding operator-valued function $[0,T] \ni t \mapsto A_f(t)$ whose values are operators $A_f(t)$ of Wick multiplication by $f(t) \in \mathcal{F}$ in the Fock space $\mathcal{F}$,*

$$\mathcal{F} \supset \mathrm{Dom}(A_f(t)) \ni g \mapsto A_f(t) g := f(t) \Diamond g \in \mathcal{F},$$

belongs to the space S_2.

Theorem 5. *Let $f \in L^2_a([0,T];\mathcal{F})$ and $\{A_f(t)\}_{t \in [0,T]}$ be the corresponding family of the operators $A_f(t)$ of Wick multiplication by $f(t) \in \mathcal{F}$ in the Fock space $\mathcal{F}$. Then*

$$\mathbb{I}(f) = \int_{[0,T]} A_f(t)\, dZ_t.$$

Taking into account Theorem 5, in what follows we will denote the Itô integral $\mathbb{I}(f)$ of $f \in L^2_a([0,T];\mathcal{F})$ by $\int_{[0,T]} f(t)\, dZ_t$. Note that this integral can be expressed in terms of the Fock space $\mathcal{F}$. Namely, for any $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L^2_a([0,T];\mathcal{F})$, we have

$$\int_{[0,T]} f(t)\, dZ_t = (0, \hat{f}_1, \ldots, \hat{f}_n, \ldots) \in \mathcal{F}, \tag{8}$$

where, for each $n \in \mathbb{N}$ and almost all $(t_1, \ldots, t_n) \in [0,T]^n$,

$$\hat{f}_n(t_1, \ldots, t_n) := \frac{1}{n} \sum_{k=1}^{n} f_{n-1}(t_k; t_1, \ldots, \not{t_k}, \ldots, t_n),$$

i.e., $\hat{f}_n$ is the symmetrization of $f_{n-1}(t; t_1, \ldots, t_{n-1})$ with respect to n variables.

5. A connection between the classical Itô integral and the Itô integral in the Fock space

As before, let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a right continuous filtration $\{\mathcal{A}_t\}_{t \in [0,T]}$, $\mathcal{A}_0$ be the trivial σ-algebra containing all P-null sets of $\mathcal{A}$ and $\mathcal{A} = \mathcal{A}_T$.

Let $N = \{N_t\}_{t\in[0,T]}$ be a normal martingale on $(\Omega, \mathcal{A}, P)$ with respect to $\{\mathcal{A}_t\}_{t\in[0,T]}$, $N_0 = 0$. It is known that the mapping

$$\mathcal{F} \ni f = (f_n)_{n=0}^{\infty} \mapsto If := \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega, \mathcal{A}, P)$$

is well-defined and isometric. Here $I_0(f_0) := f_0$ and, for each $n \in \mathbb{N}$,

$$I_n(f_n) := n! \int_0^T \int_0^{t_n} \cdots \Big(\int_0^{t_2} f_n(t_1, \ldots, t_n)\, dN_{t_1} \Big) \ldots dN_{t_{n-1}}\, dN_{t_n}$$

is an n-iterated Itô integral with respect to N. We suppose that the normal martingale N has the *chaotic representation property* (CRP). In other words, we assume that the mapping $I : \mathcal{F} \to L^2(\Omega, \mathcal{A}, P)$ is a unitary. Note that

$$N_t = IZ_t \in L^2(\Omega, \mathcal{A}, P), \quad t \in [0,T],$$

i.e., N is the I-image of the abstract martingale $[0,T] \ni t \mapsto Z_t = (0, \varkappa_{[0,t]}, 0, 0, \ldots) \in \mathcal{F}$.

The Brownian motion, the compensated Poisson process and some Azéma martingales are examples of normal martingales which possess the CRP, see e.g. [10, 11].

We note that the spaces $L^2([0,T] \times \Omega)$ and $L^2([0,T]; \mathcal{F})$ can be understood as the tensor products $L^2([0,T]) \otimes L^2(\Omega, \mathcal{A}, P)$ and $L^2([0,T]) \otimes \mathcal{F}$, respectively. Therefore,

$$1 \otimes I : L^2([0,T]; \mathcal{F}) \to L^2([0,T] \times \Omega)$$

is a unitary operator.

The next result gives a relationship between the classical Itô integral with respect to the normal martingale with CRP and the Itô integral in the Fock space $\mathcal{F}$.

Theorem 6. *We have*

$$L_a^2([0,T] \times \Omega) = (1 \otimes I) L_a^2([0,T]; \mathcal{F})$$

and, for arbitrary $f \in L_a^2([0,T]; \mathcal{F})$,

$$I\Big(\int_{[0,T]} f(t)\, dZ_t \Big) = \int_{[0,T]} If(t)\, dN_t.$$

Since N has CRP, for any $F \in L_a^2([0,T] \times \Omega)$ there exists a uniquely defined vector $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L_a^2([0,T]; \mathcal{F})$ such that

$$F(t) = If(t) = \sum_{n=0}^{\infty} I_n(f_n(t))$$

for almost all $t \in [0,T]$. Hence, using Theorem 6 and equality (8) we obtain

$$\int_{[0,T]} F(t)\, dN_t = I\Big(\int_{[0,T]} f(t)\, dZ_t \Big) = \sum_{n=1}^{\infty} I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P).$$

It should be noticed that the right hand side of the latter equality was used by Hitsuda [9] and Skorohod [14] to define an extension of the Itô integral. Namely, a function

$$F(\cdot) = \sum_{n=0}^{\infty} I_n(f_n(\cdot)) \in L^2([0,T] \times \Omega)$$

is Hitsuda-Skorohod integrable if and only if

$$\sum_{n=1}^{\infty} I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P) \quad \text{or, equivalently,} \quad \sum_{n=1}^{\infty} \|\hat{f}_n\|_{\mathcal{F}_n}^2 n! < \infty.$$

The corresponding *Hitsuda-Skorohod integral* $\mathbb{I}_{\mathrm{HS}}(F)$ of F is defined by the formula

$$\mathbb{I}_{\mathrm{HS}}(F) := \sum_{n=1}^{\infty} I_n(\hat{f}_n).$$

Acknowledgments. I am grateful to Prof. Yu. M. Berezansky for the statement of a question and helpful discussions and the referee for the useful remarks. This work has been partially supported by the DFG, Project 436 UKR 113/78/0-1 and by the Scientific Program of National Academy of Sciences of Ukraine, Project No 0107U002333.

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Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

E-mail address: tesko@imath.kiev.ua

Received 16/01/2008