The Lie-algebraic structure of the Lax-Sato integrable superanalogs for the Liouville heavenly type equations

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In the paper [1] the general Lie-algebraic approach to constructing the Lax-Sato integrable heavenly type systems has been developed. It is based on the classical Adler-Kostant-Symes (AKS) theory and $\mathcal{R}$-operator structures related with the loop Lie algebra $\widetilde{diff}(\mathbb{T}^n)$ of the vector fields on the n-dimensional torus $\mathbb{T}^n$ and adjacent Lie algebra $diff_{hol}(\mathbb{C}\times\mathbb{T}^n)\subset diff(\mathbb{C}\times\mathbb{T}^n)$ of the holomorphic in the “spectral” parameter $\lambda\in\mathbb{S}^1_\pm$ vector fields on $\mathbb{C}\times\mathbb{T}^n$. A generalization of this Lie-algebraic scheme, related with the loop Lie algebra $\widetilde{diff}(\mathbb{T}^{1|N})$ of superconformal vector fields on the $1|N$-dimensional supertorus $\mathbb{T}^{1|N}\simeq\mathbb{S}^1\times\Lambda_1^N$, where $\Lambda:=\Lambda_0\oplus\Lambda_1$ is an infinite-dimensional Grassmann algebra over $\mathbb{C}\subset\Lambda_0$, has been proposed in [2] for $n=1$ and applied to construct the Lax-Sato integrable superanalogs of the Mikhalev-Pavlov heavenly equation for every $N\in\mathbb{N}\setminus\{4;5\}$. In our report the Lax-Sato integrable superanalogs of the Liouville heavenly type equations are obtained by use the loop Lie algebra $\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})$ of the superconformal vector fields on $\mathbb{T}^{1|N}_{\mathbb{C}}\simeq\mathbb{T}^1_{\mathbb{C}}\times\Lambda_1^N$ as a result of some diffeomorphic mapping in the space of variables $(z,\vartheta)\in\mathbb{T}^{1|N}_{\mathbb{C}}$, where $\vartheta:=(\vartheta_1,\dots,\vartheta_N)^\top$, $\vartheta_i\in\Lambda_1$, $i=\overline{1,N}$.

At first one introduces the superderivatives $D_{\vartheta_i}:=\partial/\partial\vartheta_i+\vartheta_i\partial/\partial z$, $z\in\mathbb{T}^1_{\mathbb{C}}$, $\vartheta_i\in\Lambda_1$, $i=\overline{1,N}$, in the superspace $\Lambda_0\times\Lambda_1^N$. The loop Lie algebra $\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})$ are formed by the superconformal vector fields such as $\tilde{a}:=a\partial/\partial z+<Da,D>/2$, where $D:=(D_{\vartheta_1},D_{\vartheta_2},\cdots,D_{\vartheta_N})^\top$, $\vartheta:=(\vartheta_1,\dots,\vartheta_N)^\top$, $a\in C^\infty(\mathbb{T}^{1|N}_{\mathbb{C}};\Lambda_0)$, with the commutator

$$[\tilde{a},\tilde{b}]:=\tilde{c}=c\partial/\partial z+<Dc,D>/2,\quad c=a\partial b/\partial z-b\partial a/\partial z+<Da,Db>/2,$$

This loop Lie algebra $\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})$ allows the splitting $\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})=\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})_+\oplus\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})_-$. Here the Lie subalgebras $\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})_\pm$ are assumed to be formed by the vector fields $\tilde{a}(z)$ on $\mathbb{T}^{1|N}_{\mathbb{C}}$, being holomorphic in $z\in\mathbb{S}^1_\pm\subset\mathbb{C}$ respectively, where $\tilde{a}(\infty)=0$ for any $\tilde{a}(z)\in\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})_-$.

The nontrivial Casimir invariant $h^{(p_y)}\in I(\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})^*)$ on a dense subspace $\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})^*\simeq\Lambda^1(\mathbb{T}^1_{\mathbb{C}})$ of the dual space through the pairing $(\tilde{l},\tilde{a}):=\operatorname{res}_{\lambda\in\mathbb{C}}\int_{\mathbb{S}^1}z^{-1}dz\int_{\Lambda_1^N}(la)d^N\vartheta$, $\tilde{l}:=ldz\in\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})^*$, satisfies the relationship

$$(l(\nabla h^{(p_y)}(l))^2)_z-Nl((\nabla h^{(p_y)}(l))^2)_z/4=(-1)^N<Dl,D(\nabla h^{(p_y)}(l))^2>/4,\tag{1}$$

where $\nabla h^{(p_y)}(\tilde{l}):=\nabla h^{(p_y)}(l)\partial/\partial z+<D\nabla h^{(p_y)}(l),D>/2$. If the corresponding gradient has the asymptotic expansion $\nabla h^{(p_y)}(l)\simeq\sum_{j\le r}V_jz^j$, where $p_y=r$ and $V_j\in C^2(\mathbb{R}^2\times\Lambda_1^N;\Lambda_0)$, $j\in\mathbb{Z}$, $j\le r$, $r\in\mathbb{Z}_+$, are some functional parameters, as $|z|\to\infty$, we can construct the Hamiltonian flow

$$dl/dy=-l_z\nabla h_+^{(p_y)}(l)-(4-N)l(\nabla h_+^{(p_y)}(l))_z/2+(-1)^N<Dl,D\nabla h_+^{(p_y)}(l)>/2\tag{2}$$

in the framework of the classical AKS-theory. The constant Casimir invariant $h^{(p_t)}\in I(\widetilde{diff}(\mathbb{T}^{1|N}_{\mathbb{C}})^*)$ generates the trivial flow

$$dl/dt=0.\tag{3}$$

The compatibility condition of these two flows for all $y,t \in \mathbb{R}$ is equivalent to the following system of two *a priori* compatible linear vector field equations

$$\partial\psi/\partial y + V\partial\psi/\partial z + < DV, D\psi > /2 = 0, \quad \partial\psi/\partial t = 0, \tag{4}$$

where $\nabla h_+^{(p_y)}(l) := V$, $V = V(y,t,\vartheta;z) = \sum_{0\le j\le r} V_j z^j$, and $\nabla h^{(p_t)}(l) = 0$, for a smooth function $\psi \in C^2(\mathbb{R}^2 \times \Lambda_1^N; \Lambda_0)$. In this case we have the evolutions

$$dz/dy = V - < \theta, DV > /2, \; d\vartheta/dy = (DV)/2, \; dz/dt = 0, \; d\theta/dt = 0. \tag{5}$$

Under the diffeomorphic mapping $z \mapsto z - \varkappa - < \theta, \eta > := \lambda$ and $\vartheta \mapsto \vartheta + \eta := \tilde{\vartheta}$, $\eta := (\eta_1, \ldots, \eta_N)^\top$, $\tilde{\vartheta} := (\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_N)^\top$, on $\mathbb{T}_{\mathbb{C}}^{1|N}$, generated by the functions $\varkappa := \varkappa(y,t) \in C^3(\mathbb{R}^2; \Lambda_0)$ and $\eta := \eta(y,t) \in C^3(\mathbb{R}^2; \Lambda_1^N)$, the equations (4) are rewritten as

$$\partial\psi/\partial y + W\partial\psi/\partial\lambda + < \tilde{D}W, \tilde{D}\psi > /2 = 0, \quad \partial\psi/\partial t - U\partial\psi/\partial\lambda - < \tilde{D}U, \tilde{D}\psi > /2 = 0, \tag{6}$$

where $W := W(y,t,\tilde{\vartheta};\lambda) = \sum_{0\le j\le r} W_j \lambda^j$, $U := U(y,t,\tilde{\vartheta})$, $\tilde{D} := (D_{\vartheta_1}, D_{\vartheta_2}, \ldots, D_{\vartheta_N})^\top$ and $D_{\tilde{\vartheta}_i} := \partial/\partial\tilde{\vartheta}_i + \tilde{\vartheta}_i \partial/\partial\lambda$, $i = \overline{1,N}$. Taking into account the evolutions (5) and

$$d\lambda/dy = W - < \tilde{\theta}, \tilde{D}W > /2, \; d\tilde{\vartheta}/dy = (\tilde{D}W)/2, \; d\lambda/dt = -U + < \tilde{\theta}, \tilde{D}U > /2, \; d\tilde{\vartheta}/dt = -(\tilde{D}U)/2,$$

one obtains the function W such as $W = \tilde{V} + < \eta, \tilde{D}\tilde{V} > - \partial\varkappa/\partial y + < \eta, \partial\eta/\partial y >$, where $\tilde{V} := \tilde{V}(y,t,\tilde{\vartheta};\lambda) = V(y,t,\vartheta;z)|_{z=\lambda+\varkappa+<\theta,\eta>, \; \vartheta=\tilde{\vartheta}-\eta}$. Futhermore, the superderivatives transform by the rules $D_{\vartheta_i} = D_{\tilde{\vartheta}_i} - 2\eta_i \partial/\partial\lambda$, $i = \overline{1,N}$, and the functions $\varkappa$ and η obey the relationships $\partial\varkappa/\partial t - < \eta, \partial\eta/\partial t > = U$, $\partial\eta/\partial t = -(\tilde{D}U)/2$.

If $W_2 := 1$ and $U := 1/2 \exp\varphi$, $\varphi := \varphi(y,t,\vartheta)$, the compatibility condition for the first order partial differential equations (6) leads to the Lax-Sato integrable superanalogs of Liouville heavenly type equations [3]

$$\varphi_{yt} = \exp\varphi - \sum_{i=1}^N (\partial\varphi_y/\partial\tilde{\vartheta}_i)(\partial \exp\varphi/\partial\tilde{\vartheta}_i)/4, \quad W_0 := 1, \tag{7}$$

$$\varphi_{yt} - \varphi_{tt} = \exp\varphi - \sum_{i=1}^N (\partial(\varphi_y - \varphi_t)/\partial\tilde{\vartheta}_i)(\partial \exp\varphi/\partial\tilde{\vartheta}_i)/4, \quad W_0 := -1/2\exp\varphi. \tag{8}$$

Because of the relationship (1) the element $\tilde{l} \in \widetilde{diff}(\mathbb{T}_{\mathbb{C}}^{1|N})^*$ can be found explicitly. For example, in the case of $r = 2$ and $N = 1$ it has the following form

$$\tilde{l}(y,t,\vartheta_1;z) = (z^{-4}(\vartheta_1(1 - 2v_1 z^{-1} + (3v_1^2 - 2v_0)z^{-2}) + \beta_1/2 + (\beta_0/4 - 9\beta_1 v_1/8)z^{-1}))dz, \tag{9}$$

where $V_2 := 1$ and $V_j := v_j + \vartheta_1\beta_j$, $j = \overline{0,1}$. Thus, one can formulate the following proposition.

Proposition 1. *For all $N \in \mathbb{N}$ the super-Liouville heavenly type equations (7) and (8) possess the Lax-Sato vector field representations (6), being equivalent to the commutability condition of two Hamiltonian flows (2) and (3) on $\widetilde{diff}(\mathbb{T}_{\mathbb{C}}^{1|N})^*$. In the case of $N = 1$ the equations (7) and (8) are put into the AKS-scheme for the loop Lie algebra $\widetilde{diff}(\mathbb{T}_{\mathbb{C}}^{1|N})$ with the element $\tilde{l} \in \widetilde{diff}(\mathbb{T}_{\mathbb{C}}^{1|N})^*$ in the form (9).*

References

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