Hyperbolic quasiperiodic motion of charged particle on 2-sphere

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Let $\mathbb{E}^3$ be 3-D Euclidean space endowed with a scalar product $\langle\cdot,\cdot\rangle$ and cross-product $\cdot\times\cdot$, and let $\iota : \mathbb{S}^2 \hookrightarrow \mathbb{E}^3$ stands for the inclusion map of 2-D sphere $\mathbb{S}^2$ into $\mathbb{E}^3$: $\iota\left(\mathbb{S}^2\right) := \left\{\mathbf{x}\in\mathbb{E}^3 : \|\mathbf{x}\|^2 := \langle\mathbf{x},\mathbf{x}\rangle = 1\right\}$.

We consider a partial case of Newton equation

$$\nabla_{\dot{x}}\dot{x} = f(t\omega, x) + P(t\omega, x)\dot{x} \tag{1}$$

that governs the motion of quasiperiodically excited particle on $\mathbb{S}^2$. Here ∇ stands for the Levi-Civita connection of naturally induced Riemannian metric on $\mathbb{S}^2$, $\{f(\varphi,\cdot)\}_{\varphi\in\mathbb{T}^k}$ is a smooth family of vector fields on $\mathbb{S}^2$ parametrized by points of the standard k-dimensional torus $\mathbb{T}^k := \mathbb{R}^k/2\pi\mathbb{Z}^k$, $\{P(\varphi,\cdot)\}_{\varphi\in\mathbb{T}^k}$ is a smooth family of $(1,1)$-tensor fields, and $\omega\in\mathbb{R}^k$ is the basic frequency vector with rationally independent components. For Eq. (1), there naturally arise the problem of quasiperiodic response, i.e. the existence problem for ω-quasiperiodic solution $t\mapsto x(t) := u(t\omega)$ associated with a continuous mapping $u(\cdot)$:$\mathbb{T}^k \mapsto \mathbb{S}^2$. Such a solution is said to be hyperbolic if the corresponding system in variations

$$\begin{aligned}&\nabla_{\dot{x}(t)}\eta = \zeta\\&\nabla_{\dot{x}(t)}\zeta = \left[\nabla f\left(t\omega,x\right)\eta - R(\eta,\dot{x})\dot{x} + \nabla P\left(t\omega,x\right)(\eta,\dot{x}) + P\left(t\omega,x\right)\zeta\right]_{x=x(t)},\end{aligned}$$

where R is the Riemann curvature tensor, is exponentially dichotomic.

We consider the case where the charged particle of unit mass is constrained to move on $\iota\left(\mathbb{S}^2\right) := \left\{\mathbf{x}\in\mathbb{E}^3 : \|\mathbf{x}\|^2 = 1\right\}$ by the applied force $\mathbf{\Phi}$ represented in the form

$$\mathbf{\Phi}(t\omega,\mathbf{x},\dot{\mathbf{x}}) = -\frac{\mathbf{x}-\mathbf{a}}{\|\mathbf{x}-\mathbf{a}\|^3} + \mathbf{E}(t\omega) + \dot{\mathbf{x}}\times\mathbf{B}(t\omega).$$

Here $\mathbf{a}\in\mathbb{E}^3$ is a constant vector with norm $a := \|\mathbf{a}\|$; $\mathbf{E}(\cdot) : \mathbb{T}^k \mapsto \mathbb{E}^3$ and $\mathbf{B}(\cdot) : \mathbb{T}^k \mapsto \mathbb{E}^3$ are smooth mappings. The force $\mathbf{\Phi}$ can be naturally interpreted as the superposition of the Coulomb force caused by a charge placed at point $\mathbf{a}$ and the Lorentz force caused by the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. Let ι_* stands for the derivative of the inclusion map. In the case under consideration, the forces affecting the motion of the constrained particle are

$$\begin{gathered}\iota_* f(t\omega,x) = \mathbf{f}(t\omega,\mathbf{x}) = -\frac{\mathbf{x}+a\mathbf{k}}{\|\mathbf{x}+a\mathbf{k}\|^3} + \mathbf{E}(t\omega) + \left\langle\frac{\mathbf{x}+a\mathbf{k}}{\|\mathbf{x}+a\mathbf{k}\|^3} - \mathbf{E}(t\omega),\mathbf{x}\right\rangle\mathbf{x},\\ \iota_* P(t\omega,x)\dot{x} = \dot{\mathbf{x}}\times\mathbf{B}(t\omega) - \langle\dot{\mathbf{x}}\times\mathbf{B}(t\omega),\mathbf{x}\rangle\,\mathbf{x}.\end{gathered}$$

where $\mathbf{x} := \iota(x)$, $\dot{\mathbf{x}} := \iota_*\dot{x}$, $\mathbf{k} := -\mathbf{a}/a$. First consider the case where the influence of magnetic field can be neglected.

Theorem 1. *Let $\mathbf{B}(\varphi)\equiv 0$. If there holds the inequality*

$$\frac{a}{(1+a)^3} - \langle\mathbf{E}(\varphi),\mathbf{k}\rangle > 0 \quad \forall\varphi\in\mathbb{T}^k \tag{1}$$

and there exists a point $\varphi_0\in\mathbb{T}^k$ such that $\mathbf{E}(\varphi_0)\nparallel\mathbf{k}$, then the system of charged particle on $\mathbb{S}^2$ has a unique ω-quasiperiodic solution $t\mapsto x(t)$ such that $0 < \langle\mathbf{x}(t),\mathbf{k}\rangle \le 1$ for all $t\in\mathbb{R}$ where $\mathbf{x}(t) := \iota\circ x(t)$. This solution is hyperbolic.

This theorem is obtained by applying results of [1]. We essentially use the so-called U- monotonicity property of the system $\nabla_{\dot{x}}\dot{x} = f$ in the hemisphere $S^+ := \left\{x \in \mathbb{S}^2 : 0 < \langle \iota(x), \mathbf{k}\rangle \leq 1\right\}$.

Namely, to ensure such a property we have constructed a function $U(\cdot) \in C^\infty(S^+ \mapsto \mathbb{R})$ satisfying the conditions

$$\lambda_f(\varphi, x) + \frac{\langle \nabla U(x), f(\varphi, x)\rangle}{2} > 0 \quad , \quad \mu_U(x) \geq 2 \quad \forall (\varphi, x) \in \mathbb{T}^k \times S^+$$

where

$$\lambda_f(\varphi, x) := \min_{\eta \in T_x\mathbb{S}^2} \left\{ \frac{\langle \nabla f(\varphi, x)\eta, \eta\rangle}{\|\eta\|^2} \right\},$$

$$\mu_U(x) := \min_{\eta \in T_x\mathbb{S}^2} \left\{ \frac{\langle \nabla_\eta \nabla U(x), \eta\rangle}{\|\eta\|^2} - \frac{\langle \nabla U(x), \eta\rangle^2}{2\,\|\eta\|^2} \right\}.$$

When $\mathbf{B}(\varphi) \not\equiv 0$, we restrict ourselves to the case where $\langle \mathbf{E}(\varphi), \mathbf{k}\rangle = 0$ and $\mathbf{B}(\varphi) \perp \mathbf{E}(\varphi)$. We show how to establish sufficient condition for the existence of hyperbolic ω-quasiperiodic solution in the domain $\left\{x \in \mathbb{S}^2 : 0.5 < \langle \iota(x), \mathbf{k}\rangle \leq 1\right\}$. Set

$$E := \max_{\varphi \in \mathbb{T}^k} \|\mathbf{E}(\varphi)\|, \quad B := \max_{\varphi \in \mathbb{T}^k} \|\mathbf{B}(\varphi)\|$$

Define $z_+ = z_+(B, E)$ and $z_* = z_*(B, E)$, respectively, as the greatest roots of the equations

$$J(z) := z^2 - \frac{B^2}{4}z - \sqrt{3}(E+1) = 0,$$

$$I(z) := \frac{z^3}{3} - \frac{B^2}{8}z^2 - \sqrt{3}(E+1)z = I(z_+) + \sqrt{3}(E+1)z_+.$$

It turns out that the sought sufficient condition take the form

$$\frac{4a}{(1+a)^3} > \max\left\{9B^2 z_*, \frac{4E}{\sqrt{3}} + B^2\right\}.$$

References

[1] I.O. Parasyuk. Quasiperiodic extremals of nonautonomous Lagrangian systems on Riemannian manifolds. *Ukrainian Math. J.*, 66(10): 1553–1574, 2015.