Foliations with leaves of non-positive curvature and bounded total curvature on closed 3-manifolds

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Let (M, g) be a complete non-compact surface equipped with a smooth riemannian metric. The total curvature of M is the improper integral $\int_M K d\mu$ of the Gaussian curvature K with respect to the volume element $d\mu$ of (M, g). It is said that M admits total curvature if for any compact exhaustion Ω_i of M, the limit

$$\lim_{i\to+\infty} \int_{\Omega_i} K d\mu = \int_M K d\mu, \tag{1}$$

exists. In [1] Cohn-Vossen proved that $\int_M K d\mu \leq 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. Huber in [2] states that if

$$\int_M K_- < \infty, \tag{2}$$

where $K_- = max\{-K, 0\}$, then $\int_M K d\mu$ exists and M is homeomorphic to a compact Riemann surface with finitely many punctures, i.e. M has a finite topology. Hartman in [3] under the assumption (2) proved that the area of a geodesic ball of radius r at a fixed point must grow at most quadratically in r. Note also that Li proved in [4] that if M has at most quadratic area growth, finite topology and the Gaussian curvature of M is either non-positive or non-negative near infinity of each end, then M must have finite total curvature.

The following theorem describes a topological structure of riemannian 3-Manifolds admitting codimension one C^2-foliations $\mathcal{F}$ with leaves which have both non-positive curvature and bounded total curvature in the induced riemannian metric.

Theorem 1. *Let $\mathcal{F}$ be a transversaly orientable C^2-foliation of a closed orientable riemannian 3-Manifold M. Suppose, that the leaves of $\mathcal{F}$ have non-positive curvature and admit a finite total curvature in the induced riemannian metric. Then the following holds:*

(1) M is aspherical;
(2) $\mathcal{F}$ is a foliation almost without holonomy;
(3) At least one of the following holds:
  (a) $\mathcal{F}$ is a surface bundle over the circle with the fiber genus $g \geq 1$;
  (b) M is divided by a finite set of compact surfaces $\{K_i\}$, which are homeomorphic to torus T^2, into pieces $\{A_j\}$, which are fibered over the circle. This division defines a graph G of fundamental groups $\pi_1(A_j)$ and $\pi_1(K_i)$, where vertexes of G correspond to the $\{A_j\}$ and edges of G correspond to the tori $\{K_i\}$ and the fundamental group $\pi_1(M)$ is isomorphic to a fundamental group of the graph G;
(4) $\mathcal{F}$ is a flat foliation (i.e. all leaves of $\mathcal{F}$ are flat) iff M is either torical bundle or torical semi-bundle.

Conversely, let M be such as described in (3) above. Then M admits a riemannian metric and transversaly orientable foliation with leaves of non-positive curvature and finite total curvature in the induced metric.

References

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