# A note on similarity of matrices

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Let $\mathbb{F}$ be a field of characteristic 0. Denote by $\mathbb{F}_{m\times n}$ the set of $m\times n$ matrices over $\mathbb{F}$ and by $\mathbb{F}_{m\times n}[x_1, x_2, , x_n]$ the set of $m\times n$ matrices over the polynomial ring $\mathbb{F}[x_1, x_2, , x_n]$. Denote by $GL(n, \mathbb{F})$ the group of invertible matrices in $\mathbb{F}_{n\times n}$. In what follows, we denote by $I_n$ the $n \times n$ identity matrix and by $0_{n,k}$ the zero $m \times n$ matrix. The Kronecker product of matrices $A = \left[\ a_{ij}\ \right] \in \mathbb{F}_{m\times n}$ and $B$ is denoted by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

Matrices $A \in \mathbb{F}_{n\times n}$ and $B \in \mathbb{F}_{n\times n}$ are said to be similar, if there is a nonsingular matrix $P \in \mathbb{F}_{n\times n}$ such that $A = PBP^{-1}$. Two tuples of $n \times n$ matrices over $\mathbb{F}$

$$\mathbf{A} = \{A_1, A_2, \dots, A_k\} \quad \text{and} \quad \mathbf{B} = \{B_1, B_2, \dots, B_k\}$$

are said to be simultaneously similar if there exists a matrix $U \in GL(n, \mathbb{F})$ such that

$$A_i = U^{-1}B_iU \qquad \text{for all} \qquad i = 1, 2, \dots, k.$$

The task of classifying square matrices up to similarity is one of the core and oldest problems in linear algebra (see [2], [4], [5], [9], [7] and references therein), and it is generally acknowledged that it is also one of the most hopeless problems already for $k = 2$. Standard approaches for deciding similarity depend upon the Jordan canonical form, the invariant factor algorithm and the Smith form, or the closely related rational canonical form. In numerical linear algebra, this leads to deep algorithmic problems, unsolved even up to this date, that are caused by numerical instabilities in solving eigenvalue problems or by the inability to effectively compute sizes of the Jordan blocks or degrees of invariant factors, if the matrix entries are not known precisely.

The purpose of this report is to give an exposition of geometric ideas to solution of similarity of matrices over a field. We propose new necessary and sufficient conditions under which matrices $A \in \mathbb{F}_{n\times n}$ and $B \in \mathbb{F}_{n\times n}$ are similar.

Chris Byrnes and Michael Gauger [6] derived a new type of rank conditions for algebraically deciding similarity of arbitrary pairs of matrices over a field $\mathbb{F}$. Their main result is that two matrices $A, B \in \mathbb{F}_{n\times n}$ are similar if and only if the following two conditions hold:

1. The characteristic polynomials coincide, i.e. $\det(I_nx - A) = \det(I_nx - B)$.
2. $\operatorname{rank}(I_n \otimes A - A \otimes I_n) = \operatorname{rank}(I_nB - B \otimes I_n) = \operatorname{rank}(B \otimes I_n - I_n \otimes A)$.

In subsequent work by J.D. Dixon [1] it was shown that the first condition on the characteristic polynomials is superfluous, so that similarity can be solely decided based on rank computations. He proved that two matrices $A, B \in \mathbb{F}_{n\times n}$ are similar if and only if

$$r^2(A, B) = r(A, A)r(B, B).$$

Here $r(A, B) = \operatorname{rank}(B \otimes I_n - I_n \otimes A)$ and similarly for $r(A, A)$ and $r(B, B)$.

Over an algebraically closed field, S. Friedland [3] showed the closely related linear dimension inequality

$$2 \dim \operatorname{Ker}(B \otimes I_n - I_n \otimes A) \le \dim \operatorname{Ker}(A \otimes I_n - I_n \otimes A) + \dim \operatorname{Ker}(B \otimes I_n - I_n \otimes B)$$

and proved that equality holds if and only if matrices $A$ and $B$ are similar.

For given two matrices $A, B \in \mathbb{F}_{n\times n}$ we define the matrix

$$M = \left[\begin{array}{c} A\otimes I_n - I_n\otimes B \end{array}\right] \in \mathbb{F}_{n^2\times n^2}.$$

For matrix $M$ there exist a matrix $W \in GL(n^2, \mathbb{F})$ such that

$$MW = H_M = \begin{bmatrix} 0_{l,1} & 0_{l,n^2-1} \\ H_1 & 0_{m_1,n^2-1} \\ H_2 & 0_{m_2,n^2-2} \\ \dots & \dots \\ H_k & 0_{m_k,r} \end{bmatrix}$$

is the lower block-triangular matrix and $l$ is the number of first zero rows of matrix $M$. The matrices $H_i$ are defined as follows:

$$H_1 = \begin{bmatrix} 1 \\ * \end{bmatrix} \in M_{m_1,1}(\mathbb{F}), \quad H_2 = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} \in M_{m_2,2}(\mathbb{F}), \quad H_r = \begin{bmatrix} 0 & \dots & 0 & 1 \\ * & * & * & * \end{bmatrix} \in M_{m_r,n^2-r}(\mathbb{F})$$

and $l + m_1 + m_2 + \dots + m_k = n^2$.

The lower block-triangular matrix $H_M$ is called the Hermite normal form of the matrix $M$ and the form $H_M$ is uniquely determined by $M$ (see [8]). It is evident that $\operatorname{rank} H_M = n^2 - r$.

Let $x_1, x_2, \dots, x_r$ be independent variables. Consider the vector

$$W \begin{bmatrix} 0_{n^2-r,1} \\ x_1 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} W_1(x_1,\dots,x_r) \\ W_2(x_1,\dots,x_r) \\ \vdots \\ W_n(x_1,\dots,x_r) \end{bmatrix} = \begin{bmatrix} W_1(\overline{x}) \\ W_2(\overline{x}) \\ \vdots \\ W_n(\overline{x}) \end{bmatrix} \in \mathbb{F}_{n^2,1}[x_1, x_2, \dots, x_r],$$

where $W_i(\overline{x}) = W_i(x_1,\dots,x_r) \in \mathbb{F}_{n,1}[x_1, x_2, \dots, x_r]$. Put

$$W(\overline{x}) = \begin{bmatrix} W_1(\overline{x}) & W_2(\overline{x}) & \dots & W_n(\overline{x}) \end{bmatrix} \in \mathbb{F}_{n,n}[x_1, x_2, \dots, x_r].$$

**Theorem 1.** *If matrices $A, B \in \mathbb{F}_{n\times n}$ are similar over $\mathbb{F}$ then* $\operatorname{rank} M \le n^2 - n$.

**Theorem 2.** *Matrices $A, B \in \mathbb{F}_{n\times n}$ are similar over $\mathbb{F}$ if and only if matrix $W(\overline{x})$ is nonsingular.*

**Corollary 3.** *If $A = B \in \mathbb{F}_{n\times n}$ then matrix $W(\overline{x})$ is nonsingular.*

## References

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