Functions with three critical points on closed non-oriented 3-manifolds

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Let M be a closed smooth 3-manifold and $f, g : M \to \mathbb{R}$ be smooth functions.

Definition 1. Functions $f, g : M \to \mathbb{R}$ are called *topologically equivalent* if there are homeomorphisms $h : M \to M$ and $k : \mathbb{R} \to \mathbb{R}$ such that $f \circ h = k \circ g$. If k additionally preserve the orientation of $\mathbb{R}$, f and g are called *topologically conjugated* and homeomorphisms k, g by *conjugated*.

The problem of topological classification of Morse functions was solved in [1] and [2] for closed manifolds of different dimensions. The same result for arbitrary functions with isolated critical point on closed 2-manifolds was obtained in [3]. The relevance of this problem is contributed by the close connection with the Hamiltonian dynamical system's classification in dimensions 2 and 4. Local topological classification with isolated critical points and global topological classification with 3 critical points on oriented manifold were obtained in [4] The main issued of this research is to get similar results in non-oriented case.

It is known [3] that if p is an isolated critical point, $y = f(p)$, then there exists closed neighborhood $U(p)$ such that

$$f^{-1}(y) \cap U(p) = Con(\cup S^1_i).$$

Here $Con(\cup S^1_i)$ is a cone on a disjoint union of circles S^1_i, that is the union of two-dimensional disks, the centers of which are pasted together.

In order to describe the behavior of function in a neighborhood of critical point p we will construct a tree (graph without cycles) Gf_p. Let $U(p)$ be the neighborhood described above, which boundary is the sphere S^2 and $\partial(f^{-1}(y) \cap U(p)) = \cup S^1_i$ is the union of the embedded circles. To each component D_j of $S^2 \setminus \cup S^1_i$ we put in correspondence vertex v_j of the graph Gf_p and to each circle S^1_i we put an edge e_i. The vertex v_j is incident to e_i if the boundary of D_j contains S^1_i. Thus, v_i and v_j are connected by an edge if D_i and D_j are neighbor.

Theorem 2. *Let p and q be isolated critical points of smooth functions $f : \mathbb{R}^3 \to \mathbb{R}^1$ and $g : \mathbb{R}^3 \to \mathbb{R}^1$ correspondingly. Then there are neighborhoods U of p and V of q and homeomorphisms $h : U \to V$ and $k : \mathbb{R} \to \mathbb{R}$ such that $f \circ h = k \circ g$ if and only if graphs Gf_p and Gg_q are isomorphic.*

We construct a distinguishing graph Gf for the function f with 3 critical points on 3-manifold, such that it has the following properties:

1) The vertices of the graph are divided into four types: white, black, gray and non-colored. The number of vertices of each color (the first three types) is same. The non-colored vertices have degree 3. Each white vertex is equipped with the orientation number (+1 or −1).

2) If from the graph we remove vertices of one color and edges that incident to them, we obtain simply-connected graphs (tree) Gf'_i.

Theorem 3. *Let $f, g : M \to \mathbb{R}$ be smooth functions that have three critical points on a smooth closed 3-manifold. The functions f and g are conjugated if and only if their distinguishing graphs are equivalent.*

Example 4. The number of

1. (1) topologically non-equivalent and topologically non-conjugated functions defined on $S^1\tilde{\times}S^2$ equals 1;
2. (2) topologically non-equivalent functions with three critical points on $S^1\tilde{\times}S^2\sharp S^1\tilde{\times}S^2$ equals 16;
3. (3) topologically non-conjugated functions defined on $S^1\tilde{\times}S^2\sharp S^1\tilde{\times}S^2$ equals 24.

References

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