On The Second Regularized Trace Formula for a Differential Operator with Unbounded Coefficients

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Let H be an infinite dimensional separable Hilbert space. Let denote the inner product and the norm in H by $(.,.)$ and $\|.\|$, respectively and denote the set of all kernel operators from H to H by $\sigma_1(H)$. Let $H_1 = L_2([0,\pi];H)$ be the set of all strongly measurable functions f defined on $[0,\pi]$ with their values in H such that for every $g \in H$ the scalar function $(f(x),g)$ is measurable in the interval $[0,\pi]$ and

$$\int_0^\pi \|f(x)\|^2 dx < \infty.$$

In $H_1 = L_2([0,\pi];H)$ we consider the operators

$$L = L_0 + Q, \;\; L_0 = y^{'v} + Ay$$

with the same boundary conditions $y^{'}(0) = y^{'}(\pi) = y^{'''}(0) = y^{'''}(\pi) = 0$. Here the operator $A : D(A) \to H$ is a densely defined on H such that $A = A^* \geq I, \;\; A^{-1} \in \sigma_\infty(H)$ where I is identity operator on H, A^* is the adjoint operator of A and $\sigma_\infty(H)$ is the set of all compact operators from H to H. And, $Q(x)$ is an operator function satisfying the following conditions:

(a) $Q(x) : H \to H$ is a self-adjoint operator for every $x \in [0,\pi]$.
(b) $Q(x)$ is weakly measurable in the interval $[0,\pi]$ and for every $f,g \in H$, the scalar function $(Q(x)f, \; g)$ is measurable on $[0,\pi]$.
(c) The function $\|Q(x)\|$ is bounded on $[0,\pi]$.

Let $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \ldots$ be the eigenvalues of the operator A and $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ be the orthonormal eigenvectors corresponding to these eigenvalues. Here, each eigenvalue is represented as many times as its multiplicity. Moreover, let the eigenvalues of the operator L_0 and L be $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \ldots$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$, respectively.

Lemma 1. *If $\gamma_j \sim a.j^\alpha$ $(a > 0, \alpha < \infty)$ as $j \to \infty$ then the asymptotic formula*

$$\lambda_n, \; \mu_n \sim \; dn^{\frac{4\alpha}{4+\alpha}} \; as \; n \to \infty \tag{1}$$

holds where d is a constant.

Let $R_\lambda^0 = (L_0 - \lambda I)^{-1}$, $R_\lambda = (L - \lambda I)^{-1}$ be the resolvents of the operators L_0 and L , respectively. By the well known equality

$$R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0 \;\; (\lambda \in \rho(L) \cap \rho(L_0))$$

we have:

Lemma 2.

$$\sum_{q=1}^{n_p} ({\lambda_q}^2 - {\mu_q}^2 \;) = \sum_{j=1}^{s} M_{pj} + M_p^{(s)}$$

where

$$M_{pj} = \frac{(-1)^j}{\pi i j} \int_{|\lambda| = b_p} \lambda \; tr[(QR_\lambda^0)^j] d\lambda \; (j = 1, \; 2, \ldots) \tag{2}$$

$$M_p^{(s)} = \frac{(-1)^s}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 tr[R_\lambda (QR_\lambda^0)^{s+1}] d\lambda. \tag{3}$$

Theorem 3. *If the operator function* $Q(x)$ *satisfies the conditions (a), (b), (c) and* $\gamma_j \sim aj^\alpha$ $(a > 0, \alpha > \frac{8}{7}(3+\sqrt{2}))$ *as* $j \to \infty$ *then ,*

$$\lim_{p\to\infty} M_{pj} = 0 \quad (j = 2, 3, 4, \dots).$$

Theorem 4. *If the operator function* $Q(x)$ *satisfies the following conditions*

i) $Q(x)$ *has weak derative of the 8-th order in the interval* $[0,\pi]$ *and the function* $(Q^{(8)}(x)u, v)$ *is continuos for every* $u, v \in H$ *.*

ii) For every $x \in [0,\pi]$, $Q^{(i)}(x) : H \to H$ $(i = 0, 1, ..., 8)$ *are self-adjoint operators.*

iii) For every $x \in [0,\pi]$, $Q^{(8)}(x)$, $AQ^{(2i)}(x) \in \sigma_1(H)$ $(i = 0, 1, ..., 8)$ *and the functions* $\|Q^{(8)}(x)\|_{\sigma_1(H)}$, $\|AQ^{(2i)}(x)\|_{\sigma_1(H)}$ $(i = 0, 1, ..., 8)$ *are bounded and measurable in the interval* $[0,\pi]$ *.*

and if $\gamma_j \sim aj^\alpha$ $(a > 0, \alpha > \frac{8}{7}(3+\sqrt{2}))$ *as* $j \to \infty$ *then the formula*

$$\lim_{p\to\infty} \sum_{q=1}^{n_p} [{\lambda_q}^2 - {\mu_q}^2 - \frac{2}{\pi}\mu_q \int_0^\pi (Q(x)\varphi_{j_q}, \varphi_{j_q}) dx]$$

$$= \frac{1}{2}[trAQ(0) + trAQ(\pi)] + \frac{1}{32}[trQ^{(4)}(0) + trQ^{(4)}(\pi)] - \frac{1}{\pi}\int_0^\pi trAQ(x) dx \tag{4}$$

is satisfied. Here j_1, j_2, $\dots$ *are natural numbers.*

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