Problem on non-overlapping polycylindrical domains with poles on the boundary of a polydisk

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The goal of the present work is the study of the problem of a product of powers of the generalized conformal radii of polycylindrical nonoverlapping domains with poles on the boundary of a polydisk. The spatial analogs of a number of known results concerning the nonoverlapping domains on a plane were obtained in [1], where a generalization of the notion of inner radius was given. Namely, the notion of harmonic radius of the spatial domain $B \subset \mathbb{R}^n$ relative to some internal point was introduced. Work [1] was the essential break-through in the consideration of nonoverlapping domains in the spatial case. Then, work [2] advanced an approach that allowed the transfer of some results known in the case of a complex plane onto $\mathbb{C}^n$. At the same time, the problems of nonoverlapping domains in the case of a complex plane represent a sufficiently well-developed trend of the geometric theory of functions of complex variable (see, e.g., [1–6]).

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural, real, and complex numbers, respectively, and $\mathbb{R}^+ = (0, \infty)$. Let $\overline{\mathbb{C}}$ be a Riemann sphere (extended complex plane). It is well known that $\mathbb{C}^n = \underbrace{(\mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C})}_{n-\text{times}}$, $n \in \mathbb{N}$. $\overline{\mathbb{C}}^n = \underbrace{(\overline{\mathbb{C}} \times \overline{\mathbb{C}} \times \ldots \times \overline{\mathbb{C}})}_{n-\text{times}}$ is a compactification of the space $\mathbb{C}^n$ (see, e.g., [3, 4, 5]), where the set of infinitely remote points has the complex dimension $n-1$. Let $[D]^n$ (Cartesian degree of a domain $D \in \overline{\mathbb{C}}$) denote the Cartesian product $\underbrace{D \times D \times \ldots \times D}_{n-\text{times}}$, and let $[d]^n$ (Cartesian degree of a point $d \in \overline{\mathbb{C}}$) denote the point with $\overline{\mathbb{C}}^n$, which have the coordinates $\underbrace{(d, ..., d)}_{n-\text{times}}$. It is clear that $\mathbb{C}^1 = \mathbb{C}$, $\overline{\mathbb{C}}^1 = \overline{\mathbb{C}}$. The topology in $\overline{\mathbb{C}}^n$ is introduced like in a Cartesian product of topological spaces. In this topology, $\overline{\mathbb{C}}^n$ is compact (see [3, 4, 5]).

The domain $\mathbb{B} = B_1 \times B_2 \times \ldots \times B_n \subset \overline{\mathbb{C}}^n$, where each domain $B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, is called a polycylindrical domain in $\overline{\mathbb{C}}^n$ (see [3]). The domains B_k, $k = \overline{1,n}$, are called coordinate domains of the domain $\mathbb{B}$.

Let $r(B,a)$ be inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to the point $a \in B$. The generalized inner radius of the polycylindrical domain $\mathbb{B}$ relative to the point $\mathbb{A} = (a_1, a_2, \ldots, a_n) \in \mathbb{B}$, $a_k \in B_k$, $k = \overline{1,n}$, is

$$R(\mathbb{B}, \mathbb{A}) := \left(\prod_{k=1}^{n} r(B_k, a_k)\right)^{\frac{1}{n}},$$

where the quantities $r(B_k, a_k)$, $k = \overline{1,n}$, mean the inner radii of the coordinate domains B_k relative to a_k.

Let $\mathbb{U}^n = [U]^n$, where $U = \{z \in \mathbb{C} : |z| < 1\}$ (unit disk in the complex plane $\mathbb{C}$). By Γ_n we denote the skeleton of the polydisk $\mathbb{U}^n$ i.e., the set of points $\mathbb{A} = (a_1, a_2, \ldots, a_n) \subset \mathbb{C}^n$, $|a_s| = 1$, $s = \overline{1,n}$.

The system $\{\mathbb{B}_k\}_{k=1}^m$ $\left(\mathbb{B}_k = B_1^{(k)} \times \ldots \times B_n^{(k)},\ \ k = \overline{1,m}\right)$ is called a system of nonoverlapping polycylindrical domains, if, for every fixed p_0, $p_0 = \overline{1,n}$, the system of domains $\left\{B_{p_0}^{(k)}\right\}$, $k = \overline{1,m}$, is a system of nonoverlapping domains on $\overline{\mathbb{C}}$.

Let $m \in \mathbb{N}$, $m \geqslant 2$. The system of points $\Delta_m := \{a_k\}_{k=1}^m$, $a_k \in \mathbb{C}$, is called m-radial, if $|a_k| \in \mathbb{R}^+$ for $k = \overline{1,m}$, $0 = \arg a_1 < \arg a_2 < \ldots < \arg a_m < 2\pi$.

The system of points $\{\mathbb{A}_k\}_{k=1}^m$ $\left(\mathbb{A}_k = \left(a_1^{(k)}, a_2^{(k)}, \ldots, a_n^{(k)}\right) \in \mathbb{C}^n, \quad k = \overline{1,m}\right)$, is called radial in the space $\mathbb{C}^n$, if, for every fixed p_0 the sequence $\left\{a_{p_0}^{(k)}\right\}$, $k = \overline{1,m}$, is an m-radial system of points on the corresponding complex plane $\mathbb{C}$.

We will consider radial systems of points in the space $\mathbb{C}^n$ of the form

$$\{\mathbb{A}_k\}_{k=1}^m, \quad \mathbb{A}_k = \left(a_1^{(k)}, a_2^{(k)}, \ldots, a_n^{(k)}\right) \in \mathbb{C}^n, \quad k = \overline{1,m}, \quad a_{p_0}^{(1)} > 0, \quad p_0 = \overline{1,n},$$
$$\arg a_{p_0}^{(k)} < \arg a_{p_0}^{(k+1)}, \quad k = \overline{1,m-1}, \quad \arg a_{p_0}^{(m)} < 2\pi. \tag{1}$$

We introduce the following notations

$$\alpha_{p_0}^{(1)} := \frac{1}{\pi}\left(\arg a_{p_0}^{(2)} - \arg a_{p_0}^{(1)}\right), \quad \alpha_{p_0}^{(2)} := \frac{1}{\pi}\left(\arg a_{p_0}^{(3)} - \arg a_{p_0}^{(2)}\right), \ldots, \alpha_{p_0}^{(m)} := \frac{1}{\pi}\left(2\pi - \arg a_{p_0}^{(m)}\right).$$

Let

$$F_\delta(x) = 2^{x^2+6} \cdot x^{x^2+2-2\delta} \cdot (2-x)^{-\frac{1}{2}(2-x)^2}(2+x)^{-\frac{1}{2}(2+x)^2},$$
$$x \in (0,2], \quad \delta \in [0,\ 1], \quad F_\delta(x) \subset \overline{\mathbb{C}}.$$

Then the following proposition is valid.

Theorem 1. [6] *Let $m, n \in \mathbb{N}$, $m \geqslant 7$, $\gamma \in (0, \gamma_0]$, $\gamma_0 = \sqrt[3]{m}$ and $\delta \in [0;\ 0,7]$. Then, for any radial system of points of the form (1) $\{\mathbb{A}_k\}_{k=1}^m = \left\{a_p^{(k)}\right\}_{k=1}^m \in \overline{\mathbb{C}}^n$, $p = \overline{1,n}$, such that $\mathbb{A}_k \in \Gamma_n$, $k = \overline{1,m}$, and for any collection of nonoverlapping polycylindrical domains $\mathbb{B}_k$, $\mathbb{A}_k \in \mathbb{B}_k \subset \overline{\mathbb{C}}^n$, $k = \overline{0,m}$, $\mathbb{A}_0 \in \mathbb{B}_0 \subset \overline{\mathbb{C}}^n$, the following inequality holds*

$$R^\gamma(\mathbb{B}_0, \mathbb{A}_0) \prod_{k=1}^m R(\mathbb{B}_k, \mathbb{A}_k) \leqslant \gamma^{-\frac{\delta \cdot m}{2}} \cdot \left(\prod_{k=1}^m \prod_{p=1}^n \alpha_p^{(k)}\right)^{\frac{\delta}{n}} \cdot \left[F_\delta\left(\frac{2}{m}\sqrt{\gamma}\right)\right]^{\frac{m}{2}}.$$

One of the extreme systems is the system

$$\{\mathbb{B}_k\}_{k=0}^m = \left\{[B_0^{(0)}]^n, [B_1^{(0)}]^n, [B_2^{(0)}]^n, \ldots, [B_m^{(0)}]^n\right\},$$
$$\{\mathbb{A}_k\}_{k=0}^m = \left\{[0]^n, \left[b_1^{(0)}\right]^n, \left[b_2^{(0)}\right]^n, \ldots, \left[b_m^{(0)}\right]^n\right\},$$

where the domains $B_k^{(0)}$ and the points $b_k^{(0)}$, $k = \overline{1,m}$, are, respectively, the circular domains and the poles of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2.$$

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