A purely algebraic construction of Schwartz distributions

Vakhtang Lomadze
(Mathematics Department of I. Javakhishvili Tbilisi State University, Georgia)
E-mail: `vakhtang.lomadze@tsu.ge`

Let I be an interval of real axis, and let $L(I)$ be the space of all locally integrable functions defined on I. Assuming (without loss of generality) that the interval contains 0, for every locally integrable function $u \in L(I)$, let $J(u)$ denote the (absolutely) continuous function defined by

$$J(u)(x) = \int_0^x u(\alpha)d\alpha, \quad x \in I.$$

The integral operator $J : L(I) \to L(I)$ is injective, but not bijective (of course).

Define the Mikusinski space $M(I)$ to be the inductive limit of the sequence

$$L(I) \xrightarrow{J} L(I) \xrightarrow{J} L(I) \xrightarrow{J} L(I) \xrightarrow{J} \cdots .$$

Call its elements Mikusinski functions. By the very definition, a Mikusinski function is represented by a pair (u, m), where $u \in L(I)$ and $m \in \mathbb{Z}_+$. Two such pairs (u, m) and (v, n) represent the same Mikusinski function if and only if

$$J^n u = J^m v.$$

Remark. In fact, our Mikusinski functions constitute a very small portion of Mikusinski's operators defined in [1].

Obviously, the map $u \mapsto (u, 0)$ is injective. This permits us to make the identification

$$u = (u, 0).$$

We extend the integration operator J to Mikusinski functions by setting

$$J(u, m) = (Ju, m).$$

Define the differentiation operator $D : M(I) \to M(I)$ by

$$D(u, m) = (u, m + 1).$$

Notice that

$$DJ = id \quad \text{and} \quad JD = id.$$

So, both of the operators $J : M(I) \to M(I)$ and $D : M(I) \to M(I)$ are bijective, and are inverse to each other.

The iterated derivatives of constant functions are not zero, and a natural idea is to "kill" all of them. We are led to consider the quotient space

$$M(I)/N(I),$$

where $N(I)$ denotes the subspace of $M(I)$ spanned by functions $D^m 1$, $m \geq 1$.

The differential operator of $M(I)$ induces a differential operator of $M(I)/N(I)$. We shall denote it by the same letter D. Thus,

$$D(w(\text{mod } N(I))) = (Dw)(\text{mod } N(I)), \quad w \in M(I).$$

Lemma 1. $L(I) \cap N(I) = \{0\}$.

Define the canonical map $\jmath : L(I) \to M(I)/N(I)$ by the formula

$$\jmath(u) = u \ (\text{mod } N(I)).$$

It is immediate from the above lemma, that $\jmath$ is injective.

Theorem 2. *$M(I)/N(I)$ is canonically isomorphic to $\mathcal{D}'_{fin}(I)$, the space of Schwartz distributions of finite order.*

Proof. This is easy. Indeed, for each $u \in L(I)$, let T_u be the corresponding Schwartz distribution. One can show that if $u \in L(I)$, then $D\jmath(u) = 0$ if and only if u is a constant function. This implies that $D^m\jmath(u) = 0$ if and only if u is a polynomial function of degree $\leq m$. It follows that the mapping

$$D^m\jmath(u) \;\mapsto\; D^m T_u$$

is well-defined and injective. The surjectivity is clear.

Remarks. 1) Mikusinski functions admit multiplication by all rational functions. Due to this property of Mikusinski functions, the representation of distribution space as $M(I)/N(I)$ provides a simple foundation of Heaviside's operational calculus.

2) As is known, Schwartz distributions defined on a compact interval have finite order. Therefore, the Schwartz space $\mathcal{D}'(I)$, can be defined as the projective limit

$$\lim_{\leftarrow} M([\alpha,\beta])/N([\alpha,\beta]),$$

where $[\alpha,\beta]$ runs over all compact subintervals of I that contain 0.

References

[1] Mikusinski, J., *Operational Calculus.* London: Pergamon Press 1959.

[2] Schwartz, L., Généralisation de la notion de fonction, dérivation, de transformation de Fourier et applications mathématiques et physiques. *Annales Univ. Grenoble* 21 (1945) 57–74.