# Representing trees of finite ultrametric spaces and weak similarities

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An *ultrametric* on a set $X$ is a function $d\colon X\times X\to\mathbb{R}^+$, $\mathbb{R}^+=[0,\infty)$, such that for all $x,y,z\in X$:

(i) $d(x,y)=d(y,x)$,
(ii) $(d(x,y)=0)\Leftrightarrow(x=y)$,
(iii) $d(x,y)\le\max\{d(x,z),d(z,y)\}$.

The pair $(X,d)$ is called an *ultrametric space.* If condition (iii) is omitted, then $(X,d)$ is a *semimetric space*, see [1]. The *spectrum* of a semimetric space $(X,d)$ is the set

$$\operatorname{Sp}(X)=\{d(x,y)\colon x,y\in X\}.$$

Recall that a *graph* is a pair $(V,E)$ consisting of a nonempty set $V$ and a (probably empty) set $E$ elements of which are unordered pairs of different points from $V$. For a graph $G=(V,E)$, the sets $V=V(G)$ and $E=E(G)$ are called *the set of vertices* and *the set of edges*, respectively. A connected graph without cycles is called a *tree.* A tree $T$ may have a distinguished vertex called the *root*; in this case $T$ is called a *rooted tree.* With every finite ultrametric space $(X,d)$ it is possible to associate a labeled rooted tree $T_X$, which is called a *representing tree* of the space $X$, see, for example, [2, P. 109].

**Definition 1.** Let $T_1$ and $T_2$ be rooted trees with the roots $v_1$ and $v_2$ respectively. A bijective function $\Psi\colon V(T_1)\to V(T_2)$ is an isomorphism of $T_1$ and $T_2$ if

$$(\{x,y\}\in E(T_1))\Leftrightarrow(\{\Psi(x),\Psi(y)\}\in E(T_2))$$

for all $x,y\in V(T_1)$ and $\Psi(v_1)=v_2$. If there exists an isomorphism of rooted trees $T_1$ and $T_2$, then we will write $T_1\simeq T_2$.

**Definition 2.** Let $(X,d)$ and $(Y,\rho)$ be semimetric spaces. A bijective mapping $\Phi\colon X\to Y$ is a *weak similarity* if there exists a strictly increasing bijection $f\colon \operatorname{Sp}(X)\to\operatorname{Sp}(Y)$ such that the equality

$$f(d(x,y))=\rho(\Phi(x),\Phi(y))$$

holds for all $x$, $y\in X$. If $\Phi\colon X\to Y$ is a weak similarity, then we write $X\overset{w}{\simeq}Y$ and say that $X$ and $Y$ are *weakly similar.*

The notion of weak similarity of semimetric spaces was introduced in [3] is a slightly different form, where also some properties of these mappings were studied.

Denote by $\tilde{\mathfrak{R}}$ the class of finite ultrametric spaces $X$ for which $T_X$ has exactly one inner node at each level except the last level. The rooted tree $T_X$ without the labels we will denote by $\overline{T}_X$.

The next theorem gives a description of finite ultrametric spaces for which the isomorphism of representing trees implies the weak similarity of the spaces.

**Theorem 3** ([2]). *Let $X$ be a finite ultrametric space. Then the following statements are equivalent.*

*(i) The implication $(\overline{T}_X\simeq\overline{T}_Y)\Rightarrow(X\overset{w}{\simeq}Y)$ holds for every finite ultrametric space $Y$.*
*(ii) $X\in\tilde{\mathfrak{R}}$.*

Denote by $\mathfrak{D}$ the class of all finite ultrametric spaces $X$ such that the different internal nodes of $T_X$ have the different labels. It is clear that $\tilde{\mathfrak{R}}$ is a subclass of $\mathfrak{D}$. A question arises whether there exist finite ultrametric spaces $X$, $Y\in\mathfrak{D}$ which do not belong to the class $\tilde{\mathfrak{R}}$ and for which the isomorphism of $\overline{T}_X$ and $\overline{T}_Y$ implies $X\overset{w}{\simeq}Y$.

Let us define a rooted tree $T$ with $n$ levels by the following two conditions:

(A) There is only one inner node at the level $k$ of $T$ whenever $k < n-1$.

(B) If $u$ and $v$ are different inner nodes at the level $n-1$ then the numbers of offsprings of $u$ and $v$ are equal.

Denote by $\mathfrak{T}$ the class of all finite ultrametric spaces $X$ for which $T_X$ satisfies conditions (A) and (B).

**Theorem 4** ([2]). *Let $X \in \mathfrak{D}$ be a finite ultrametric space. Then the following statements are equivalent.*

*(i) The implication $(\overline{T}_X \simeq \overline{T}_Y) \Rightarrow (X \overset{w}{=} Y)$ holds for every finite ultrametric space $Y \in \mathfrak{D}$.*

*(ii) $X \in \mathfrak{T}$.*

## References


[1] L. M. Blumenthal. *Theory and applications of distance geometry*, Oxford, Clarendon Press, 1953.

[2] E. Petrov. Weak similarities of finite ultrametric and semimetric spaces. *p-Adic Numbers, Ultrametric Analysis and Applications*, 10(2): 108–117, 2018.

[3] O. Dovgoshey, E. Petrov. Weak similarities of metric and semimetric spaces. *Acta Mathematica Hungarica* 141(4): 301–319, 2013.