Asymptotic behavior of solutions to a nonlinear Beltrami equation

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Let D be a domain in $\mathbb{C}$ and $\mu\colon D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. in D. The linear PDE

$$f_{\overline{z}} = \mu(z) f_z. \tag{1}$$

is called the Beltrami equation; here $z = x + iy$,

$$f_{\overline{z}} = \frac{1}{2}(f_x + i f_y), \quad f_z = \frac{1}{2}(f_x - i f_y).$$

The function μ is called the complex dilatation.

Let $\sigma\colon D \to \mathbb{C}$ be a measurable function, and $m \geq 0$. We consider the following nonlinear equation

$$f_r = \sigma(re^{i\theta})\, |f_\theta|^m\, f_\theta, \tag{2}$$

written in the polar coordinates (r, θ). Here f_r and f_θ are the partial derivatives of f in r and θ, respectively, satisfying

$$r f_r = z f_z + \bar{z} f_{\bar{z}}\,,\ f_\theta = i(z f_z - \bar{z} f_{\bar{z}}).$$

The equation (2) in the Cartesian coordinates has the form

$$f_{\bar{z}} = \frac{A(z)|z f_z - \bar{z} f_{\bar{z}}|^m - 1}{A(z)|z f_z - \bar{z} f_{\bar{z}}|^m + 1}\, \frac{z}{\bar{z}}\, f_z\,, \tag{3}$$

where $A(z) = \sigma(z)\,|z|\,i$.

Note that in the case $m = 0$, the equation (2) is the usial Beltrami equation (1) with the complex dilatation

$$\mu(z) = \frac{z}{\bar{z}}\, \frac{A(z) - 1}{A(z) + 1}\,.$$

Let $\mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem 1. *Let $f\colon \mathbb{B} \to \mathbb{B}$ be a regular homeomorphic solution of the equation (2) which belongs to Sobolev class $W^{1,2}_{\mathrm{loc}}$, and normalized by $f(0) = 0$. Assume that the coefficient $\sigma : \mathbb{B} \to \mathbb{C}$ satisfies the following condition*

$$\liminf_{r \to 0} \left(\frac{1}{\pi r^2} \iint\limits_{|z|<r} \frac{dxdy}{|z|\,(\operatorname{Im}\overline{\sigma}(z))^{\frac{1}{m+1}}} \right)^{m+1} \leq \sigma_0 < \infty\,.$$

Then

$$\liminf_{z \to 0} \frac{|f(z)|}{|z|} \leq c_m\, \sigma_0^{\frac{1}{m}} < \infty\,,$$

where c_m is a positive constant depending on the parameter m.

References

[1] A. Golberg, R. Salimov and M. Stefanchuk. Asymptotic dilation of regular homeomorphisms. — arXiv:1805.00981v1. — 2018.